Asymptotical behaviour of roots in infinite Coxeter groups

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Joint work with Christophe Hohlweg (UQÀM) and Jean-Philippe Labbé (FU Berlin)
What is a root system? (in this talk)

- $V$: a real vector space, of finite dimension $n$
- $B$: a symmetric bilinear form on $V$

Construction of a root system in $(V, B)$:

1. Start with a simple system $\Delta$:
   - $\Delta$ is a basis for $V$;
   - $\forall \alpha \in \Delta, B(\alpha, \alpha) = 1$;
   - $\forall \alpha \neq \beta \in \Delta$:
     - either $B(\alpha, \beta) = -\cos\left(\frac{\pi}{m}\right)$ for some $m \in \mathbb{Z}_{\geq 2}$,
     - or $B(\alpha, \beta) \leq -1$. 
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2. For each $\alpha \in \Delta$, define the $B$-reflection $s_{\alpha}$:

$$s_{\alpha} : V \rightarrow V$$

$$v \mapsto v - 2B(\alpha, v) \alpha.$$

Check: $s_{\alpha}(\alpha) = -\alpha$, and $s_{\alpha}$ fixes pointwise $\alpha^\perp$.

Notation: $S = \{ s_{\alpha}, \alpha \in \Delta \}$.

3. Construct the $B$-reflection group $W := \langle S \rangle$.

4. Act by $W$ on $\Delta$ to construct the root system

$$\Phi := W(\Delta).$$

Note: if $\rho = w(\alpha)$ (with $\alpha \in \Delta$), $ws_{\alpha}w^{-1}$ is the $B$-reflection associated to the root $\rho$. 
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Coxeter group and root system

Proposition (Krammer)

- \((W, S)\) is a Coxeter system.
- The order of \(s_\alpha s_\beta\) is \(m\) if \(B(\alpha, \beta) = -\cos(\pi / m)\), and \(\infty\) if \(B(\alpha, \beta) \leq -1\).
- Let \(\Phi^+ := \Phi \cap \text{cone}(\Delta)\). Then: \(\Phi = \Phi^+ \sqcup (-\Phi^+)\).

Note: Conversely, from any Coxeter system it is possible to construct a root system, using the classical geometric representation [Tits].
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Infinite root systems

Finite root systems are well studied:
\( \Phi \) is finite \( \iff \) \( W \) is finite (\( \iff \) \( B \) is positive definite).

What happens when \( \Phi \) is infinite?

Simplest example in rank 2:

Matrix of \( B \) in the basis \((\alpha, \beta)\):
\[
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}.
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What is a root system?

\[ \rho'_n = n\alpha + (n+1)\beta \]

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\[ s_\beta(\alpha) = \rho'_2 \]

\[ s_\beta(\alpha) = \rho'_3 \]

\[ s_\beta(\alpha) = \rho'_4 \]

\[ \beta = \rho'_1 \]

\[ \alpha = \rho_1 \]

\[ (a) \quad B(\alpha, \beta) = -1 \]

\[ s_\alpha(v) = v - 2B(v, \alpha)\alpha. \]
Observations

- The **norms** of the roots tend to $\infty$;
- The **directions** of the roots tend to the direction of the isotropic cone $Q$ of $B$:

\[
Q := \{ v \in V, \ B(v, v) = 0 \}.
\]

(in the example the equation is $v_\alpha^2 + v_\beta^2 - 2v_\alpha v_\beta = 0$, and $Q = \text{span}(\alpha + \beta)$.)
What if $B(\alpha, \beta) < -1$?

- Matrix of $B$: $\begin{bmatrix} 1 & \kappa \\ \kappa & 1 \end{bmatrix}$ with $\kappa < -1$. We write $\infty(\kappa)$.

