# A BRIEF INTRODUCTION TO ENRIQUES SURFACES

IGOR V. DOLGACHEV

*To Shigeru Mukai on the occasion of his 60th birthday*

Abstract. This is a brief introduction to the theory of Enriques surfaces over arbitrary algebraically closed fields.

## Contents



## 1. Introduction

This is a brief introduction to the theory of Enriques surfaces. Over $\mathbb{C}$, this theory can be viewed as a part of the theory of K3-surfaces, namely, the theory of pairs $(X, \iota)$ consisting of a K3-surface $X$ and a fixed-point-free involution $\iota$ on $X$. It also can be viewed as the theory of lattice polarizes K3 surfaces, where the lattice $M$ is the lattice $U(2) \oplus E_8(2)$ with the standard notation of quadratic hyperbolic lattices [17]. The account of this theory can be found in many places, for example, in [33]. We intentionally omit this theory and try to treat the theory of Enriques surfaces without appeal to their K3-covers. This makes more sense when we do not restrict ourselves with the basic field of complex numbers and take for the ground field an algebraically closed field of arbitrary characteristic $p \geq 0$. This approach to Enriques surfaces follows the book of Cossec and the author [12], the new revised, corrected and extended version is in preparation [13].

The author shares his love of Enriques surfaces with Shigeru Mukai and is happy to dedicate this survey to him. He is also grateful to the organizers of the conference, and especially to S. Kondō, for the

invitation and opportunity to give a series of lectures on this topic. Discussions with D. Allcock, S. Mukai and H. Ohashi during the conference were very helpful for writing this survey.

## 2. History

Let $S$ be a smooth projective surface over an algebraically closed field $\Bbbk$. We use the customary notations from the theory of algebraic surfaces. Thus we reserve $D$ to denote a divisor on $S$ and very often identify it with the divisor class modulo linear equivalence. The group of such divisor classes is the Picard group $\mathrm{Pic}(S)$. The group of divisor classes with respect to numerical equivalence is denoted by $\mathrm{Num}(S)$. We denote by $|D|$ the linear system of effective divisors linearly equivalent to $D$. We set

$$h^i(D) = \dim_{\Bbbk} H^i(S, \mathcal{O}_X(D)),\ p_g = h^0(K_S) = h^2(\mathcal{O}_S),\ q = h^1(\mathcal{O}_S).$$

We use the Riemann-Roch Theorem

$$h^0(D) - h^1(D) + h^2(D) = \tfrac{1}{2}(D^2 - D \cdot K_S) + 1 - q + p_g$$

and Serre's duality $h^i(D) = h^{2-i}(K_S - D)$.

The theory of minimal models provides us with a birational morphism $f : S \to S'$ such that either the canonical class $K_{S'}$ is nef (i.e. $K_{S'} \cdot C \geq 0$ for any effective divisor $C$), or $S'$ is a projective bundle over $\mathrm{Spec}\, \Bbbk$, or over a smooth projective curve $B$.

If the latter happens, the surface $S$ is called *ruled* and, if $S' = \mathbb{P}^2$ or $B \cong \mathbb{P}^1$, it is called *rational*. It is easy to see that a rational surface has $p_g = q = 0$. In 1894, G. Castelnuovo tried to prove that the converse is true. He could not do it without assuming additionally that $h^0(2K_S) = 0$ [6]. He used the so called the *termination of adjoints* (showing that, under this assumption $|C + mK_S| = \emptyset$ for any curve $C$ and large $m$, and, if $m$ is minimal with this property, the linear system $|C + (m-1)K_S|$ gives a pencil of rational curves on $S$ that implies the rationality).

The modern theory of minimal models provides us with a simple proof of Castlelnuovo's Theorem. First use that $D^2 \geq 0$ for any nef divisor $D$.[1] Thus, if $S$ is not rational, then we may assume that $K_S$ is nef, hence $K_S^2 \geq 0$. By Riemann-Roch, $h^0(-K) + h^0(2K) \geq K_S^2 + 1$ implies $h^0(-K) \geq 1$, thus $-K_S \geq 0$ cannot be nef unless $K_S = 0$ in which case $p_g = 1$.

[1] In fact, take any positive number $N$ and an ample divisor $A$, then $ND + A$ is ample and $(ND+A)^2 = N^2D^2 + 2NA \cdot D + A^2$ must be positive, however if $D^2 < 0$ and $N$ is large enough, we get a contradiction.

Still not satisfied, Castelnuovo tried to avoid the additional assumption $h^0(2K_S) > 0$. He discussed this problem with F. Enriques during their walks under arcades of Bologna. Each found an example of a surface with $p_g = q = 0$ with some effective multiple of $K_S$. Since the termination of the adjoint is a necessary condition for rationality, the surfaces are not rational.

The example of Enriques is a smooth normalization of a non-normal surface $X$ of degree 6 in $\mathbb{P}^3$ that passes with multiplicity 2 through the edges of the coordinate tetrahedron. Its equation can be reduced by changing coordinates to the form

$$F = x_1^2x_2^2x_3^2 + x_0^2x_2^2x_3^2 + x_0^2x_1^2x_3^2 + x_0^2x_1^2x_2^2 + x_0x_1x_2x_3q(x_0, x_1, x_2, x_3) = 0, \tag{1}$$

where $q$ is a non-degenerate quadratic form.

The surface $X$ has *ordinary singularities*: a double curve $\Gamma$ with ordinary triple points that are also triple points of the surface, and a number of pinch points. The completion of a local ring of $X$ at a general point of $\Gamma$ is isomorphic to $\Bbbk[[t_1, t_2, t_3]]/(t_1t_2)$, at triple points $\Bbbk[[t_1, t_2, t_3]]/(t_1t_2t_3)$, and at pinch points $\Bbbk[[t_1, t_2, t_3]]/(t_1^2 + t_2^2t_3)$. Let $\pi : S \to X$ be the normalization. The pre-image of a general point on $\Gamma$ consists of two points, the pre-image of a triple point consists of three points, and the pre-image of a pinch point consists of one point.

Let $\mathfrak{c}_0 = \mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_S, \mathcal{O}_X)$. It is an ideal in $\mathcal{O}_X$, called the *conductor ideal*. It is equal to the annihilator ideal of $\pi_*\mathcal{O}_S/\mathcal{O}_X$. Let $\mathfrak{c} = \mathfrak{c}_0\mathcal{O}_S$. This is an ideal in $\mathcal{O}_S$ and $\pi_*(\mathfrak{c}) = \mathfrak{c}_0$. The duality theorem for finite morphisms gives an isomorphism

$$\omega_S = \mathfrak{c} \otimes \pi^*\mathcal{O}_X(d - 4), \tag{2}$$

where $\omega_S$ is the canonical sheaf on $S$ and $d = \deg X$ (in our case $d = 6$). In particular, it implies that $\mathfrak{c}$ is an invertible sheaf isomorphic to $\mathcal{O}_S(-\Delta)$, where $\Delta$ is an effective divisor on $S$. Under the assumption on singularities, $\mathfrak{c}_0 \cong \mathcal{O}_\Gamma$, hence $\Delta = \pi^{-1}(\Gamma)$.

Returning to our sextic surface, we find that $\deg \Gamma = 6$, the number $t$ of triple points is equal to 4 and each edge contains 4 pinch points. The canonical class formula shows that $\omega_S \cong \pi^*\mathcal{O}_X(2)(-\Delta)$. The projection formula gives $\pi_*\omega_S \cong \mathcal{O}_X(2) \otimes \mathcal{I}_\Gamma$ (we use that $\mathfrak{c}_0$ annihilates $\pi_*\mathcal{O}_S/\mathcal{O}_X$). Since $\deg \Gamma = 6$, $\omega_S$ has no sections, i.e. $p_g(S) = 0$. Also, the exact sequence

$$0 \to \mathcal{J}_\Gamma(2) \to \mathcal{O}_X(2) \to \mathcal{O}_\Gamma(2) \to 0$$

allows us to check that $H^1(S, \omega_S) \cong H^1(X, \mathcal{J}_\Gamma(2)) = 0$ (we use that the curve $\Gamma$ is projectively normal), i.e. $q = 0$.

Now,

$$\omega_S^{\otimes 2} \cong \mathcal{O}_S(2K_S) \cong \mathfrak{c}^{\otimes 2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^3}(2d-8)$$
$$\cong \pi^*\mathcal{O}_X(4)(-2\Delta) = \pi^*(\mathcal{O}_X(4) \otimes \mathcal{J}_\Gamma^{<2>}),$$

where $\mathcal{J}_\Gamma^{<2>}$ is the second symbolic power of the ideal sheaf $\mathfrak{c}_0 = \mathcal{I}_\Gamma$, the sheaf of functions vanishing with order $\geq 2$ at a general point of each irreducible component of $\Gamma$. The global section of the right-hand side defined by the union of four coordinate planes shows that $h^0(2K_S) > 0$, in fact, $\omega_S^{\otimes 2} \cong \mathcal{O}_S$.

It follows from the description of singularities of the sextic that the pre-image of each edge of the tetrahedron, i.e. an irreducible component of the double curve $\Gamma$, is an elliptic curve. The pre-image of the section of the surface with a face of the tetrahedron is the sum of three elliptic curves $F_1 + F_2 + F_3$, where $F_i \cdot F_j = 1, i \neq j$ and $F_i^2 = 0$. The pre-images of the opposite edges are two disjoint elliptic curves $F_i + F_i'$. The preimage of the pencil of sections of the sextic by the planes containing one edge is an elliptic pencil on $S$ of the form $|2F_i| = |2F_i'|$.

This example of Enriques was included in Castelnuovo's paper [6] and was very briefly mentioned in Enriques foundational paper [23], n.39. Enriques returned to his surface only much later, in a paper of 1906 [24], where he proved, besides other things, that any nonsingular surface with $q = p_g = 0, 2K_S \sim 0$ is birationally isomorphic to a sextic surface as above or its degeneration. Modern proofs of Enriques' results were given in the sixties, in the dissertations of Boris Averbuch from Moscow and Michael Artin from Boston.

In his paper Castelnuovo considers the birational transformation of $\mathbb{P}^3$ defined by the formula

$$T : [x_0, x_1, x_2, x_3] = [y_2y_3, y_0y_1, y_0y_2, y_0y_3].$$

Plugging in this formula in the equation of the sextic, we obtain

$$F(x_0, x_1, x_2, x_3) = y_0^4y_2^2y_3^2\big((y_0y_1)^2 + (y_2y_3)^2 + (y_1y_3)^2 + (y_1y_2)^2\big)$$
$$+y_0^3y_1y_2^2y_3^2q(y_2y_3, y_0y_1, y_0y_2, y_0y_3).$$

After dividing by $y_0^3y_2^2y_3^2$, we obtain that the image of $X$ is a surface of degree 5 in $\mathbb{P}^3$ given by the equation

$$G = y_0\big((y_0y_1)^2+(y_2y_3)^2+(y_1y_3)^2+(y_1y_2)^2\big)+y_1q(y_2y_3, y_0y_1, y_0y_2, y_0y_3) = 0.$$

It has four singular points $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$. The local computations show that the first two points are double points locally isomorphic to a singularity $z^2 + f_4(x, y) = 0$, where $f_4(x, y)$ is a binary form of degree 4 without multiple roots. Classics called such a singularity an *ordinary tacnode*. The plane $y = 0$ is called the tacnodal tangent plane. Nowadays we call such singularities *simple* *elliptic singularities* of degree 2. Their minimal resolution has a smooth elliptic curve as the exceptional curve with self-intersection equal to $-2$. The other two singular points are ordinary triple points (= simple elliptic singularities of degree 3).

