Sign elements in symmetric groups

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Introduction

Work in progress

- Question by G. Navarro about characters in symmetric groups, related to a paper by him and I.M. Isaacs. (March 2008)
- Answer to question was surprisingly elegant and inspired the general definition of sign elements and sign classes in finite groups.
- Content of talk:
  - Generalities about group characters
  - The Isaacs-Navarro question
  - Sign elements/classes in finite groups and their relation to the question
  - Generalities about characters of symmetric groups
  - Special types of sign classes in symmetric groups
  - Answering the Isaacs-Navarro question
  - A general result about sign classes in symmetric groups

Properties of irreducible characters of finite groups

Background

G finite group. G has a set $\mathrm{Irr}(G)$ of irreducible characters. We list some fundamental properties of the irreducible characters.

- Any $\chi \in \mathrm{Irr}(G)$ is a class function, i.e. constant on the conjugacy classes of G.
- $|\mathrm{Irr}(G)| = k(G)$, the number of conjugacy classes of G.
- Character values are algebraic integers.
- The character table of G is a square $k(G)$-matrix. Rows indexed by irreducible characters and columns by conjugacy classes. Entry (i, j) is the value $\chi_i(g_j)$ of character χ_i on an element g_j in conjugacy class K_j. The first column contains the degrees of the irreducible characters $\chi_i(1)$.
- The ring of generalized characters:

$$\mathcal{R}(G) = \{\sum_{i=1}^{k(G)} z_i\chi_i \mid z_i \in \mathbb{Z}\}.$$

Orthogonality relations for characters

- Row orthogonality (First orthogonality relation):

$$\sum_{i=1}^{k(G)} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs}.$$

- Column orthogonality (Second orthogonality relation):

$$\sum_{i=1}^{k(G)} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|.$$

Question of Isaacs and Navarro

- Background for the question may be found in their preprint entitled "Character Sums and Double Cosets" from 2008.
  It is known the the 2-Sylow subgroups of symmetric groups are self-normalizing.
- **Question:** Let P be 2-Sylow subgroup of S_n and $Irr_{2'}(S_n)$ be the set of odd degree irreducible characters of S_n. Does there exist signs e_χ for $\chi \in Irr_{2'}(S_n)$ such that the generalized character

$$\Theta = \sum_{\chi \in Irr_{2'}(S_n)} e_\chi \chi$$

satisfies:

$$(i) \quad \Theta(x) \text{ is divisible by } |P/P'| \text{ for all } x \in S_n.$$

and

$$(ii) \quad \Theta(x) = 0 \text{ for all } x \in S_n \text{ of odd order.}$$

Sign classes

- A sign class in a finite group G is a conjugacy class on which all irreducible characters of G take one of the values 0, 1 or -1.
- Elements in sign classes are called sign elements.
- **Example:** Let $G = S_3$. Character table

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

The classes 2 and 3 are sign classes.

Examples of sign elements

- In abelian groups only elements satisfying $g^2 = 1$ are sign elements.
- Non-central involutions in dihedral groups are sign elements
- In $\mathrm{SL}(2, 2^n)$ there is an involution on which all irreducible characters except the Steinberg character take the values 1 or -1. Thus this is a sign element. This is a very interesting example.
- Sign elements of odd prime order p *may* occur when you have a self-centralizing p-Sylow subgroup of order p in G. This occurs for example for $p = 7$ in the simple group M_{11} which also has sign elements of order 6.
- Question: Are there other examples of involutions as sign elements in quasisimple groups than the $\mathrm{SL}(2, 2^n)$? (None in symmetric groups for $n \geq 5$, as we shall see.)

Support of sign elements

- The support of a sign element $s \in G$ is defined as

$$\mathrm{supp}(s) = \{\chi \in \mathrm{Irr}(G) \mid \chi(s) \neq 0\}.$$

- Column orthogonality shows that for a sign element $s \in G$ we have that $|\mathrm{supp}(s)| = |C_G(s)|$.

