Minimal transitive factorizations of a permutation of type \((p, q)\)

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Given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$,

$$\alpha_\lambda = (1\ldots\lambda_1) (\lambda_1 + 1\ldots\lambda_1 + \lambda_2) \ldots (n - \lambda_\ell + 1\ldots n).$$

$\mathcal{F}_\lambda :=$ the set of all $(n + \ell - 2)$-tuples $(\eta_1, \ldots, \eta_{n+\ell-2})$ of transpositions such that

1. $\eta_1 \cdot \cdot \cdot \eta_{n+\ell-2} = \alpha_\lambda$
2. $\langle \eta_1, \ldots, \eta_{n+\ell-2} \rangle = S_n$.

Such tuples are called minimal transitive factorizations of $\alpha_\lambda$, which are related to the branched covers of the sphere suggested by Hurwitz.
Table: The elements of $F_{(4)}$ where $\alpha_{(4)} = (1\ 2\ 3\ 4)$

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Question.

Find the cardinality of $\mathcal{F}_\lambda$.

1. Goulden and Jackson (1997) proved that if $(\lambda_1, \ldots, \lambda_\ell) \vdash n$,

$$|\mathcal{F}(\lambda_1, \ldots, \lambda_\ell)| = (n + \ell - 2)! n^{\ell-3} \prod_{i=1}^\ell \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}.$$ 

Their proof is done for arbitrary $\lambda$, but this is not bijective.

2. Bousquet-Mélou and Schaeffer (2000) obtained the above formula by using the inclusion-exclusion principle.
## Known bijective proofs

| $\lambda$   | $|\mathcal{F}_\lambda|$ | Bijective proof            |
|-------------|---------------------------|-----------------------------|
| $(n)$       | $n^{n-2}$                 | Biane et al.                |
| $(1, n - 1)$| $(n - 1)^n$               | Kim-Seo (2003)              |
| $(2, n - 2)$| $4(n - 1)(n - 2)^{n-1}$   | Seo(2004), Rattan (2006)   |
| $(3, n - 3)$| $\frac{27}{2}(n - 1)(n - 2)(n - 3)^{n-2}$ | Rattan (2006)               |
Enumeration of the Case $\lambda = (p, q)$

Recall that Goulden and Jackson’s formula is

$$|\mathcal{F}(\lambda_1, \ldots, \lambda_\ell)| = (n + \ell - 2)! \ n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. $$

In case of $\lambda = (p, q)$,

$$|\mathcal{F}_{(p,q)}| = \frac{pq}{p+q} \binom{p+q}{p} p^p q^q.$$
A signed permutation is a permutation $\sigma$ on $\{\pm 1, \ldots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \ldots, n\}$.

The hyperoctohedral group $B_n$ is the group of signed permutations on $\{\pm 1, \ldots, \pm n\}$.

We will use the two notations

$[a_1 \ a_2 \ldots \ a_k] = (a_1 \ a_2 \ldots \ a_k - a_1 - a_2 \ldots - a_k)$, zero cycle

$((a_1 \ a_2 \ldots \ a_k)) = (a_1 \ a_2 \ldots \ a_k) (-a_1 - a_2 \ldots - a_k)$, paired nonzero cycle

$\epsilon_i := [i] = (i - i)$ and $((i \ j))$, transpositions of type $B$, satisfies

$\epsilon_i((i \ j)) = ((i \ j))\epsilon_j = ((i - j))\epsilon_i = \epsilon_j((i - j))$
The absolute order on $B_n$ is

$$\pi \leq \sigma \iff \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1} \sigma),$$

where $\ell(\pi)$ is the absolute length for $\pi \in B_n$.

The poset $S^{B}_{nc}(p, q)$ of annular noncrossing permutations of type $B$ is defined by the interval poset of $B_{p+q}$ as follows:

$$S^{B}_{nc}(p, q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \leq \sigma \leq \gamma_{p,q}\} \subseteq B_{p+q},$$

where $\epsilon$ is the identity and $\gamma_{p,q} = [1 \ldots p][p+1 \ldots p+q]$. 
\[ \gamma_{2,1} = [1 \ 2][3] \]

**Figure:** The Hasse diagram for \( S_{nc}^B(2, 1) \).
Drawing permutations on annulus with noncrossing arrows

\[ \pi = ((1\ 5\ 6))((2\ 3\ 4)) \in \mathcal{S}_{nc}^{B}(4, 3) \]

\[ \sigma = [1\ 5\ -7\ 2]((3\ 4)) \in \mathcal{S}_{nc}^{B}(4, 3) \]
1. Nica and Oancea (2009) showed that $S_{nc}^B(p, q)$ is poset-isomorphic to $NC^{(B)}(p, q)$ of annular noncrossing partitions of type $B$.

2. Goulden-Nica-Oancea (2011) showed that the number of maximal chains in the poset $NC^{(B)}(p, q)$ is

$$
\binom{p+q}{q} p^p q^q + \sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}.
$$

3. It turns out that half of the 2nd term is equal to $|F_{(p,q)}|$.

$$
\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q.
$$
A paired nonzero cycle \(((a_1 \ a_2 \ldots \ a_k))\) touching both the interior and exterior circles is called connected.

