Crystal energy via charge in types A and C

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and work joint with Naito, Sagaki, Shimozono (in progress)
Outline

Crystals

Energy function

Charge

Arbitrary type
Outline

Crystals

Energy function

Charge

Arbitrary type
A $U_q(g)$-crystal is a nonempty set $B$ with maps

\[\text{wt}: B \rightarrow P\]
\[e_i, f_i: B \rightarrow B \cup \{\emptyset\} \quad \text{for all } i \in I\]

Write $b \xrightarrow{i} b'$ for $b' = f_i(b)$.
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Write $\bullet_i \over\rightarrow b \, \bullet$ for $b' = f_i(b)$
Kashiwara–Nakashima tableaux

embed $B(1^N) \hookrightarrow B(\lambda) \otimes |\lambda|$

Type $A_r$:

Example

Type $A_3$

- strictly increasing in columns
Kashiwara–Nakashima tableaux

\[ \text{embed } B(1^N) \hookrightarrow B(\square) \otimes |\lambda| \]

**Type** \( A_r \):

\[
\begin{array}{c}
1 \quad 2 \quad \ldots \quad r - 1 \quad r \quad r + 1
\end{array}
\]

**Example**

**Type** \( A_3 \)

\[
\begin{array}{c}
1 \\
3 \\
4
\end{array} \quad \longleftrightarrow \quad 
\begin{array}{c}
4 \\
3 \\
1
\end{array} \otimes 
\begin{array}{c}
3 \\
1
\end{array}
\]

- strictly increasing in columns
Kashiwara–Nakashima tableaux

Embed $B(1^N) \hookrightarrow B([\square]) \otimes |\lambda|$

Type $A_r$:

Example

Type $A_3$

- strictly increasing in columns
**Kashiwara–Nakashima tableaux**

Embed $B(1^N) \hookrightarrow B(\lambda) \times |\lambda|$

Type $C_r$:

```
 1 1 → 2 2 → ... r − 1 r → −r r − 1 → ... 1 → −1
```

Example

Type $C_3$:

```
1
3
3
```

```
3 × 3 × 1
```

- alphabet $[\bar{r}] := \{1 < 2 < ... < r < \bar{r} < \bar{r} − 1 < ... < \bar{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:
  
  $$z = b(p), \quad \bar{z} = b(q), \quad q − p \leq k − z.$$
Kashiwara–Nakashima tableaux

Type $C_r$:

$B(1^N) \hookrightarrow B(\square) \otimes |\lambda|$

Example

Type $C_3$

| alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \bar{r-1} < \ldots < \bar{1}\}$ |
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Example

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Kashiwara–Nakashima tableaux

eMBED $B(1^N) \hookrightarrow B([\square]) \otimes |\lambda|$

Type $C_r$:

Example

- alphabet $[\vec{r}] := \{1 < 2 < \ldots < r < \vec{r} < \vec{r} - 1 < \ldots < \vec{1}\}$
- strictly increasing in columns
- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \vec{z})$ s.t.:
  
  $$z = b(p), \quad \vec{z} = b(q), \quad q - p \leq k - z.$$
Column KR crystals for types $A_n^{(1)}$ and $C_n^{(1)}$

Example

$B^{2,1}_{2,1}$ of type $A_3^{(1)}$  
$B^{2,1}_{2,1}$ of type $C_2^{(1)}$
Outline

Crystals

Energy function

Charge

Arbitrary type
Energy function

\[ B := B_\mu = B^{\mu_1,1} \otimes B^{\mu_2,1} \otimes \cdots, \] connected by \( f_0 \) arrows.

The energy \( D : B \to \mathbb{Z} \) originates from exactly solvable lattice models (computed via local energies and the combinatorial \( R \)-matrix).

