Combinatorial aspects of the box-ball systems

Reiho Sakamoto
University of Tokyo

Contents

- Kirillov—Reshetikhin crystals
- Box-Ball Systems (BBS)
- Rigged configuration bijection ϕ
- ϕ = Inverse scattering transform for BBS
- Crystal interpretation of ϕ
- Ultradiscrete tau functions
- Relation with Kostka—Macdonald polynomials

Kirillov—Reshetikhin crystals (Type A)

- KR crystals $B^{r,s}$
- As the set,

$$B^{r,s} = \{\text{all rectangular column strict semistandard tableaux, height } r\text{, width } s\}$$

Type $A_n^{(1)}$ case, letters run over $\{1, 2, \cdots, n+1\}$.

- Algebraic structure is given by the Kashiwara operarors.
- There is nicely defined tensor product $B^{r,s} \otimes B^{r',s'}$.

Combinatorial R

- There is the unique crystal isomorphism called the combinatorial R;

$$R : B^{r,s} \otimes B^{r',s'} \simeq B^{r',s'} \otimes B^{r,s}$$

- There is concrete algorithm due to Mark Shimozono (2002) in terms of row insertions/deletions.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 4 \\ \hline 2 & 4 & 5 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & 6 \\ \hline \end{array} \simeq \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 4 & 4 & 4 & 6 \\ \hline \end{array}$$

Affine crystals & energy function

- Affinization of crystals: as the set, it is

$$\mathrm{Aff}(B) = \{b[d] \mid b \in B,\ d \in \mathbb{Z}\}$$

- Affine combinatorial R.

$$b[d] \otimes b'[d'] \simeq \tilde{b}'[d' - H(b \otimes b')] \otimes \tilde{b}[d + H(b \otimes b')]$$

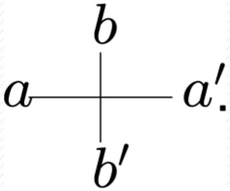
- H is called energy function.
- Normalization: $H = 0$ on tensor products of the highest weight crystals.

$$H\left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 4 \\ \hline 2 & 4 & 5 & 6 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & 6 \\ \hline \end{array} \right) = 3$$

Box-Ball Systems (1)

- Vertex diagrams of the combinatorial R.

$$a \otimes b \simeq b' \otimes a' \longrightarrow$$



- Highest element $u_{l,0}^{(a)} = u_l^{(a)} \in B^{a,l}$

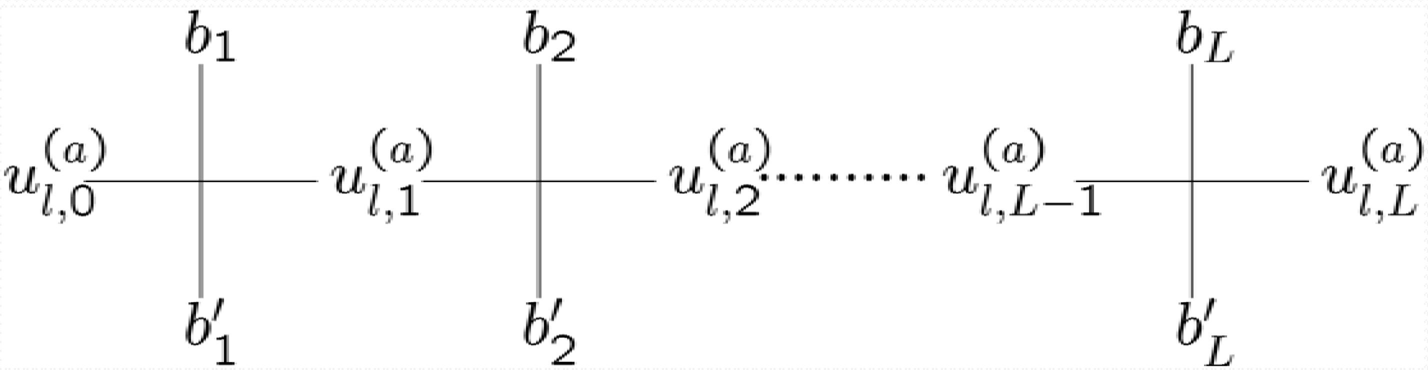
$$u_4^{(3)} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline \end{array}$$

Box-Ball Systems (2)

- Definition of time evolution operator $T_l^{(a)}$

$$T_l^{(a)}(b) := b'$$

where,



$$b = b_1 \otimes \cdots \otimes b_L, \qquad b' = b'_1 \otimes \cdots \otimes b'_L$$