A generalized character

- Suppose that you have a sign element s. For $\chi \in \mathrm{Irr}(G)$ put $e_\chi = \chi(s)$. If $e_\chi \neq 0$ then e_χ is 1 or -1.
- Consider the generalized character

$$\Theta_s = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$$

- Column orthogonality shows that Θ_s vanishes on all conjugacy classes except the class of s.
- On the class of s Θ_s takes the value $|C_G(s)|$.

A very small example

- Column orthogonality : $\sum_{i=1}^{k(G)} \chi_i(g_u)\overline{\chi_i(g_v)} = \delta_{uv}|C_G(g_u)|$.
- Let $G = S_3$.

$$\left[\begin{array}{rrr} (1^3) & (2,1) & (3) \\ \hline\hline 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{array}\right]$$

The second and third conjugacy class are sign classes.

- Choose $v = 2$. Get that $\Theta_{(2,1)} = \chi_1 - \chi_3$. Thus

$$\chi_1(g_u) - \chi_3(g_u) = 0 \quad \text{for} \quad u \neq v$$

$$\chi_1(g_u) - \chi_3(g_u) = 2 \quad \text{for} \quad u = v = 2.$$

- Choose $v = 3$. Get that $\Theta_{(3)} = \chi_1 - \chi_2 + \chi_3$. Thus

$$\chi_1(g_u) - \chi_2(g_u) + \chi_3(g_u) = 0 \quad \text{for} \quad u \neq 3$$

$$\chi_1(g_u) - \chi_2(g_u) + \chi_3(g_u) = 3 \quad \text{for} \quad u = v = 3.$$

Relevance to Isaacs-Navarro question

- Suppose that you can find a 2-element s in S_n which is a sign element with support $Irr_{2'}(S_n)$. Consider $\Theta = \Theta_s$.
- Condition (ii) stating that $\Theta(x) = 0$ for all $x \in S_n$ of odd order is trivially fulfilled, since s is a 2-element
- Condition (i) stating that $\Theta(x)$ is divisible by $|P/P'|$ for all $x \in S_n$ requires only a simple calculation in the case $x = s$.

Remark on block orthogonality

- When $s \in G$ is a sign element, the generalized character

$$\Theta_s = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$$

is the difference between two disjoint multiplicity-free characters Θ_s^+ and Θ_s^- which coincide on all conjugacy classes except the class of s.

- Here

$$\Theta_s^+ = \sum_{\chi \in \mathrm{Irr}(G), e_\chi = 1} \chi, \quad \Theta_s^- = \sum_{\chi \in \mathrm{Irr}(G), e_\chi = -1} \chi$$

- Block orthogonality shows that if p is a prime number dividing the order of the sign element s and if you split Θ_s^+ and Θ_s^- into components according to the p-blocks of characters of G, then the values of these components for a given p-block still coincide on all p-regular elements in G. This has consequences for the decomposition numbers.

Characters and partitions

- We will discuss from now on sign classes in the symmetric groups S_n.
- The irreducible characters and the conjugacy classes of S_n are labelled canonically by the partitions of n.
- A partition λ of n is a sequence of natural numbers

$$\lambda = (a_1, a_2, \ldots, a_m)$$

such that

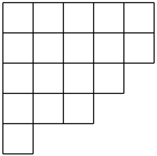
$$a_1 \geq a_2 \geq \ldots \geq a_m \text{ and } a_1 + a_2 + \ldots + a_m = n$$

Characters of symmetric groups

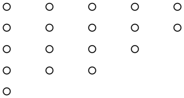
- The irreducible characters of S_n are all integer valued.
- Let $\mathcal{P}(n)$ be the set of partitions of n.
- We write the entries of the character table of S_n as $[\lambda](\mu)$, for $\lambda, \mu \in \mathcal{P}(n)$. This is the value of the irreducible character of S_n, labelled by λ on the conjugacy class labelled by μ.