One can show the converse: a minimal resolution of a normal quintic with four singular points at the vertices of the coordinate tetrahedron such that two or the singular points are ordinary triple points and another two singular points are ordinary tacnodes with faces of the tetrahedron as the tacnodal tangent planes is an Enriques surface. In modern times, the quintic birational models of an Enriques surfaces were studied in [30], [43], and [44].

Consider the birational transformation of $\mathbb{P}^3$ given by the formula

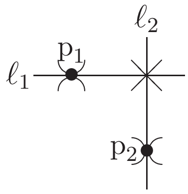
$$(x_0 : x_1 : x_2 : x_3) = (x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2).$$

It transforms the sextic surface $V(F_6)$ to a birationally isomorphic a sextic surface $V(G_6)$. The two birational morphisms $S \to \mathbb{P}^3$ are defined by linear system $|H|$ and $|H + K_S|$.

Since $[0, 0, 0, 1]$ is a triple point of the quintic surface $V(G)$, we can write its equation in the form

$$G = x_3^2 A_3(x_0, x_1, x_2) + 2x_3 B_4(x_0, x_1, x_2) + C_5(x_0, x_1, x_2) = 0,$$

Projecting from the triple point $[0, 0, 0, 1]$, we get a rational double cover $V(G) \dashrightarrow \mathbb{P}^2$. Its branch curve is a curve of degree 8 given by the equation $B_4^2 - C_5A_3 = 0$. The projections of the tacnodal planes $y_0 = 0$ and $y_1 = 0$ are line components of this octic curve. The residual sextic curve has a double point at the intersection of these lines and two tacnodes with tacnodal tangent planes equal to the lines. This is an *Enriques octic*.




In 1906 Enriques proved that any Enriques surface is birationally isomorphic to the double cover of $\mathbb{P}^2$ with branch curve as above or its degeneration [24].

Castelnuovo gave also his own example of a non-rational surface with $q = p_g = 0$. It differs from Enriques' one by the property that $h^0(2K) = 2$. In this example, $S$ is given as a minimal resolution of a surface $X$ of degree 7 in $\mathbb{P}^3$ with the following singularities:

- a triple line $\ell$;
- a double conic $C$ disjoint from $\ell$;
- 3 tacnodes $p, q, r$ with tacnodal tangent planes $\alpha = 0, \beta = 0, \gamma = 0$ containing $\ell$.

The equation is

$$F_7 = f_3^2 h + \alpha\beta\gamma f_4 = 0,$$

where $h = 0$ is any plane containing the line $\ell$, $f_3 = 0$ is a cubic surface containing $\ell$ and $C$. The tacnodal planes are tangent planes to $f_3 = 0$ at the points $p, q, r$. The quartic surface $f_4 = 0$ contains $C$ as a double conic.

The pencil of planes through the line $\ell$ cuts out a pencil of quartic curves on $X$ with 2 nodes on $C$. Its members are birationally isomorphic to elliptic curves. On the minimal resolution $S'$ of $X$, we obtain an elliptic fibration with a 2-section defined by the pre-image of the double conic. Each tacnodal tangent plane cuts out a double conic, and the pre-image of it on $S'$ is a divisor $2E_i + 2F_i$, where $E_i$ is a $(-1)$-curve and $F_i$ is an elliptic curve. Blowing down $E_1, E_2, E_3$, we obtain a minimal elliptic surface $S$ with three double fibers. The canonical class is equal to $-F + F_1 + F_2 + F_3$. It is not effective. However, $2K_S \sim F$, so $h^0(2K) = 2$.

## 3. Generalities

Recall Noether's Formula

$$12(1 - q + p_g) = K_S^2 + c_2,$$

where $c_2 = \sum(-1)^i b_i(S)$ is the Euler characteristic in the usual topology if $\Bbbk = \mathbb{C}$ or étale topology otherwise.

In *classical definition*, an Enriques surface is a smooth projective surface with $q = p_g = 0$ and $2K_S = 0$. It is known that $q = h^1(\mathcal{O}_S)$ is equal to the dimension of the tangent space of the Picard scheme $\mathbf{Pic}_{S/\Bbbk}$. Thus its connected component $\mathbf{Pic}^0_{S/\Bbbk}$ is trivial. The usual computation, based on the Kummer exact sequence, gives that $b_1 = 2\dim \mathbf{Pic}^0_{S/\Bbbk}$. Thus $b_1 = 0$. Noether's Formula implies that $c_2 = 12$, hence $b_2 = 10$. Also, since $2K_S = 0$, the Kodaira dimension of $S$ must be equal to 0. A *modern definition* of an Enriques surface is the following (see [3]:

**Definition**. An Enriques surface is a smooth projective algebraic surface of Kodaira dimension 0 satisfying $b_1 = 0$ and $b_2 = 10$.

Other surfaces of Kodaira dimension 0 are abelian surfaces with $q = 2, K_S = 0$, K3-surfaces with $q = 0, K_S = 0$, and hyperelliptic surfaces with $q = 1, p_g = 0$.

Let $S$ be an Enriques surface. Since the Kodaira dimension is zero, we obtain that $K_S^2 = 0$. Also, since $h^0(mK_S)$ is bounded, $p_g \le 1$. Noether's Formula gives $q = p_g$.

Recall that $\mathbf{Pic}^0_{S/\Bbbk}$ parameterizes divisor classes algebraically equivalent to zero. It is an open and closed subscheme of the Picard scheme. Another closed and open group subscheme $\mathbf{Pic}^\tau_{S/\Bbbk}$ parameterizing divisor classes numerically equivalent to zero. The group $\mathbf{Pic}_{S/\Bbbk}(\Bbbk)$ is the Picard group $\mathrm{Pic}(S)$ of divisor classes modulo linear equivalence. The group $\mathrm{Pic}^0(S) := \mathbf{Pic}^0_{S/\Bbbk}(\Bbbk)$ is the subgroup of divisor classes algebraically equivalent to zero. The group $\mathrm{Pic}^\tau(S) := \mathbf{Pic}^\tau_{S/\Bbbk}(\Bbbk)$ is the subgroup of numerically trivial divisor classes. The quotient group $\mathrm{NS}(S) = \mathrm{Pic}(S)/\mathrm{Pic}^0(S)$ is a finitely generated abelian group, the Néron-Severi group of $S$. The quotient group $\mathrm{Pic}(S)^\tau/\mathrm{Pic}^\tau(S)$ is the torsion subgroup $\mathrm{Tors}(\mathrm{NS}(S))$ of the Néron-Severi group and the quotient $\mathrm{Pic}(S)/\mathrm{Pic}^\tau(S)$ is isomorphic $\mathrm{NS}(S)/\mathrm{Tors}(S)$. It is a free abelian group denoted by $\mathrm{Num}(S)$.

If $p = 0$, all group schemes are reduced, hence $h^1(\mathcal{O}_S) = \dim \mathbf{Pic}^0_{S/\Bbbk}$. In our case, this implies that $h^1(\mathcal{O}_S) = 0$. In fact, it is known that $\mathbf{Pic}^0_{S/\Bbbk}$ is reduced if $h^2(\mathcal{O}_S) = 0$ and, for Enriques surfaces, this always happens if $p \neq 2$ [4]. If $p = 2$ and $h^2(\mathcal{O}_S) = h^1(\mathcal{O}_S) = 1$, the group scheme $\mathbf{Pic}^0_{S/\Bbbk}$ coincides with $\mathbf{Pic}^\tau_{S/\Bbbk}$. It is a finite non-reduced group scheme of order 2 isomorphic to the group schemes $\boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$. In the first case, an Enriques surface is called a $\boldsymbol{\mu}_2$-surface, and in the second case it is called an $\boldsymbol{\alpha}_2$-surface, or *supersingular surface* (because in this case the Frobenius acts trivially on $H^2(S, \mathcal{O}_S)$).

If $h^2(\mathcal{O}_S) = h^1(\mathcal{O}_S) = 0$, the Enriques surface $S$ is called *classical Enriques surface*. In this case $\mathbf{Pic}^0_{S/\Bbbk} = 0$, $\mathrm{Pic}(S) = \mathrm{NS}(S)$ and $\mathbf{Pic}^\tau_{S/\Bbbk}$ is a constant group scheme defined by the group $\mathrm{Tors}(\mathrm{NS}(S))$. By Riemann-Roch, for any torsion divisor class $D \neq 0$ in $\mathrm{Pic}(S)$, we have $h^0(D) + h^0(K_S - D) \ge 1 - q + p_g = 1$. This implies that either $D$ or $K_S - D$ is effective. Since a non-trivial torsion divisor class cannot be effective, we have $D \sim K_S$.

Since $h^0(2K_S) \neq 0$, and $K_S$ is numerically trivial, $2K_S = 0$. Thus $\mathrm{Tors}(\mathrm{Pic}(S)) = \langle K_S \rangle$. By Riemann-Roch, $h^0(-K_S) + h^0(2K_S) \ge 1$ so $h^0(-K_S) > 0$ or $h^0(2K_S) > 0$. In the first case, $p_g = q = 0$, so $h^0(2K_S) > 0$ (since, by definition, the Kodaira dimension is 0, so $S$ is not rational). So, we always have $h^0(2K_S) > 0$. It is known that $K_S$ is numerically trivial if the Kodaira dimension is equal to 0 (this is highly non-trivial result, the core of the classification of algebraic surfaces). Thus $2K_S = 0$. So, $\mathrm{Tors}(\mathrm{Pic}(S))$ is of order $\le 2$. It is trivial if $q = p_g = 1$ and of order 2 otherwise.

For any finite group scheme $G$ over $\Bbbk$, one has a natural isomorphism

$$\mathrm{Hom}_{\text{gr-sch}/\Bbbk}(D(G), \mathbf{Pic}_{S/\Bbbk}) \cong H^1_{\mathrm{fl}}(S, G),$$

where $D(G)$ is the Cartier dual of $G$ and the right-hand-side is the group of flat cohomology with coefficients in the sheaf represented by $G$. This group is isomorphic to the group of isomorphism classes of $G$-torsors over $S$. In our case, by taking $G = (\mathbb{Z}/2\mathbb{Z})_\Bbbk, \boldsymbol{\mu}_2, \boldsymbol{\alpha}_2$, we obtain $D(G) = \boldsymbol{\mu}_2, (\mathbb{Z})_S, \boldsymbol{\alpha}_2$, respectively. If $p \neq 2$ the groups $\boldsymbol{\mu}_2$ and $(\mathbb{Z})_\Bbbk$ are isomorphic. Hence we have a non-trivial $(\mathbb{Z}/2\mathbb{Z})_\Bbbk$-torsor if $p \neq 2$, or $S$ is a $\boldsymbol{\mu}_2$-surface. The corresponding degree 2 finite étale cover $\pi : X \to S$ is a K3-surface. The cover is known as the *K3-cover* of an Enriques surface. The Galois group of the cover is a group of order 2, acting freely on $X$ with the quotient isomorphic to $S$. Conversely, any such involution $\iota$ on a K3-surface, defines, after passing to the quotient map $X \to X/\langle\iota\rangle$ the K3-cover of the Enriques surface $S \cong X/\langle\iota\rangle$. So, in this case the theory of Enriques surfaces becomes a chapter in the theory of K3 surfaces. This has been overused in the modern literature by applying transcendental methods, in particularly, the theory of periods, to solve some problems on Enriques surfaces of pure geometrical nature.

If $p = 2$ and $S$ is not classical, the non-trivial $\boldsymbol{\mu}_2$ or $\boldsymbol{\alpha}_2$-torsor defines an inseparable degree 2 cover $X \to S$, also called the K3-cover. However, the surface $X$ is not isomorphic to a K3-surface. It could be birationally isomorphic to a K3-surface or a non-normal rational surface.