Alternative construction (S., Tingley) as affine grading on \( B \):
- constant on classical components (\( f_0 \) arrows removed)
- increases by 1 along \( f_0 \) arrows which are not at the end of a 0-string (Demazure arrows)

Remark

In most cases, \( B \) is still connected upon removal of non-Demazure \( f_0 \) arrows.
\( \Rightarrow \) \( D \) is well-defined up to constant.
Notable exception: type \( C \).
**Energy function**

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Energy function

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Notable exception: type \( \mathcal{C} \)
Energy function

\[ B := B_\mu = B^{\mu_1,1} \otimes B^{\mu_2,1} \otimes \cdots, \text{ connected by } f_0 \text{ arrows.} \]

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Notable exception: type \( C \).
Energy function

\[ B := B_\mu = B^{\mu_1,1}_1 \otimes B^{\mu_2,1}_2 \otimes \cdots, \] connected by \( f_0 \) arrows.

The energy \( D : B \to \mathbb{Z} \) originates from exactly solvable lattice models (computed via local energies and the combinatorial \( R \)-matrix).

Alternative construction (S., Tingley) as affine grading on \( B \):

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Outline

Crystals

Energy function

Charge

Arbitrary type
Charge type $A$

Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$

**Example**

$$\mu = (3, 3, 3, 1)$$

1132214323

charge(1132214323) = 1 + 2 + 3 = 6
Charge type $A$

Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$

**Example**

$\mu = (3, 3, 3, 1)$

$$\text{charge}(1132214323) = 1 + 2 + 3 = 6$$
Charge type $A$

Charge à la Lascoux and Schützenberger:

$w$ word of partition content $\mu$

**Example**

$\mu = (3, 3, 3, 1)$

1132214323 charge contribution 1

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$
Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$

Example

$\mu = (3, 3, 3, 1)$

\[
\begin{array}{c}
1132214323 \\
112323
\end{array}
\]

charge(1132214323) = 1 + 2 + 3 = 6
Charge type $A$

Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$

**Example**

$\mu = (3, 3, 3, 1)$

\[
\begin{array}{c}
1132214323 \\
11 2 323
\end{array}
\]

charge(1132214323) = 1 + 2 + 3 = 6
Charge type \( A \)

Charge à la Lascoux and Schützenberger: \( w \) word of partition content \( \mu \)

**Example**

\[
\mu = (3, 3, 3, 1)
\]

\[
\begin{align*}
1132214323 & \quad \text{charge contribution 1} \\
11 \ 2 \ 323 & \quad \text{charge contribution 2}
\end{align*}
\]

\[
\text{charge}(1132214323) = 1 + 2 + 3 = 6
\]
Charge type \( A \)

Charge à la \textbf{Lascoux} and \textbf{Schützenberger}:
\( w \) word of partition content \( \mu \)

Example

\[ \mu = (3, 3, 3, 1) \]

\[
\begin{align*}
11 & 32214323 & \text{charge contribution 1} \\
11 & 2 & 323 & \text{charge contribution 2} \\
1 & 2 & 3
\end{align*}
\]

\[
\text{charge}(1132214323) = 1 + 2 + 3 = 6
\]
Charge type \( A \)

Charge à la Lascoux and Schützenberger: \( w \) word of partition content \( \mu \)

**Example**

\[
\mu = (3, 3, 3, 1)
\]

\[
\begin{align*}
1132214323 & \quad \text{charge contribution 1} \\
11 & 2 & 323 & \quad \text{charge contribution 2} \\
1 & 2 & 3
\end{align*}
\]

charge\((1132214323) = 1 + 2 + 3 = 6\)
Charge type $A$

Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$ 

Example

$\mu = (3, 3, 3, 1)$


classic partition: $1132214323$

charge contribution 1

$11 \ 2 \ 323$

charge contribution 2

$1 \ 2 \ 3$

charge contribution 3

charge$(1132214323) = 1 + 2 + 3 = 6$
Charge type $A$

Charge à la Lascoux and Schützenberger: $w$ word of partition content $\mu$

Example

$\mu = (3, 3, 3, 1)$

\begin{align*}
1132214323 & \quad \text{charge contribution 1} \\
11 & 2 & 323 & \quad \text{charge contribution 2} \\
1 & 2 & 3 & \quad \text{charge contribution 3}
\end{align*}

$\text{charge}(1132214323) = 1 + 2 + 3 = 6$
Charge on KN tableaux - type A

\[ B_\mu := \bigotimes_{i=1}^{\mu_1} B_{\mu_i';1} \]

circular order \( \prec_i \): \( i \prec_i i + 1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i - 1 \)

construct reordered \( c \) from \( b \in B_\mu \)

