Classification of Ehrhart polynomials of integral simplices

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Abstract. Let $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ be the δ-vector of an integral convex polytope $\mathcal{P}$ of dimension d. First, by using two well-known inequalities on δ-vectors, we classify the possible δ-vectors with $\sum_{i=0}^{d} \delta_i \leq 3$. Moreover, by means of Hermite normal forms of square matrices, we also classify the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$. In addition, for $\sum_{i=0}^{d} \delta_i \geq 5$, we characterize the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i$ is prime.

Résumé. Soit $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ le δ-vecteur d'un polytope intégrante de dimension d. Tout d'abord, en utilisant deux bien connus des inégalités sur δ-vecteurs, nous classons les δ-vecteurs possibles avec $\sum_{i=0}^{d} \delta_i \leq 3$. En outre, par le biais de Hermite formes normales, nous avons également classer les δ-vecteurs avec $\sum_{i=0}^{d} \delta_i = 4$. De plus, pour $\sum_{i=0}^{d} \delta_i \geq 5$, nous caractérisons les δ-vecteurs des simplex inégalités lorsque $\sum_{i=0}^{d} \delta_i$ est premier.

Keywords: Ehrhart polynomial, δ-vector, integral convex polytope, integral simplex.

1 Introduction

One of the most attractive problems on enumerative combinatorics of convex polytopes is to find a combinatorial characterization of the Ehrhart polynomials of integral convex polytopes. In particular, the δ-vectors of integral simplices play an important and interesting role.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* polytope, i.e., a convex polytope any of whose vertices has integer coordinates, of dimension d, and let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n, we define

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N| \quad \text{and} \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$, $n(\mathcal{P} \setminus \partial\mathcal{P}) = \{n\alpha : \alpha \in \mathcal{P} \setminus \partial\mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X. The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [2], who established the following fundamental properties:

(0.1) $i(\mathcal{P}, n)$ is a polynomial in n of degree d;

(0.2) $i(\mathcal{P}, 0) = 1$;

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(0.3) (loi de réciprocité) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. We refer the reader to [1, Chapter 3], [3, Part II] or [10, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \dots$ of integers by the formula

$$(1-\lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n = \sum_{i=0}^{\infty} \delta_i \lambda^i. \tag{1}$$

Then the basic facts (0.1) and (0.2) on $i(\mathcal{P}, n)$ together with a fundamental result on generating function [10, Corollary 4.3.1] guarantee that $\delta_i = 0$ for every $i > d$. We say that the sequence (resp. the polynomial)

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d) \quad \left(\text{resp. } \delta_{\mathcal{P}}(t) = \sum_{i=0}^{d} \delta_i t^i\right)$$

which appears in 1 is called the *δ-vector* (resp. the *δ-polynomial*) of $\mathcal{P}$. Alternate names of δ-vectors are, for example, *Ehrhart h-vector*, *Ehrhart δ-vector* or *h^*-vector*. By the reciprocity law (0.3), one has

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n)\lambda^n = \frac{\sum_{i=0}^{d} \delta_{d-i}\lambda^{i+1}}{(1-\lambda)^{d+1}}. \tag{2}$$

The following properties on δ-vectors are well known:

- By 1, one has $\delta_0 = i(\mathcal{P}, 0) = 1$ and $\delta_1 = i(\mathcal{P}, 1) - (d+1) = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$.
- By 2, one has $\delta_d = i^*(\mathcal{P}, 1) = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. In particular, we have $\delta_1 \geq \delta_d$.
- Each δ_i is nonnegative [9].
- If $\delta_d \neq 0$, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i \leq d-1$ [4].
- It follows from 2 that

$$\max\{j : \delta_j \neq 0\} + \min\{k : k(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset\} = d + 1.$$

- When $d = N$, the leading coefficient of $i(\mathcal{P}, n)$, which coincides with $\sum_{i=0}^{d} \delta_i / d!$, is equal to the usual volume of $\mathcal{P}$ [10, Proposition 4.6.30]. In general, the positive integer $\sum_{i=0}^{d} \delta_i$ is called the *normalized volume* of $\mathcal{P}$.

On the classification problem of the δ-vectors of integral convex polytopes, when we consider the possible δ-vectors of small dimensions, all of them are essentially given in [8] when $d = 2$. However, the possible δ-vectors are presumably open when $d \geq 3$.