Let $\rho = \mathrm{rank}\ \mathrm{Pic}(S) = \mathrm{rank}\ \mathrm{NS}(S)$. If $\Bbbk = \mathbb{C}$, the Hodge decomposition $H^2(S,\mathbb{C}) = H^{20} \oplus H^{11} \oplus H^{02}$ implies that $H^{11} = b_2(S) = 10$. By Lefschetz Theorem, all integral 2-cohomology classes are algebraic, hence $H^2(S,\mathbb{Z}) = \mathrm{Pic}(S)$ and $H^2(X,\mathbb{Z})/\mathrm{Tors} = \mathrm{Num}(S) = \mathbb{Z}^{10}$. The Poincaré Duality implies that the intersection form on $\mathrm{Num}(S)$ defines a quadratic form on $\mathrm{Num}(S)$ whose matrix has determinant $\pm 1$. Also, the Hodge Index Theorem gives that the lattice is of signature $(1,9)$. We say that $\mathrm{Num}(S)$ is a quadratic unimodular lattice. The adjunction formula $D^2 = 2\chi(\mathcal{O}_S) - 2$ for any irreducible effective divisor $D$ implies that $D^2$ is always even. The Hodge Index Theorem gives that the signature of the real quadratic space $\mathrm{Num}(S) \otimes \mathbb{R}$ is equal to $(1,9)$. Finally, Milnor's Theorem about even unimodular indefinite integral quadratic forms implies that $\mathrm{Num}(S) = U \oplus E_8$,[2] where $U$ is a hyperbolic plane over $\mathbb{Z}$ and $E_8$ is a certain negative definite unimodular even quadratic form of rank 8.

[2]Here we abuse the notation to use $\oplus$ to denote the orthogonal sum of quadratic lattices.

If $p \neq 0$, more subtle techniques, among them the duality theorems in étale and flat cohomology imply the same result provided one proves first that $\rho = b_2 = 10$. There are two proofs of this fact one by E. Bombieri and D. Mumford [4] and another by W. Lang [34]. The first proof uses the existence of a genus one fibration on $S$, the second uses the fact that an Enriques surface with no global regular vector fields can be lifted to characteristic 0. Another that $\mathrm{Num}(S)$ is isomorphic to the lattice $U \oplus E_8$ was given by L. Illusie [28], 7.3. It uses the crystalline cohomology.

One can use the following description of the lattice $U \oplus E_8$ which we denote by $\mathbb{E}_{10}$, sometimes it is called the *Enriques lattice*. Let $\mathbb{Z}^{1,10}$ be the standard hyperbolic lattice with an orthonormal basis $e_0, e_1, \ldots, e_{10}$ satisfying $(e_i, e_j) = 0, (e_0, e_0) = 1, (e_i, e_i) = -1, i > 0$. Then $\mathbb{E}_{10}$ is isomorphic as a quadratic lattice to the orthogonal complement of the vector $k_{10} = 3e_0 - e_1 - \cdots - e_{10}$. One can take as its basis the vectors $\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_i = e_i - e_{i+1}, i = 1, \ldots, 9$. Such a basis is called the *root basis*. The matrix of the symmetric bilinear form with respect to this basis is equal to $-2I + A$, where $A$ is the incidence matrix of the graph:

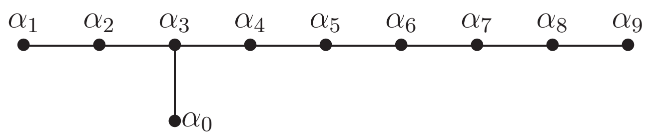



Figure 1. Enriques lattice

The quadratic lattice $\mathbb{E}_{10}$ is isomorphic to the orthogonal complement of the canonical class of a rational surface obtained by blowing up 10 points in the projective plane. In fact, if we denote by $e_0$ the class of the pre-image of a line on the plane and by $e_i$ the classes of the exceptional divisors, we obtain that the canonical class is equal to $-3e_0 + e_1 + \cdots + e_{10}$, hence the claim. This explains the close relationship between the theory of Enriques surfaces and the theory of rational surfaces. In fact, if we take the 10 points in the special position, namely to be the double points of an irreducible rational curve of degree 6, the rational surface, called a *Coble surface*, lies on the boundary of a partial compactification of the moduli space of Enriques surfaces.

We set

$$f_i = e_i + k_{10}, \ i = 1, \ldots, 10.$$

Since $(f_i, k_{10}) = 0$ these vectors belong to $\mathbb{E}_{10}$. We have

$$f_i \cdot f_j = 1 - \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker symbol. Also, adding them up, we obtain

$$f_1 + \ldots + f_{10} = 9k_{10} + 3e_0 = 3(10e_0 - 3e_1 - \cdots - 3e_{10}).$$

We set

$$\Delta = \frac{1}{3}(f_1 + \cdots + f_{10}) = 10e_0 - 3e_1 - \cdots - 3e_{10}.$$

We have

$$(\Delta, \Delta) = 10,\ (\Delta, f_i) = 3,\ (f_i, f_j) = 1 - \delta_{ij}.$$

A sequence of $k$ isotropic vectors in $\mathbb{E}_{10}$ satisfying this property is called an *isotropic $k$-sequence*. The maximal $k$ possible is 10. An ordered isotropic 10-sequence defines a root basis in $\mathbb{E}_{10}$ as follows. To see this, we consider the sublattice $L$ of $\mathbf{E}_{10}$ spanned by $f_1, \ldots, f_{10}$. The direct computation shows that its discriminant is equal to 9, this it is a sublattice of index 3 in $\mathbf{E}_{10}$. The vector $\delta = \frac{1}{3}(f_1 + \cdots + f_{10})$ has integer intersection with each $f_i$, hence it defines an element in the dual lattice $L^*$ such that $3\delta \in L$. This implies that $\delta \in \mathbf{E}$ and we may set

$$\alpha_i^* = \delta,\ \alpha_1^* = \Delta - f_1,\ \alpha_2 = 2\Delta - f_1 - f_2,\ \alpha_k^* = 3\delta - \sum_{i=1}^{k} f_i,\ k \geq 3. \quad (3)$$

The vectors $(\alpha_1^*, \ldots, \alpha_{10}^*)$ form a basis of $\mathbf{E}_{10}$ and its dual basis is a root basis with the intersection graph as in Figure 1.

The following matrix is the intersection matrix of the vectors $\alpha_i^*$. It was shown to me first by S. Mukai during our stay in Bonn in 1983.

$$\begin{pmatrix}
10 & 7 & 14 & 21 & 18 & 15 & 12 & 9 & 6 & 3 \\
7 & 4 & 9 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\
14 & 9 & 18 & 28 & 24 & 20 & 16 & 12 & 8 & 4 \\
21 & 14 & 28 & 42 & 36 & 30 & 24 & 18 & 12 & 6 \\
18 & 12 & 24 & 36 & 30 & 25 & 20 & 15 & 10 & 5 \\
15 & 10 & 20 & 30 & 25 & 20 & 16 & 12 & 8 & 4 \\
12 & 8 & 16 & 24 & 20 & 16 & 12 & 9 & 6 & 3 \\
9 & 6 & 12 & 18 & 15 & 12 & 9 & 6 & 4 & 2 \\
6 & 4 & 8 & 12 & 10 & 8 & 6 & 4 & 2 & 1 \\
3 & 2 & 6 & 6 & 5 & 4 & 3 & 2 & 1 & 0
\end{pmatrix}$$

Let $\mathrm{O}(\mathbb{E}_{10})$ be the orthogonal group of the lattice $\mathbb{E}_{10}$, i.e. the group of automorphisms of $\mathbb{E}_{10}$ preserving the quadratic form. We have

$$\mathrm{O}(\mathbb{E}_{10}) = W(\mathbb{E}_{10}) \times \{\pm 1\},$$

where $W(\mathbb{E}_{10})$ is the Weyl group of $\mathbb{E}_{10}$ generated by reflections $s_{\alpha_i} : x \mapsto x + (x, \alpha_i)\alpha_i$. It is known that it coincides with the full *reflection group* of $\mathbb{E}_{10}$ (see [18]). We denote by $W_S$ the Weyl group of $\mathrm{Num}(S)$.

## 4. Polarized Enriques surfaces

The moduli space of Enriques surfaces exists as a stack only. If $p \neq 2$, it is an irreducible smooth unirational Artin stack of dimension 10. Over $\mathbb{C}$, it admits a coarse moduli space isomorphic to an arithmetic quotient of a symmetric domain of type IV. If $p = 2$, it consists of two irreducible unirational components intersecting along a 9-dimensional substack. One components correspond to classical Enriques surfaces and another to $\boldsymbol{\mu}_2$-surfaces. The intersection corresponds to $\boldsymbol{\alpha}_2$-surfaces. This is a recent result of Christian Liedtke [35].

To consider a quasi-projective moduli spaces one has to polarize the surface. A *polarized surface* is a pair $(S, D)$, where $D$ is a nef divisor class with $D^2 > 0$ and $|D|$ is base-point-free. An isomorphism of polarized surfaces $(S, D) \to (S', D')$ is an isomorphism $f : S \to S'$ such that $f^*(D') \sim D$. Let us discuss such classes.

Let $D$ be any irreducible curve on $S$. By adjunction formula, $D^2 \geq -2$. If $D^2 = -2$, then $D \cong \mathbb{P}^1$. An Enriques surface containing a smooth rational curve is called *nodal* and *unnodal* otherwise. If $D^2 \geq 0$, by Riemann-Roch, $h^0(D) > 0$. Let $W_S^n$ be the subgroup of $W_S$ generated by the reflections $s_\alpha$, where $\alpha$ is the class of a smooth rational curve. Applying elements of $W_S^n$, we obtain that $D \sim D_0 + \sum R_i$, where $D_0$ is a nef effective divisor, and $R_i \cong \mathbb{P}^1$. If $D$ is nef and $D^2 \geq 2$, then $h^1(D) = 0$ and $\dim |D| = \frac{1}{2}D^2$. If $D^2 = 0$, then $D = kE$, where $h^0(E) = 1$ but $h^0(2E) = 2$. A nef divisor $E$ such that $h^0(E) = 1$ but $h^0(2E) = 2$ is called a *half-fiber*. The linear system $|2E|$ is a pencil of curves of arithmetic genus 1. It defines a genus one fibration $f : S \to \mathbb{P}^1$ whose general fiber is a curve of arithmetic genus 1. It is nonsingular if $p \neq 2$ but could have a cusp if $p = 2$. If $K_S \neq 0$ the genus one fibration defined by the pencil $|D|$ has two fibers of the form $2F_1$ and $2F_2$. Conversely, if $p \neq 2$, any surface admitting an elliptic fibration over $\mathbb{P}^1$ with two double fibers is birationally isomorphic to an Enriques surface. If $p = 2$ and $K_S = 0$, the genus one fibration has only one double fiber, and any surface admitting a genus one fibration over $\mathbb{P}^1$ with one double fiber is birationally isomorphic to an Enriques surface or to a rational surface.

For any nef divisor $D$ with $D^2 > 0$, let $\Phi(D) = \min\{D \cdot E\}$, where $E$ is a half-fiber. The function $\Phi$ satisfied the inequality (see [12], Corollary 2.7.1)

$$\Phi(D)^2 \leq D^2.$$

We have $\Phi(D) = 1$ if and only if $|D|$ has base-points (in fact, there are two of them, counting with multiplicity). Also $\Phi(D) = 2$ if and only if $|D|$ defines a double cover of a normal surface, or a birational morphism onto a non-normal surface, or $D^2 = 4$ and the map is of degree 4 onto $\mathbb{P}^2$. In the first case, the linear system is called *superelliptic* (renamed to *bielliptic* in [13]). Finally, $\Phi(D) \geq 3$ if and only if $|D|$ defines a birational morphism onto a normal surface with at most rational double points as singularities.

