1 Consider the following vectors of $\mathbb{R}^4$

$$\mathbf{s} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} \ , \ \mathbf{t} = \begin{pmatrix} -3 \\ 1 \\ -7 \\ 5 \end{pmatrix} \ , \ \mathbf{u} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 5 \end{pmatrix} \ , \ \mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \ , \ \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \ , \ \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Define subsets U and V of $\mathbb{R}^4$ by

$$U = \mathbf{a} + \langle\{\mathbf{s}, \mathbf{t}\}\rangle_{\text{span}} \ , \ V = \mathbf{b} + \langle\{\mathbf{u}, \mathbf{v}\}\rangle_{\text{span}}$$

Here $\langle S\rangle_{\text{span}}$ denotes the subvectorspace generated by S for any subet S of $\mathbb{R}^4$, and for a vector $\mathbf{c} \in \mathbb{R}^4$, and a subvectorspace $W \subset \mathbb{R}^4$, $\mathbf{c} + W = \{\mathbf{c} + \mathbf{w} \mid \mathbf{w} \in W\}$.

(1) Find equations defining $\langle U\rangle_{\text{span}}$, i.e. find a system of equations of degree one whose solution set is $\langle U\rangle_{\text{span}}$.

(2) Find all $\mathbf{x} \in V$ satisfying $\langle\{\mathbf{x}\}\rangle_{\text{span}} \cap U \neq \emptyset$.

2 Let $\langle \cdot, \cdot \rangle$ be the canonical Euclidean inner product of $\mathbb{R}^3$, and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be a normal orthogonal basis with respect to this inner product. Define the linear map $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ by

$$f(\mathbf{x}) = \mathbf{x} - \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_1 - \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_2.$$

Answer the following questions:

(1) Find the representing matrix of f with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(2) Find the eigenvalues and eigenvectors of the linear map f.

(3) Show that the representing matrix A of the linear map f with respect to the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is symmetric.

(4) Find the eigenvalues and eigenvectors of A.

3 Answer the following questions.

(1) Find the limit

$$\lim_{x \to 0} (\cos x)^{1/x^2}.$$

(2) Let $D = \{(x, y) | x \geq 0,\ y \geq 0,\ x^3 + y^3 \leq 1\}$. Find the value of the integral

$$\iint_D x^8 y^5 \, dx dy.$$

4

(1) Express the Taylor expansion of the function $f(x, y) = \log(x^2+y^2)$ at $x = y = 1$ in the form

$$f(x, y) = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) + R_2(x, y)$$

(Express the error term $R_2(x, y)$ in a suitable form).

(2) Draw the shape of the surface $z = \log(x^2 + y^2)$.

(3) Let $x = y = 1$. Determine one effective digit of the change of $z = \log(x^2 + y^2)$ if x increases by 0.003 and y decreases by 0.002. Also, explain the estimate of the error term on which your calculation is based.