Young diagrams

The Young diagram of the partition $\lambda = (a_1, a_2, \ldots, a_m)$ of n is obtained by arranging n boxes/nodes as the following example shows:
$\lambda = (5^2, 4, 3, 1)$:



or



Hooks

- A hook in (the Young diagram of) λ is a subdiagram as marked by bullets below.

$$\begin{array}{ccccc}
\circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \bullet & \bullet & \bullet \\
\circ & \bullet & \circ & \circ & \\
\circ & \bullet & \circ & & \\
\circ & & & &
\end{array}$$

- This is the (2,2)-hook $\mathcal{H}_{2,2}(\lambda)$. Its corner node is in position (2,2). The length $h_{2,2}(\lambda)$ of the hook is the number of bullets, i.e. 6. The leg length $b_{2,2}(\lambda)$ of the hook is the number of bullets below the corner node i.e. 2.

Hook diagram

Each node is the “corner” of a hook and has an associated hook length.

○	○	○	○	○
○	**6**	○	○	○
○	○	○	○	
○	○	○		
○				

Here is the complete hook diagram of our example:

9	7	6	4	2
8	**6**	5	3	1
6	4	3	1	
4	2	1		
1				

Hook removal

You remove the hook $\mathcal{H}_{i,j}(\lambda)$ from λ by deleting the nodes of the hook and pushing the diagram together. The result is denoted $\lambda \backslash \mathcal{H}_{i,j}(\lambda)$.

Example:
$\lambda = (5^2, 4, 3, 1), (i, j) = (2, 2)$.

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & - & - & - & - \\ \circ & - & * & * & \\ \circ & - & * & & \\ \circ & & & & \end{array}$$

$\lambda \backslash \mathcal{H}_{2,2}(\lambda)$:

$$\begin{array}{ccccc} \circ & \circ & \circ & \circ & \circ \\ \circ & * & * & & \\ \circ & * & & & \\ \circ & & & & \\ \circ & & & & \end{array}$$

The $-$'s are removed and the $*$'s are moved.

The Murnaghan-Nakayama formula

If $\lambda = (a_1, a_2, ..., a_m)$ and $h \in \mathbb{N}$ define

$$\mathcal{Y}(\lambda) = \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq a_i\}$$

and

$$\mathcal{Y}(\lambda)_h = \{(i,j) \in \mathcal{Y}(\lambda) \mid h_{i,j}(\lambda) = h\}.$$

Theorem:(Murnaghan-Nakayama formula) Let $\lambda, \mu \vdash n$ with $\mu = (l_1, l_2, \ldots, l_k)$.
For all $r, 1 \leq r \leq k$ we have

$$[\lambda](\mu) = \sum_{(i,j)\in\mathcal{Y}(\lambda)_{l_r}} (-1)^{b_{i,j}^{\lambda}}[\lambda \backslash \mathcal{H}_{i,j}(\lambda)](\mu_r),$$

where $\mu_r = (l_1, l_2, \ldots, l_{r-1}, l_{r+1}, \ldots, l_k)$.

Consequence of Murnaghan-Nakayama formula

In his book "The Representation Theory of the Symmetric Groups", Springer Lecture Notes, 1978, G. James lists a useful consequence of the MN-formula.

Theorem: Let ν be a partition of $n-h$. The generalized character

$$X(\nu, n) = \sum_{\lambda} (-1)^{b_\lambda} [\lambda]$$

vanishes on all $\mu \vdash n$ which do not contain a part equal to h. Here λ runs through all partitions of n for which $\nu = \lambda \backslash \mathcal{H}_{i,j}(\lambda)$ for some $(i,j) \in \mathcal{Y}(\lambda)_h$ and then $b_\lambda = b_{i,j}(\lambda)$.

Example: Suppose that $\nu = (2), h = 3, n = 5$. Then

$$X(\nu, 5) = [5] - [2^2, 1] + [2, 1^3].$$

Sign partitions

- We call $\mu \in \mathcal{P}(n)$ a sign partition if the the corresponding conjugacy class is a sign class, ie. if $[\lambda](\mu) \in \{0, 1, -1\}$ for all $\lambda \in \mathcal{P}(n)$. The support of a sign partition μ is defined as

$$\mathrm{supp}(\mu) = \{\lambda \in \mathcal{P}(n) \mid [\lambda](\mu) \neq 0\}$$

- (n) is always a sign partition
- By the MN-formula $[\lambda](n) \neq 0$ if and only if $\lambda = (n-k, 1^k)$ is a hook partition and then $[\lambda](n) = (-1)^k$.

