1 For a real number t, let A and B be the matrices

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -3 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2+t & 4+2t \\ 1 & 1 & -1 \\ -2 & -2+t & 2+2t \end{pmatrix}.$$

By multiplying column vectors from the left with matrices, view A as a linear map from $\mathbb{R}^4$ to $\mathbb{R}^3$, and B as a linear map from $\mathbb{R}^3$ to $\mathbb{R}^4$.

(1) Determine the dimensions of the kernel $\operatorname{Ker} A$ of A and the image $\operatorname{Im} B$ of B.

(2) Determine the dimension of the subspace $\operatorname{Ker} A + \operatorname{Im} B$ of $\mathbb{R}^4$.

(3) Determine the set of t for which $\operatorname{Ker} A = \operatorname{Im} B$.

2 For an integer $n \geq 2$, let A be a degree n symmetric matrix and $b \in \mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ be the canonical inner product on $\mathbb{R}^n$ and f the function on $\mathbb{R}^n$ given by

$$f(x) = \frac{1}{2}\langle Ax, x \rangle - \langle b, x \rangle \qquad (x \in \mathbb{R}^n)$$

(1) Show that if f has an extremum at the point $x^* \in \mathbb{R}^n$, then $Ax^* = b$.

(2) Show that if all eigenvalues of A are positive, then f has exactly one minimum.

3 Answer the following questions:

(1) Find the minima and maxima of the function $f(x, y) = xy$ on the set of real numbers x, y satisfying $x^2 + xy + y^2 - 1 = 0$.

(2) Find the Taylor series of the function $f(x) = \sqrt{1 + \frac{1}{x}}$ at $x = 1$ up to the term $(x - 1)^2$.

4 Answer the following questions:

(1) Use integration by parts to show that for $q > p > 0$,

$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| \leq \frac{2}{p}.$$

(2) Show that the improper integral

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

converges.

(3) Determine if the improper integral

$$\int_0^\infty \frac{|\sin x|}{x}\, dx$$

converges. Give a reason.