Example

\[
\begin{array}{c}
\begin{array}{cccc}
3 & 2 & 1 & 2 \\
5 & 3 & 2 \\
6 & 4 & 4 \\
\end{array}
& \quad \text{and} & 
\begin{array}{cccc}
3 & 3 & 4 & 2 \\
5 & 2 & 2 \\
6 & 4 & 1 \\
\end{array}
\end{array}
\]

\[ \text{cw}(b) = \begin{pmatrix}
6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \\
\end{pmatrix} \]
Charge on KN tableaux - type A

\[ B_\mu := \bigotimes_{i=1}^{\mu_1} B_{\mu_i';1} \]

circular order \( \preceq_i \): \( i \preceq_i i+1 \preceq_i \cdots \preceq_i n \preceq_i 1 \preceq_i \cdots \preceq_i i-1 \)

construct reordered \( c \) from \( b \in B_\mu \)

**Example**

\[ b = \begin{array}{cccc} 3 & 2 & 1 & 2 \\ 5 & 3 & 2 \\ 6 & 4 & 4 \end{array} \quad \text{and} \quad c = \begin{array}{cccc} 3 & 3 & 4 & 2 \\ 5 & 2 & 2 \\ 6 & 4 & 1 \end{array} \]

\[ \text{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix} \]
Charge on KN tableaux - type $A$

$$B_{\mu} := \bigotimes_{i=1}^{\mu_1} B_{\mu_i}^{\mu_i'} \cdot 1$$

circular order $\prec_i$: $i \prec_i i + 1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i - 1$

Construct reordered $c$ from $b \in B_{\mu}$

**Example**

\[
b = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
5 & 3 & 2 & \\
6 & 4 & 4 & \\
\end{array}
\quad \text{and} \quad c = \begin{array}{cccc}
3 & 3 & 4 & 2 \\
5 & 2 & 2 & \\
6 & 4 & 1 & \\
\end{array}
\]

\[
cw(b) = \begin{pmatrix}
6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \\
\end{pmatrix}
\]
Charge on KN tableaux - type A

Example

\[
\begin{align*}
    b &= \begin{array}{ccc}
        3 & 2 & 1 \\
        5 & 3 & 2 \\
        6 & 4 & 4 \\
    \end{array} \\
    \text{and} \quad c &= \begin{array}{ccc}
        3 & 3 & 4 \\
        5 & 2 & 2 \\
        6 & 4 & 1 \\
    \end{array} \\
    cw(b) &= \begin{pmatrix}
        6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
        1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3
    \end{pmatrix}
\end{align*}
\]

\[
\sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \text{charge}(cw_2(b))
\]

Remark

A similar construction works in type C.
Charge on KN tableaux - type A

Example

\[ b = \begin{array}{cccc} 3 & 2 & 1 & 2 \\ 5 & 3 & 2 & \\ 6 & 4 & 4 \end{array} \quad \text{and} \quad c = \begin{array}{cccc} 3 & 3 & 4 & 2 \\ 5 & 2 & 2 & \\ 6 & 4 & 1 \end{array} \]

\[ \text{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix} \]

\[ \sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \text{charge}(\text{cw}_2(b)) \]

Remark

A similar construction works in type C.
Charge on KN tableaux - type $A$

**Example**

\[
b = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
5 & 3 & 2 \\
6 & 4 & 4 \\
\end{array} \quad \text{and} \quad c = \begin{array}{cccc}
3 & 3 & 4 & 2 \\
5 & 2 & 2 \\
6 & 4 & 1 \\
\end{array}
\]

\[
cw(b) = \left( \begin{array}{cccc}
6 & 5 & 4 & 4 \\
1 & 1 & 3 & 2 \\
1 & 4 & 3 & 2 \\
\end{array} \right)
\]

\[
\sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \text{charge}(cw_2(b))
\]