A signed permutation with at least one connected paired nonzero cycle is called connected.

A maximal chain of \(S_{nc}^B(p, q)\) with at least one connected signed permutation is called connected.
Proposition

The number of disconnected maximal chains of $S_{nc}^B(p, q)$ is equal to

$$(p + q)^p q^q$$

and the number of connected maximal chains of $S_{nc}^B(p, q)$ is equal to

$$2 \frac{pq}{p + q} \binom{p + q}{q} p^p q^q.$$
\( \gamma_{2,1} = [1\ 2][3] \)

**Figure:** Connected maximal chains in \( S^B_{nc}(2,\ 1) \).
Theorem (Kim-Seo-Shin, 2012)

There is a 2-1 map from the set $\mathcal{CM}(S^B_{nc}(p,q))$ of connected maximal chains in $S^B_{nc}(p,q)$ to the set $\mathcal{F}(p,q)$ of minimal transitive factorizations of $\alpha_{p,q}$.

Proof.

The composition of three maps $|\varphi^+| := |\cdot| \circ (\cdot)^+ \circ \varphi$

$$
\begin{align*}
\mathcal{CM}(S^B_{nc}(p,q)) & \xrightarrow{\varphi} \mathcal{F}^{(B)}(p,q) & \xrightarrow{(\cdot)^+} \mathcal{F}^+(p,q) & \xrightarrow{|\cdot|} \mathcal{F}(p,q)
\end{align*}
$$

is the desired 2-1 map.
\[
\{ \epsilon < ((1\ 3)) < ((1\ 3 - 2)) < [1\ 2][3] \} \in C\mathcal{M}(S^B_{nc}(2, 1))
\]

\[
\tau_i = \pi_{i-1}^{-1} \pi_i
\]

\[
( ((1\ 3)), ((2\ - 3)), ((1\ - 3)) ) \in \mathcal{F}^{(B)}(2, 1)
\]

\[
\sigma_i = \tau_i^+
\]

\[
( ((1\ 3)), ((2\ 3)), ((1\ 3)) ) \in \mathcal{F}^+(2, 1)
\]

\[
\eta_i = |\sigma_i|
\]

\[
( (1\ 3), (2\ 3), (1\ 3) ) \in \mathcal{F}(2, 1)
\]
Why is the map \((\cdot)^+\) surjective?

Given a minimal transitive factorization of \(\beta_{3,2} = ((1\ 2\ 3))((4\ 5))\)

\[
( ( (1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ 5)), ((3\ 4)) ) \in \mathcal{F}_ {3,2}^+,
\]

since \(\gamma_{3,2} = [1\ 2\ 3][4\ 5] = \epsilon_4\ \epsilon_1\ \beta_{3,2},\)

\[
\gamma_{3,2} = \epsilon_4\ \epsilon_1\ ((1\ 2))\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4))
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\]

we have one minimal transitive factorization of \(\gamma_{3,2}\)

\[
( ( (1\ 2)), ((2 - 5)), ((2\ 3)), ((4 - 5)), ((3 - 4)) ) \in \mathcal{F}_{3,2}^{(B)}.\]
Why is the map \((\cdot)^+\) two-to-one?

\[(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = ( ((1\ 2)), ((2\ -\ 5)), ((2\ 3)), ((4\ -\ 5)), ((3\ -\ 4)) ) \in \mathcal{F}^{(B)}_{(3,2)} \]

\[\tau_i' = \begin{cases} 
\tau_i & \text{if } \tau_i \text{ is disconnected} \\
((a\ -\ b)) & \text{if } \tau_i = ((a\ b)) \text{ is connected.}
\end{cases} \]

\[(\tau_1', \tau_2', \tau_3', \tau_4', \tau_5') = ( ((1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ -\ 5)), ((3\ 4)) ) \in \mathcal{F}^{(B)}_{(3,2)} \]

It satisfies

\[(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^+ = (\tau_1', \tau_2', \tau_3', \tau_4', \tau_5')^+ \in \mathcal{F}^+_{(3,2)}.\]
### Summary

| $\lambda$ | $|\mathcal{F}_\lambda|$ | bijective proof |
|-----------|-----------------|----------------|
| $(n)$     | $n^{n-2}$       | Dénes et al.   |
| $(1, n-1)$| $(n-1)^n$       | Kim-Seo (2003) |
| $(2, n-2)$| $4(n-1)(n-2)^{n-1}$ | Seo(2004), Rattan (2006) |
| $(3, n-3)$| $\frac{27}{2}(n-1)(n-2)(n-3)^{n-2}$ | Rattan (2006) |
| $(p, q)$ | $\frac{pq}{p+q} \binom{p+q}{p} p^p q^q$ | Kim-Seo-Shin (2012) |
| $(p, q, r)$ | $(p+q+r+1)pqr \binom{p+q+r}{p,q,r} p^p q^q r^r$ | open |
| $(1^n)$  | $(2n-2)!n^{n-3}$ | open |