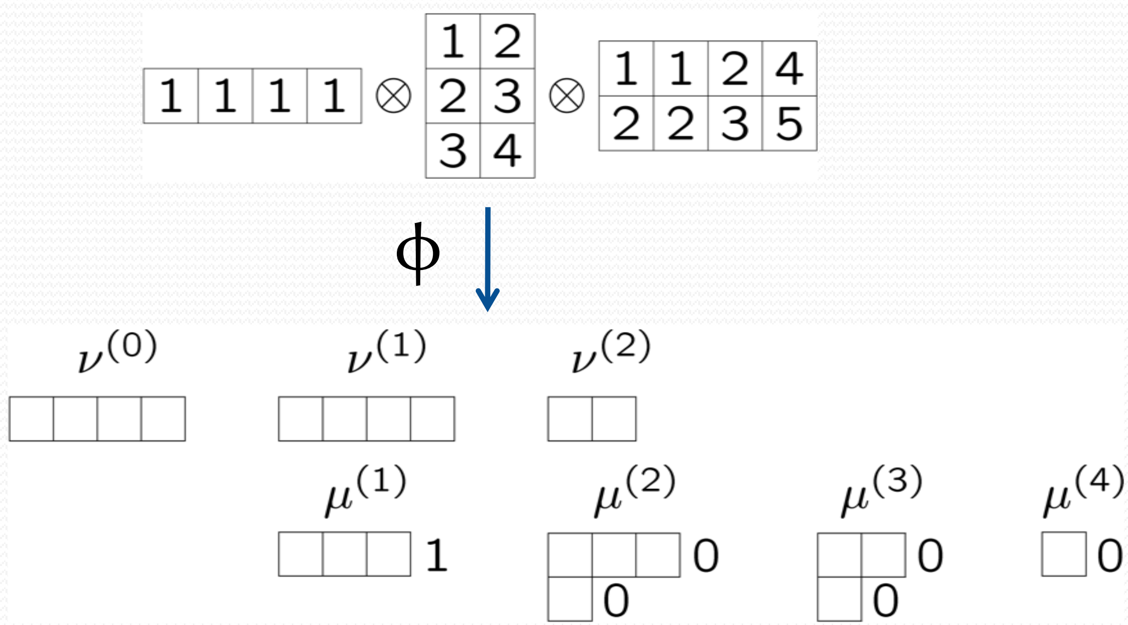
Example of BBS

- Evolution under $T_\infty^{(1)}$.
  Shape of paths $(B^{1,1})^{\otimes 28}$

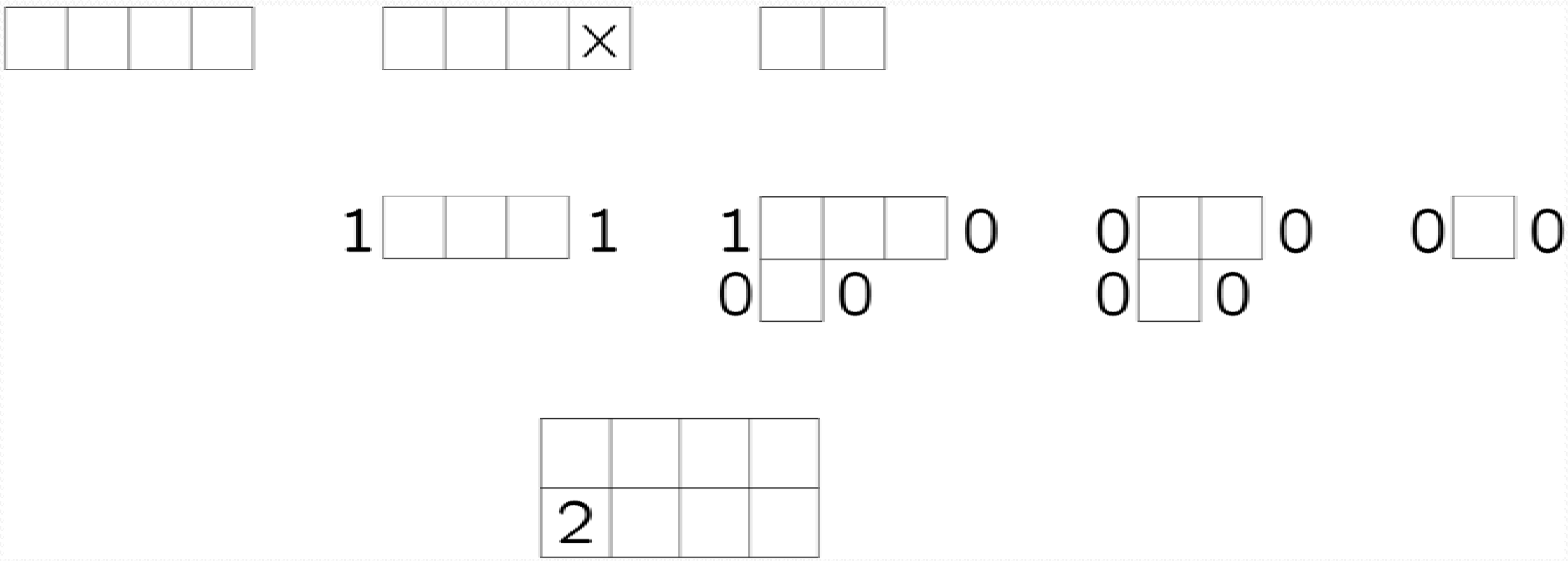
- t=0: 1222211133243111111111111111
- t=1: 1111222211324331111111111111
- t=2: 1111111122213224331111111111
- t=3: 1111111111112211322433211111
- t=4: 1111111111111111221113221433 21

Rigged configuration bijection (1)

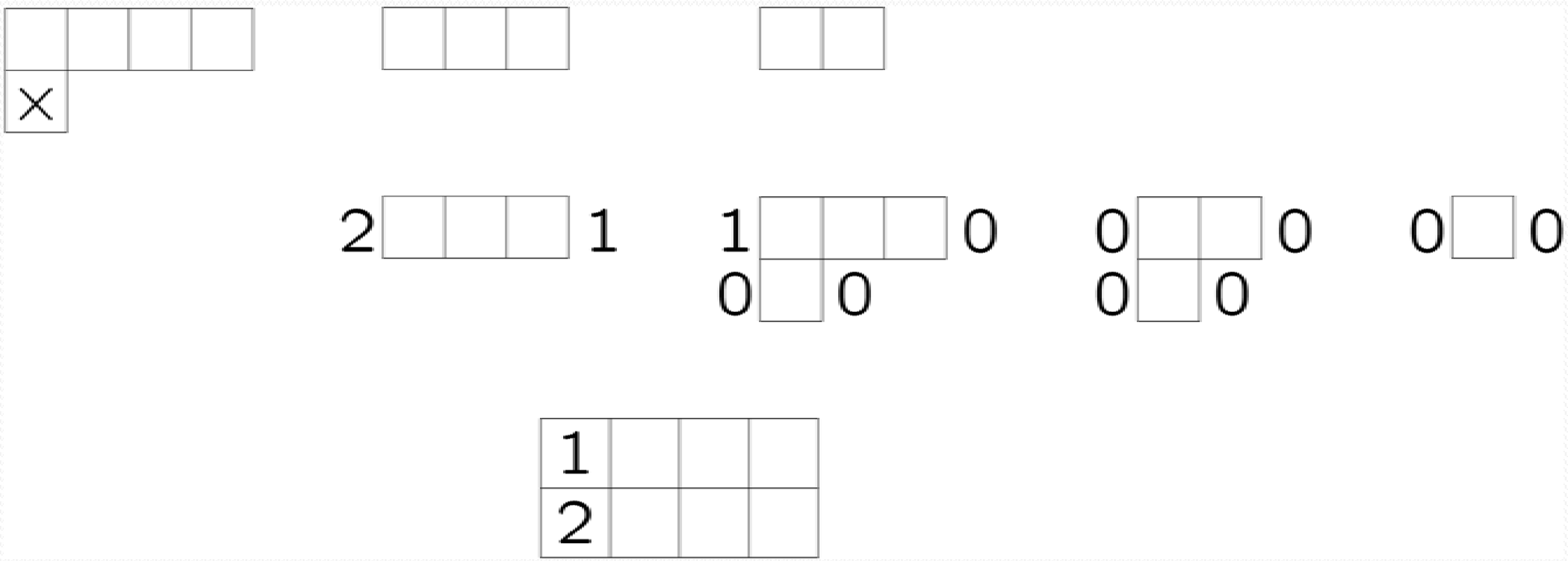
- There is one-to-one correspondence between
  (1) tensor products of the KR crystals
  (2) combinatorial objects called rigged configurations