In this article, we discuss the δ-vectors of small normalized volumes. In particular, the δ-vectors of integral simplices play a crucial role when $\sum_{i=0}^{d} \delta_i \leq 4$.

A brief overview of this article is as follows. After reviewing some well-known technique how to compute the δ-vectors of integral simplices in Section 1, we study the possible δ-vectors of $\sum_{i=0}^{d} \delta_i \leq 3$

by using two well-known inequalities on δ-vectors in Section 2. In Section 3, we consider the case where $\sum_{i=0}^{d} \delta_i = 4$ by considering all the δ-vectors of all the integral simplices up to some equivalence. In Section 4, we discuss the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i$ is prime and we classify the possible δ-vectors of integral simplices of $\sum_{i=0}^{d} \delta_i = 5$ and 7.

2 Review on the computation of the δ-vectors of integral simplices

Before proving our theorems, we recall a combinatorial technique to compute the δ-vector of an integral simplex.

Given an integral simplex $\mathcal{F} \subset \mathbb{R}^N$ of dimension d with the vertices $v_0, v_1, \ldots, v_d$, we set

$$S = \left\{ \sum_{i=0}^{d} r_i (v_i, 1) \in \mathbb{R}^{N+1} : 0 \leq r_i < 1 \right\} \cap \mathbb{Z}^{N+1} \text{ and}$$

$$S^* = \left\{ \sum_{i=0}^{d} r_i (v_i, 1) \in \mathbb{R}^{N+1} : 0 < r_i \leq 1 \right\} \cap \mathbb{Z}^{N+1}.$$

We define the degree of an integer point $(\alpha, n) \in S$ $((\alpha, n) \in S^*)$ with $\deg(\alpha, n) = n$, where $\alpha \in \mathbb{Z}^N$ and $n \in \mathbb{Z}_{\geq 0}$. Let $\delta_i = |\{\alpha \in S : \deg \alpha = i\}|$ and $\delta_i^* = |\{\alpha \in S^* : \deg \alpha = i\}|$. Then we have

Lemma 2.1 *Work with the same notation as above. Then we have*

(a)

$$\sum_{n=0}^{\infty} i(\mathcal{F}, n)\lambda = \frac{\delta_0 + \delta_1 \lambda + \cdots + \delta_d \lambda^d}{(1-\lambda)^{d+1}};$$

(b)

$$\sum_{n=0}^{\infty} i(\mathcal{F}^*, n)\lambda = \frac{\delta_1^* \lambda + \cdots + \delta_{d+1}^* \lambda^{d+1}}{(1-\lambda)^{d+1}};$$

(c)

$$\delta_i^* = \delta_{d+1-i} \textit{ for } 1 \leq i \leq d+1.$$

We also recall the following

Lemma 2.2 [1, Theorem 2.4] *Suppose that* $(\delta_0, \delta_1, \ldots, \delta_d)$ *is the* δ*-vector of an integral convex polytope of dimension* d*. Then there exists an integral convex polytope of dimension* $d+1$ *whose* δ*-vector is* $(\delta_0, \delta_1, \ldots, \delta_d, 0)$.

Note that the required δ-vector is obtained by forming the pyramid over the integral convex polytope.

3 Two well-known inequalities on δ-vectors

In this section, we present two well-known inequalities on δ-vectors. By using them, we give the complete classification of the possible δ-vectors of integral convex polytopes with $\sum_{i=0}^{d} \delta_i \leq 3$.

Let $s = \max\{i : \delta_i \neq 0\}$. Stanley [11] shows the inequalities

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq [s/2] \tag{3}$$

by using the theory of Cohen–Macaulay rings. On the other hand, the inequalities

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_{i+1}, \quad 0 \leq i \leq [(d-1)/2] \tag{4}$$

appear in [4, Remark (1.4)]. A proof of the inequalities 4 is given by using combinatorics on convex polytopes.