Here are examples.

If $D^2 = 2$, then $D \sim E_1 + E_2$ or $D \sim 2E_1 + R$, where $|2E_i|$ are genus 1 pencils and $R \cong \mathbb{P}^1$ such that $E_i \cdot E_2 = 1$ and $R \cdot E_1 = 1$. The linear system $|D|$ is a pencil of curves of arithmetic genus 2.

Assume $D^2 = 4$ and $\phi(D) = 1$, then, after blowing up the two base points, we obtain a degree 2 map to $\mathbb{P}^2$ with the branch divisor equal to an Enriques octic (it may degenerate if $S$ is nodal). If $\Phi(D) = 2$, and $S$ is unnodal, then $D \sim E_1 + E_2$, where $E_i$ are half-fibers with $E_1 \cdot E_2 = 2$. The map given by $|D|$ is a finite map of degree 4 onto $\mathbb{P}^2$. If $p \neq 2$, its branch locus is a curve of degree 12, the image of the dual of a nonsingular cubic curve under a map $\mathbb{P}^2 \to \mathbb{P}^2$ given by conics [45]. If $p = 2$, the map could be inseparable. If $S$ is a $\boldsymbol{\mu}_2$-surface, the map is separable and its branch curve is a plane cubic. If $p \neq 2$ or $S$ is a $\boldsymbol{\mu}_2$-surface, the preimage $\tilde{D}$ of $D$ on the K3-cover $X$ defines a linear system $|\tilde{D}|$ on $X$ that maps $X$ onto a complete intersection of three quadrics in $\mathbb{P}^5$.

Assume $D^2 = 6$ and $\Phi(D) = 2$. Again, if $S$ is unnodal, then $D \sim E_1+E_2+E_3$, where $|2E_i|$ are genus one pencils and $E_i \cdot E_j = 1$. The map is a birational map onto an Enriques sextic in $\mathbb{P}^3$. The moduli space of polarized surfaces $(S, D)$ admits a compactification, a GIT-quotient of sextic surfaces passing through the edges of the tetrahedron.

Assume $D^2 = 8$ and $\Phi(D) = 2$. If $S$ is unnodal, then $D \sim 2E_1 + 2E_2$ or $D \sim 2E_1+2E_2+K_S|$, where $|2E_i|$ are genus one pencils and $E_1 \cdot E_2 = 1$. In the first case, the map given by the linear system $|D|$ is a double cover $\phi : S \to \mathcal{D}_4$, where $\mathcal{D}_4$ is a 4-nodal quartic del Pezzo surface. It is isomorphic to a complete intersection of two quadrics in $\mathbb{P}^4$ with equations

$$x_0x_1 + x_2^2 = 0, \ x_3x_4 + x_2^2 = 0.$$

The surface contains 4 lines forming a quadrangle. The vertices of the quadrangle are the four singular points. Its minimal resolution is isomorphic to the blow-up $\tilde{\mathcal{D}}_4$ of 5 points in the projective planes equal to the singular points of the Enriques octic. The rational map $\tilde{\mathcal{D}}_4 \dashrightarrow \mathcal{D}_4$ is given by the anti-canonical linear system. The cover ramifies over the singular points and a curve from $|\mathcal{O}_{\mathcal{D}}(2)|$. Thus, birationally, the cover is isomorphic to the double octic. If $S$ is nodal the degree 8 polarization can be also given by the linear system $|4E + 2R|$ or

$|4E+2R+K_S|$, where $|2E|$ is a genus one pencil and $R$ is a $(-2)$-curve with $E \cdot E = 1$. In the first case, the linear system $|4E+2R|$ defines a degree 2 cover of a degenerate 4-nodal quartic del Pezzo surface $\mathcal{D}_4'$. Its equation is

$$x_0x_1 + x_2^2 = 0, \;\; x_3x_4 + x_0^2 = 0.$$

It has two ordinary nodes and one rational double point of type $A_3$. Its minimal resolution is isomorphic to the blow-up of 5 points in the plane equal to singular points of a degenerate Enriques octic.

The following is the picture of the branch curve of the rational map $S \dashrightarrow \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is a minimal resolution of singularities of $\mathcal{D}_4$. The second We assume here that $p \neq 2$.

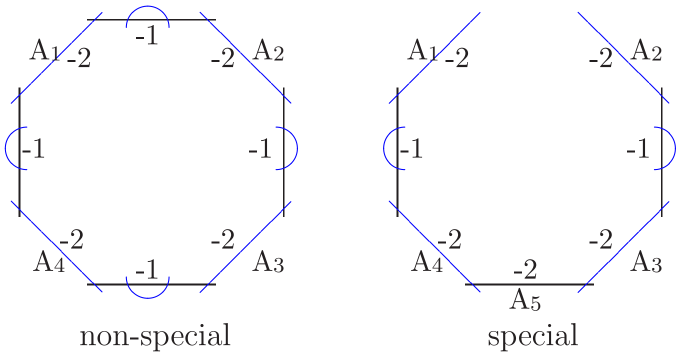



FIGURE 2. Branch curve of a bielliptic map ($p \neq 2$)

If $S$ admits a K3-cover, then the preimage of the linear system $|D|$ on the cover defines a degree 2 map onto $\mathbb{P}^1 \times \mathbb{P}^1$ with branch curve of type $(4,4)$ invariant with respect to an involution of the quadric with four isolated fixed points. This is sometimes referred to as the *Horikawa model*.

The linear systems $|D+K_S|$, where $|D|$ as above, maps $S$ birationally onto a non-normal surface of degree 8 in $\mathbb{P}^4$. So, we see that the type of polarization depends on the linear but not the numerical equivalence class of the divisor.

Note that all such linear systems exist on any Enriques surface, nodal or not.

Finally, assume $D^2 = 10$ and $\Phi(D) = 3$. If $S$ is unnodal, the linear system $|D|$ defines a closed embedding onto a surface $\mathsf{F}$ of degree 10 in $\mathbb{P}^5$. Its homogeneous ideal is generated by 10 cubics. This model was first studied by G. Fano in [26] and [25]. One can represent the divisor class $3D$ as the sum of 10 divisor classes $E_1 + \cdots + E_{10}$ whose numerical classes form an isotropic 10-sequence in $\mathrm{Num}(S)$. The images of $E_i$ and $E_i \in |E_i + K_S|$ are plane cubics contained in $\mathsf{F}$. The linear system $|D - E_i - E_j|, i \neq j$, consists of an isolated genus one curve $E_{ij}$ which is mapped onto a curve of degree 4 on $\mathsf{F}$. The linear system $|E_i + E_j + E_k|, k \neq i \neq j$ maps $S$ onto an Enriques sextic $S'$ in $\mathbb{P}^3$, and the image of $E_{ij}$ is an elliptic quintic, and the images of $E_i$ and $E_j$ are coplanar edges of the tetrahedron. The images of the 7 curves $E_s, s \neq i, j, k$ and 21 curves $E_{ab}, a, b \neq i, j, k$ are plane cubic curves on $S'$. If $p \neq 2$, we also have the adjoint 28 curves $E_i'$ and $E_{ab}'$ numerically equivalent to $E_i$ and $E_{ab}$. The image of each pair $E_i + E_i', E_{ab} + E_{ab}'$ is a plane section of $S'$.

Note that the numerical equivalence classes of the curves $E_i$ are determined uniquely by the choice of the Fano polarization $|D|$. A choice of an ordered representatives $E_i$ of these classes such that $3D \sim E_1 + \cdots + E_{10}$ defines a *supermarking* of $S$, i.e. a splitting of the projection $\mathrm{Pic}(S) \to \mathrm{Num}(S)$ preserving the intersection forms. A *marking* of $S$ is just an isomorphism of quadratic lattices $\mathrm{Num}(S) \to \mathbb{E}_{10}$. So, there are $2^9$ supermarkings liftings a given marking. A supermarking of $S$ defines a choice of 10 planes in $\mathbb{P}^5$ cutting out 10 plane cubics on the Fano model $\mathsf{F}$. One can show that the moduli space of supermarked unnodal Enriques surfaces is irreducible and it is mapped into an irreducible component of the variety of ordered 10-tuples of mutually intersecting planes in $\mathbb{P}^5$ (see [21]).

If $S$ is nodal and $K_S \neq 0$, a Fano polarization $|D|$ or $|D + K_S|$ maps $S$ into a nonsingular quadric in $\mathbb{P}^5$. If we identify the quadric with the Grassmann variety $G(2,4)$ of lines in $\mathbb{P}^3$, then the image of $S$ is isomorphic to the *Reye congruence* of lines, the set of lines in a web of quadrics in $\mathbb{P}^3$ that are contained in a pencil from the web. Such a polarization of a nodal surface is called a Reye polarization [9]. If $|D|$ is a Reye polarization, then $|D + K_S|$ maps $S$ into $\mathbb{P}^5$ that can be identified with a general 5-dimensional linear system of quadrics. The image of $S$ is the locus of reducible quadrics.

An interesting open problem is to determine the Kodaira dimension of the moduli space of polarized Enriques surface. If $D^2 = 4$ and $D = |E_1 + E_2|$ with $E_1 \cdot E_2 = 2$, then the moduli space is rational [5]. $|D| = |E_1 + E_2 + E_3|$ is an Enriques sextic polarization, then, up to a projective transformation, a sextic model is defined uniquely by the quadratic form $q$ from (1). This shows that the moduli space is also rational.

The moduli space of Enriques surfaces with polarization of degree 8 and type $|2E_1 + 2E_2|$ is birationally isomorphic to the GIT-quotient $|\mathcal{O}_{D_4}(2)|/\mathrm{Aut}(\mathcal{D}_4)$. It can be shown to be rational (I. Dolgachev). The moduli space of Fano polarizations is birationally covered by the space of quintic elliptic curves in $\mathbb{P}^3$. It was shown by A. Verra that the latter space is rational and of dimension 10 [46]. Thus the moduli space of Fano polarized Enriques surfaces is unirational.

One can show that it is covered by the moduli space of supermarked Enriques surfaces, a $2^9$-cover of the unirational moduli space of Fano polarized Enriques surfaces. However, I do not know its Kodaira dimension.

Note that over $\mathbb{C}$, the coarse moduli space of Enriques surfaces is rational [32], also the moduli space of nodal Enriques surfaces is rational [22].

## 5. Nodal Enriques surfaces

Recall that a nodal Enriques surface is an Enriques surface $S$ containing a smooth rational curve. Over $\mathbb{C}$, a smooth rational curve $R$ on $S$ splits under the K3-cover $\pi : X \to S$ into the disjoint sum of two smooth rational curves $R_+$ and $R_-$. The Picard group $\mathrm{Pic}(X)$ contains the divisor class $R_+ - R_-$ that does not belong to $\pi^*(\mathrm{Pic}(S)$. The theory of periods of lattice polarized K3 surfaces shows that the nodal surfaces form an irreducible hypersurface in the coarse moduli space of Enriques surfaces. Over any algebraically closed field of characteristic $p \neq 2$ one can show that a nodal Enriques surface is isomorphic to a Reye congruence of lines in $\mathbb{P}^3$. The moduli space of Reye congruences is an irreducible variety of dimension 9. On the other hand, the moduli space of Enriques sextics is of dimension 10. This shows that a general Enriques surface is unnodal.

There are two invariants that measure how nodal an Enriques surface is. The first is the *degeneracy invariant* $d(S)$ defined in [12]. It is equal to a minimal $k$ such that there exists an isotropic $k$ sequence $(f_1, \ldots, f_k)$ in $\mathrm{Num}(S)$ such that each $f_i$ is a nef numerical class. If $S$ is unndodal, then $d(S) = 10$, maximal possible. It is known that $d(S) \geq 3$ if $p \neq 2$. However, no example of a surface with $d(S) = 3$ is known to me. Note that this result implies, if $p \neq 2$, that any Enriques surface admits a non-degenerate Enriques sextic model or a double plane model with Enriques octic curve as the branch curve.

