From alternating sign matrices to Painlevé VI

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Outline

1. Three-coloured chessboards
2. Combinatorial results
3. Symmetric polynomials
4. Painlevé VI
5. Future problems
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1. Three-coloured chessboards
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Three-coloured chessboards

Chessboard of size \((n + 1) \times (n + 1)\). Paint squares with three colours 0, 1, 2 mod 3.

- Adjacent squares have distinct colour.
- “Domain wall boundary conditions” (DWBC).

Read entries mod 3.
Three-coloured chessboards

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Paint squares with three colours 0, 1, 2 mod 3.

- Adjacent squares have distinct colour.
- “Domain wall boundary conditions” (DWBC).
Read entries mod 3.
Example

When $n = 3$ there are seven chessboards. $0 = \text{black, } 1 = \text{red, } 2 = \text{yellow.}$
Other descriptions

Chessboard

Alternating sign matrix

Ice graph
Bijection to alternating sign matrices

Represent colours (residue classes mod 3) by integers so that neighbours differ by 1.

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 \\
3 & 2 & 1 & 0 \\
\end{array}
\]

Contract each block \[[a \ b] \ c \ d] \to (b + c - a - d)/2 \in \{-1, 0, 1\}.

\[
\begin{array}{ccc}
0 & 1 & 0 \\
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\end{array}
\]

Gives bijection to \(n \times n\) alternating sign matrices. Non-zero entries in each row and column alternate in sign and add up to 1.
Bijection to alternating sign matrices

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Bijection to ice graphs  
(states of six-vertex model)

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Put arrows between adjacent squares. Larger entry to the right, $0 < 1 < 2 < 0$.

- Each vertex has two incoming and two outgoing edges.
- Domain wall boundary conditions.

Vertex = Oxygen, Incoming arrow = Hydrogen bond.
Bijection to ice graphs
(states of six-vertex model)

0 ↑ 1 2 0

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2 1 2 1

0 2 1 0

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\[ \binom{6}{k} + \binom{4}{k-2} = \]
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\[ \binom{k}{n} = \binom{l}{n} + \binom{l}{n-2} \]
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$1 \ 6 \ 15 \ 20 \ 15 \ 6 \ 1 = \binom{6}{k}$
Example

For $n = 5$, the number of chessboards with exactly $k$ squares of colour 0 and $l$ squares of colour 2 are as follows.

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\[
\begin{align*}
1 & \quad 6 & \quad 15 & \quad 20 & \quad 15 & \quad 6 & \quad 1 = \binom{6}{k} \\
1 & \quad 4 & \quad 7 & \quad 8 & \quad 7 & \quad 4 & \quad 1 = \binom{4}{k} + \binom{4}{k-2} + \frac{1 \quad 4 \quad 6 \quad 4 \quad 1}{1 \quad 4 \quad 7 \quad 8 \quad 7 \quad 4 \quad 1}
\end{align*}
\]
Generating function (partition function)

\[ Z_n(t_0, t_1, t_2) = \sum_{\text{chessboards of size } (n+1) \times (n+1)} t_0^\# \text{ squares coloured 0} \cdot t_1^\# \text{ squares coloured 1} \cdot t_2^\# \text{ squares coloured 2} \]

\[ Z_5(t_0, t_1, t_2) = t_0^{14} t_1^{14} t_2^8 + 4 t_0^{13} t_1^{14} t_2^9 + \cdots . \]

Partition function for three-colour model with DWBC. Studied by Baxter (1970) for periodic boundary conditions.
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Partition function for three-colour model with DWBC. Studied by Baxter (1970) for periodic boundary conditions.
Three questions

- How many states are there (for fixed $n$)?
  What is $Z_n(1, 1, 1)$?

- How common are the various colors?
  What is
  \[
  \frac{\partial Z_n}{\partial t_j}(1, 1, 1) = \sum_{\text{chessboards}} \#\text{squares of colour } j
  \]

- What is the joint distribution of the three colours?
  What is $Z_n(t_0, t_1, t_2)$?
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Question 1: Enumeration

Alternating sign matrix theorem:

\[ \text{#chessboards} = \frac{1! \cdot 4! \cdot 7! \cdots (3n - 2)!}{n! (n + 1)! (n + 2)! \cdots (2n - 1)!}. \]


Much simpler proof by Kuperberg (1996), using six-vertex model.

We generalize Kuperberg’s work using eight-vertex-solid-on-solid (8VSOS) model.
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8VSOS model (no details)


Generalizes both three-colour model and six-vertex model. Same states, but more general weight function.

It gives a “nice” way to put $2n$ extra parameters into three-colour model (or 2 extra parameters into six-vertex model).