Somewhat surprisingly, when $\sum_{i=0}^{d} \delta_i \leq 3$, the above inequalities 3 together with 4 give a characterization of the possible δ-vectors. In fact,

Theorem 3.1 *Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$, which satisfies $\sum_{i=0}^{d} \delta_i \leq 3$, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $(\delta_0, \delta_1, \ldots, \delta_d)$ satisfies all inequalities 3 and 4. Moreover, all integral convex polytopes can be chosen to be simplices.*

Note that the "Only if" part of Theorem 3.1 is obvious. Thus we may show the "If" part. Moreover, when $\sum_{i=0}^{d} \delta_i = 1$, it is obvious that the possible sequence is only $(1, 0, \ldots, 0)$ and this is a δ-vector of some integral convex polytope, in particular, integral simplex. A sketch of a proof of the "If" part with $\sum_{i=0}^{d} \delta_i = 2$ or 3 is as follows:

- When $\sum_{i=0} \delta_i = 2$, the possible integer sequence looks like $(1, 0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$, where $\underbrace{1}_{i}$ means that $\delta_i = 1$. On the other hand, we have $i \leq \lfloor (d+1)/2 \rfloor$ by 4. Hence we may find an integral convex polytope, in particular, an integral simplex, whose δ-vector coincides with that. Note that we may construct such simplex with $i = \lfloor (d+1)/2 \rfloor$ by virtue of Lemma 2.2.
- When $\sum_{i=0} \delta_i = 3$, we have two candidates of the possible integer sequences.
  - When $(1, 0, \ldots, 0, \underbrace{2}_{i}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$, similar discussions to the previous case can be applied.
  - When $(1, 0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$, from 3 and 4, we have the inequalities
    $$1 \leq i < j \leq d, \quad 2i \leq j \ \text{ and } \ i + j \leq d + 1.$$
    Once we can find an integral simplex whose δ-vector coincides with that with $2i = j$ and $i + j = d + 1$, we can also construct an integral simplex whose δ-vector is that for any integers i and j with the above inequalities.

On the other hand, the following example shows that Theorem 3.1 is no longer true for the case where $\sum_{i=0}^{d} \delta_i = 4$.

Example 3.2 The integer sequence $(1,0,1,0,1,1,0,0)$ cannot be the δ-vector of any integral convex polytope of dimension 7, although this satisfies the inequalities 3 and 4. In fact, suppose, on the contrary, that there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension 7 with $(\delta_0, \delta_1, \dots, \delta_7) = (1,0,1,0,1,1,0,0)$ its δ-vector. Since $\delta_1 = 0$, we know that $\mathcal{P}$ is a simplex. Let $v_0, v_1, \dots, v_7$ be the vertices of $\mathcal{P}$. By using Lemma 2.1, one has $S = \{(0,\dots,0), (\alpha, 2), (\beta, 4), (\gamma, 5)\}$ and $S^* = \{(\alpha', 3), (\beta', 4), (\gamma', 6), (\sum_{i=0}^{7} v_i, 7)\}$. Write $\alpha' = \sum_{i=0}^{7} r_i v_i$ with each $0 < r_i \leq 1$. Since $(\alpha', 3) \notin S$, there is $0 \leq j \leq 7$ with $r_j = 1$. If there are $0 \leq k < \ell \leq 7$ with $r_k = r_\ell = 1$, say, $r_0 = r_1 = 1$, then $0 < r_q < 1$ for each $2 \leq q \leq 7$ and $\sum_{i=2}^{7} r_i = 1$. Hence $(\alpha' - v_0 - v_1, 1) \in S$, a contradiction. Thus there is a unique $0 \leq j \leq 7$ with $r_j = 1$, say, $r_0 = 1$. Then $\alpha = \sum_{i=1}^{7} r_i v_i$ and $\gamma = \sum_{i=1}^{7} (1 - r_i) v_i$. Let $\mathcal{F}$ denote the facet of $\mathcal{P}$ whose vertices are $v_1, v_2, \dots, v_7$ with $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \dots, \delta'_6) \in \mathbb{Z}^7$. Then $\delta'_2 = \delta'_5 = 1$. Since $\delta'_i \leq \delta_i$ for each $0 \leq i \leq 6$, it follows that $\delta(\mathcal{F}) = (1,0,1,0,0,1,0)$. This contradicts the inequalities 3.

4 Hermite normal forms with a given δ-vector

In this section, we give the complete classification of the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$ by means of Hermite normal forms. Moreover, it turns out that all the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$ can be chosen to be integral simplices.