**Remark**

A similar construction works in type $C$. 
Charge on KN tableaux - type A

**Example**

\[
b = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
5 & 3 & 2 & \\
6 & 4 & 4 & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
3 & 3 & 4 & 2 \\
5 & 2 & 2 & \\
6 & 4 & 1 & \\
\end{array}
\]

\[
w_c(b) = \left( \begin{array}{ccccccccccc}
6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
1 & 2 & 1 & 2 & 3 & 1 & 2 & 1 & 4 & 3 & 2 & 3 \\
\end{array} \right)
\]

\[
\sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \text{charge}(w_c(b))
\]

**Remark**

A similar construction works in type C.
Charge on KN tableaux - type $A$

**Example**

\[
\begin{array}{c}
\begin{array}{c}
3 & 2 & 1 & 2 \\
5 & 3 & 2 \\
6 & 4 & 4 \\
\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
3 & 3 & 4 & 2 \\
5 & 2 & 2 \\
6 & 4 & 1 \\
\end{array}
\end{array}
\]

\[
cw(b) = \left( \begin{array}{cccccccc}
6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
1_2 & 1_2 & 3_1 & 2_1 & 2 & 1 & 4 & 3 & 2 & 3 \\
\end{array} \right)
\]

\[
\sum_{\gamma \in \text{Des}(c)} \text{arm} (\gamma) = \text{charge}(cw_2(b))
\]

**Remark**

A similar construction works in type $C$. 
Charge on KN tableaux - type A

Example

\[ b = \begin{array}{ccc}
3 & 2 & 1 \\
5 & 3 & 2 \\
6 & 4 & 4 \\
\end{array} \quad \text{and} \quad c = \begin{array}{ccc}
3 & 3 & 4 \\
5 & 2 & 2 \\
6 & 4 & 1 \\
\end{array} \]

\[ cw(b) = \left( \begin{array}{ccccccc}
6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\
1_2 & 1_2 & 3_1 & 2_1 & 2 & 1 & 4 & 3 & 2 & 3 \\
\end{array} \right) \]

\[ \sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \text{charge}(cw_2(b)) \]

Remark

A similar construction works in type C.
**Relation between charge and energy**

**Theorem (Lenart, S. 2011)**

\[
B = B_{r_N,1} \otimes \cdots \otimes B_{r_1,1} \text{ of type } A_n^{(1)} \text{ or type } C_n^{(1)}
\]

Then for \( b \in B \)

\[
D(b) = \text{charge}(b)
\]

**Idea of proof:** Verify that \( \text{charge} \) satisfies the recursive relations of the energy function.
Relation between charge and energy

**Theorem (Lenart, S. 2011)**

\[ B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1} \text{ of type } A_n^{(1)} \text{ or type } C_n^{(1)} \]

Then for \( b \in B \)

\[ D(b) = \text{charge}(b) \]

**Idea of proof:** Verify that charge satisfies the recursive relations of the energy function.
Outline

Crystals

Energy function

Charge

Arbitrary type
Generalizing the charge to arbitrary root systems

Key concept: quantum Bruhat graph (QBG).

In type $A_{n-1}$, it is the graph on $S_n$ with directed edges

$$w \rightarrow wt_{ij},$$

where

$$\ell(w_{ij}) = \ell(w) + 1 \quad \text{(Bruhat graph)},$$

or

$$\ell(w_{ij}) = \ell(w) - \ell(t_{ij}) = \ell(w) - 2(j - i) + 1.$$
Generalizing the charge to arbitrary root systems

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In type $A_{n-1}$, it is the graph on $S_n$ with directed edges

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$$\ell(wt_{ij}) = \ell(w) - \ell(t_{ij}) = \ell(w) - 2(j - i) + 1.$$
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Generalizing the charge to arbitrary root systems

**Key concept:** quantum Bruhat graph (QBG).

In type $A_{n-1}$, it is the graph on $S_n$ with directed edges $w \rightarrow wt_{ij}$, where

$$
\ell(wt_{ij}) = \ell(w) + 1 \quad \text{(Bruhat graph)}, \quad \text{or} \quad
\ell(wt_{ij}) = \ell(w) - \ell(t_{ij}) = \ell(w) - 2(j - i) + 1.
$$
Quantum Bruhat graph for $S_3$:
The key ingredient

**Fact.** Fix two column strict fillings (in type $A$)

\[
\begin{array}{c}
\begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_k \\
\end{array} \\
\begin{array}{c}
b_1 \\
b_2 \\
\vdots \\
b_k \\
\end{array}
\end{array}
\]

and where the second one is reordered according to the first.

