Triangulations of Cayley and Tutte polytopes

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July 30, 2012
Cayley’s theorem and Braun’s conjecture

Theorem (Cayley, 1857)
The number of integer sequences \((a_1, \ldots, a_n)\) such that \(1 \leq a_1 \leq 2\) and \(1 \leq a_i \leq 2a_{i-1}\) for \(i = 2, \ldots, n\), is equal to the total number of partitions of integers \(N \in \{0, 1, \ldots, 2^n - 1\}\) into parts \(1, 2, 4, \ldots, 2^{n-1}\).
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Conjecture (Braun, 2011)
Define the Cayley polytope \(C_n \subseteq \mathbb{R}^n\) by inequalities

\[
1 \leq x_1 \leq 2, \quad \text{and} \quad 1 \leq x_i \leq 2x_{i-1} \quad \text{for} \quad i = 2, \ldots, n.
\]

Then \(n! \text{vol } C_n\) is equal to the number of connected graphs on \(n + 1\) nodes.
Main result

**Theorem (K-Pak)**

Define the Tutte polytope $T_n(q, t) \subseteq \mathbb{R}^n$ (by inequalities or by vertices), $T_n(0, 1) = C_n$. Then

$$n! \text{vol} T_n(q, t) = \sum q^{k(G)-1} t^{|E(G)|},$$

where the sum is over all graphs on $n + 1$ nodes, and $k(G)$ is the number of connected components of $G$. 
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In other words, $n! \text{vol} T_n(q, t) = t^n T_{K_{n+1}}(1 + q/t, 1 + t)$, where $T_H(x, y)$ denotes the Tutte polynomial of the graph $H$. 
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We call $n! \text{ vol } P$ the normalized volume of $P \subseteq \mathbb{R}^n$. 
Triangulation of Cayley polytope

Conjecture (Braun, 2011)

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We will define:

- a map from connected graphs to (labeled) trees
- a map from trees to simplices

so that:

- the simplices triangulate $C_n$
- the normalized volume of each simplex is equal to the number of graphs that map into the corresponding tree
Connected graphs to trees: neighbors first search

- \( T \) labeled connected graph

  At each step, visit the previously unvisited neighbors of the active node in decreasing order of their labels; make the one with the smallest label the new active node.

  If all the neighbors of the active node have been visited, backtrack to the last visited node that has not been an active node, and make it the new active node.

The result is an ordering of the nodes and a search tree.

This is a variant of the neighbors first search introduced by Gessel and Sagan (1996).
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Example
Example
Example
Example
Example
Example
Cane paths

A cane path is an up-up-\ldots-up-down right path.
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**Fact**

Number of graphs with search tree $T$ is $2^\alpha(T)$, where $\alpha(T)$ is the number of cane paths in $T$. 
Fact

If the node \( v \) is visited \( i \)-th in the neighbors first search and \( j \) is the number of cane paths starting in \( v \), then the coordinate of \( v \) is \( \frac{x_i}{2^j} \).
Fact

If the node $v$ is visited $i$-th in the neighbors first search and $j$ is the number of cane paths starting in $v$, then the coordinate of $v$ is $x_i/2^j$. 
Trees to simplices

\[
\begin{align*}
1 & \leq \frac{x_8}{16} \leq \frac{x_{10}}{4} \leq \frac{x_7}{8} \leq \frac{x_9}{2} \leq x_{11} \leq \frac{x_3}{4} \leq \frac{x_5}{8} \leq \frac{x_4}{4} \leq \frac{x_6}{8} \leq \frac{x_2}{2} \leq x_1 \leq 2.
\end{align*}
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Trees to simplices

The result is a Schläfli orthoscheme with normalized volume equal to $2^{\alpha(T)}$. 

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The result is a Schläfli orthoscheme with normalized volume equal to $2^{\alpha(T)}$.

The resulting simplices triangulate Cayley’s polytope. So this proves Braun’s conjecture.
Triangulation of $C_3$
Another subdivision of $C_3$
Gayley polytope

Cayley polytope $\mathbf{C}_n$:

$$1 \leq x_1 \leq 2, \text{ and } 1 \leq x_i \leq 2x_{i-1} \text{ for } i = 2, \ldots, n$$

Its normalized volume is the number of connected graphs on $n + 1$ nodes.
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Gayley polytope $G_n$:

$$0 \leq x_1 \leq 2, \text{ and } 0 \leq x_i \leq 2x_{i-1} \text{ for } i = 2, \ldots, n$$

It is an orthoscheme with sides $2, 4, \ldots, 2^n$, so its normalized volume is $2^{\binom{n+1}{2}}$, i.e. the number of all graphs on $n+1$ nodes.
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Charles Mills Gayley (1858 – 1932), professor of English and Classics at UC Berkeley
Triangulation of Gayley polytope

