1 Let V be a complex vector space of dimension n, let $I\colon V \to V$ be the identity map, and let $O\colon V \to V$ be the zero map. Please answer the following problems.

(1) Let $p, q\colon V \to V$ be linear maps that satisfy

$$p + q = I$$

$$p^2 = p, \;\; q^2 = q.$$

Show that

$$pq = qp = O.$$

Show, in addition, that V is the direct sum of the subspaces $U = \ker(p - I)$ and $W = \ker(q - I)$.

(2) Suppose that the linear map $f\colon V \to V$ is diagonalizable, but that f only has two different eigenvalues α and β. Show that there exists p, q that satisfy the conditions in (1) such that $f = \alpha p + \beta q$.

2 Let A be a real symmetric $n \times n$ matrix all of whose eigenvalues are positive, and let Q_A be the function defined by

$$Q_A(x_1, \ldots, x_n) = (x_1 \ldots x_n) A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Given a real number α, consider the improper integral

$$\int_{0<x_1^2+\cdots+x_n^2\leq 1} \frac{Q_A(x_1, \ldots, x_n)}{(x_1^2 + \cdots + x_n^2)^{\frac{\alpha}{2}}} dx_1 \cdots dx_n.$$

Please answer the following problems.

(1) For $n = 2$ and $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, determine all α for which the improper integral above converges and find the value of the integral.

(2) For general n, determine all α for which the improper integral above converges.

3 Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^2} < \infty.$$

Please answer the following problems.

(1) Show that the sequence $\{\alpha_n\}_{n=1}^{\infty}$ has no accumulation point in the complex plane.

(2) Show that if $|z| < \frac{1}{3}$, then

$$|\log(1+z) - z| \leq 2|z|^2.$$

Here $\log(1+z)$ is the principal value of the logarithm.

(3) Show that for all $z \in \mathbb{C}$,

$$f(z) = \prod_{n=1}^{\infty} (1 - \frac{z}{\alpha_n}) e^{\frac{z}{\alpha_n}}$$

converges, and that this defines a holomorphic function on all of $\mathbb{C}$.

4 Please answer the following problems.

(1) Let x be a real number. Show that the double limit

$$\lim_{m\to\infty}\big(\lim_{n\to\infty}(\cos(\pi\, m!\, x))^{2n}\big)$$

exists.

(2) Let $f(x)$ be a real valued function defined on $\mathbb{R}$. State the ϵ-δ definition of what it means for $f(x)$ *not* to be continuous at $x = a$.

(3) Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \lim_{m\to\infty}\big(\lim_{n\to\infty}(\cos(\pi\, m!\, x))^{2n}\big).$$

Using the ϵ-δ definition of continuity, decide whether or not $g(x)$ is continuous on all of $\mathbb{R}$.