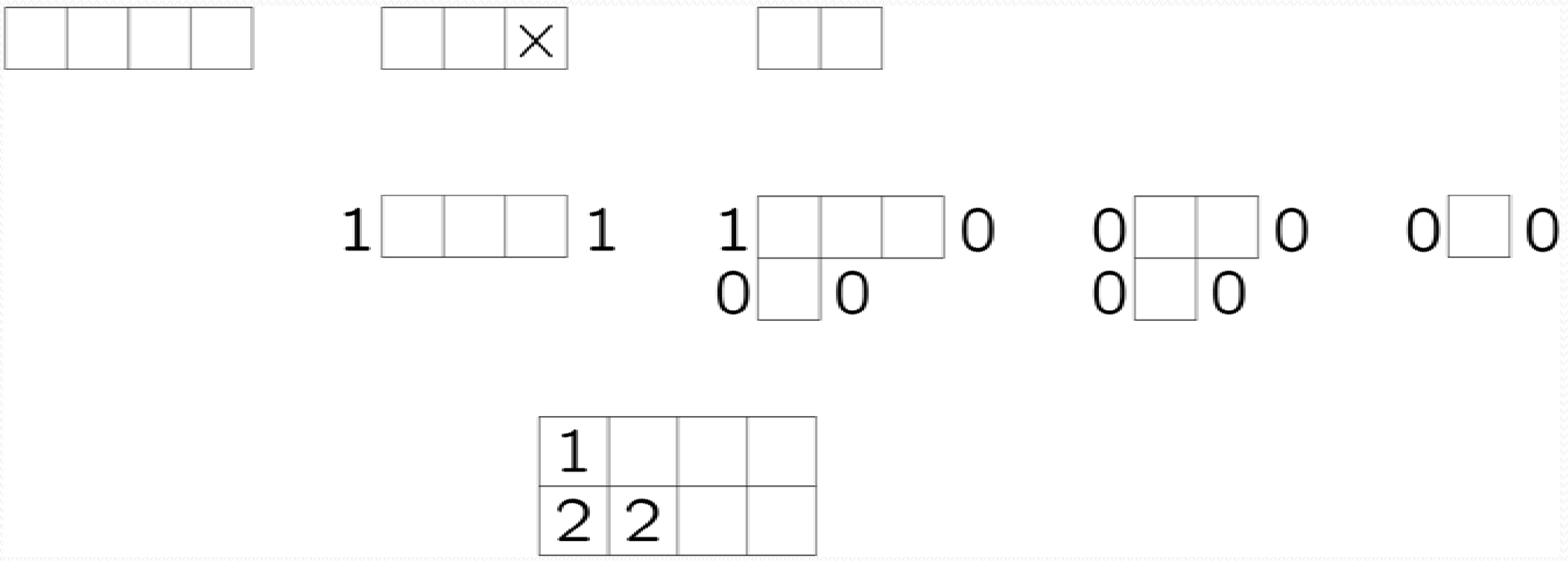
Example



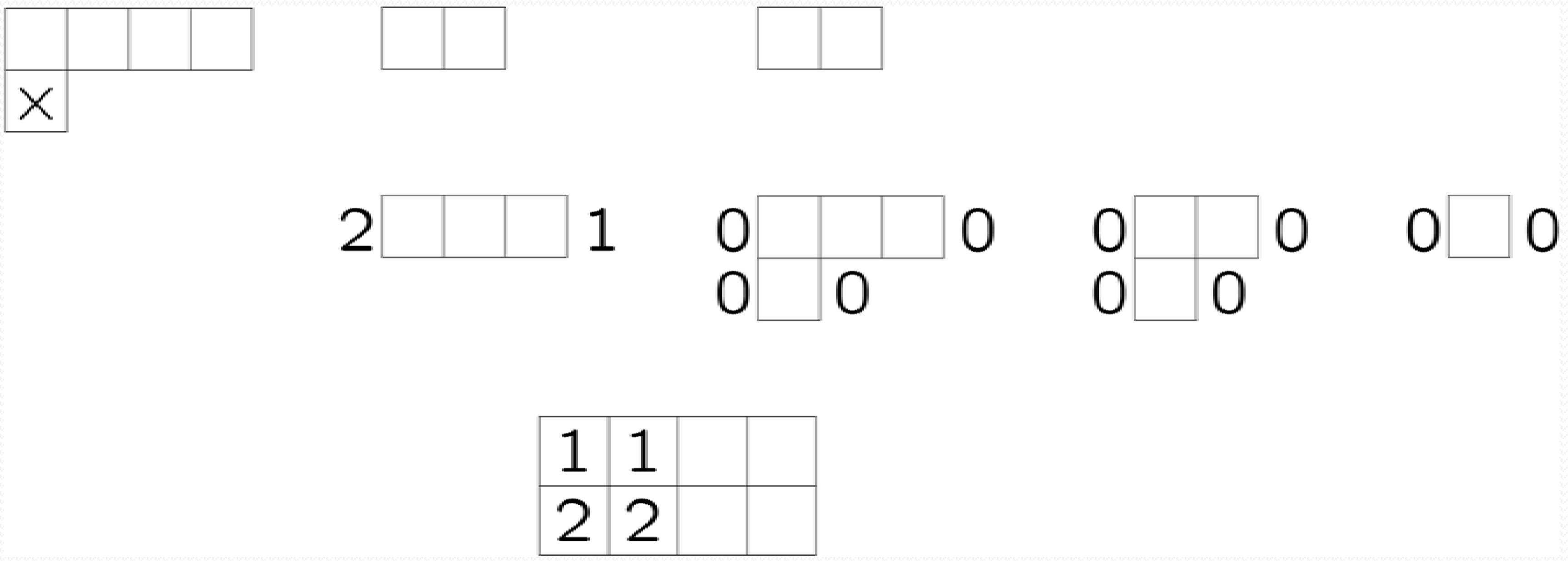
Example



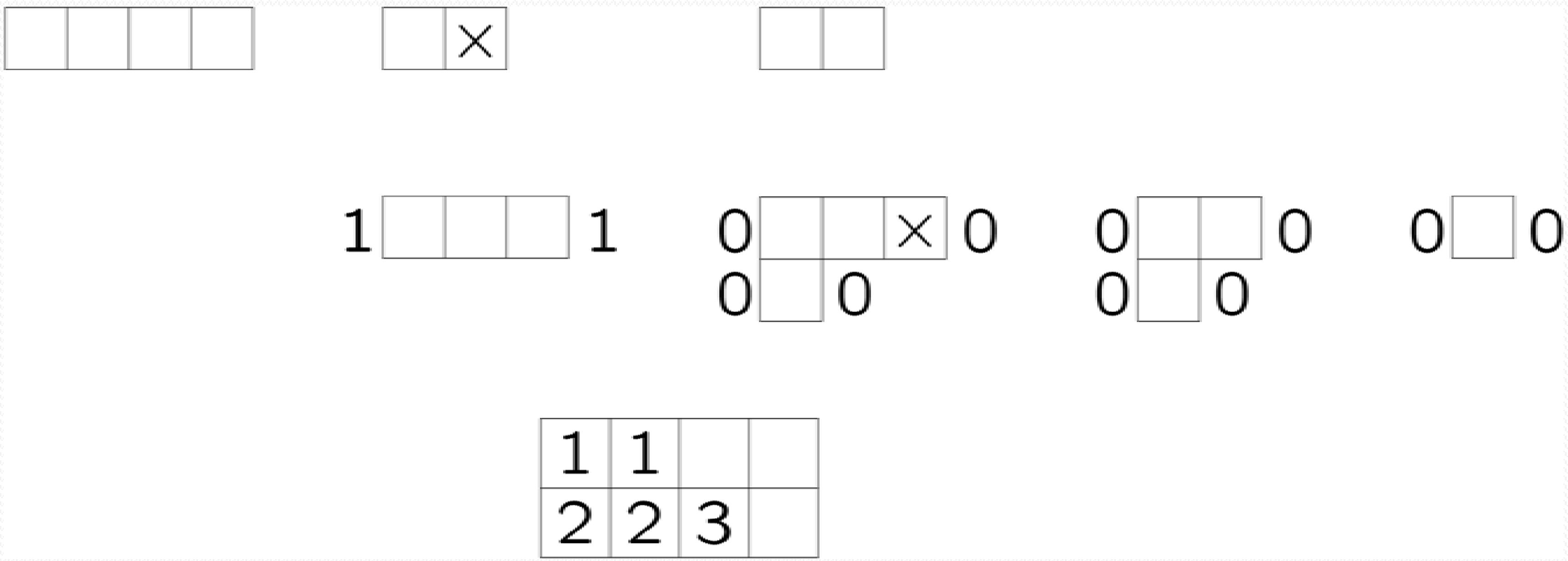
Example



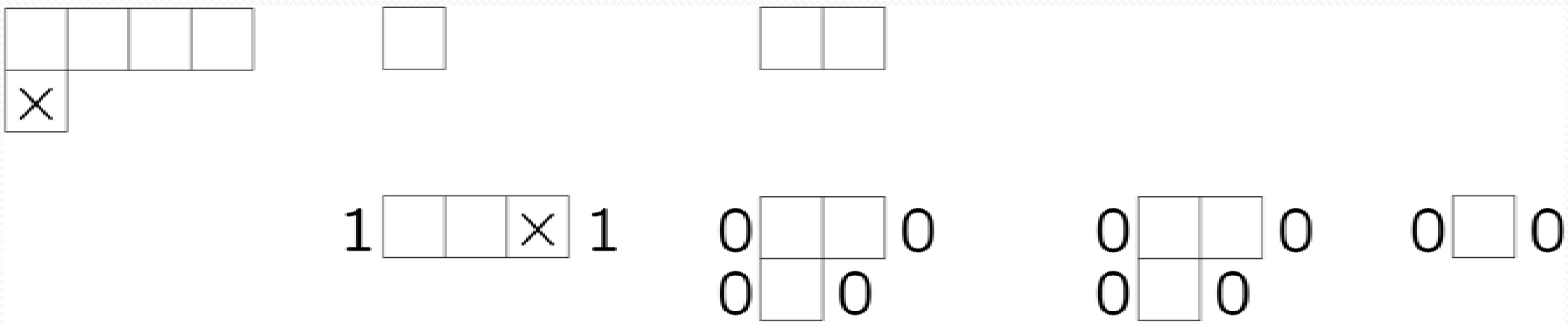
Example



Example

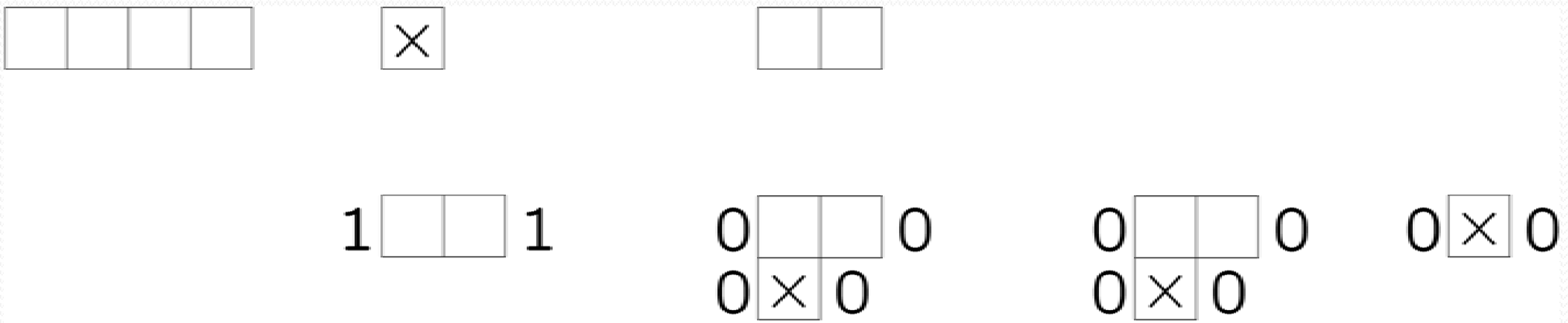


Example



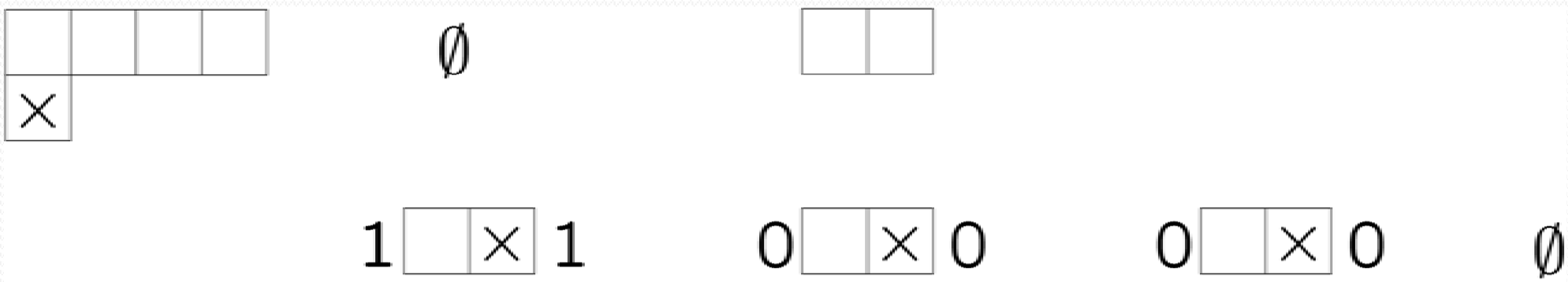
1	1	2	
2	2	3	

Example



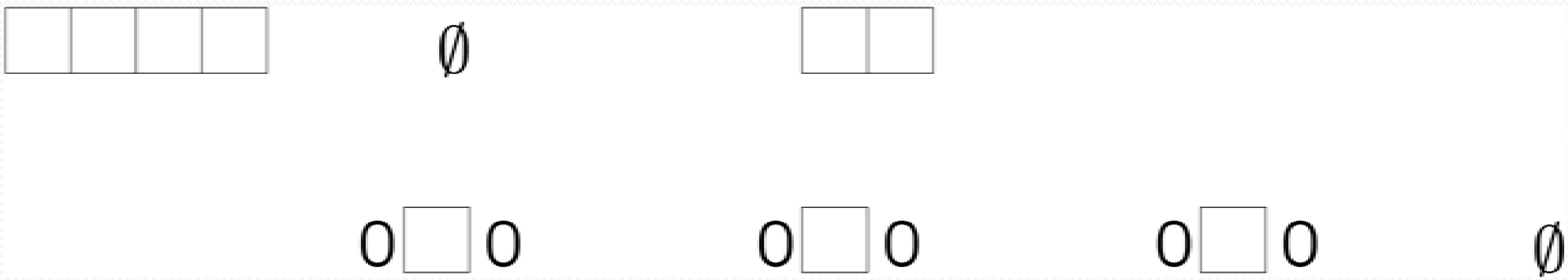
1	1	2	
2	2	3	5

Example



1	1	2	4
2	2	3	5

Example



Continue……..

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&2&3&5\\\hline\end{array}$$

Rigged configuration bijection (2)

- The bijection ϕ was originally introduced in order to prove the fermionic formula for the Kostka—Foulkas polynomials.
