1 Let a be a real number, and $A = A_a$ be the matrix

$$A_a = \begin{pmatrix} a & 2-2a & -2+2a \\ -1-a & -3+2a & 2-2a \\ -1-a & -3+a & 2-a \end{pmatrix}.$$

(1) Show that A is diagonalizable if $a = 1$.

(2) Determine the Jordan normal form of A for $a \neq 1$.

(3) Let $\langle\ ,\ \rangle$ be the canonical Euclidean inner product. Determine for which a the sequence $\{\langle \mathbf{x}, A^n \mathbf{y}\rangle\}_{n=1}^{\infty}$ is bounded for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

2 Let $f(x,y)$ be a C^2-function, and $p(x,y)$ be a polynomial of degree at most 2. Assume that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-p(x,y)}{x^2+y^2} = 0$$

(1) Express $p(x,y)$ in terms of f and its (higher) partial derivatives (the answer suffices).

(2) Calculate $\frac{1}{2\pi}\int_0^{2\pi} p(r\cos\theta,\, r\sin\theta)\, d\theta$ for fixed $r > 0$.

(3) Show that if $f(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta,\, r\sin\theta)\, d\theta$ holds for every $r > 0$, then $f_{xx}(0,0) + f_{yy}(0,0) = 0$.

3 Given a natural number $n \geq 2$ and a real number $R > 0$, let $\gamma_{1,R} = [0, R]$, $\gamma_{2,R} = \{Re^{i\theta} \,|\, 0 \leq \theta \leq 2\pi/n\}$, $\gamma_{3,R} = \{re^{2\pi i/n} \,|\, 0 \leq r \leq R\}$. Let γ_R be the closed oriented curve given by running through $\gamma_{1,R}$, $\gamma_{1,R}$, and $\gamma_{1,R}$ counterclockwise.

(1) For $R > 1$, calculate the integral $\displaystyle\int_{\gamma_R} \frac{dz}{z^n + 1}$.

(2) Show that $\displaystyle\lim_{R\to\infty} \int_{\gamma_{2,R}} \frac{dz}{z^n + 1} = 0$.

(3) Calculate the integral $\displaystyle I_n = \int_0^\infty \frac{dx}{x^n + 1}$.

(4) Calculate $\displaystyle\lim_{n\to\infty} I_n$.

4 Given real numbers a_0, a_1, a_2 and a positive real number a_3 let $f(x,y) = f_{a_0,a_1,a_2,a_3}(x,y)$ be the polynomial

$$f(x,y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3.$$

Let $C = C(a_0, a_1, a_2, a_3)$ be the subset of $\mathbb{R}^2$ given by

$$C = \{(x,y) \,|\, x, y \geq 0,\ f(x,y) = 1\}$$

(1) If C is bounded, then the function $x^2 + y^2$ achieves its maximum on C. Explain why.

(2) Let $\alpha \geq 0$. Determine, for which $a_0, a_1, a_2, a_3, \alpha$, the set C intersects the line $y = \alpha\, x$. Determine the intersection point(s) if any exists.

(3) Let $\mathcal{L}$ be the set of all lines $y = \alpha\, x$ with positive slope $\alpha \geq 0$. Show that C intersects every line $L \in \mathcal{L}$ if and only if C is bounded.

(4) Let A be the subset $\mathbb{R}^4$ consisting of those (a_0, a_1, a_2, a_3) such that C is bounded. Show that A is an open subset of $\mathbb{R}^4$.