1 Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & -1 \\ 2 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix}$$

and let $f_A : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear map defined by A. Please answer the following questions.

(1) Find the dimension of the image $U = f_A(\mathbb{R}^3)$. Show your work.

(2) Let $(\mathbf{x}, \mathbf{u})$ denote the inner product of $\mathbf{x}$ and $\mathbf{u}$ in $\mathbb{R}^4$. Find a basis of the subspace $U^\perp = \{\mathbf{x} \in \mathbb{R}^4 |\ (\mathbf{x},\ \mathbf{u}) = 0\ (\forall \mathbf{u} \in U)\}$.

(3) For given real numbers a, b, define $\mathbf{c} = \begin{pmatrix} 1 \\ a \\ -b \\ 1 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} a \\ 0 \\ b - 3a \\ 2 \end{pmatrix}$. Find all a, b such that U, $\mathbf{c}$, and $\mathbf{d}$ span $\mathbb{R}^4$.

2 In this problem, all matrices and vectors have real entries. The norm of the vector $\mathbf{v}$ is denoted by $||\mathbf{v}||$. Please answer the questions below.

(1) Find all eigenvalues and eigenvectors of the following matrix. Eigenvectors should be given as unit vectors with non-negative first component.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$$

(2) Show that if $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{x} \neq \mathbf{0}$, then $A\mathbf{x} \neq \mathbf{0}$.

(3) Let $\lambda_1, \lambda_2, \lambda_3$ ($\lambda_1 > \lambda_2 > \lambda_3$) be the eigenvalues and $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ the corresponding eigenvectors found in (1), and for $\mathbf{x} \neq \mathbf{0}$, let

$$F(\mathbf{x}) = \frac{A\mathbf{x}}{||A\mathbf{x}||}.$$

Given $a_1,\ a_2,\ a_3 \in \mathbb{R} \setminus \{0\}$, let $\{\mathbf{x}_n\}_{n=0}^{\infty}$ be the sequence of vectors recursively defined by

$$\mathbf{x}_0 = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n) \quad (n \ \geq 0).$$

Express the coefficients in writing $\mathbf{x}_n$ as a linear combination of $\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$ in terms of $a_1,\ a_2,\ a_3$.

(4) Show that the limit $\lim_{n\to\infty} \mathbf{x}_n$ of the sequence $\{\mathbf{x}_n\}_{n=0}^{\infty}$ in (3) exists and is an eigenvector for A.

3 Please answer the following questions.

(1) Find the 3rd order Taylor expansion at the origin and with respect to x, y of the two-variable function $(1 + x \sin y)^{-1}$.

(2) Let $f(x, y, z) = \dfrac{3}{1+x^2} \log(1+e^y+z^2) - y \quad (x, y, z \in \mathbb{R})$, let a be a real number, and define

$$F(t) = f(a\cos t, a\sin t, t^2) \quad (t \in \mathbb{R}).$$

Find all a for which $F'(0) = 0$.

(3) Find the value of the double integral

$$\iint_D \frac{|x|}{(x^2+y+1)^2}\, dxdy$$

over the region $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le x^2 \le 1\}$ in the plane.

4 Let $f(x, y)$ be the function defined on $\mathbb{R}^2$ by the formula

$$f(x, y) = e^{-x^2-y^2}(1 - x - y).$$

Please answer the following questions.

(1) A critical point for $f(x, y)$ is a point (a, b) such that $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. Find all critical points for $f(x, y)$.

(2) If $R > 0$, and if $D_R = \{(x, y) |\ x^2 + y^2 \leq R^2\}$ is the closed disc of radius $\mathbb{R}$, then show that the maximum value and minimum value of $f(x, y)$ on D_R exist. Show also that for R sufficiently large, these also the maximum value and minimum value of $f(x, y)$ on $\mathbb{R}^2$.

(3) Find the maximum value and minumim value of $f(x, y)$ on $\mathbb{R}^2$ and find the points where these are attained.