- Starting points:
  Kerov—Kirillov—Reshetikhin (1986)
  Kirillov—Reshetikhin (1986)
- Established by:
  Kirillov—Schilling—Shimozono (2002)

Inverse scattering formalism

- New application of ϕ
- The bijection ϕ gives inverse scattering transform for the box-ball systems.
  (Kuniba—Okado—S—Takagi—Yamada 2006).
- Time evolutions of the paths b are linearlized on the corresponding rigged configuration $\phi(b)$.

Crystal interpretation of ϕ (1)

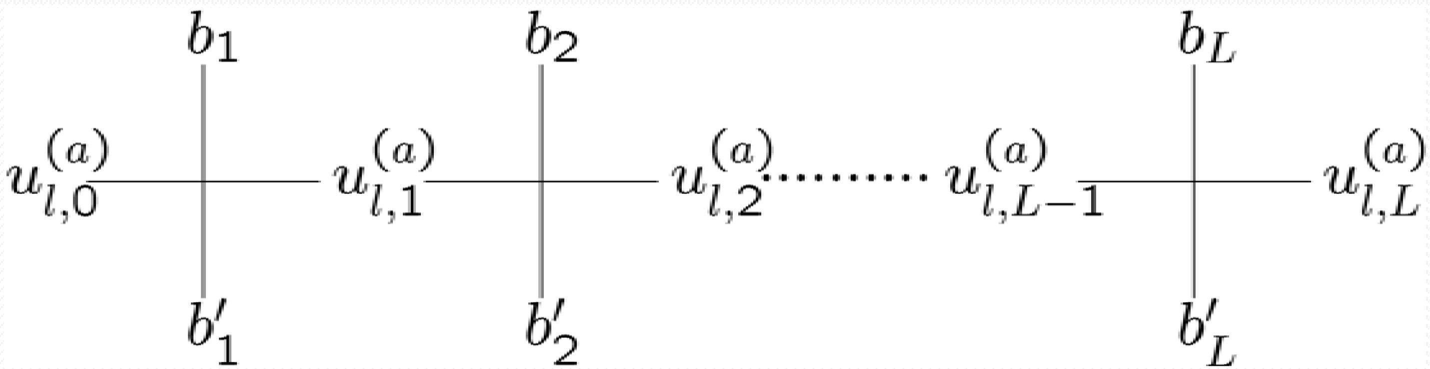
- Original definition of the algorithm for ϕ is given by elaborate combinatorial procedures.
- We want to clarify representation theoretic origin of ϕ.
