Proofs of two conjectures of Kenyon and Wilson on Dyck tilings

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Basic definitions

- **matching** on \( \{1, 2, \ldots, 2n\} \)
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  \[
  \begin{array}{cccccccc}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 9 & 10 \end{array}
  \]

- **noncrossing matching** on \( \{1, 2, \ldots, 2n\} \)

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Basic definitions

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- **noncrossing matching** on \( \{1, 2, \ldots, 2n\} \)

- **Dyck path** of length \( 2n \) ↔ **partition** contained in \((n - 1, n - 2, \ldots, 1)\)
Basic definitions

- **matching** on \{1, 2, \ldots, 2n\}

- **noncrossing matching** on \{1, 2, \ldots, 2n\}

- **Dyck path** of length \(2n \leftrightarrow\) **partition** contained in \((n - 1, n - 2, \ldots, 1)\)

- These objects are counted by Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\).
Motivation: Double-dimer model
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\[ M: \text{matrix whose rows and columns are indexed by Dyck paths of length } 2n: \]
\[ M_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda \succeq \mu, \\ 0, & \text{otherwise}. \end{cases} \]
Motivation: Double-dimer model

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\]

Kenyon and Wilson showed that one can compute \( P(\pi) \) using the inverse of \( M \).
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\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

- \(P(\pi)\): probability that a random double-dimer has matching \(\pi\)
- \(M\): matrix whose rows and columns are indexed by Dyck paths of length \(2n\):

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1, & \text{if } \lambda \succ \mu, \\
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- Kenyon and Wilson showed that one can compute \(P(\pi)\) using the inverse of \(M\).

**Theorem (Kenyon and Wilson, 2010)**

\[ (M^{-1})_{\lambda,\mu} = (-1)^{|\lambda/\mu|} \times (\# \text{ cover-inclusive Dyck tilings of shape } \lambda/\mu) \]
Dyck tilings

- Dyck tile
(cover-inclusive) Dyck tilings

- Dyck tile
- Dyck tiling of shape $\lambda/\mu$
(cover-inclusive) Dyck tilings

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Cell A is **covered** by cell B:
(cover-inclusive) Dyck tilings

- Dyck tile
- Dyck tiling of shape \( \lambda / \mu \)

Cell A is **covered** by cell B:

cover-inclusive Dyck tiling: if tile A has a cell covered by a cell of tile B, then A is **completely covered** by B.
Notations

\( \mathcal{D}(\lambda/\mu) \): set of Dyck tilings of shape \( \lambda/\mu \)

\[
\mathcal{D}(\lambda/\ast) = \bigcup_{\nu \in \text{Dyck}(2n)} \mathcal{D}(\lambda/\nu),
\]

\[
\mathcal{D}(\ast/\mu) = \bigcup_{\nu \in \text{Dyck}(2n)} \mathcal{D}(\nu/\mu).
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- $|\mathcal{D}(\lambda/\ast)| = \# \text{Dyck tilings with lower path } \lambda = \text{row sum of } |M^{-1}|$
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Main Problem
Notations

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Main Problem

- Find $|\mathcal{D}(\lambda/\star)|$ and $|\mathcal{D}(\star/\mu)|$. 
Notations

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Main Problem

- Find $|\mathcal{D}(\lambda/\ast)|$ and $|\mathcal{D}(\ast/\mu)|$.
- Find $q$-analogs of $|\mathcal{D}(\lambda/\ast)|$ and $|\mathcal{D}(\ast/\mu)|$: Kenyon and Wilson’s conjectures
Chords of Dyck paths

A **chord** is a matching pair of up step and down step.
Chords of Dyck paths

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- The **length** $|c|$ and the **height** $ht(c)$ are defined as follows:
$\mathcal{D}(\lambda/\star)$: fixed lower path

There are 12 Dyck tilings with fixed lower path
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The fixed lower path has half length $n = 4$ with chords of length 1,1,1,2
\( \mathcal{D}(\lambda/\ast) \): fixed lower path

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- The fixed lower path has half length \( n = 4 \) with chords of length 1,1,1,2

\[
12 = \frac{4!}{1 \cdot 1 \cdot 1 \cdot 2} = \frac{n!}{\prod_{c \in \text{Chord}(\lambda)} |c|}
\]

