1 With respect to the matrices

$$A = \begin{pmatrix} 1 & 2 & -1 & -2 & 2 \\ -1 & -2 & 2 & 3 & -2 \\ -1 & -2 & 0 & 1 & -2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 3 & 2 & 0 & 3 \\ 1 & 4 & 4 & 1 & 4 \end{pmatrix}$$

we define the linear maps $T_A : \mathbb{C}^5 \to \mathbb{C}^3$ and $T_B : \mathbb{C}^5 \to \mathbb{C}^2$ between complex linear spaces, such that $T_A(\boldsymbol{x}) = A\boldsymbol{x}$ and $T_B(\boldsymbol{x}) = B\boldsymbol{x}$. Here,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{C}^5.$$

Their respective kernels are $V = \operatorname{Ker} T_A$ and $W = \operatorname{Ker} T_B$.

Answer the following questions:

(1) Calculate $\dim V$ and $\dim W$.

(2) For each of $V \cup W$ and $V + W$, decide whether it is a subspace of $\mathbb{C}^5$.

(3) For the subspaces found in (2), calculate their dimensions.

2 We define

$$V = \{p + qx + rx^2 \mid p, q, r \in \mathbb{R}\}$$

as the linear space over $\mathbb{R}$ composed of all real polynomials in x of degree 2 or less. With respect to $f = p + qx + rx^2 \in V$ $(p, q, r \in \mathbb{R})$, we define its norm as

$$\|f\| = \sqrt{p^2 + q^2 + r^2}.$$

We consider a linear transformation $T : V \to V$ such that, for some real number a, we have

$$T(1) = 1 + x, \qquad T(x) = -1 + x^2, \qquad T(1 + ax - x^2) = 0.$$

A is the representation matrix of T with respect to the basis $1, x, x^2$.

Answer the following questions:

(1) Calculate A.

(2) Find all the eigenvalues of A, and their respective eigenspaces.

(3) Using a, express a necessary and sufficient condition for A to be diagonalizable.

(4) We suppose that A is diagonalizable. $T^n : V \to V$ being $T : V \to V$ composed n times, find the dimension, as a subspace of V, of $W = \{f \in V \mid \lim_{n\to\infty} \|T^n(f)\| = 0\}$.

3 Answer the following questions:

(1) Applying the Taylor theorem to $f(x) = \log(\cos x)$, prove the following inequality when $|x| \leq \dfrac{\pi}{4}$:

$$\left| f(x) + \frac{x^2}{2} \right| \leq \frac{2}{3}|x|^3.$$

(2) Let $f(x, y)$ be a function over $\mathbb{R}^2$ of class C^2. The function

$$g(x, t) = \int_0^t f(x, y)\, dy$$

over $\mathbb{R}^2$ is known to be of class C^2. Let $F(x) = g(x, x)$. Find the constants a, b, c such that the following equation is true:

$$F''(x) = a\, f_x(x, x) + b\, f_y(x, x) + c\, g_{xx}(x, x).$$

(3) Calculate the following integral:

$$\int_0^{\pi} \left(\int_0^{\sqrt{\pi^2 - x^2}} \sin \sqrt{x^2 + y^2}\, dy \right) dx.$$

4 We define the function $f(x, y)$ of two variables as

$$f(x, y) = 3x^4 + 5y^3 - 3x^2y^2 - 6y^2.$$

Find the extrema of $f(x, y)$.