There is a unique path in the quantum Bruhat graph of the following form:
The key ingredient

Fact. Fix two column strict fillings (in type $A$)

\[
\begin{array}{c|c|c}
  a_1 & b_1 \\
  a_2 & b_2 \\
  \vdots & \vdots \\
  a_k & b_k \\
\end{array}
\]

where the second one is reordered according to the first.

There is a unique path in the quantum Bruhat graph of the following form:
\((k,k+1),\ldots,(k,n)\)

\((2,k+1),\ldots,(2,n)\)

\((1,k+1),\ldots,(1,n)\)
Fillings as chains of permutations

\[ b = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 3 \end{pmatrix} \quad \rightarrow \quad c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 4 & 2 \end{pmatrix} \quad \rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots). \]

\[ \begin{pmatrix} 3 \\ 4 \end{pmatrix} > \begin{pmatrix} 3 \\ 1 \end{pmatrix} < \begin{pmatrix} 3 \\ 2 \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \]

\[(2, 3), (2, 4), (1, 3), (1, 4)\]
Fillings as chains of permutations

\[ b = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
4 & 3 \\
\end{array} \quad \rightarrow \quad c = \begin{array}{cccc}
3 & 3 & 1 & 2 \\
4 & 2 \\
\end{array} \quad \rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots) .

\[
\begin{array}{ccc}
3 & 3 & 3 \\
4 & 1 & 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 4 & 4 \\
2 & 2 & 1 \\
\end{array}
\]

\[
( (2, 3), (2, 4), (1, 3), (1, 4) )
\]
Fillings as chains of permutations

\[ b = \begin{array}{cccc}
3 & 2 & 1 & 2 \\
4 & 3 \\
\end{array} \quad \rightarrow \quad c = \begin{array}{cccc}
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\end{array} \quad \rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots). \]

\[ \begin{array}{c|c|c|c|c|c}
3 & 3 & 3 & 4 & 1 \\
4 & 1 & 2 & 2 & 1 \\
\end{array} \quad \begin{array}{c|c|c|c|c|c}
1 & 4 & 4 & 2 & 2 \\
2 & 1 & 1 & 3 & 3 \\
\end{array} \quad \left. \begin{array}{c|c|c|c|c|c}
(2, 3), (2, 4), (1, 3), (1, 4) \quad | \quad (1, 2), (1, 3), (1, 4) \quad | \\
\right. \]
Fillings as chains of permutations

\[ b = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 3 \end{pmatrix} \quad \rightarrow \quad c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 4 & 2 \end{pmatrix} \quad \rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots). \]

\[
\begin{array}{cccccccc}
3 & 3 & 3 & 4 & 1 & 1 & 2 \\
4 & 1 & 2 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 4 & 2 & 1 \\
2 & 4 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
2 & 2 & 2 & 1 \\
3 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
2 & 2 & 1 \\
3 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 \\
3 & 4 \\
\end{array}
\]

\[
( (2, 3), (2, 4), (1, 3), (1, 4) \ | \ (1, 2), (1, 3), (1, 4) \ | \ (1, 2), (1, 3), (1, 4) )
\]
**Fillings as chains of permutations**

\[ b = \begin{array}{ccc}
3 & 2 & 1 \\
4 & 3 & 2 \\
\end{array} \quad \Rightarrow \quad c = \begin{array}{ccc}
3 & 3 & 1 \\
4 & 2 & \end{array} \quad \Rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots).
\]

\[ ((2, 3), (2, 4), (1, 3), (1, 4) \mid (1, 2), (1, 3), (1, 4) \mid (1, 2), (1, 3), (1, 4)) \]

\[ l_r = \text{arm(descent)} \]

\[ \text{charge}(b) = \sum_{\gamma \in \text{Des}(c)} \text{arm}(\gamma) = \sum_{\pi_r > \pi_{r+1}} l_r =: \text{level}(\Pi). \]
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\[ b = \begin{array}{cccc}
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4 & 3 \\
\end{array} \quad \rightarrow \quad c = \begin{array}{cccc}
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4 & 2 \\
\end{array} \quad \rightarrow \quad \Pi = (\pi_1, \pi_2, \ldots). \]

\[ \begin{array}{cccc}
3 & 3 & 3 & 4 \\
4 & 1 & 2 & 2 \\
\end{array} \quad \begin{array}{cccc}
1 & 1 & 1 & 2 \\
\end{array} \]