(weak) Conjecture 1 of KW
$D(*/\mu)$: fixed upper path

There are 12 Dyck tilings with fixed upper path
$\mathcal{D}(*/\mu)$: fixed upper path

- There are 12 Dyck tilings with fixed upper path

- The fixed upper path has chords of height 3, 2, 2, 1
**$D(\ast/\mu)$**: fixed upper path

- There are 12 Dyck tilings with fixed upper path

![Diagram of Dyck tilings]

- The fixed upper path has chords of height 3, 2, 2, 1

![Diagram showing heights of chords]

- $12 = 3 \cdot 2 \cdot 2 \cdot 1 = \prod_{c \in \text{Chord}(\mu)} \text{ht}(c)$ (weak) Conjecture 2 of KW
$q$-analogs?

- We need nice statistics of Dyck tilings.
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For \( T \in \mathcal{D}(\lambda/\mu) \) define

\[
|T| = \text{number of tiles in } T
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\[
\text{art}(T) = \frac{|\lambda/\mu| + |T|}{2}
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We need nice statistics of Dyck tilings.

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- $|T| = \text{number of tiles in } T$
- $\text{art}(T) = \frac{|\lambda/\mu| + |T|}{2} = \frac{\text{area+tiles}}{2}$

- $\text{area}(T) = 5$, $\text{tiles}(T) = 3$, $\text{art}(T) = \frac{5 + 3}{2} = 4$
q-analogs?

- We need nice statistics of Dyck tilings.
- For $T \in \mathcal{D}(\lambda/\mu)$ define

$$|T| = \text{number of tiles in } T$$

$$\text{art}(T) = \frac{|\lambda/\mu| + |T|}{2} = \frac{\text{area} + \text{tiles}}{2}$$

- area$(T) = 5$, tiles$(T) = 3$, art$(T) = \frac{5 + 3}{2} = 4$

- The usual $q$-integers, $q$-factorials:

$$[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = [1]_q [2]_q \cdots [n]_q.$$
Conjecture (Kenyon and Wilson, 2010)

\[
\sum_{T \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\vert c \vert]_q}
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Two conjectures of Kenyon and Wilson

Conjecture (Kenyon and Wilson, 2010)

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\sum_{T \in D(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\text{ht}(c)]_q}
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Conjecture (Kenyon and Wilson, 2010)

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\sum_{T \in D(\ast/\mu)} q^{|T|} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
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$$\sum_{T \in D(\ast/μ)} q^{|T|} = \prod_{c \in \text{Chord}(μ)} [\text{ht}(c)]_q$$

Conjecture 1 has been proved by Kim (non-bijectively) and Kim, Mészáros, Panova, Wilson (bijectively).
Two conjectures of Kenyon and Wilson

Conjecture (Kenyon and Wilson, 2010)

\[ \sum_{T \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q !}{\prod_{c \in \text{Chord}(\lambda)} [\lvert c \rvert]_q} \]

Conjecture (Kenyon and Wilson, 2010)

\[ \sum_{T \in \mathcal{D}(\ast/\mu)} q^{\lvert T \rvert} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q \]

- Conjecture 1 has been proved by Kim (non-bijectively) and Kim, Mészáros, Panova, Wilson (bijectively).
- Conjecture 2 has been proved bijectively by Kim and Konvalinka independently.
Inductive Proof of Conjecture 1

- $\mathcal{D}(\lambda/\ast; a, b)$: set of generalized Dyck tilings
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- $\mathcal{D}(\lambda/\ast; a, b)$: set of generalized Dyck tilings
- The upper path starts $a$ units above the starting point of $\lambda$ and ends $b$ units above the ending point of $\lambda$. 

![Diagram of generalized Dyck tilings](image)
Inductive Proof of Conjecture 1

- $D(\lambda/\ast; a, b)$: set of generalized Dyck tilings
- The upper path starts $a$ units above the starting point of $\lambda$ and ends $b$ units above the ending point of $\lambda$.