- For the cases $B^{1,s_1} \otimes \cdots \otimes B^{1,s_L}$, there is an interpretation of ϕ^{-1} (S 2006).
- We want to improve this theory, because
  (1) It is not easy to generalize to

  $$B^{r_1,s_1} \otimes \cdots \otimes B^{r_L,s_L}$$

  (2) It does not clarify the meanings of combinatorial procedures in the definition of ϕ.

Crystal interpretation of ϕ (2)

- Start from the definition of BBS:



- Label each columns of $b_j \in B^{\alpha_j,\beta_j}$ as:

$$b_j = c_{\beta_j} \cdots c_2 c_1$$

- Define sub-tableaux by:

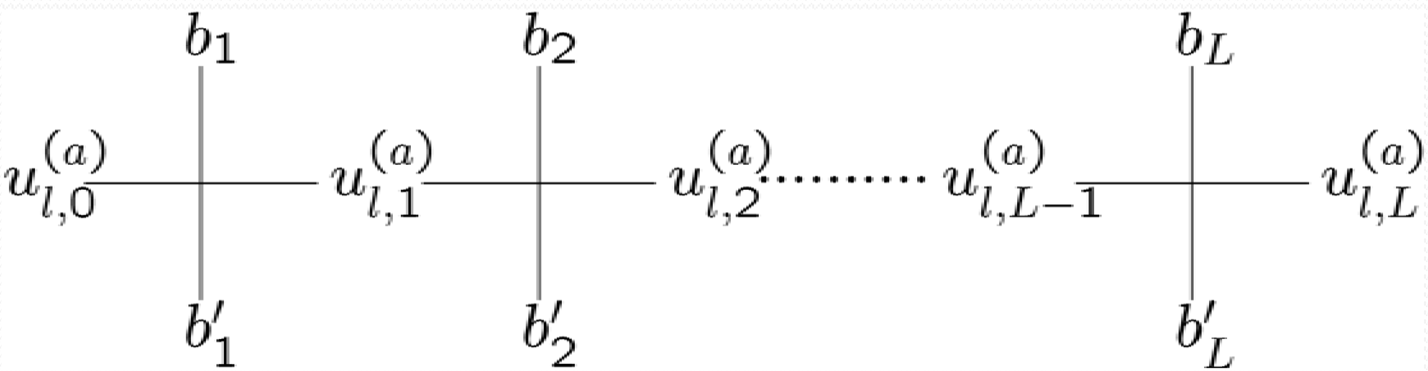
$$b_{j,k} = c_k \cdots c_2 c_1$$

Crystal interpretation of ϕ (3)