The Yang–Baxter equation implies non-trivial and useful symmetries in these extra parameters.
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The **Yang–Baxter equation** implies non-trivial and useful symmetries in these extra parameters.
Trigonometric case

The 8VSOS model involves elliptic functions. Using the trigonometric limit case, we can prove that, with $\omega = e^{2\pi i/3}$,

$$Z_n\left(\frac{1}{(1 - \lambda)^3}, \frac{1}{(1 - \lambda\omega)^3}, \frac{1}{(1 - \lambda\omega^2)^3}\right) = \frac{(1 - \lambda\omega^2)^2(1 - \lambda\omega^{n+1})^2\left(A_n(1 + \omega^n\lambda^2) + (-1)^n C_n\omega^{2n}\lambda\right)}{(1 - \lambda^3)^{n^2+2n+3}}$$

where $A_n$ are alternating sign matrix numbers and

$$C_n = \prod_{j=1}^{n} \frac{(3j - 1)(3j - 3)!}{(n + j - 1)!}$$

count cyclically symmetric plane partitions.
Consequence

The case $\lambda = 0$ is the ASM Theorem: $Z_n(1, 1, 1) = A_n$. 

Applying $\frac{\partial}{\partial \lambda} \bigg|_{\lambda=0}$ gives expressions for the first moments

$$\sum \text{#squares of colour } j,$$

which answers Question 2.
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Applying $\partial / \partial \lambda \bigg|_{\lambda=0}$ gives expressions for the first moments

$$\sum \# \text{squares of colour } j,$$

which answers Question 2.
Answer to Question 2: How common are the colours?

Suppose \( n \equiv 0 \mod 6 \) and consider the colour 0 (all other cases are similar).

Probability that random square from random chess-board (chosen uniformly) has colour 0 is

\[
\frac{1}{3} + \frac{2}{9(n+1)^2} \frac{2 \cdot 5 \cdots (3n-1)}{1 \cdot 4 \cdots (3n-2)} + \frac{4}{9(n+1)^2}
\]

\[
= \frac{1}{3} + \frac{2}{9} \frac{\Gamma(1/3)}{\Gamma(2/3)} n^{-5/3} + \frac{4}{9} n^{-2} + O(n^{-3}), \quad n \to \infty.
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Question 3: What is $Z_n$ in general?

$Z_n$ can be split as a sum of two parts.

Each part is a specialized affine Lie algebra character, and a tau function of Painlevé VI. Moreover, each part satisfies a Toda-type recursion.

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Free energy

Suppose \( t_0, t_1, t_2 \) are positive. **Conjecture:**

\[
\lim_{n \to \infty} \frac{\log Z_n(t_0, t_1, t_2)}{n^2} = \frac{1}{3} \log(t_0 t_1 t_2) + \log \left( \frac{(\zeta + 2)^3}{2^{2/3} \zeta^{1/12} (\zeta + 1)^{4/3}} \right),
\]

where \( \zeta \) is determined by

\[
\frac{(t_0 t_1 + t_0 t_2 + t_1 t_2)^3}{(t_0 t_1 t_2)^2} = \frac{2(\zeta^2 + 4\zeta + 1)^3}{\zeta(\zeta + 1)^4}, \quad \zeta \geq 1.
\]

Compare Baxter’s formula for *periodic* boundary conditions:

\[
\frac{1}{3} \log(t_0 t_1 t_2) + \log \left( \frac{2^{5/3} \zeta^{1/3} (\zeta + 1)^{4/3}}{(2\zeta + 1)^{5/2}} \right).
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Outline

1. Three-coloured chessboards
2. Combinatorial results
3. Symmetric polynomials
4. Painlevé VI
5. Future problems
Symmetric polynomials

Let

\[ S_n(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = \frac{\prod_{i,j=1}^{n} G(x_i, y_j)}{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)} \det_{1 \leq i,j \leq n} \left( \frac{F(x_i, y_j, z)}{G(x_i, y_j)} \right), \]

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\[ G(x, y) = (\zeta + 2)xy(x + y) - \zeta (x^2 + y^2) - 2(\zeta^2 + 3\zeta + 1)xy + \zeta (2\zeta + 1)(x + y). \]

This is a symmetric (!) polynomial in all \(2n + 1\) variables, depending on parameter \(\zeta\).

It can be identified with a character of \(A_{4n-3}^{(2)}\) affine Lie algebra. “Cauchy-type”: all minors have the same form. Very useful!
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Relation to three-colour model

Let

\[ p_n(\zeta) = \text{elementary factor} \]

\[ \times S_n\left(\underbrace{2\zeta + 1, \ldots, 2\zeta + 1}_{n+1}, \underbrace{\frac{\zeta}{\zeta + 2}, \ldots, \frac{\zeta}{\zeta + 2}}_{n}\right). \]

This is a polynomial in \( \zeta \) of degree \( n(n + 1)/2 \).