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\[
\begin{array}{cccccc}
3 & 3 & 3 & 4 & 1 & 1 & 2 \\
4 & 1 & 2 & < & \rightarrow & \rightarrow & > \\
1 & 4 & > & \rightarrow & 4 & \rightarrow & 2 \\
2 & 2 & < & \rightarrow & 1 & < & 3 \\
\end{array}
\]

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Construction of level statistic

Step 1. Fix a partition \( \mu \).

Step 2. Associate with \( \mu \) a sequence (\( \mu \)-chain) \( \Gamma \) of pairs \((i_r, j_r)\) (i.e., roots in type \( A \)) — several choices possible, but not explained.

Example. For \( \mu = (4, 2, 0) \), we considered
\[ \Gamma = ((2, 3), (2, 4), (1, 3), (1, 4)|(1, 2), (1, 3), (1, 4)|(1, 2), (1, 3), (1, 4)) \, . \]

Step 3. Define \( l_r = \#\{s \geq r : (i_s, j_s) = (i_r, j_r)\} \).

Step 4. Define admissible subsets:
\[ A(\Gamma) = A(\mu) = \#\{\text{subsets} \, \Pi \, \text{of} \, \Gamma \, \text{giving rise to paths in the QBG}\} \, . \]

Step 5. Given \( \Pi = (\pi_1, \pi_2, \ldots) \in A(\mu) \) as a path in the QBG, define
\[ \text{level}(\Pi) = \sum_{\pi_r > \pi_{r+1}} l_r \, . \]
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Remarks.

1. The above construction works for any finite root system, as all the ingredients apply to the general case.

2. The level statistic originates in the Ram-Yip formula for Macdonald polynomials of arbitrary type:

\[ (*) \quad P_\mu(x; q, 0) = \sum_{\Pi \in A(\mu)} q^{\text{level}(\Pi)} x^{\text{weight}(\Pi)}. \]

In fact, we can rewrite (*) via the bijection between \( A(\mu) \) and fillings explained before (which also works in type C).

Theorem (L.)

In types A and C, we have

\[ P_\mu(x; q, 0) = \sum_{b \in B_{\mu_1}^{1,1} \otimes B_{\mu_2}^{1,1} \otimes \ldots} q^{\text{charge}(b)} x^{\text{weight}(b)}. \]
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**Theorem (L.)**

*In types A and C, we have*

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Main results (in arbitrary type)

Construction. (L. and Lubovsky) On $A(\mu)$ was defined the structure of an affine crystal (purely combinatorially) — the quantum alcove model.

Conjecture. (L. and Lubovsky)

1. There is a bijection between $A(\mu)$ in type $X_n$ and the KR crystal $B_\mu := B_{\mu_1}^{1} \otimes B_{\mu_2}^{1} \otimes \ldots$ of type $X_n^{(1)}$ under which the arrows of $A(\mu)$ correspond to arrows of $B_\mu$.

2. If $\Pi \in A(\mu) \leftrightarrow b \in B_\mu$ under this bijection, then

$$E(b) = \text{level}(\Pi).$$
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Status of the conjecture. (L., Naito, Sagaki, S., Shimozono)

- The KR crystal and its energy function are realized in terms of quantum Lakshmibai-Seshadri (LS) paths.
- For $\mu$ regular (in type $A$: partitions with distinct parts), the quantum LS paths are in bijection with $A(\Gamma)$ for a special $\mu$-chain $\Gamma$. The conjecture is verified in this case.
- It remains to:
  1. relate quantum LS-paths and the quantum alcove model for arbitrary $\mu$;
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