- Define the local energy by

$$E_{l,j,k}^{(a)} = H\left(u_{l,j-1}^{(a)} \otimes b_{j,k}\right)$$

for $(l \in \mathbb{Z}_{>0}, 1 \leq j \leq L, 1 \leq k \leq \beta_j)$.



Crystal interpretation of ϕ (4)

- Energy spectrum for type $A_n^{(1)}$.
- Collection of tables labeled by $1 \leq a \leq n$, whose i-th row, (j,k)-th column is given by

$$\varepsilon_{l,j,k}^{(a)} := \left(E_{l,j,k}^{(a)} - E_{l,j,k-1}^{(a)}\right) - \left(E_{l-1,j,k}^{(a)} - E_{l-1,j,k-1}^{(a)}\right).$$

- Note: column (j,k) obey lexicographic ordering.

Example

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&2&3&5\\\hline\end{array}$$

$$a = 1: \quad \left|\begin{array}{cccc|cc|cccc} 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right|$$

$$a = 2: \quad \left|\begin{array}{cccc|cc|cccc} 0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right|$$

$$a = 3: \quad \left|\begin{array}{cccc|cc|cccc} 0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right|$$

$$a = 4: \quad \left|\begin{array}{cccc|cc|cccc} 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right|$$

Crystal interpretation of ϕ (5)

- Meanings of the energy spectrum.
- Original definition of ϕ: box-adding procedure.

$\updownarrow$

Integers in the energy spectrum represent the box-adding procedure:

$\varepsilon^{(a)}_{l,j,k}$ = number of added box at specific position.

- We can construct the rigged configuration ϕ(b) from the energy spectrum.

Example

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 3 & 5 \\ \hline \end{array}$$

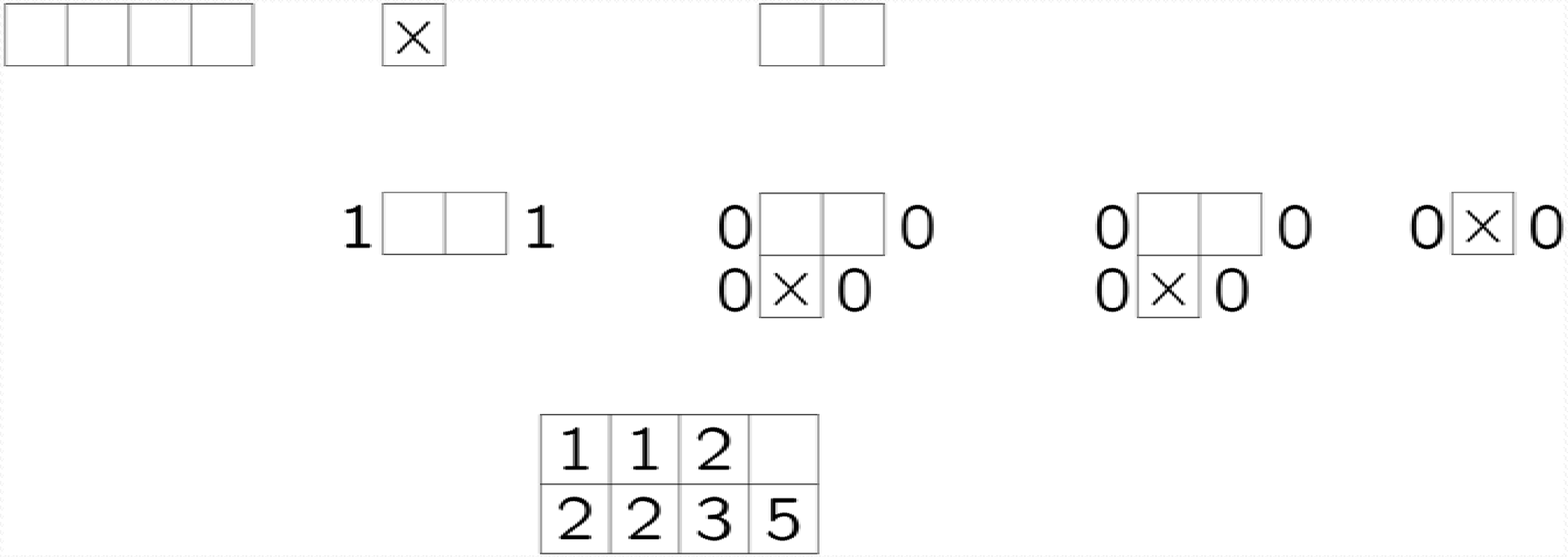
$$a = 1 : \quad \left[\begin{array}{cccc|cc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$a = 2 : \quad \left[\begin{array}{cccc|cc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

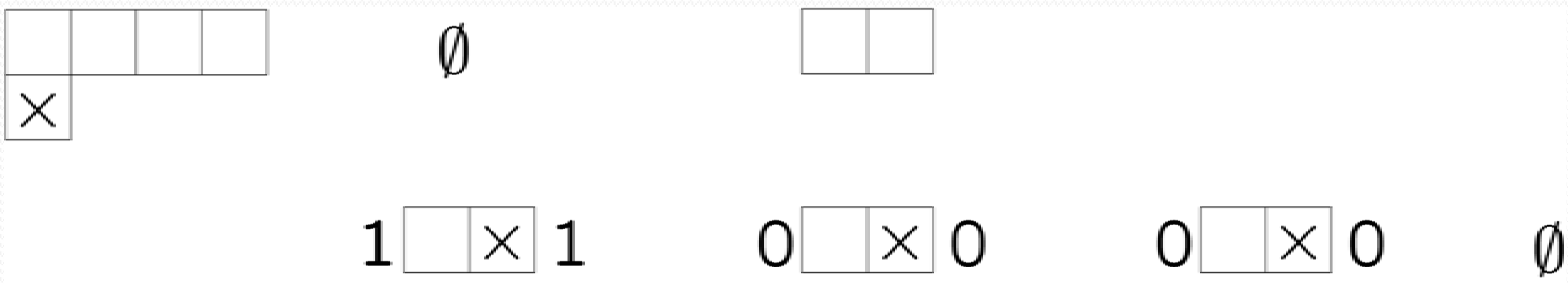
$$a = 3 : \quad \left[\begin{array}{cccc|cc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$a = 4 : \quad \left[\begin{array}{cccc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example



