Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2012 Admission

Part 1 of 2

Tuesday, February 7, 2012, 9:00 a.m.∼12:00 noon

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let $f_A \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $f_A(x) = Ax$, where A is the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

Please answer the following questions.

(1) Find a basis for the kernel $\operatorname{Ker} f_A$ and a basis for the image $\operatorname{Im} f_A$.

(2) Find a 3×3-matrix P which satisfies that for all $x \in \operatorname{Ker} f_A$, $Px = 0$ and for all $x \in \operatorname{Im} f_A$, $Px = x$.

2 Decide if each of the following statements is true or false. If true, then please give a proof. If false, then please give a counterexample and prove that this is so.

(1) Let P and Q be planes in the Euclidean space $\mathbb{R}^3$ and suppose that an invertible linear transformation f of $\mathbb{R}^3$ maps P to Q. Then f maps a normal vector to the plane P to a normal vector to the plane Q.

(2) Suppose that all eigenvalues of a real 2×2-matrix A have absolute value at most 1. Then, for all $x \in \mathbb{R}^2$, the subset $\{A^n x \mid n = 1, 2, \ldots\}$ of $\mathbb{R}^2$ is bounded.

(3) Suppose that $\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\}$ is a linearly independent subset of the complex vector space that consists of all complex 2×2-matrices. Then the subset $\left\{ \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} \right\}$ of $\mathbb{C}^2$ is linearly independent.

3 Please answer the following questions.

(1) Find the value of the integral

$$\int_1^\infty \left\{ \left(\log \frac{x}{x+1} \right) + \frac{1}{x+1} \right\} dx.$$

(2) Show that if $0 < \theta < \frac{\pi}{2}$, then $2\theta < \sin\theta + \tan\theta$.

(3) Consider the following system of differential equations with $f(x, y)$ a function of class C^1 defined over $\mathbb{R}^2$:

$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0, \qquad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$$

Express the system of differential equations in polar coordinates and find all solutions.

4 Let D be the subset of the plane defined by

$$D = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x + y \leq 2, \ \ -1 \leq xy \leq 0\}.$$

Please answer the following questions.

(1) Make a sketch of the subset D.

(2) Find the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ associated with the change of variables $u = x + y$ and $v = xy$, where $x, y \in \mathbb{R}$.

(3) Find the value of the double integral

$$\iint_D |x^3 - y^3| \, dxdy.$$