Theorem (K., 2011)

$$\sum_{T \in D(\lambda/\ast; a, b)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\lfloor c \rfloor]_q} \sum_{T \in D(\Delta_n/\ast; a, b)} q^{\text{art}(T)}$$
Inductive Proof of Conjecture 1

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- \( \Delta_n \): the highest path = \( \triangle \)
Inductive Proof of Conjecture 1

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- $\Delta_n$: the highest path =

- $\mathcal{D}(\lambda/\ast; 0, 0) = \mathcal{D}(\lambda/\ast)$
Inductive Proof of Conjecture 1

- $\mathcal{D}(\lambda/\ast; a, b)$: set of **generalized Dyck tilings**
- The upper path starts $a$ units above the starting point of $\lambda$ and ends $b$ units above the ending point of $\lambda$.

\[ \sum_{T \in \mathcal{D}(\lambda/\ast; a, b)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\lvert c \rvert]_q} \sum_{T \in \mathcal{D}(\Delta_n/\ast; a, b)} q^{\text{art}(T)} \]

- $\Delta_n$: the highest path =

- $\mathcal{D}(\lambda/\ast; 0, 0) = \mathcal{D}(\lambda/\ast)$
- $\mathcal{D}(\Delta_n/\ast; 0, 0)$ has only one tile, the empty tiling of $\Delta_n/\Delta_n$. 

Why are the generalized Dyck tilings easier?
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Theorem (K., Mészáros, Panova, Wilson, 2011)  
There is a bijection $\phi$ from Dyck tilings to increasing ordered trees such that the lower path of $T$ corresponds to the shape of the tree $\phi(T)$ and  

$$\text{art}(T) = \text{inv}(\phi(T)).$$
Bijective Proof of Conjecture 1

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There is a bijection $\phi$ from Dyck tilings to increasing ordered trees such that the lower path of $T$ corresponds to the shape of the tree $\phi(T)$ and
\[
\text{art}(T) = \text{inv}(\phi(T)).
\]

Theorem (Björner and Wachs, 1989)
\[
\sum_{\text{sh}(P) = \lambda} q^{\text{inv}(P)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [c]_q}
\]
A Hermite history of shape $\mu$ is a labeling $H$ of the down steps such that
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![Diagram of a Hermite history labeling]
A **Hermite history** of shape $\mu$ is a labeling $H$ of the down steps such that a down step of height $i$ has label in $\{0, 1, \ldots, i - 1\}$.

$H(\mu)$: set of Hermite histories of shape $\mu$
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$\mathcal{H}(\mu)$: set of Hermite histories of shape $\mu$

$\|H\|$ = sum of labels

$$
\sum_{H \in \mathcal{H}(\mu)} q^{\|H\|} = \prod_{c \in \text{Chord}(\mu)} [ht(c)]_q.
$$
A **Hermite history** of shape $\mu$ is a labeling $H$ of the down steps such that a down step of height $i$ has label in $\{0, 1, \ldots, i - 1\}$.

- $\mathcal{H}(\mu)$: set of Hermite histories of shape $\mu$
- $\|H\|$: sum of labels

$\sum_{H \in \mathcal{H}(\mu)} q^{\|H\|} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q$.

It is sufficient to find a bijection from $\mathcal{D}(*/\mu)$ to $\mathcal{H}(\mu)$. 
Hermite Histories

- A **Hermite history** of shape $\mu$ is a labeling $H$ of the down steps such that
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$\mathcal{H}(\mu)$: set of Hermite histories of shape $\mu$

$\|H\|$: sum of labels

$$\sum_{H \in \mathcal{H}(\mu)} q^{\|H\|} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q.$$

It is sufficient to find a bijection from $\mathcal{D}(\ast/\mu)$ to $\mathcal{H}(\mu)$.

There is a simple bijection between **Hermite histories** and **matchings**:
A Hermite history of shape $\mu$ is a labeling $H$ of the down steps such that a down step of height $i$ has label in $\{0, 1, \ldots, i - 1\}$.

$\mathcal{H}(\mu)$: set of Hermite histories of shape $\mu$

$\|H\| = \text{sum of labels}$

$$\sum_{H\in \mathcal{H}(\mu)} q^{\|H\|} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q.$$

It is sufficient to find a bijection from $\mathcal{D}(*/\mu)$ to $\mathcal{H}(\mu)$.

