1 Let $V \subset \mathbb{R}^4$ be the subspace spanned by the vectors $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$, and let $W_t \subset \mathbb{R}^4$ be the subspace spanned by the vectors $\begin{pmatrix} t+1 \\ t-1 \\ t+1 \\ t-1 \end{pmatrix}$, $\begin{pmatrix} t+2 \\ t \\ t+1 \\ t+1 \end{pmatrix}$, $\begin{pmatrix} t+3 \\ t+1 \\ t+2 \\ t+1 \end{pmatrix}$. Here t is a real number. Please answer the following questions.

(1) Solve the system of linear equations

$$\begin{pmatrix} a & b & c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

with respect to the real numbers a, b, c, d.

(2) Find a system of linear equations whose space of solutions is equal to V.

(3) Find a system of linear equations whose space of solutions is equal to W_t.

(4) Find a basis for the subspace $V \cap W_t$ and find its dimension.

2 Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the complex linear map defined by $T(v) = Av$, where

$$A = \begin{pmatrix} -5 & 6 & 2 \\ -8 & 8 & 1 \\ 8 & -6 & 1 \end{pmatrix}.$$

Please answer the following questions.

(1) Find the eigenvalues and eigenvectors for T.

(2) Let $V_\lambda \subset \mathbb{C}^3$ be the eigenspace corresponding to the eigenvalue λ for T. Show that $\mathbb{C}^3$ decomposes as the direct sum of the subspaces V_λ.

(3) Let $v_\lambda \in V_\lambda$ be the summand of $v \in \mathbb{C}^3$ corresponding to the eigenvalue λ, and let $p_\lambda\colon \mathbb{C}^3 \to \mathbb{C}^3$ be the linear map defined by $p_\lambda(v) = v_\lambda$. For each eigenvalue λ, find the matrix that represents p_λ with respect to the standard basis of $\mathbb{C}^3$.

3 Please answer the following questions.

(1) Find the 4th order Taylor expansion at the origin and with respect to x, y of the two-variable function $\cos\left(\dfrac{x}{1+y^2}\right)$.

(2) Decide whether or not the function $f(x,y) = x^2 + 2x^2y - xy^2$ $(x, y \in \mathbb{R})$ has any extrema.

(3) Find the value of the integral

$$\iint_D \frac{1}{1+(x^2+y^2)^2}\, dxdy,$$

where D is the planar region defined by

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq y\}.$$

4 Let a, b, n be positive integers with $a < b$ and define

$$S_n(a,b) = \frac{1}{an+1} + \frac{1}{an+2} + \cdots + \frac{1}{bn-1} + \frac{1}{bn}.$$

Please answer the following questions.

(1) Show that $\log \dfrac{bn+1}{an+1} < S_n(a,b) < \log \dfrac{b}{a}$.

(2) Find the value of $S(a,b) = \lim_{n\to\infty} S_n(a,b)$.

(3) With $S(a,b)$ as in (2), decide whether the series

$$\sum_{k=1}^{\infty} S(k^2, k^2+1)$$

converges or diverges.