Example



1	1	2	4
2	2	3	5

Crystal interpretation of ϕ (6)

- CONCLUSION
- The original algorithm for ϕ is equivalent to the computation of the energy spectrum:

$$\varepsilon_{l,j,k}^{(a)} := \left(E_{l,j,k}^{(a)} - E_{l,j,k-1}^{(a)}\right) - \left(E_{l-1,j,k}^{(a)} - E_{l-1,j,k-1}^{(a)}\right).$$

- Crystal interpretation of the map ϕ (S 2007).

Is there other link between BBS and combinatorics?

- It will be interesting to search links between BBS and other mathematical structures.

- One progress in this direction is relationship between tau function of the BBS and the Kostka—Macdonald polynomials.

Ultradiscrete tau function (1)

- From now, we exclusively consider the case

$$B^{1,1} \otimes B^{1,1} \otimes \cdots \otimes B^{1,1}$$

- Path $p = a_1 \otimes a_2 \otimes \cdots \otimes a_L$
- Energy statistics on p

$$\mathrm{maj}(p) = \sum_{i=1}^{L-1} (L-i)\chi(a_i < a_{i+1})$$

where χ(True)=1 and χ(False)=0.

- Tau function

$$\tau(p) = \mathrm{maj}(1 \otimes p)$$

Ultradiscrete tau function (2)

- $\tau(p)$ is special case of tau functions introduced by [Kuniba—S—Yamada 2006].
- Tau functions gives general solutions to the BBS.
- Proof based on
  - Ultradiscretization of the KP hierarchy
  - Yang—Baxter relation for affine crystals
  - Recursive & algebraic reformulation of the map ϕ^{-1} [S 2006]
    (see also [Kuniba—Okado—S—Takagi—Yamada 2006])

Ultradiscrete tau function (3)

- There are 3 expressions for $\tau(p)$.
  - Explicit formula in terms of the charge function of the fermionic formula for the Kostka—Foulkas polynomials.
  - Energy function of affine crystals (in the former slide).
  - Counting balls in dynamics of the BBS.

Tau functions with partitions

- According to composition $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$ divide a path

$$p = a_{[1]} \otimes a_{[2]} \otimes \cdots \otimes a_{[n]}$$

$$a_{[i]} = a_{\mu_{[i-1]}+1} \otimes a_{\mu_{[i-1]}+2} \otimes \cdots \otimes a_{\mu_{[i]}} \in (B^{1,1})^{\otimes \mu_i}$$

where $\mu_{[i]} = \sum_{j=1}^{i} \mu_j$.

- Define

$$\tau_\mu(p) = \sum_{i=1}^{n} \tau(a_{[i]})$$

Example

- Path $p = a \otimes b$
  where $a = 4312111,\ b = 4321111\ (\mu = (7,7))$
- Time evolution of BBS for a and b:

$$\begin{pmatrix} 4 & 3 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 3 & 2 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 3 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- Therefore $\tau_{(7,7)}(p) = 11 + 7 = 18.$

Generating functions of tau functions

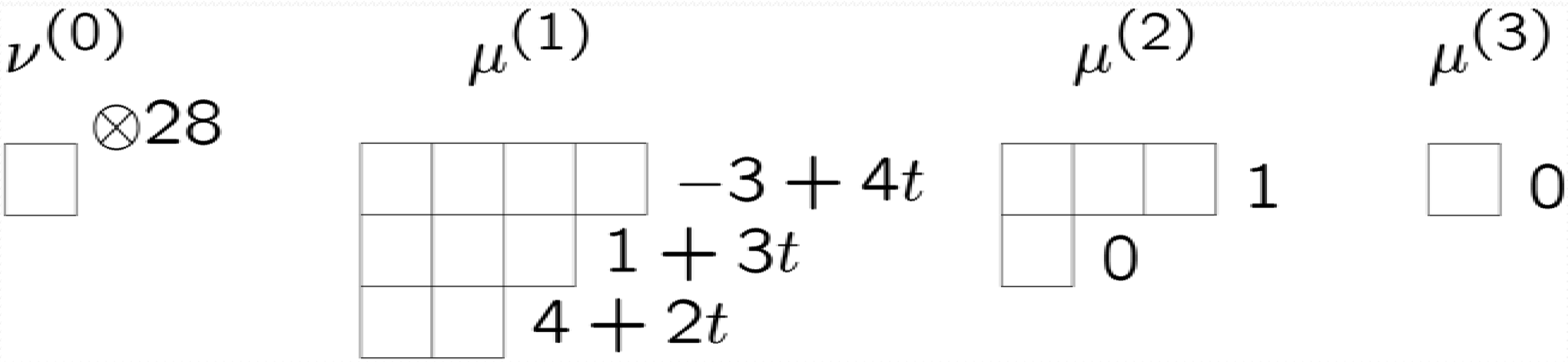
- Theorem [A. N. Kirillov—S 2008]
  Let α be a composition and μ be a partition of the same size. Then,
  $$\sum_{p \in \mathcal{P}(\alpha)} q^{\tau_\mu(p) - \tau_{(1^{|\mu|})}(p)} = \sum_{\eta \vdash |\mu|} K_{\eta,\alpha} \tilde{K}_{\eta,\mu}(q,1).$$
  $\mathcal{P}(\alpha)$: set of all paths with weight α.
- We can prove this by elementary argument.
- However relationship between our tau statistics and Haglund's statistics is not clear.