There is a simple bijection between Hermite histories and matchings:

If $H \leftrightarrow \pi$, then $\|H\| = \text{cr}(\pi)$.
A bijection between Dyck tilings and Hermite histories

- The **entry** and the **exit** of a Dyck tile are defined:
A bijection between Dyck tilings and Hermite histories

- The **entry** and the **exit** of a Dyck tile are defined:

- The label of a down step is **the number of tiles traveled**:
A bijection between Dyck tilings and Hermite histories

- The **entry** and the **exit** of a Dyck tile are defined:

- The label of a down step is the **number of tiles traveled**:

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The **entry** and the **exit** of a Dyck tile are defined:

- The label of a down step is **the number of tiles traveled**:

If \( T \leftrightarrow H \), then \(|T| = |H|\).

\[
\sum_{T \in \mathcal{D}(*/\mu)} q^{|T|} = \sum_{H \in \mathcal{H}(\mu)} q^{|H|} = \prod_{c \in \text{Chord}(\mu)} [ht(c)]_q.
\]
How to recover the Dyck tiling
How to recover the Dyck tiling
How to recover the Dyck tiling

\[
\begin{array}{c}
\text{1} \\
\text{1} \\
\text{3} \\
\text{0} \\
\text{4} \\
\text{1} \\
\text{3} \\
\text{1} \\
\text{0} \\
\end{array}
\]
How to recover the Dyck tiling
How to recover the Dyck tiling
How to recover the Dyck tiling

\[
\begin{array}{ccccccc}
5 & 1 & 0 & 2 & 4 & 0 & 2
\end{array}
\rightarrow
\begin{array}{ccccccc}
5 & 1 & 2 & 4 & 0 & 2 & 1
\end{array}
\]
How to recover the Dyck tiling
How to recover the Dyck tiling
There are three bijections on Dyck tilings.
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\[ \text{Bij}_0 : \{\text{Dyck tilings}\} \rightarrow \{\text{matchings}\} \]
Summery

- There are three bijections on Dyck tilings.
  - $\text{Bij}_0 : \{\text{Dyck tilings}\} \rightarrow \{\text{matchings}\}$
  - $\text{Bij}_1 : \{\text{Dyck tilings}\} \rightarrow \{\text{increasing ordered trees}\}$
There are three bijections on Dyck tilings.

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- $\text{Bij}_2 : \{\text{Dyck tilings}\} \rightarrow \{\text{increasing ordered trees}\}$
**Summary**

- There are three bijections on Dyck tilings.
  - $\text{Bij}_0 : \{\text{Dyck tilings}\} \rightarrow \{\text{matchings}\}$
  - $\text{Bij}_1 : \{\text{Dyck tilings}\} \rightarrow \{\text{increasing ordered trees}\}$
  - $\text{Bij}_2 : \{\text{Dyck tilings}\} \rightarrow \{\text{increasing ordered trees}\}$

- $\text{Bij}_0$ sends **upper path** $\mu$ to shape (of matching), and **tiles** to **crossings**

$$
\sum_{T \in \mathcal{D}(\ast \mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
$$
There are three bijections on Dyck tilings.

- **Bij**₀ : \{Dyck tilings\} → \{matchings\}
- **Bij**₁ : \{Dyck tilings\} → \{increasing ordered trees\}
- **Bij**₂ : \{Dyck tilings\} → \{increasing ordered trees\}

**Bij**₀ sends upper path \(\mu\) to shape (of matching), and tiles to crossings

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\sum_{T \in \mathcal{D}(\ast/\mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
\]

**Bij**₁ sends lower path \(\lambda\) to shape (of plane tree), and art to inv

\[
\sum_{T \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\|c\|]_q}
\]
There are three bijections on Dyck tilings.

- **Bij₀**: \( \{ \text{Dyck tilings} \} \rightarrow \{ \text{matchings} \} \)
- **Bij₁**: \( \{ \text{Dyck tilings} \} \rightarrow \{ \text{increasing ordered trees} \} \)
- **Bij₂**: \( \{ \text{Dyck tilings} \} \rightarrow \{ \text{increasing ordered trees} \} \)

**Bij₀** sends **upper path** \( \mu \) to shape (of matching), and **tiles** to **crossings**

\[
\sum_{T \in \mathcal{D}(\ast / \mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
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\]

When lower path \( \lambda = \text{zigzag}_n \)
There are three bijections on Dyck tilings.

