1 Let V be a finite dimensional complex vector space, let $f\colon V \to V$ and $g\colon V \to V$ be linear maps, and assume that g is a bijection and that the identity $g \circ f \circ g^{-1} \circ f = \mathrm{id}_V$ holds. Here id_V is the identity map of V. Please answer the following questions.

(1) Show that f is a bijection.

(2) Suppose that λ is an eigenvalue of f. Show that λ is non-zero and that λ^{-1} is an eigenvalue of f.

(3) Suppose that $\dim V = 3$ and that f has an eigenvalue $\mu \neq \pm 1$. Find the Jordan canonical form of f. In addition, find the matrix that represents g with respect to the basis for V determined by the Jordan canonical form of f.

2 Let $f(x, y) = x^3 + y^3 - 3xy$ $(x, y \in \mathbb{R})$, and let C be the planar curve defined by the equation $f(x, y) = 0$. Please answer the following questions.

(1) Find the parametrization of C with respect to the parameter t given by the intersection of C and the line $y = tx$.

(2) Find the area of the region enclosed by the part of C that corresponds to the parameter values $0 \leq t < \infty$. (This region is equal to the bounded connected component of the complement of C.)

Hint: The area of the region enclosed by the closed curve C_0 oriented counter-clockwise is given by the line integral $\int_{C_0} x dy = -\int_{C_0} y dx$.

3 Please answer the following questions:

(1) Let $\zeta \leq 0$ be a real number and consider the meromorphic function

$$f(z) = \frac{\exp(-i\zeta z)}{1+z^2}, \quad z \in \mathbb{C}.$$

Let C be the closed path that consists of the interval $C_1 = [-R, R]$ in the real axis and the half-circle $C_2 = \{Re^{i\theta} \mid 0 \leq \theta \leq \pi\}$. The path C is given the counter-clockwise orientation and it is assumed that $R > 1$. Show that

$$\int_C f(z)dz = \pi \exp(\zeta).$$

(2) Show that it $\zeta \leq 0$, then $\displaystyle\int_{-\infty}^{\infty} \frac{\exp(-i\zeta t)}{1+t^2} dt = \pi \exp(\zeta)$.

(3) Find the value of $\displaystyle\int_{-\infty}^{\infty} \frac{\exp(-i\zeta t)}{1+t^2} dt$ for $\zeta > 0$.

4 Let $\| \quad \| \colon \mathbb{R}^2 \to [0, \infty)$ be the standard length function. Please answer the following questions.

(1) Show that the image by $\| \quad \|$ of an open subset of $\mathbb{R}^2$ is an open subset of $[0, \infty)$.

(2) Show that the image by $\| \quad \|$ of a closed subset of $\mathbb{R}^2$ is a closed subset of $[0, \infty)$.