1 Let V be a linear space of dimension n over $\mathbb{C}$. We assume that the linear transformation f over V has distinct eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$, with respective eigenspaces $W_1, W_2, \ldots, W_n$. I being the identity transformation over V, we define the linear transformations $f_j = f - \alpha_j I$ $(j = 1, 2, \ldots, n)$. The composition of all these transformations except f_j is $g_j = f_1 \cdots f_{j-1} f_{j+1} \cdots f_n$.

Answer the following questions:

(1) Prove that the restrictions of f_j to each of the W_k's $(k = 1, 2, \ldots, n)$ are linear transformations over W_k.

(2) Prove that $g_j \neq 0$.

(3) Let $\operatorname{Im} g_j$ be the image of g_j. Prove that $\operatorname{Im} g_j \subset W_j$.

(4) Prove that $\operatorname{Im} g_j = W_j$.

2 Let $f(x)$ be a continuous function over $\mathbb{R}$, null outside of a compact set $K \subset \mathbb{R}$, and $\varphi(x)$ be a positive continuous function over $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1.$$

For some $t \in \mathbb{R}$ ($t \neq 0$), we define

$$g_t(x) = \int_{-\infty}^{\infty} \varphi_t(x-y) f(y)\, dy \qquad \text{and} \qquad \varphi_t(x) = \frac{1}{|t|}\varphi\Big(\frac{x}{t}\Big).$$

Answer the following questions:

(1) Show that $f(x) = \displaystyle\int_{-\infty}^{\infty} \varphi_t(x-y) f(x)\, dy$.

(2) Show that, for any $\delta > 0$,

$$\lim_{t\to 0} \int_{|x|\geq\delta} \varphi_t(x)\, dx = 0.$$

(3) $f(x)$ is bounded over $\mathbb{R}$, and uniformly continuous. Explain why it is so.

(4) Let $\|g_t - f\| = \sup_{x\in\mathbb{R}} |g_t(x) - f(x)|$. Prove that

$$\lim_{t\to 0} \|g_t - f\| = 0.$$

3 Let Γ be the circle in the complex plane of center at the origin, and radius 1. It is directed counter-clockwise. Answer the following questions:

(1) When α is a complex number such that $|\alpha| < 1$, calculate the integral

$$\int_{\Gamma} \frac{\log(2+z)}{(z-\alpha)(z-\alpha^{-1})}\, dz.$$

For $z \neq 0$, we select the value of $\log z = \log|z| + i \arg z$ such that $-\pi < \arg z \leq \pi$.

(2) Representing Γ as $z = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, calculate the integral

$$\int_0^{2\pi} \frac{\log(5+4\cos\theta)}{5+4\cos\theta}\, d\theta.$$

4 Answer the following questions:

(1) Let V and W be 2 sets, and $f : V \to W$ and $g : W \to V$ functions between them. We assume that their composition $g \circ f : V \to V$ is the identity function. For each of the following statements, prove the statement when true, or give a counter-example when false:

(a) f is injective.

(b) f is surjective.

(c) g is injective.

(d) g is surjective.

(2) For any non-empty subset E of $\mathbb{R}$, $k \in \mathbb{R}$ is an upper bound of E when, for any $x \in E$, the inequality $x \leq k$ is true. The smallest upper bound of E is called its least upper bound. For any non-empty subset M of $\mathbb{R}$, we define $M' = \{x + 1 \mid x \in M\}$. For each of the following statements, prove the statement when true, or give a counter-example when false:

(a) When α is an upper bound of M, $\alpha + 1$ is an upper bound of M'.

(b) When α is the least upper bound of M, $\alpha + 1$ is the least upper bound of M'.