The next invariant was introduced by V. Nikulin [42]. Let

$$\overline{\mathrm{Num}}(S) := \mathrm{Num}(S)/2\mathrm{Num}(S) \cong \overline{\mathbb{E}}_{10} := \mathbb{E}_{10}/2\mathbb{E}_{10} \cong \mathbb{F}_2^{10}.$$

We equip the vector space $\overline{\mathbb{E}}_{10}$ with the quadratic form $q : \overline{\mathbb{E}}_{10} \to \mathbb{F}_2$ defined by

$$q(x + 2\mathbb{E}_{10}) = \tfrac{1}{2}x^2 \mod 2.$$

One can show that the quadratic form is non-degenerate and is of even type, i.e. equivalent to the direct sum of 5 hyperbolic planes $x_1x_2+x_3x_4+\cdots+x_9x_{10}$. Its orthogonal group is denoted by $\mathrm{O}^+(10,\mathbb{F}_2)$. It contains a simple subgroup of index 2. We denote by $(x,y)$ the value of the associated symmetric form $(x,y)=q(x+y)+q(x)+q(y)$ on a pair $x,y\in\overline{\mathrm{Num}}(S)$.

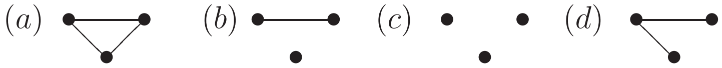
Let $\mathcal{R}_S$ be the set of smooth rational curves on $S$. Slightly modifying Nikulin's definition we set

$$\Delta_+ = \{R+2\mathbb{E}_{10}, R\in\mathcal{R}_S\}\subset q^{-1}(1)$$

and define the *r-invariant* of $S$ as the smallest subset $\mathsf{r}(S)$ of $\Delta_+$ such that any $x\in\Delta_+$ can be written as a sum of elements from $\mathsf{r}(S)$. We picture $\mathsf{r}(S)$ as a graph with vertices in $\mathsf{r}(S)$ and the edges connecting two elements $x,y$ in $\mathsf{r}(S)$ such that $(x,y)=1$.

$\sharp\mathsf{r}(S)=1$ : 

$\sharp\mathsf{r}(S)=2$ : 


$\sharp\mathsf{r}(S)=3$ : 


In the case when the K3-cover $\pi: X\to S$ is étale (i.e. when $p\neq 2$, or $S$ is a $\boldsymbol{\mu}_2$-surface), Nikulin defines another root invariant $\mathsf{R}(S)$ as follows.

Let $N^+$ ($N^-$) be the subgroup of $\mathrm{Pic}(X)$ that consists of $\iota^*$-invariant (anti-invariant) divisor classes. It is clear that $N^-$ is contained in the orthogonal complement $(N^+)^\perp$ in $\mathrm{Pic}(S)$. Also, since $G=\langle\iota\rangle$ acts freely, $N^+=\pi^*(\mathrm{Pic}(S))$. Since $N^+$ contains an ample divisor, $N^-$ does not contain $(-2)$-curves. By the Hodge Index Theorem, $N^-$ is negative definite. The quotient group $N^-/\mathrm{Im}(\iota^*-1)=\mathrm{Ker}(\iota^*+1)/\mathrm{Im}(\iota^*-1)$ is isomorphic to the group cohomology $H^1(G,\mathrm{Pic}(X))$, and the Hochshild-Serre spectral sequence in étale cohomology gives a boundary map $d_2: E_2^{1,1}=H^1(G,\mathrm{Pic}(X))\to E_2^{3,0}=H^3(G,\mathbb{G}_m)\cong\mathrm{Hom}(G,\mathbb{G}_m)$. Its kernel coincides with the kernel of the homomorphism of the Brauer group $\pi^*:\mathrm{Br}(S)\to\mathrm{Br}(X)$, see [2]. It is shown in loc.cit. that $d_2$ coincides with the norm map $\mathrm{Nm}:\mathrm{Pic}(X)\to\mathrm{Pic}(S)$[3] restricted to $\mathrm{Ker}(\iota^*+1)$ and its image is contained in $\mathrm{Ker}(\pi^*)=\langle K_S\rangle$. Thus, the order of $N^-/\mathrm{Im}(\iota^*-1)$ is at most 4, and in the case when $S$ is a $\boldsymbol{\mu}_2$-surface, the group is trivial.

[3]Recall that the norm map is defined on invertible sheaves by setting $\mathrm{Nm}(\mathcal{L})=\det\pi_*\mathcal{L}$.

Consider the subgroup $N_0^+ = \mathrm{Im}(\iota^* - 1)$ of $N^-$. For any $x \in \mathrm{Pic}(X)$, we have $\iota^*(x)+x \in \pi^*(\mathrm{Pic}(S))$, hence $(\iota^*(x)+x)^2 = 2x^2+2x\cdot\iota^*(x) \equiv 0 \mod 4$, and we obtain $x\cdot\iota^*(x)$ is even. This implies that $(x-\iota^*(x))^2 \equiv 0 \mod 4$. Thus the lattice $N_0^-(\frac{1}{2})$ are integral even lattices.

Consider elements $\delta_- = (\iota^* - 1)(x)$ in $N_0^-$ such that $\delta_-^2 = -4$ and $\delta_+ := (\iota^* + 1)(x) = \pi^*(y)$, where $y^2 = -2$ and $y \mod 2\mathrm{Num}(S) \in \Delta_+$. Note that, if $\delta_- = (\iota^* - 1)(x')$, then $x' = x + z$, where $\iota^*(z) = z$, hence $\delta_- = \iota(x) + x + 2z = \pi^*(y')$, where $y' = y + 2z \equiv y \mod 2\mathrm{Num}(S)$. Also, $2x = (x + \iota^*(x) + (x - \iota^*(x)$ implies that $x^2 = -2$, hence, by Riemann-Roch, $x$ is effective, and thus $y$ is effective. Conversely, if $y = R + 2z$ for some $R \in \mathcal{R}_S$, then $\pi^*(R)$ is the disjoint union of two $(-2)$-curves $R'$ and $\iota(R')$ and we get $\pi^*(y) = x + \iota^*(x)$, where $x = R' + \pi^*(z)$. This gives $x - \iota^*(x) = R - \iota^*(R)$, so that $x - \iota(x) = \delta_-$ and $x + \iota(x) = \delta_+$. In particular, we see that elements $\delta_-$ are equal to $R' - \iota^*(R')$, where $R' + \iota(R') \in \pi^*(\mathcal{R}_S)$.

Following Nikulin, we define $K$ to be the sublattice of $N_0^-(\frac{1}{2})$ spanned by the classes $\delta_-$. It is a negative definite lattice spanned by vectors of norm $-2$. It follows that it is a root lattice, the orthogonal sum of root lattice of types $A, D, E$. We also have a homomorphism $\phi : K/2K \to \overline{\mathrm{Num}}(S)$ that sends $\delta_-$ to $y + 2\mathrm{Num}(S)$, where $\pi^*(y) = \delta_+$. Note that the image of $\phi$ is the linear subspace $\langle\Delta_+\rangle$ spanned by $\Delta_+$ and the image of the cosets of the $\delta_-$'s is the set $\Delta_+$. The *Nikulin* $\mathsf{R}$*-invariant* is the pair $(K, H)$, where $H = \mathrm{Ker}(\phi)$. The $\mathsf{r}$-invariant of $S$ is equal to the image of the cosets of a root basis of $K$.

Over $\mathbb{C}$, one can use the theory of periods of K3 surfaces to show that Enriques surfaces with rank $K = r$ form a codimension $r$ subvariety in the moduli space.

A slightly different definition of the $R$-invariant was given by S. Mukai [39]. He considers the kernel of the norm map $\mathrm{Nm} : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ and defines the *root system* of $S$ as a sublattice of $\mathrm{Ker}(\mathrm{Nm})(\frac{1}{2})$ generated by vectors with norm $(-2)$.

An Enriques surface is called *general nodal* if $\sharp\mathsf{r}(S) = 1$, i.e. any two $(-2)$-curves are congruent modulo $2\mathrm{Num}(S)$. In terms of $R$-invariant, it means that $(K, H) = (A_1, \{0\})$. The following theorem gives equivalent characterizations of a general nodal surface.

**Theorem 1.** *The following properties are equivalent.*

(i) *$S$ is a general nodal Enriques surface.*
(ii) *Any genus one fibration on $S$ contains at most one reducible fiber that consists of two irreducible components. Any half-fiber is irreducible.*

(ii') *Any genus one fibration on $C$ contains at most one reducible fiber that consists of two irreducible components.*

(iii) *Any isotropic* 10*-sequence* $(f_1, \ldots, f_{10})$ *contains at least 9 nef classes* $f_i$*'s.*

(iv) *Any two* $(-2)$*-curves are* $f$*-equivalent.*

(v) *For any Fano polarizations* $h$*, the set* $\Pi_h = \{R \in \mathcal{R}_S : R{\cdot}h \leq 4\}$ *consists of one element.*

(vi) *For any* $d \leq 4$*,* $S$ *admits a Fano polarization* $\Delta$ *such that* $\Pi_h = \{R\}$*, where* $R \cdot h = d$*.*

(vii) *A genus one fibration that admits a smooth rational curve as a 2-section does not contain reducible fibres.*

Here two $(-2)$-curves $R$ and $R'$ are called $f$*-equivalent* if there exists a sequence of genus one fibrations $|2E_1|, \ldots, |2E_{k-1}|$ and a sequence of $(-2)$-curves $R_1 = R, \ldots, R_k = R'$ such that

$$R_1 + R_2 \in |2E_1|, R_2 + R_3 \in |2E_2|, \ldots, R_{k-1} + R_k \in |2E_{k-1}|.$$

Obviously, the $f$-equivalence is an equivalence relation on the set of nodal curves.

## 6. Automorphisms of Enriques surfaces

One of the main special features of Enriques surfaces is the richness of its symmetry group, i.e the group $\mathrm{Aut}(S)$ of birational automorphisms. Since $S$ is a minimal model, this group coincides with the group of biregular automorphisms. The group of biregular automorphisms of any projective algebraic variety over $\Bbbk$ is the group of $\Bbbk$-points of a group scheme $\mathbf{Aut}_{S/\Bbbk}$ of locally finite type. This means that the connected component of the identity $\mathbf{Aut}^0_{S/\Bbbk}$ of $\mathbf{Aut}_{S/\Bbbk}$ is an algebraic group over $\Bbbk$, and the group of connected components is countable. Since $S$ is not an abelian surface, $\mathbf{Aut}^0_{S/\Bbbk}$ is a linear algebraic group. Since $S$ is not uniruled, its dimension is zero. The tangent space of $\mathbf{Aut}^0_{S/\Bbbk}$ is isomorphic to the space $H^0(S, \Theta_{S/\Bbbk})$ of regular vector fields on $S$. If $p \neq 2$, this space is trivial, but it could be non-trivial if $p = 2$. In any case, we obtain that the group $\mathrm{Aut}(S) = \mathbf{Aut}_{S/\Bbbk}(\Bbbk)$ is a countable discrete group.

Enriques himself realized this fact. In his paper [24] of 1906 he remarks that any $S$ containing a general pencil of elliptic curves has infinite automorphism group. The paper ends with the question whether there exists a special degeneration of the sextic model such that the group of automorphisms is finite. We will return to this question later.

A usual way to investigate $\mathrm{Aut}(S)$ is to consider its natural representation by automorphisms of some vector space or of an abelian group.