Let $\mathbb{Z}^{d \times d}$ denote the set of $d \times d$ integer matrices. Recall that a matrix $A \in \mathbb{Z}^{d \times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral convex polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$ of dimension d, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and an integral vector $w \in \mathbb{Z}^d$, such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$. Clearly, if $\mathcal{P}$ and $\mathcal{Q}$ are unimodularly equivalent, then $\delta(\mathcal{P}) = \delta(\mathcal{Q})$. Conversely, given a vector $v \in \mathbb{Z}_{\geq 0}^{d+1}$, it is natural to ask what are all the integral convex polytopes $\mathcal{P}$ under unimodular equivalence, such that $\delta(\mathcal{P}) = v$. We focus on this problem for simplices with one vertex at the origin. In addition, we do not allow any shifts in the equivalence, i.e., integral convex polytopes $\mathcal{P}$ and $\mathcal{Q}$ of dimension d are equivalent if there exists a unimodular matrix U, such that $\mathcal{Q} = f_U(\mathcal{P})$.

For discussing the representative under this equivalence of the integral simplices with one vertex at the origin, we consider Hermite normal forms of square matrices.

Let $\mathcal{P}$ be an integral simplex in $\mathbb{R}^d$ of dimension d with the vertices $v_0, v_1, \dots, v_d$, where $v_0 = (0,\dots,0)$. Define $M(\mathcal{P}) \in \mathbb{Z}^{d \times d}$ to be the matrix with the row vectors $v_1, \dots, v_d$. Then we have the following connection between the matrix $M(\mathcal{P})$ and the δ-vector of $\mathcal{P}$: $|\det(M(\mathcal{P}))| = \sum_{i=0}^{d} \delta_i$. In this setting, $\mathcal{P}$ and $\mathcal{P}'$ are equivalent if and only if $M(\mathcal{P})$ and $M(\mathcal{P}')$ have the same Hermite normal form, where the *Hermite normal form* of a nonsingular integral square matrix B is a unique nonnegative lower triangular matrix $A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{d \times d}$ such that $A = BU$ for some unimodular matrix $U \in \mathbb{Z}^{d \times d}$ and $0 \leq a_{ij} < a_{ii}$ for all $1 \leq j < i$. (See, e.g., [7, Chapter 4].) In other words, we can pick the Hermite normal form as the representative in each equivalence class. By considering the δ-vectors of all the integral simplices arising from the Hermite normal forms M with $\det(M) = 4$, we obtain the following

Theorem 4.1 *Let $1+t^{i_1}+t^{i_2}+t^{i_3}$ be a polynomial in t with $1 \leq i_1 \leq i_2 \leq i_3 \leq d$. Then there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-polynomial coincides with $1+t^{i_1}+t^{i_2}+t^{i_3}$ if and only if (i_1, i_2, i_3) satisfies*

$$i_3 \leq i_1 + i_2, \; i_1 + i_3 \leq d+1, \; i_2 \leq \lfloor (d+1)/2 \rfloor \tag{5}$$

and an additional condition

$$2i_2 \leq i_1 + i_3 \quad or \quad i_2 + i_3 \leq d+1. \tag{6}$$

Moreover, all integral convex polytopes can be chosen to be simplices.

Note that the inequalities 5 follow from the inequalities 3 and 4, that is to say, the condition 5 is automatically a necessary condition. Thus, the condition 6 is the new necessary condition on δ-vectors when $\sum_{i=0}^{d} \delta_i = 4$.

A sketct of a proof of this theorem is as follows:

- On the “If” part, we may construct integral simplices whose δ-polynomials look like $1+t^{i_1}+t^{i_2}+t^{i_3}$ satisfying 5 and 6. By characterizing the possible δ-vectors of all the integral simplices arising from the Hermite normal forms M with $\det(M) = 4$, we can find such integral simplices.
- On the “Only if” part, we may show that if a polynomial $1 + t^{i_1} + t^{i_2} + t^{i_3}$ satisfies 5 but does not satisfy 6, then $i_1 > 1$, i.e., an integral convex polytope with this δ-polynomial is always a simplex. By characterizing the possible δ-vectors of all the integral simplices arising from the Hermite normal forms M with $\det(M) = 4$, we can say that there exists no integral simplex whose δ-polynomial is equal to $1 + t^{i_1} + t^{i_2} + t^{i_3}$ not satisfying 6.

Example 4.2 As we see in Example 3.2, the integer sequence $(1, 0, 1, 0, 1, 1, 0, 0)$ cannot be the δ-vector of any integral convex polytope of dimension 7. In fact, since $8 = 2i_2 > i_1 + i_3 = 7$ and $9 = i_2 + i_3 > 8$, there exists no integral convex polytope of dimension 7 whose δ-polynomial is $1+t^2+t^4+t^5$. On the other hand, there exists an integral convex polytope of dimension 8 whose δ-vector is $(1, 0, 1, 0, 1, 1, 0, 0, 0)$ since $9 = i_2 + i_3 = d + 1$.