Example of $T_{\infty}^{(1)}$ (1)

- t=0: 12222111332431111111111111111
- t=1: 11111222211324331111111111111
- t=2: 11111111122213224331111111111
- t=3: 11111111111122113224332111111
- t=4: 11111111111111122111322143321

Example of $T_\infty^{(1)}$ (2)

- Corresponding rigged configuration with respect to each time t:



- Now we are going to look at the energy spectrum with respect to each t:

Energy spectrum (t=0)

1	2	2	2	2	1	1	1	3	3	2	4	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	c							b			a																
		c							b			a															
			c							b																	
				c																							
								e			d																
									e																		
												e															
											f																

Energy spectrum (t=1)

1	1	1	1	1	2	2	2	2	1	1	3	2	4	3	3	1	1	1	1	1	1	1	1	1	1	1	1
					c						a		b														
						c						a		b													
							c								b												
								c																			
											d		e														
														e													
															e												
													f														

Energy spectrum (t=2)

1	1	1	1	1	1	1	1	1	2	2	2	1	3	2	2	4	3	3	1	1	1	1	1	1	1	1	1
									c				a			b											
										c				a			b										
											c							b									
															c												
													d			e											
																	e										
																		e									
																f											

Energy spectrum (t=3)

1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1	3	2	2	4	3	3	2	1	1	1	1	1
												a				b			c								
													a				b			c							
																		b			c						
																						c					
																d			e								
																				e							
																					e						
																			f								

Energy spectrum (t=4)

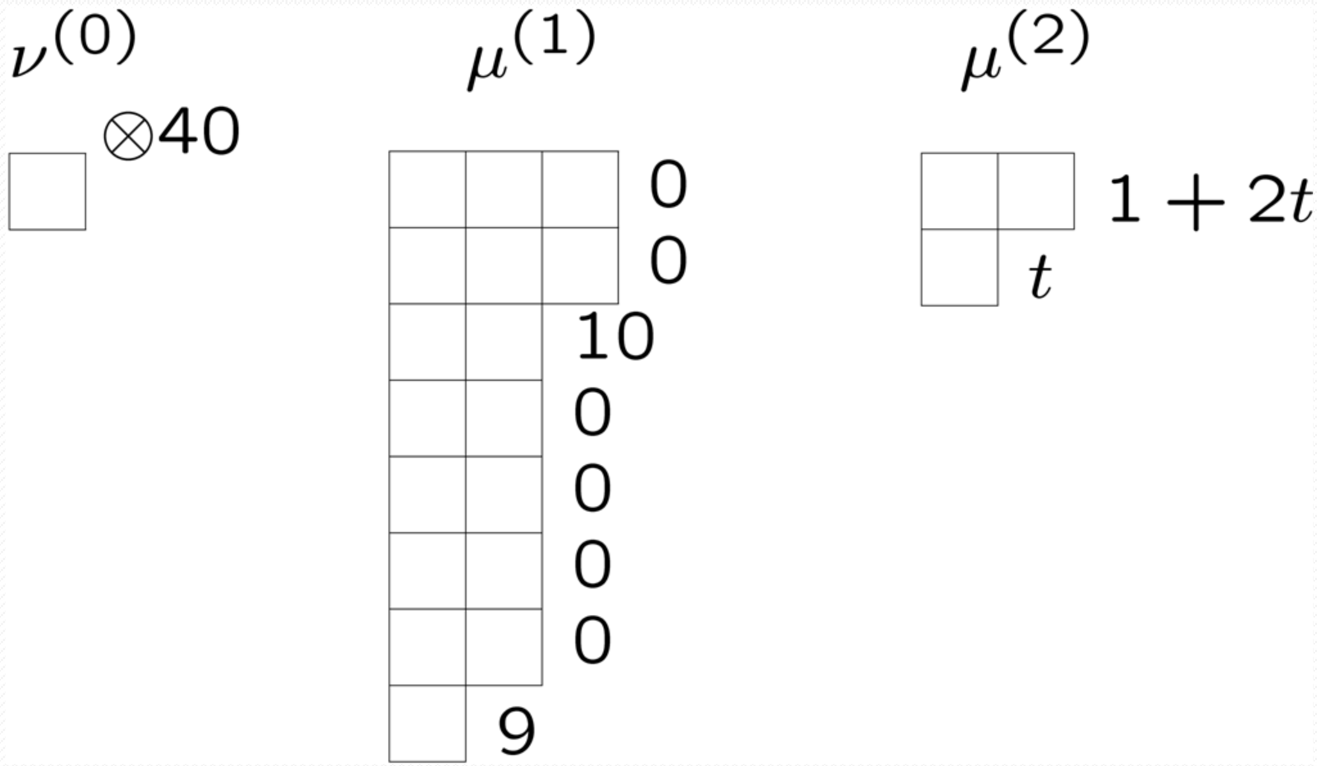
1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	1	1	3	2	2	1	4	3	3	2	1
														a					b				c				
															a					b				c			
																					b				c		
																										c	
																			d				e				
																								e			
																									e		
																							f				

Example of $T_{\infty}^{(2)}$ (1)

- t=0: 1122132132132111212221112221111111122111
- t=1: 1122112213213211312221112221111111122111
- t=2: 1122112211221321123321112221111111122111
- t=3: 1122112211221122131122213321111111122111
- t=4: 1122112211221122112132211122211111133111

Example of $T_\infty^{(2)}$ (2)

- Corresponding rigged configuration with respect to each time t:



Energy spectrum (t=0)

1	1	2	2	1	3	2	1	3	2	1	3	2	1	1	1	2	1	2	2	2	1	1	1	2	2	2	1	1	1	1	1	1	1	1	2	2	1	1	1
		h			g			f			e					d		c						b											a				
			h			g			f			e							c						b											a			
																				c						b													
					j						i																												
								j																															

Energy spectrum (t=1)

1	1	2	2	1	1	2	2	1	3	2	1	3	2	1	1	3	1	2	2	2	1	1	1	2	2	2	1	1	1	1	1	1	1	1	2	2	1	1	1
		h				g			f			e				d		c						b											a				
			h				g			f			e						c						b											a			
																				c						b													
									j							i																							
												j																											

Energy spectrum (t=2)

1	1	2	2	1	1	2	2	1	1	2	2	1	3	2	1	1	2	3	3	2	1	1	1	2	2	2	1	1	1	1	1	1	1	1	2	2	1	1	1
		h				g				f			e				d	c						b											a				
			h				g				f			e					c						b											a			
																				c						b													
													i					j																					
																			j																				

Energy spectrum (t=3)

1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	3	1	1	2	2	2	1	3	3	2	1	1	1	1	1	1	1	1	2	2	1	1	1
		h				g				f				e			d			c				b											a				
			h				g				f				e						c				b											a			
																						c				b													
																	i							j															
																									j														

Energy spectrum (t=4)

1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	1	3	2	2	1	1	1	2	2	2	1	1	1	1	1	1	3	3	1	1	1
		h				g				f				e				d		c						b									a				
			h				g				f				e						c						b									a			
																						c						b											
																				i															j				
																																				j			