Neighbors first search on a general graph: arrange connected components so that their maximal labels are decreasing from left to right, perform neighbors first search on each tree from left to right.
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Coordinates:
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Coordinates:

\[
\begin{align*}
0 & \leq \frac{x_{11}}{4} - x_8 \\
& \leq \frac{x_6}{4} - 1 \\
& \leq \frac{x_{10}}{2} - x_8 \\
& \leq \frac{x_5}{2} - 1 \\
& \leq x_7 - 1 \\
& \leq \frac{x_3}{4} - 1 \\
& \leq x_9 - x_8 \\
& \leq \frac{x_4}{4} - 1 \\
& \leq x_8 \\
& \leq \frac{x_2}{2} - 1 \\
& \leq x_1 - 1 \leq 1.
\end{align*}
\]
$t$-Cayley and $t$-Gayley polytope

Replace powers of 2 by powers of $1 + t$, $t > 0$:

- $t$-Cayley polytope $C_n(t)$:
  
  \[ 1 \leq x_1 \leq 1 + t, \quad \text{and} \quad 1 \leq x_i \leq (1 + t)x_{i-1} \quad \text{for} \quad i = 2, \ldots, n \]

- $t$-Gayley polytope $G_n(t)$:
  
  \[ 0 \leq x_1 \leq 1 + t, \quad \text{and} \quad 0 \leq x_i \leq (1 + t)x_{i-1} \quad \text{for} \quad i = 2, \ldots, n \]

- Coordinates of the form $x_i/2^j$ become $x_i/(1 + t)^j$.
- Coordinates of the form $x_l$ (for roots) become $tx_l$.
Normalized volumes

**Theorem**

The normalized volume of $C_n(t)$ is

$$\sum t^{|E(G)|},$$

where the sum is over all connected graphs $G$ on $n + 1$ nodes.

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where the sum is over all graphs $G$ on $n + 1$ nodes, i.e. $(1 + t)^{\binom{n+1}{2}}$. 
Tutte polytope: hyperplanes

Take $0 < q \leq 1$ and $t > 0$. Define Tutte polytope $T_n(q, t)$ by

$$x_n \geq 1 - q,$$

$$qx_i \leq q(1 + t)x_{i-1} - t(1 - q)(1 - x_{j-1}),$$

where $1 \leq j \leq i \leq n$ and $x_0 = 1$.

**Theorem**

The normalized volume of Tutte polytope is

$$\sum q^{k(G)-1} t|E(G)|,$$

where the sum is over all graphs on $n + 1$ nodes.
Define $V_n(t)$ as the set of points with properties $x_1 \in \{1, 1 + t\}$, $x_i \in \{1, (1 + t)x_{i-1}\}$ for $i = 2, \ldots, n$. 
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\[
\begin{array}{ccc}
1 + t & (1 + t)^2 & (1 + t)^3 \\
1 + t & (1 + t)^2 & 1 \\
1 + t & 1 & 1 + t \\
1 + t & 1 & 1 \\
1 & 1 + t & (1 + t)^2 \\
1 & 1 + t & 1 \\
1 & 1 & 1 + t \\
1 & 1 & 1 \\
\end{array}
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1 & 1 + t & 1 \\
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1 & 1 & 1 \\
\end{array}
\]

It is easy to see that $V_n(t)$ is the set of vertices of $C_n(t)$. 
Tutte polytope: vertices

Replace the trailing 1’s of each point in $V_n(t)$ by $1 - q$, denote the resulting set $V_n(q, t)$.

\[
\begin{array}{ccc}
1 + t & (1 + t)^2 & (1 + t)^3 \\
1 + t & (1 + t)^2 & 1 - q \\
1 + t & 1 & 1 + t \\
1 + t & 1 - q & 1 - q \\
1 & 1 + t & (1 + t)^2 \\
1 & 1 + t & 1 - q \\
1 & 1 & 1 + t \\
1 - q & 1 - q & 1 - q \\
\end{array}
\]

Then $V_n(q, t)$ is the set of vertices of $T_n(q, t)$. 
Triangulation of $T_2(q, t)$

$$(1 + t)^2$$

$1 + t$

$1$

$1 - q$

$1 - q$  $1$  $1 + t$
Theorem
Define polynomials $r_n(t)$, $n \geq 0$, by

$$r_0(t) = 1, \quad r_n(t) = - \sum_{j=1}^{n} \binom{n}{j} (1 + t)^{(j+1)/2} r_{n-j}(t).$$

Then

$$\sum t^{|E(G)|} = \sum_{j=0}^{n} \binom{n}{j} (1 + t)^{(j+1)/2} r_{n-j}(t),$$

where the sum is over connected graphs on $n + 1$ nodes.