1 Let $V \subset \mathbb{R}^4$ be the subspace generated by the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

and for every real number t, let $W \subset \mathbb{R}^4$ be the subspace generated by the three vectors

$$\begin{pmatrix} t+4 \\ t \\ t+2 \\ t \end{pmatrix}, \qquad \begin{pmatrix} t+3 \\ t \\ t+1 \\ t+1 \end{pmatrix}, \qquad \begin{pmatrix} t+2 \\ t \\ t+2 \\ t+2 \end{pmatrix}$$

Please answer the following problems.

(1) Find the dimension of W.

(2) Find the dimension of $V + W$.

(3) Find the dimension of $V \cap W$.

2 Consider the set

$$V = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R},\ a + d = 0\}$$

as a real vector space with respect to matrix sum and scalar product. Please answer the following problems.

(1) Show that $\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$ is a basis of V.

(2) Let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ be an element of V. Show that the formula

$$F_A(X) = AX - XA$$

defines a linear map $F_A \colon V \to V$. In addition, find the matrix that represents F_A with respect to the basis in (1).

(3) For $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$, find the eigenvalues and eigenvectors of F_A.

3 Please answer the following problems.

(1) Find the value of the double integral

$$\int_D xy\,dx\,dy,$$

where $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ \sqrt{x} + \sqrt{y} \leq 1\}$.

(2) Let $z = f(x, y)$ be the real valued function of class C^1 defined implicity by the equation

$$x^2y + yz + z^3x = 3$$

on the open disc of radius $\frac{1}{2}$ centered at $(1, 1)$. Find the partial derivatives

$$\frac{\partial f}{\partial x}(1, 1), \quad \frac{\partial f}{\partial y}(1, 1)$$

of $f(x, y)$ at $(x, y) = (1, 1)$.

(3) Find the Taylor expansion of the function $g(x) = \dfrac{1}{\cos x}$ to the 4th order around $x = 0$.

4 Let a be a real number and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = xy + ay^2 - x^3.$$

Find the maximum value of $f(x, y)$ on $\mathbb{R}^2$ and find the point(s) where this maximum value is attained.