Remark 4.3 We see that all the possible δ-vectors can be obtained by integral simplices when $\sum_{i=0}^{d} \delta_i \leq 4$. However, the δ-vector $(1, 3, 1)$ cannot be obtained from any integral simplex, while this is a possible δ-vector of some integral convex polytope of dimension 2. In fact, suppose that $(1, 3, 1)$ can be obtained from a simplex. Since $\min\{i : \delta_i \neq 0, i > 0\} = 1$ and $\max\{i : \delta_i \neq 0\} = 2$, one has $\min\{i : \delta_i \neq 0, i > 0\} = 3 - \max\{i : \delta_i \neq 0\}$, which implies that the assumption of [5, Theorem 2.3] is satisfied. Thus the δ-vector must be shifted symmetric, a contradiction.

5 Ehrhart polynomials of integral simplices with prime volumes

From the previous two sections, we know that all the possible δ-vectors with $\sum_{i=0}^{d} \delta_i \leq 4$ can be obtained by integral simplices, while this does not hold when $\sum_{i=0}^{d} \delta_i = 5$. Therefore, for the further classifications of the δ-vectors with $\sum_{i=0}^{d} \delta_i \geq 5$, it is natural to investigate the δ-vectors of integral simplices. In this section, we establish the new equalities and inequalities on δ-vectors for integral simplices when $\sum_{i=0}^{d} \delta_i$ is prime. Moreover, by using them, we classify all the possible δ-vectors of integral simplices with $\sum_{i=0}^{d} \delta_i = 5$ and 7.

The following equalities or inequalities are new constraints on the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i$ is prime.

Theorem 5.1 *Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral simplex of dimension d and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$ its δ-vector. Suppose that $\sum_{i=0}^{d} \delta_i = p$ is an odd prime number. Let $i_1, \dots, i_{p-1}$ be the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{p-1}}$ with $1 \le i_1 \le \cdots \le i_{p-1} \le d$. Then*

(a)

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \le d + 1;$$

(b)

$$i_k + i_\ell \ge i_{k+\ell} \ \ \textit{for} \ \ 1 \le k \le \ell \le p - 1 \ \ \textit{with} \ \ k + \ell \le p - 1.$$

Example 5.2 When $\sum_{i=0}^{d} \delta_i$ is not prime, Theorem 5.1 is not true. In fact, by virtue of Theorem 4.1, $(1, 1, 0, 2, 0, 0)$ is the δ-vector of some integral simplex of dimension 5. However, one has $2 = i_1 + i_1 < i_2 = 3$.

A proof of this theorem is given by considering the additive group S (appeared in Section 2) associated with an integral simplex with prime normalized volume. Since the order of S is equal to the normalized volume of $\mathcal{P}$, S is nothing but a cyclic group $\mathbb{Z}/p\mathbb{Z}$. By studying S and the degrees of its elements, we obtain the statements (a) and (b). Note that (b) follows from [6, Theorem 2.2], known as *Cauchy–Davenport theorem*.

As an application of Theorem 5.1, we give a complete characterization of the possible δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i = 5$ and 7.

Corollary 5.3 *Given a finite sequence $(\delta_0, \delta_1, \dots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$, which satisfies $\sum_{i=0}^{d} \delta_i = 5$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \dots, \delta_d)$ if and only if $i_1, \dots, i_4$ satisfy*

$$i_1 + i_4 = i_2 + i_3 \le d + 1, \ \ 2i_1 \ge i_2 \ \ \textit{and} \ \ i_1 + i_2 \ge i_3,$$

where $i_1, \dots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \le i_1 \le \cdots \le i_4 \le d$.

Corollary 5.4 *Given a finite sequence $(\delta_0, \delta_1, \dots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$, which satisfies $\sum_{i=0}^{d} \delta_i = 7$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \dots, \delta_d)$ if and only if $i_1, \dots, i_6$ satisfy*

$$i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \le d + 1, \ \ i_1 + i_\ell \ge i_{\ell+1} \ \ \textit{for} \ \ 1 \le \ell \le 3 \ \ \textit{and} \ \ 2i_2 \ge i_4,$$

where $i_1, \dots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ with $1 \le i_1 \le \cdots \le i_6 \le d$.

By virtue of Theorem 5.1, the "Only if" parts of Corollary 5.3 and Corollary 5.4 are obvious.

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