In our case, this would be $\mathrm{Num}(S)$. Since automorphisms preserve the intersection form, we have a homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Num}(S)) \cong \mathrm{O}(\mathbb{E}_{10}).$$

From now on, we fix an isomorphism $\mathrm{Num}(S) \cong \mathbb{E}_{10}$ and identify these two lattices. Since automorphisms preserve the ample cone in $\mathrm{Num}(S)$, the image does not contain $-1_{\mathbb{E}_{10}}$, hence it is contained in the reflection group $W(\mathbb{E}_{10})$. The kernel of $\rho$ preserves any ample divisor class, hence it is contained in a group of projective automorphisms of some projective space $\mathbb{P}^n$. It must be a linear algebraic group, hence it is a finite group.

The reduction homomorphism $\mathbb{E}_{10} \to \overline{\mathbb{E}}_{10} = \mathbb{E}_{10}/2\mathbb{E}_{10}$ defines a surjective homomorphism $W(\mathbb{E}_{10}) \to \mathrm{O}^+(10, \mathbb{F}_2)$ [7], [27]. Let

$$W(\mathbb{E}_{10})(2) := \mathrm{Ker}(W(\mathbb{E}_{10}) \to \mathrm{O}^+(10, \mathbb{F}_2)).$$

It is called the *2-level congruence subgroup* of $W(\mathbb{E}_{10})$.

**Proposition 2** (A. Coble)**.** *The subgroup $W(\mathbb{E}_{10})(2)$ is the smallest normal subgroup containing the involution $\sigma = 1_U \oplus (-1)_{E_8}$ for some (hence any) orthogonal decomposition $\mathbb{E}_{10} = U \oplus E_8$.*

*Proof.* Coble's proof is computational. It is reproduced in [12], Chapter 2, §10. The following nice short proof is due to Daniel Allcock.

Let $\Gamma$ be the minimal normal subgroup containing $\sigma$. It is generated by the conjugates of $\sigma$ in $W = W(\mathbb{E}_{10})$. Let $(f, g)$ be the standard basis of the hyperbolic plane $U$. If $\alpha_0, \ldots, \alpha_7$ is the basis of $E_8$ corresponding to the subdiagram of type $E_8$ of the Dynkin diagram of the Enriques lattice $\mathbb{E}_{10}$, then we may take

$$f = 3\alpha_0 + 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8,\ g = f + \alpha_9.$$

The stabilizer $W_f$ of $f$ in $W$ is the semi-product $E_8 \rtimes W(E_8) \cong W(E_9)$, where $E_9$ is the affine group of type $E_8$ and $W(E_9)$ is its Weyl group, the reflection group with the Dynkin diagram of type $T_{2,3,6}$. The image of $v \in E_8 = U^\perp$ in $W_f$ is the transformation

$$\phi_v : x \mapsto x + [(f - v) \cdot x]f + (x \cdot f)v.$$

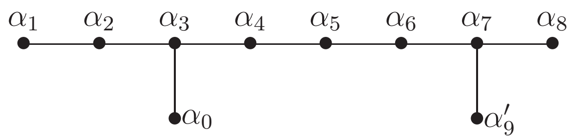
The inclusion of $W(E_8)$ in $W(E_9)$ is natural, it consists of compositions of the reflections in the roots $\alpha_0, \ldots, \alpha_8$. In particular, any $w \in W(E_8)$ acts identically on $E_8^\perp = U$. The image of $\sigma$ in $W(E_9)$ is equal to $(-1)_{E_8} \in W(E_8) \subset W(E_9)$. Let us compute the $\phi(v)$-conjugates of $\sigma$. If $x \in E_8$, we have

$$\phi_v \circ \sigma \circ \phi_{-v}(x) = \phi_v(\sigma(x + (v \cdot x)f)$$
$$= \phi_v(-x + (v \cdot x)f) = (-x + (v \cdot x)f) + (v \cdot x)f = -x + (2v \cdot x)f.$$

Thus the intersection of $\Gamma$ with $W_f$ is equal to $\phi(2E_8) : 2$. The quotient $W_f/\Gamma \cap W_f$ injects into $\mathrm{O}(\bar{E}_{10}) \cong \mathrm{O}(10, \mathbb{F}_2)^+$.

Let us consider the subgroup $H$ generated by $W_f$ and $\Gamma$. Since $W_f$ normalizes $\Gamma \cap W_f$, the kernel of the homomorphism $H \to \mathrm{O}(\bar{E}_{10})$ coincides with $\Gamma$. To finish the proof it suffices to show that $H$ coincides with the preimage $W_{\bar{f}}$ in $W$ of the stabilizer subgroup of the image $\bar{f}$ of $f$ in $\mathrm{O}(\bar{E}_{10})$. Indeed, the kernel of $W_{\bar{f}} \to \mathrm{O}(\bar{E}_{10})$ is equal to $W(E_{10})(2)$ and hence coincides with $\Gamma$.

Consider the sublattice $L$ of $\mathbb{E}_{10}$ generated by the roots $\alpha_0, \ldots, \alpha_7, \alpha_8$ and $\alpha_9' = \alpha_8 + 2g - f$. The Dynkin diagram is the following




Here all the roots, except $\alpha_9'$, are orthogonal to $f$. So, $H$ contains the reflections defined by these roots. Also the root $\alpha_8 - f$ is orthogonal to $f$, and $\alpha_9'$ is transformed to it under the group $2E_8 \subset \Gamma$ stabilizing $g$. So $H$ contains $s_{\alpha_9'}$ too. The Dynkin diagram contains 3 subdiagrams of affine types $\tilde{E}_8, \tilde{E}_8$ and $\tilde{D}_8$. The Weyl group is a crystallographic group with a Weyl chamber being a simplex of finite volume with 3 vertices at the boundary. This implies that $H$ has at most 3 orbits of ($\pm$ pairs) of primitive isotropic vectors in $L$. On the other hand, $W_{\bar{f}}$ contains $H$ and has at least three orbits of them, because the stabilizer of $\bar{f}$ in $\mathrm{O}(\bar{E}_{10})$ has three orbits of isotropic vectors (namely, $\{\bar{f}\}$, the set of isotropic vectors distinct from $\bar{f}$ and orthogonal to $\bar{f}$, and the set of isotropic vectors not orthogonal to $\bar{f}$). This implies that the set of orbits of $H$ is the same as the set of orbits of $W_{\bar{f}}$. Since the stabilizers of $\bar{f}$ in these two groups are both equal to $W_f$, it follows that $H = W_{\bar{f}}$. □

**Theorem 3.** *Let $S$ be an unnodal Enriques surface. Then $\rho : \mathrm{Aut}(S) \to W_S$ is injective and the image contains $W_S(2)$.*

*Proof.* If $\Bbbk = \mathbb{C}$, the fact that $\mathrm{Ker}(\rho) = \{1\}$ follows from the classification of automorphisms that act identically on $\mathrm{Num}(S)$ due to S. Mukai and Y. Namikawa [36], [37]. Without assumption on the characteristic, one can deduce it from the arguments in [20].

Consider the linear system $|D| = |2E_1 + 2E_2|$ with $D^2 = 8$. As was explained in section 4, it defines a degree 2 map $S \to \mathcal{D}_4 \subset \mathbb{P}^4$, where $\mathcal{D}_4$ is a 4-nodal del Pezzo quartic surface. Let $\sigma$ be the deck transformation of the cover and $\sigma_* = \rho(\sigma) \in W(\mathrm{Num}(S)$.[4] It is immediate that $\sigma_*$ leaves invariant the divisor classes of $E_1$ and $E_2$, and acts as the minus identity on the orthogonal complement of the sublattice $\langle E_1, E_2\rangle$ generated by $E_1, E_2$. The latter is isomorphic to the hyperbolic plane $U$ and $U^\perp \cong E_8$. Now any conjugate of $\sigma_*$ in $W_S$ is also realized by some automorphisms. In fact, $w \cdot \sigma_* \cdot w^{-1}$ leaves invariant $w(\langle E_1, E_2\rangle)$, and the deck transformation corresponding to the linear system $|2w(E_1) + 2w(E_2)|$ realizes $w \cdot \sigma_* \cdot w^{-1}$.

Now we invoke the previous proposition that says that $W(\mathbb{E}_{10})(2)$ is the minimal normal subgroup of $W(\mathbb{E}_{10})$ containing $\sigma_*$.

□

Here is the history of the theorem. We followed the proof of A. Coble in the similar case when $S$ is a rational surface obtained by blowing up of the plane at the 10 nodes of an irreducible curve of degree 6, assuming the blow-up does not contain $(-2)$-curves (*unnodal Coble surface*) [8]. This surface is a degeneration of an Enriques surface, although apparently the latter fact cannot be used to deduce the result from Coble's theorem.

Over $\mathbb{C}$, the proof follows immediately from the *Global Torelli Theorem* for K3-surfaces. It was first given by V. Nikulin [41] and, independently, by W. Barth and C. Peters [1]. Also the Global Torelli Theorem implies that the subgroup generated by $\mathrm{Aut}(S)$ and $W_S^n$ is of finite index in $W(\mathbb{E}_{10}$. In particular, $\mathrm{Aut}(S)$ is finite if and only $W_S^n$ is of finite index in $W_S$.

If $\mathbb{C}$, one can show that for a general, in the sense of moduli, Enriques surface, $\mathrm{Aut}(S) \cong W(\mathbb{E}_{10})(2)$. I do not know how to prove this fact if $p > 0$ (curiously, except when $p = 2$ and $S$ is a $\boldsymbol{\mu}_2$-surface).

The interesting case is when $S$ is a nodal Enriques surface. Over $\mathbb{C}$, Nikulin deduces the following theorem from the Global Torelli Theorem for K3-surfaces [42].

**Theorem 4.** *[V. Nikulin] Let $W(S;\mathcal{R}_S)$ be the subgroup of the group of $W_S$ leaving invariant the set of $(-2)$-curves on $S$. Then the homomorphism $\rho : \mathrm{Aut}(S) \to W(\mathrm{Num}(S);\mathcal{R}_S)/W_S^n$ has a finite kernel and a finite cokernel.*

Let $\langle\Delta_+\rangle$ be the subspace of $\overline{\mathrm{Num}}(S)$ used to define the $r$-invariant of an Enriques surface and $\mathsf{R}_S$ be the preimage of $(\langle\Delta\rangle)^\perp$ under the reduction modulo 2 map. This is called the *Reye lattice* of $S$. An

[4]Since $S$ has no smooth rational curves, the cover is a finite separable map of degree 2.

equivalent definition is

$$\mathsf{R}_S = \{x \in \mathrm{Num}(S) : x{\cdot}R \equiv 0 \mod 2 \text{ for any smooth rational curve } R\}.$$

Obviously, the action of $\mathrm{Aut}(S)$ on $S$ preserves the set of $(-2)$-curves, hence preserves the Reye lattice. Thus we have a homomorphism

$$\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathsf{R}_S).$$

Let $A_{\mathsf{R}_S} = \mathsf{R}_S^\vee/2\mathsf{R}_S$ be the discriminant group of $\mathsf{R}_S$. Since any element in the image of $\rho$ lifts to an isometry of the whole lattice $\mathbb{E}_{10}$, it must be contained in the kernel of the natural homomorphism $\mathrm{O}(\mathsf{R}_S) \to \mathrm{O}(A_{\mathsf{R}_S})$.