- Then $Q$ is the union of 2 lines.
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\[ s_{\alpha} \quad s_{\beta} \]

\[ \alpha = \rho_1 \quad \beta = \rho'_1 \]

\[ \infty(-1.01) \]

\[ s \quad t \]
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$V = \{ v \in V \mid \ldots \}$

$\{ a \in \mathbb{R}^n \mid a \cdot v = 1 \}$

$\rho_n = (n+1)(n+1)$

How to see examples of higher rank?
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Affine hyperplane
\[ V_1 = \{ v \in V \mid \sum_{\alpha \in \Delta} v_{\alpha} = 1 \} \]

Normalized isotropic cone: \( \hat{Q} := Q \cap V_1 \)

Normalized roots
\[ \hat{\rho} := \rho / \sum_{\alpha \in \Delta} \rho_{\alpha} \]

(a) \( B(\alpha, \beta) = -1 \)
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Other examples of infinite root systems in rank 3 and 4
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(a) $B(\alpha, \beta) = -1$
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$\dim 2$

$\dim 3$

$\dim 4$

$\text{conv}(\Delta)$
The displayed size of a normalized root (in red in this last picture) is decreasing as the depth of the root is increasing.

\[ dp(\rho) = 1 + \min \{k \mid \rho = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} (\alpha_{k+1}), \alpha_1, \ldots, \alpha_k, \alpha_{k+1} \in \Delta \}. \]
The “limit roots” lie in the isotropic cone $Q$

**Theorem (Hohlweg-Labbé-R.)**

Let $\Phi$ be a root system for an (infinite) Coxeter group, and $(\rho_n)_{n \in \mathbb{N}}$ an injective sequence in $\Phi$. Then:

1. $||\rho_n||$ tends to $\infty$ (for any norm on $V$);
2. if the sequence of normalized root $\hat{\rho}_n$ has a limit $\ell$, then
   $$\ell \in \hat{Q} \cap \text{conv}(\Delta).$$

Known in other contexts:

- Root systems of Lie algebras (Kac, 1990)
- Imaginary cone for Coxeter groups (Dyer, 2011)

∽∽ **Problem:** understand the set of possible limits, i.e., the accumulation points of $\hat{\Phi}$:

$$E(\Phi) := \text{Acc} \left( \hat{\Phi} \right) \quad (\text{“limit roots”}).$$
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How to construct some particular limit roots

Take two roots $\rho_1, \rho_2$ in $\Phi$ get a rank 2 reflection subgroup of $W$, and a root subsystem $\Phi'$. Note:

- $\hat{\Phi}' \subset L(\hat{\rho}_1, \hat{\rho}_2)$;
- the isotropic cone for $\Phi'$ is $Q \cap \text{span}(\rho_1, \rho_2)$;
- $\Rightarrow$ Limit roots for $\Phi'$: $E(\Phi') = Q \cap L(\hat{\rho}_1, \hat{\rho}_2)$ (0, 1 or 2 points).
The dihedral limit roots

Definition
We define the set $E_2(\Phi)$ of dihedral limit roots for the root system $\Phi$ as the subset of $E(\Phi)$ formed by the union of the $E(\Phi')$, for $\Phi'$ a root subsystem of rank 2 of $\Phi$. Equivalently,

$$E_2(\Phi) := \bigcup_{\rho_1, \rho_2 \in \Phi} L(\hat{\rho}_1, \hat{\rho}_2) \cap Q.$$ 

Note: $E_2$ is countable.

Theorem (Hohlweg-Labbé-R.)
The set of dihedral limit roots $E_2$ is dense in $E$.

- $E$ is closed, so $E = \overline{E_2}$;
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Other properties, further questions

- How does $E$ behave in regard to restriction to parabolic subgroups ($E(\Phi_I) \neq E(\Phi) \cap V_I$ in general!)
- Natural action of $W$ on $E$, easy to describe geometrically... Faithfulness?
- Explain the fractal, self-similar shapes of the pictures! We can use the action to interpret this, but we only have conjectures.
- Take $x \in E$. Is it true that $\overline{W \cdot x} = E$?
- Study $\text{conv}(E)$, which equals the closure of Dyer’s “imaginary cone”.

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A fractal phenomenon?
(conjectures/questions, work in progress with Ch. Hohlweg)

\[ \text{If } \hat{Q} \subseteq \text{conv}(\Delta), \text{ then } E(\Phi) = \hat{Q} \ ? \]

\[ \text{In general: } E(\Phi) = \hat{Q} \setminus \text{all the images by the action of } W \text{ of the parts of } \hat{Q} \text{ outside the simplex, i.e.:} \]

\[ E(\Phi) = \hat{Q} \cap \bigcap_{w \in W} w \cdot \text{conv}(\Delta) \ ? \]