1 Consider the following three functions $u_1(t) = e^t$, $u_2(t) = t\,e^t$, $u_3(t) = \dfrac{t^2}{2}\,e^t$ defined on $\mathbb{R}$.

(1) Let V be the real vector space V of real valued C^∞-functions on $\mathbb{R}$. Show that $\{u_1(t),\ u_2(t),\ u_3(t)\}$ are linear independent as elements V.

(2) Let W be the $\mathbb{R}$-subvectorspace of V generated by $u_1(t)$, $u_2(t)$, $u_3(t)$. Verify that $\dfrac{d}{dt}$ is a linear map from W to W, and calculate the representing matrix A with respect to the basis $\{u_1(t), u_2(t), u_3(t)\}$.

(3) Prove that the solution space of the differential equation $\dfrac{d^3u}{dt^3} - 3\dfrac{d^2u}{dt^2} + 3\dfrac{du}{dt} - u = 0$ contains the 3-dimensional vector space spanned by $u_1(t)$, $u_2(t)$, $u_3(t)$.

(4) Prove that if $u(t) = C(t)e^t$ is a solution of the differential equation $\dfrac{d^3u}{dt^3} - 3\dfrac{d^2u}{dt^2} + 3\dfrac{du}{dt} - u = 0$, then $C(t)$ is a polynomial of degree at most 2.

(5) Determine the space of solutions of the differential equation $\dfrac{d^3u}{dt^3} - 3\dfrac{d^2u}{dt^2} + 3\dfrac{du}{dt} - u = 0$.

2 Define the functions ϕ_n $(n = 1, 2, \dots)$ on $[0, \infty)$ by $\phi_n(x) = n^2 x e^{-nx}$.

(1) Calculate $\displaystyle\int_0^\infty \phi_n(x)\,dx$.

(2) Show that, for any $\delta > 0$, the functions $\{\phi_n\}$ converge uniformly to 0 on $[\delta, \infty)$.

(3) Show that for any bounded, continuous function f on $[0, \infty)$, $\displaystyle\lim_{n\to\infty} \int_0^\infty f(x)\phi_n(x)\,dx = f(0)$ holds.

3 Answer the following questions

(1) Assume that the function $f(z)$ is regular on a domain containing the disk $D_R = \{|z \in \mathbb{C}, |z| \leq R\}$. Prove that if $z \in \mathbb{C}$ lies in the disc D_R, then

$$f'(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

(2) Use (1) to prove that a regular function $f(z)$, which is bounded on the whole complex plane, satisfies $f'(z) \equiv 0$.

(3) Determine the subset of the z-plane which maps under the regular function $w = e^z$ to the domain $\{w \in \mathbb{C} \,|\, |w| < a\}$ $(a > 0)$ of the w-plane, and graph it.

(4) Show that a regular function defined on the whole complex plane whose real part is non-positive is a constant function.

4 For a subset M of $\mathbb{R}^n$ and a point x of $\mathbb{R}^n$ define

$$d(x, M) = \inf\{|x - y| \mid y \in M\}.$$

Here $|x|$ is the Eucidean norm, i.e. for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define $|x| = \sqrt{\sum_{i=1}^n x_i^2}$.

(1) Show that $d(x, M) = 0$ is a necessary and sufficient condition for $x \in \overline{M}$.

(2) Show that $d(x, M) \leq |y - z| + |x - y|$ for any two points x, y in $\mathbb{R}^n$, and any point z in M.

(3) Show that for fixed M, the function $x \mapsto d(x, M)$ is continuous on $\mathbb{R}^n$.

(4) Show that if M is closed, then for any $x \in \mathbb{R}^n$ there is a $y^* \in M$ such that

$$|x - y^*| = d(x, M) \ .$$