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When lower path \(\lambda = \text{zigzag}_n\)

- Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
There are three bijections on Dyck tilings.

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- **Bij**₂ : \{Dyck tilings\} \rightarrow \{increasing ordered trees\}

**Bij**₀ sends **upper path** \(\mu\) to shape (of matching), and **tiles** to **crossings**

\[
\sum_{T \in D(\ast/\mu)} q^{tiles(T)} = \prod_{c \in Chord(\mu)} [ht(c)]_q
\]

**Bij**₁ sends **lower path** \(\lambda\) to shape (of plane tree), and **art** to **inv**

\[
\sum_{T \in D(\lambda/\ast)} q^{art(T)} = \frac{[n]_q!}{\prod_{c \in Chord(\lambda)} [\left|c\right|]_q}
\]

When lower path \(\lambda = \text{zigzag}_n\)

- Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
- increasing ordered trees = permutations
There are three bijections on Dyck tilings.

- \( \text{Bij}_0 : \{ \text{Dyck tilings} \} \rightarrow \{ \text{matchings} \} \)
- \( \text{Bij}_1 : \{ \text{Dyck tilings} \} \rightarrow \{ \text{increasing ordered trees} \} \)
- \( \text{Bij}_2 : \{ \text{Dyck tilings} \} \rightarrow \{ \text{increasing ordered trees} \} \)

\( \text{Bij}_0 \) sends **upper path** \( \mu \) to shape (of matching), and **tiles** to **crossings**

\[
\sum_{T \in \mathcal{D}(\ast/\mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
\]

\( \text{Bij}_1 \) sends **lower path** \( \lambda \) to shape (of plane tree), and **art** to **inv**

\[
\sum_{T \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\lvert c \rvert]_q}
\]

When lower path \( \lambda = \text{zigzag}_n \)

- Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
- increasing ordered trees = permutations
- \( \text{Bij}_1 \) sends **art** to **inv**
There are three bijections on Dyck tilings.

- **Bij**₀ : \{Dyck tilings\} → \{matchings\}
- **Bij**₁ : \{Dyck tilings\} → \{increasing ordered trees\}
- **Bij**₂ : \{Dyck tilings\} → \{increasing ordered trees\}

**Bij**₀ sends **upper path** \(\mu\) to shape (of matching), and **tiles** to **crossings**

\[
\sum_{T \in D(\ast/\mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
\]

**Bij**₁ sends **lower path** \(\lambda\) to shape (of plane tree), and **art** to **inv**

\[
\sum_{T \in D(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\vert c \vert]_q}
\]

**When lower path** \(\lambda = \text{zigzag}_n\)

- Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
- increasing ordered trees = permutations
- **Bij**₁ sends **art** to **inv**
- **Bij**₂ sends **art** to **mad** (mahonian statistic, Clarke, Steingrímsson, Zeng, 1997)
There are three bijections on Dyck tilings.

- **Bij\(_0\)**: \{Dyck tilings\} → \{matchings\}
- **Bij\(_1\)**: \{Dyck tilings\} → \{increasing ordered trees\}
- **Bij\(_2\)**: \{Dyck tilings\} → \{increasing ordered trees\}

**Bij\(_0\)** sends **upper path** \(\mu\) to shape (of matching), and **tiles** to **crossings**

\[
\sum_{T \in \mathcal{D}(\ast/\mu)} q^{\text{tiles}(T)} = \prod_{c \in \text{Chord}(\mu)} [\text{ht}(c)]_q
\]

**Bij\(_1\)** sends **lower path** \(\lambda\) to shape (of plane tree), and **art** to **inv**

\[
\sum_{T \in \mathcal{D}(\lambda/\ast)} q^{\text{art}(T)} = \frac{[n]_q!}{\prod_{c \in \text{Chord}(\lambda)} [\mid c \mid]_q}
\]

**When lower path** \(\lambda = \text{zigzag}_n\)

- Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
- increasing ordered trees = permutations
- **Bij\(_1\)** sends **art** to **inv**
- **Bij\(_2\)** sends **art** to **mad** (mahonian statistic, Clarke, Steingrímsson, Zeng, 1997)
- **Bij\(_2\)** reduces to the bijection of Aval, Boussicault, Dasse-Hartaut
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Thank you for your attention!