Assume that $S$ is a general nodal. In this case the Reye lattice $\mathsf{R}_S$ is a sublattice of $\mathrm{Num}(S)$ of index 2. It is isomorphic to the lattice $U \oplus E_7 \oplus A_1$. One can choose a basis defined by $\Delta, E_1, \ldots, E_{10}$ as above such that the coset is equal to the coset of the divisor class $\alpha = \Delta - E_{10}$. Then $\mathsf{R}_S = \{x \in \mathbb{E}_{10} : x \cdot \alpha \in 2\mathbb{Z}\}$ has a basis formed by the divisor classes

$$\beta_0 = -4\Delta + E_1 + \cdots + E_4,\ \beta_i = E_i - E_{i+1},\ i = 1, \ldots, 9.$$

The matrix of the quadratic lattice is equal to $-2I + B$, where $B$ is the incidence matrix of the graph:

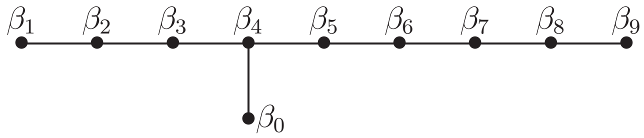
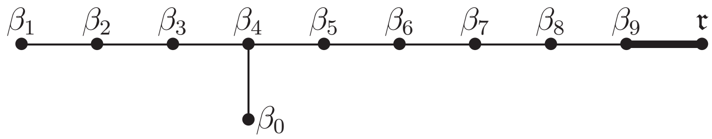



FIGURE 3. Reye lattice

We denote this quadratic lattice by $E_{2,4,6}$ and call it the *Reye lattice.*

Note that the Reye lattice is 2-reflective, i.e. the subgroup $\mathrm{Ref}_2(E_{2,4,6})$ generated by reflections in vectors $\alpha$ with $\alpha^2 = -2$ is of finite index in the orthogonal group (see [18], Example 4.11). However, it is larger than the Weyl group $W(E_{2,4,6})$ generated by reflections in the vectors $\beta_i$. The former group is generated by the reflections $s_{\beta_i}$ and the vector $\mathfrak{r} = f - \beta_9$, where $f = 2\beta_0 + \beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4 + 3\beta_5 + 2\beta_6 + \beta_7$ is an isotropic vector. We have $\mathfrak{r} \cdot \beta_9 = 2$ and $\mathfrak{r} \cdot \beta_i = 0, i \neq 9$. The Coxeter graph of the full 2-reflection group of $E_{2,4,6}$ is the following:



Note that $\mathrm{Ref}(E_{2,4,6})$ is of finite index in $\mathrm{O}(E_{2,4,6})$, and since $E_{2,4,6}$ is of finite index in $\mathrm{O}(\mathbb{E}_{10})$, it is of finite index in $W(\mathbb{E}_{10})$. In particular, the 2-level congruence subgroup $\mathrm{Ref}(E_{2,4,6})(2)$ is of finite index in $W(\mathbb{E}_{10})$. If we choose an isomorphism $\mathsf{R}_S$ such that $\mathfrak{r}$ represents the class of a $(-2)$-curve, then $W_S^n$ is a normal subgroup of $\mathrm{Ref}(E_{2,4,6})(2)$ and the quotient is isomorphic to the 2-level subgroup of $W(E_{2,4,6})$. Applying Nikulin's Theorem 4, we obtain, that, up to a finite group, $\mathrm{Aut}(S)$ is isomorphic to $W(E_{2,4,6})$.

The next theorem gives a much more precise result about the structure of the group of automorphisms of a general nodal Enriques surface.

Since $\mathsf{R}_S$ has discriminant equal to 4, its reduction modulo $2\mathsf{R}_S$ is a degenerate quadratic vector space. Its radical is generated by the coset $\bar{\alpha}$ of $\alpha$ and its orthogonal group is isomorphic to $G = 2^8 \rtimes \mathrm{Sp}(8, \mathbb{F}_2)$. It is known that the homomorphism $W(\mathsf{R}_S) \to G$ is surjective [7], [27]. Let $W(\mathsf{R})(2)'$ be equal to the pre-image of the subgroup $2^8$. Obviously it contains the 2-level congruence subgroup $W(\mathsf{R}_S)(2)$ as a subgroup of index $2^8$. The following theorem was announced in [11], its proof will be found in [13].

**Theorem 5.** *Let $S$ be a general Enriques surface and $\rho : \mathrm{Aut}(S) \to W(\mathsf{R}_S)$ be its natural representation. Then the kernel of $\rho$ is trivial, and the image is equal to $W(\mathsf{R}_S)(2)'$.*

*Proof.* Consider the following sublattices of $E_{2,4,6}$.

- $L_1 = \langle \beta_0, \dots, \beta_6 \rangle$. It isomorphic to $E_7$ and $L_1^\perp \cong A_1 \oplus U$.
- $L_2 = \langle \beta_0, \beta_2, \dots, \beta_7 \rangle$. It is isomorphic to $E_7$ and $L_2^\perp \cong A_1 \oplus U(2)$.
- $L_3 = \langle \beta_0, \beta_2, \dots, \beta_8 \rangle$. It is isomorphic to $E_8$ and $L_3^\perp \cong A_1 \oplus A_1(-1)$.

Define the following involutions of $E_{2,4.6}$:

- $B = -1_{L_1} \oplus 1_{L_1^\perp}$;
- $G = -1_{L_2} \circ s_{\beta_9}$;
- $K = -1_{L_3} \oplus 1_{L_3^\perp}$;

One can prove that the minimal normal subgroup $\langle\langle B, K \rangle\rangle$ containing $B, K$ coincides with $W(\mathsf{R}_S)(2)$ and together with $G$, the three involutions $K, B, G$ normally generate $W(\mathsf{R}_S)(2)'$ [12]. It remains to show that all of these involutions and their conjugate can be realized by automorphisms of $S$.

The involution $K$ is realized by the deck transformation of the double cover $S \to C_3$ of a cubic surface $C_3$ defined by a linear system $|2E_2 + 2E_2 - R|$, where $E_1, E_2$ are half-fibers with $E_1 \cdot E_2 = 1$ and $R$ is a smooth rational curve with $R \cdot E_1 = R \cdot E_2 = 0$. One can show that it always exists.

The involution $B$ is realized by the deck transformation of the double cover $S \to \mathcal{D}'$ onto a degenerate 4-nodal del Pezzo quartic defined by the linear system $|4E_1 + 2R|$, where $E_1$ is a half-fiber and $R$ is a smooth rational curve with $E \cdot R = 1$.

Finally, the involution $G$ is realized by the double cover $S \to \mathcal{D}_4$ defined by the linear system $|2E_1 + 2E_2|$ which we considered in the previous section. Note that $G \in W(\mathbb{E}_{10})(2)$ but does not belong to $W(\mathsf{R}_S)(2)$. □

Here the letters $K, B$ and $G$ stand for S. Kantor, E. Bertini and C. Geiser. The K3-cover of a general nodal Enriques surface is birationally isomorphic to a quartic symmetroid $Y$. This is a quartic surface in $\mathbb{P}^3$ with 10 nodes, its equation is given by the determinant of a symmetric matrix with entries linear forms in the projective coordinates. The surface $X$ admits a birational map $\pi : X \to Y$, so it is a minimal resolution of $Y$. Let $Q_1, \ldots, Q_{10}$ be the exceptional curves and $H$ is the class of a pre-image of a plane section of $Y$. For a general $X$, the Picard group of $X$ is generated by $H, Q_i$ and $H'$ such that $2H' \sim 3H - Q_1 - \cdots - Q_{10}$. The orthogonal complement to the divisor class of $2H - Q_1 - \cdots - Q_{10}$ is isomorphic to the $\pi^*(\mathsf{R}_S) \cong \mathsf{R}_S(2)$, i.e. the Reye lattice with intersection form multiplied by 2. The involutions K,B,G are induced by a Cremona involution of $\mathbb{P}^3$ that leave $Y$ invariant. Let us describe them.

The Kantor involution is defined by the linear system $|Q|$ of quartics through the first 7 nodes $p_1, \ldots, p_7$ of the symmetroid. This linear system defines a degree 2 rational map $\mathbb{P}^3 \dashrightarrow \Sigma \subset \mathbb{P}^6$, where $\Sigma$ is a cone over the Veronese surface in $\mathbb{P}^5$. We consider the elliptic fibration on the blow-up $\tilde{\mathbb{P}}^3$ of 7 points of $\mathbb{P}^3$ defined by the net of quadrics through the seven points. It has the negation involution defined by the eighth base point. If we take an quartic elliptic curve $E$ through $p_1, \ldots, p_7$ passing through a point $p$. Then a quartic surface from the linear system from above passing through $p$ intersects $E$ at one more point $p'$. This defines an involution on $E$, $p \mapsto p'$. The set of fixed points is the set of quartics from $|Q|$ that has an eight node. Thus the three remaining nodes of the symmetroid are fixed and this makes the symmetroid invariant. The birational involution of $\mathbb{P}^3$ is an analog of the Bertini involution of the plane.

The other two involutions are *dilated* Bertini and Geiser involutions of the plane. They extend these involutions to the three dimensional space.

Over $\mathbb{C}$ one can use the coarse moduli space of Enriques surfaces to show that a general (resp. general unnodal) Enriques surface has automorphism group isomorphic to $W(\mathbb{E}_{10})$ (resp. $W(E_{2,4,6})(2)')$. I believe that the same is true for any $\Bbbk$ but I cannot prove it (except in the case of $\boldsymbol{\mu}_2$-surfaces).

In any case if $p \neq 2$, the image of the group automorphism group $\mathrm{Aut}(S)$ in $W(\mathbb{E}_{10})$ is not the whole group. This is because $W(\mathbb{E}_{10})$ contains a subgroup isomorphic to $W(E_8)$ and the known information about finite groups of automorphisms of K3-surfaces shows that the order of this group is too large to be realized as an automorphism group of a K3-surface and hence of an Enriques surface.

When the nodal invariant of an Enriques surface becomes large, the automorphism group may become a finite group. The first example of an Enriques surface with a finite automorphism isomorphic to $\mathfrak{S}_3$ belongs to G. Fano [26]. However, I fail to understand Fano's proof. An example of an Enriques surface with automorphism group isomorphic to the dihedral group $D_4$ of order 4 was given in my paper [14]. At that time I did not know about Fano's example. Later on all complex Enriques surfaces with finite automorphisms were classified by Nikulin [41] (in terms of their root invariant and by S. Kondō [31] (by explicit construction). There are seven classes of such surfaces with automorphisms groups

$$D_4,\ \mathfrak{S}_3,\ \mathbb{Z}/2\mathbb{Z})^4 \rtimes D_4, (\mathbb{Z}/2\mathbb{Z})^2 \rtimes (\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}),\ \mathfrak{S}_4 \times \mathbb{Z}/2\mathbb{Z}, \mathfrak{S}_5,\ \mathfrak{S}_5.$$

Their root invariants $\mathsf{R}_S$ are, respectively,

$$(E_8 \oplus A_1, \{0\}),\ (D_9, \{0\}), (D_8 \oplus A_1^{\oplus 2}, \mathbb{F}_2),$$

$$(D_5 \oplus D_5, \mathbb{F}_2), (E_7 \oplus A_2 \oplus A_1, \mathbb{Z}/2\mathbb{Z}),\ (E_6 \oplus A_4, \{0\}), (A_9 \oplus A_1, \{0\}).$$

Note that Kondō's classification works in any characteristic $\neq 2$ and there are more examples in characteristic 2.

The classification of finite subgroups of $\mathrm{Aut}(S)$ is far from being complete. We refer for the latest works in progress of S. Mukai, H.Ohashi and H. Ito on the classification of finite groups of automorphisms of Enriques surfaces [29], [38], [39], [40]. We do not even know what is the list of possible automorphism groups of an unnodal Enriques surface. However, we can mention the following result (explained to me by Daniel Allcock).