**Result:** \( Z_n(t_0, t_1, t_2) \) is a linear combination (with elementary coefficients) of \( p_{n-1}(\zeta) \) and \( p_{n-1}(1/\zeta) \), where

\[ \frac{(t_0 t_1 + t_0 t_2 + t_1 t_2)^3}{(t_0 t_1 t_2)^2} = \frac{2(\zeta^2 + 4\zeta + 1)^3}{\zeta(\zeta + 1)^4}. \]
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More precisely . . .

Suppose $n \equiv 0 \mod 6$ and

$$\frac{(t_0 t_1 + t_0 t_2 + t_1 t_2)^3}{(t_0 t_1 t_2)^2} = \frac{2(\zeta^2 + 4\zeta + 1)^3}{\zeta(\zeta + 1)^4}.$$ 

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Note that only asymmetry comes from $t_0$. 

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Table of $p_n$

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<thead>
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<tr>
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</tr>
<tr>
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Plot of the 105 complex zeroes of $p_{14}$. 
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Painlevé VI

PVI is the nonlinear ODE
(Painlevé, Fuchs, Gambier 1900–1910)

\[
\frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2
\]

\[ - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \]

\[ + \frac{y(y-1)(y-t)}{2t^2(t-1)^2} \left( \alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right). \]

Most general second order ODE such that all movable singularities are poles.

Special functions of the 21st Century?
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Bäcklund transformations

If \( y = y(t) \) solves PVI, then so does \( t/y \), with parameters \( \alpha \leftrightarrow -\beta, \gamma \leftrightarrow \frac{1}{2} - \delta \).

Such Bäcklund transformations \( y \mapsto F(t, y, y') \) generate group (extended affine Weyl group) containing \( \mathbb{Z}^4 \).

Given a “seed” solution \( y = y_{0000} \), acting with \( \mathbb{Z}^4 \) gives new solutions \( y_{k_1k_2k_3k_4} \).
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Picard’s seed solution

When \( \alpha = \beta = \gamma = 0, \delta = 1/2 \), PVI can be solved in terms of elliptic functions (Picard, 1889).

Picard’s solutions include the algebraic solution

\[
y^4 - 4ty^3 + 6ty^2 - 4ty + t^2 = 0,
\]

which is parametrized by

\[
y = \frac{\zeta(\zeta + 2)}{2\zeta + 1}, \quad t = \frac{\zeta(\zeta + 2)^3}{(2\zeta + 1)^3}.
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If we choose \( y_{0000} = y \), what is \( y_{k_1k_2k_3k_4} \)?
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Example

\[
y_{1,2,-3,1} = \frac{\zeta(\zeta + 2)(\zeta^3 + 3\zeta^2 + 3\zeta + 5)(5\zeta^3 + 15\zeta^2 + 7\zeta + 1)}{(2\zeta + 1)(\zeta^3 + 7\zeta^2 + 15\zeta + 5)(5\zeta^3 + 3\zeta^2 + 3\zeta + 1)}
\]

The non-trivial factors are called \textit{tau functions}.

Note that

\[
5\zeta^3 + 15\zeta^2 + 7\zeta + 1 = p_2(\zeta).
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\textbf{Result:} There is a 4-dim cone in \(\mathbb{Z}^4\) where tau functions are given by specializations of \(S_n\).
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Then, if \(k_i\) are non-negative integers with \(k_0 + k_1 + k_2 + k_3 = 2n - 1\),

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Tau functions in a cone

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Tau functions satisfy bilinear recursions. For instance, it follows that

\[ p_{n+1}(\zeta)p_{n-1}(\zeta) = A_n(\zeta)p_n(\zeta)^2 + B_n(\zeta)p_n(\zeta)p'_n(\zeta) + C_n(\zeta)p'_n(\zeta)^2 + D_n(\zeta)p_n(\zeta)p''_n(\zeta). \]

with explicit coefficients.

Conjectured by Bazhanov and Mangazeev 2010.

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\[ p_{n+1}(\zeta)p_{n-1}(\zeta) = A_n(\zeta)p_n(\zeta)^2 + B_n(\zeta)p_n(\zeta)p'_n(\zeta) \]
\[ + C_n(\zeta)p'_n(\zeta)^2 + D_n(\zeta)p_n(\zeta)p''_n(\zeta). \]

with explicit coefficients.

Conjectured by Bazhanov and Mangazeev 2010.

Gives fast way of computing \( Z_n \).

Possibly, it can be used to prove our conjectured expression for the free energy.
Outline

1. Three-coloured chessboards
2. Combinatorial results
3. Symmetric polynomials
4. Painlevé VI
5. Future problems
Symmetry classes of three-coloured chessboards. For the six-vertex model, various classical Lie algebra characters appear (Kuperberg, Okada, ...). For the three-colour model, we expect various affine Lie algebras.

Macroscopic boundary effects. Arctic curves.

New phenomenon: Some tau functions for Painlevé VI are specialized affine Lie algebra characters, given by Cauchy-type determinant formulas. Does this happen for other solutions to Painlevé equations?
Questions

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