**Theorem 6.** *Let $G$ be a finite subgroup contained in $W_S(2)$. Then it is a group of order 2, and its generator is a deck transformation of a degree 2 finite map $S \to \mathcal{D}_4$, where $\mathcal{D}_4$ is a 4-nodal Enriques surface.*

*Proof.* We identify $\mathrm{Num}(S)$ with the lattice $\mathbb{E}_{10}$. Suppose $G$ contains an element $\sigma$ of order 2. Then $V = \mathbb{E}_{10} \otimes \mathbb{Q}$ splits into the orthogonal direct sum of eigensubspaces $V_+$ and $V_-$ with eigenvalues 1 and $-1$. For any $x = x_+ + x_- \in \mathbb{E}_{10}, x_+ \in V_+, x_- \in V_-$, we have

$$\sigma(x) \pm x = (x_+ - x_-) \pm (x_+ + x_-) \in 2\mathbb{E}_{10}.$$

This implies $2x_\pm \in 2\mathbb{E}_{10}$, hence $x_\pm \in \mathbb{E}_{10}$. This shows that the lattice $\mathbb{E}_{10}$ splits into the orthogonal direct sum of sublattices $V_+ \cap \mathbb{E}_{10}$ and $V_- \cap \mathbb{E}_{10}$. Since $\mathbb{E}_{10}$ is unimodular, the sublattices must be unimodular. This gives $V_+ \cap \mathbb{E}_{10} \cong U$ or $E_8$ and $V_+ \cap \mathbb{E}_{10} \cong E_8$ or $U$, respectively. Since $\sigma = g_*$ leaves invariant an ample divisor, we must have $V_+ \cap \mathbb{E}_{10} \cong U$. Thus $\sigma = 1_U \oplus (-1)_{E_8}$ and hence $\sigma = g_*$ for some deck transformation $S \to \mathcal{D}_4$.

Suppose $G$ contains an element $\sigma$ of odd order $m$. Then $\sigma^m - 1 = (\sigma - 1)(1 + \sigma + \cdots + \sigma^{m-1}) = 0$, hence, for any $x \in \mathbb{E}_{10} \setminus 2\mathbb{E}_{10}$, we have

$$x + \sigma(x) + \cdots + \sigma^{m-1}(x) \equiv mx \mod 2\mathbb{E}_{10}.$$

Since $m$ is odd, this gives $x \in 2\mathbb{E}_{10}$, a contradiction.

Finally, we may assume that $G$ contains an element of order $2^k, k > 1$. Then it contains an element $\sigma$ of order 4. Let $M = \mathrm{Ker}(\sigma^2 + 1) \subset \mathbb{E}_{10}$. Since $\sigma^2 = -1_{E_8} \oplus 1_U$ for some direct sum decomposition $\mathbb{E}_{10} = E_8 \oplus U$, we obtain $M \cong E_8$. The equality $(\sigma^2 + 1)(\sigma(x)) = \sigma^3(x) + \sigma(x) = -(\sigma^2 + 1)(x)$, implies that $\sigma(M) = M$. Consider $M$ as a module over the principal ideal domain $R = \mathbb{Z}[t]/(t^2 + 1)$. Since $M$ has no torsion, it is isomorphic to $R^{\oplus 4}$. This implies that there exists $v, w \in M$ such that $\sigma(v) = w$ and $\sigma(w) = -v$. However, this obviously contradicts our assumption that $\sigma \in W(\mathbb{E}_{10})(2)$. □

A similar proof shows that any element of finite order in $W(E_{2,4,6})(2)'$ is conjugate to either a Bertini, or Geiser, or Kantor involution. Thus we obtain that under the homomorphism $W_S \to \mathrm{O}^+(10, \mathbb{F}_2)$ (resp. $W(\mathsf{R}_S) \to \mathrm{Sp}(8, \mathbb{F}_2)$) the image of any finite subgroup of $\mathrm{Aut}(S)$ has a kernel of order at most 2 and its image is a finite subgroup of $\mathrm{O}^+(10, \mathbb{F}_2)$ (resp. $\mathrm{Sp}(8, \mathbb{F}_2)$).

To conclude our survey let me refer to my earlier surveys of the subject [15], [16]. Sadly, many of the problems of the theory discussed in these surveys remain unsolved.

## References

[1] W. Barth, C. Peters, *Automorphisms of Enriques surfaces*, Inv. Math., **73** (1983), 383–411.

[2] A. Beauville, *On the Brauer group of Enriques surfaces*. Math. Res. Lett. **16** (2009), 927-934.

[3] E. Bombieri, D. Mumford, *Enriques classification in char. p, II*, Complex Analysis and Alg. Geometry, Iwanami-Shoten, Tokyo. 1977, 23–42.

[4] E. Bombieri, D. Mumford, *Enriques classification in char. p, III*, Invent. Math. **35** (1976), 197–232.

[5] G. Casnati, *The moduli space of Enriques surfaces with a polarization of degree 4 is rational*. Geom. Dedicata **106** (2004), 185-194.

[6] G. Castelnuovo, *Sulle superficie di genere zero*, Mem. delle Soc. Ital. delle Scienze, ser. III, **10** (1895), 103–123.

[7] A. Coble,*Theta Modular Groups Determined by Point Sets*. Amer. J. Math. **40** (1918), 317-340.

[8] A. Coble, *The ten nodes of the rational sextic and of the Cayley symmetroid*, Amer. J. Math. **41** (1919), 243–265.

[9] F. Cossec, *Reye congruences*, Trans. Amer. Math. Soc. **280** (1983),737–751.

[10] F. Cossec, I. Dolgachev, *Rational curves on Enriques surfaces*, Math. Ann. **272** (1985), 369–384.

[11] F. Cossec, F.( I. Dolgachev, *On automorphisms of nodal Enriques surfaces*. Bull. Amer. Math. Soc. (N.S.) **12** (1985), 247-249.

[12] F. Cossec, I. Dolgachev, *Enriques surfaces. I*. Progress in Mathematics, 76. Birkhäuser Boston, Inc., Boston, MA, 1989.

[13] F. Cossec, I. Dolgachev, C. Liedtke, *Enriques surfaces*, in preparation.

[14] I. Dolgachev,*On automorphisms of Enriques surfaces*. Invent. Math. **76** (1984), 163-177.

[15] I. Dolgachev, *Enriques surfaces: what is left?*, Problems in the theory of surfaces and their classification (Cortona, 1988), 81-97, Sympos. Math., XXXII, Academic Press, London, 1991

[16] I. Dolgachev, *Enriques surfaces: old and new*. Geometry and complex variables (Bologna, 1988/1990), 165-176, Lecture Notes in Pure and Appl. Math., 132, Dekker, New York, 1991

[17] I. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*. J. Math. Sci. **81** (1996), 2599-2630.

[18] I. Dolgachev, *Reflection groups in algebraic geometry*. Bull. Amer. Math. Soc. **45** (2008), 1-60.

[19] I. Dolgachev, *Classical algebraic geometry. A modern view*. Cambridge University Press, Cambridge, 2012.

[20] I. Dolgachev, *Numerical automorphisms of Enriques surfaces in arbitrary characteristic*, Arithmetic and geometry of K3 surfaces and Calabi-Yau threefold, ed. R. Lazu, M. Schütt, N. Yui, Fields Institute Communications, vol. 67, Springer. 2013, pp.267–284.

[21] I. Dolgachev, D. Markushevich, *Lagrangian tens of planes, Enriques surfaces and holomorphic symplectic fourfolds*, in preparation.

[22] I. Dolgachev, S. Kondō, *Rationality of moduli spaces of Coble surfaces and general nodal Enriques surfaces*, Izv. Russ. Akad. Nauk, Ser. Mat. **77** (2013), 77–92.

[23] F. Enriques, *Introduzione alla geometria sopra le superficie algebriche*, Mem. Soc. Ital. delle Scienze, **10** (1896), 1–81.

[24] F. Enriques, *Sopra le superficie algebriche di bigenere uno*, Mem. Soc. Ital. delle Scienze, ser. 3a, **14** (1906), 39–366

[25] G. Fano, *Superficie algebriche di genere zero e bigenere uno e loro case particolari*, Rendiconti Mat. di Palermo, **29** (1910), 98–118.

[26] G. Fano, *Superficie regolari di genere zero e bigenere uno*, Revista de matematica y fisica Teoretica Univ.de Tucuman Argentina, **4** (1944), 69–79.

[27] R. Griess, *Quotients of infinite reflection groups*. Math. Ann. **263** (1983), 267-278.

[28] L. Illusie, *Complexe de De Rham-Witt et cohomologie crystalline*. Ann. Sci. Éc. Norm. Sup, 4e ser., **12** (1979), 501–661.

[29] H. Ito, H. Ohashi, *Classification of involutions on Enriques surfaces*. math.AG. arXiv:1302.2316.

[30] Y. Kim, *Normal quintic Enriques surfaces*. J. Korean Math. Soc. **36** (1999), no. 3, 545-566.

[31] S. Kondō, *Enriques surfaces with finite automorphism groups*. Japan. J. Math. (N.S.) **12** (1986), 191-282.

[32] S. Kondō, *The rationality of the moduli space of Enriques surfaces*. Compositio Math. **91** (1994), 159-173.

[33] S. Kondō, *K3 and Enriques surfaces*, Arithmetic and geometry of K3 surfaces and Calabi-Yau threefold, ed. R. Lazu, M. Schütt, N. Yui, Fields Institute Communications, vol. 67, Springer. 2013, pp.3–28.

[34] W. Lang, *Two theorems on the De Rham cohomology*, Comp. Math. **40** (1980), 417–423.

[35] C. Liedtke, *Moduli and Lifting of Enriques Surfaces*, arXiv:1007.0787.

[36] S. Mukai, Y. Namikawa, *Automorphisms of Enriques surfaces which act trivially on the cohomology groups*. Invent. Math. **77** (1984), 383-397.

[37] S. Mukai, *Numerically trivial involutions of Kummer type of an Enriques surface*. Kyoto J. Math. **50** (2010), 889-902.

[38] S. Mukai, *Kummer's quartics and numerically reflective involutions of Enriques surfaces*. J. Math. Soc. Japan **64** (2012), 231-246.

[39] S. Mukai, *Lecture notes on K3 and Enriques surfaces*. Contributions to algebraic geometry, 389-405, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich, 2012.

[40] S. Mukai, H. Ohashi, *Enriques surfaces of Hutchinson-Göpel type and Mathieu automorphisms*, Arithmetic and geometry of K3 surfaces and Calabi-Yau threefold, ed. R. Lazu, M. Schütt, N. Yui, Fields Institute Communications, vol. 67, Springer. 2013, pp. 429–453.

[41] V. Nikulin, *On the quotient groups of the automorphism groups of hyperbolic forms modulo subgroups generated by 2- reflections, Algebraic geometric applications*, Current Problems of Mathematics, t. 18, VINITI, Moscow, 1981, pp. 3–114 [English translation:J. Soviet Math. **22** (1983), 1401–1476].

[42] V. Nikulin, *Description of automorphism groups of Enriques surfaces*. Dokl. Akad. Nauk SSSR **277** (1984), 1324-1327.

[43] E. Stagnaro, *Constructing Enriques surfaces from quintics in* $P_k^3$, Algebraic Geometry-Open Problems, Lect. Notes in Math. vol. 997, Springer-Verlag, 1983, pp. 400–403.

[44] Y. Umezu, *Normal quintic surfaces which are birationally Enriques surfaces*. Publ. Res. Inst. Math. Sci. **33** (1997), no. 3, 359-384.

[45] A. Verra, *On Enriques surfaces as a fourfold cover of* $P^2$, Math. Anal. **266** (1983), 241–250.

[46] A. Verra, *A short proof of the unirationality of* $A_5$. Nederl. Akad. Wetensch. Indag. Math. **46** (1984), 339-355.

Department of Mathematics, University of Michigan, 525 E. University Av., Ann Arbor, Mi, 49109, USA
*E-mail address*: idolga@umich.edu