Sign partitions for small values of n

- $n = 2: \ (2), (1^2)$
  $n = 3: \ (3), (2, 1)$
  $n = 4: \ (4), (3, 1), (2, 1^2)$
- $n = 5: \ (5), (4, 1), (3, 2), (3, 1^2)$
  $n = 6: \ (6), (5, 1), (4, 2), (4, 1^2), (3, 2, 1)$
  $n = 7: \ (7), (6, 1), (5, 2), (5, 1^2), (4, 3), (4, 2, 1), (3, 2, 1^2)$
- $n = 8: \ (8), (7, 1), (6, 2), (6, 1^2), (5, 3), (5, 2, 1), (4, 3, 1)$
- $n = 9:$
  $(9), (8, 1), (7, 2), (7, 1^2), (6, 3), (6, 2, 1), (5, 4), (5, 3, 1), (5, 2, 1^2)$
- $n = 10: (10), (9, 1), (8, 2), (8, 1^2), (7, 3), (7, 2, 1), (6, 4), (6, 3, 1),$
  $(6, 2, 1^2), (5, 4, 1), (4, 3, 2, 1)$

Unique path partitions

- Consider the *unique path*-partitions (for short *up*-partitions). They are defined as follows.
- If $\mu = (l_1, l_2, ..., l_k)$ and λ are partitions of n, then a μ-path in λ is a sequence $\lambda = \lambda_0, \lambda_1, ..., \lambda_k = (0)$, where for $i = 1...k$ λ_i is obtained by removing an l_i-hook in λ_{i-1}. Then we call μ is an *up*-partition if for all λ the number of μ-paths in λ is at most 1.
- A *up*-partition is also a sign partition.
- If $\mu = (l_1, l_2, ..., l_k)$ is an *up*-partition with $k \geq 2$, then also $\mu' = (l_2, ..., l_k)$ is an *up*-partition.
- $(3, 2, 1)$ is s sign partition, but not a *up*-partition, since there are two (3,2,1)-paths in $\lambda = (3, 2, 1)$.

Extending partitions

- **Proposition:** *Let $m > n$. If $\mu' = (a_1, a_2, ..., a_k)$ is a partition of n, and $\mu = (m, a_1, a_2, ..., a_k)$ then μ' is a sign partition (respectively a up-partition) of n if and only if μ is a sign partition (respectively a up-partition) of $m + n$.*
- The key fact used in the proof is: Let λ be a partition of $m + n$. Since $2m > m + n$ λ cannot contain more than at most one hook of length m. Thus the *up*-statement is obvious.

sd-partitions

- We call a partition $\mu = (a_1, a_2, ..., a_k)$ *strongly decreasing* (for short *sd*-partition) if for $i = 1, ..., k-1$ we have $a_i > a_{i+1} + ... + a_k$.
- If $\mu = (a_1, a_2, ..., a_k)$ is an *sd*-partition with $k \geq 2$ then $\mu' = (a_2, ..., a_k)$ is also an *sd*-partition.
- $(3, 1^2)$ is an *up*-partition, but not an *sd*-partition
- **Proposition:** *An sd-partition is a up-partition and thus also sign partition.*

sd-partitions and non-squashing partitions I

- Hirschhorn and Sellers defined a partition $\mu = (a_1, a_2, ..., a_k)$ to be *non-squashing* if for $i = 1, ..., k-1$ we have $a_i \geq a_{i+1} + ... + a_k$. For *sd*-partitions the condition is $a_i > a_{i+1} + ... + a_k$.
- It was shown by Hirschhorn and Sellers that the number of non-squashing partitions of n equals the number of *binary* partitions of n, ie. partitions whose parts are powers of 2. A bijection was given by Sloane and Sellers.

sd-partitions and non-squashing partitions II

Let $s(n)$ denote the number of *sd*-partitions of n. Put $s(0) = 1$. Ordering the set of *sd*-partitions according to their largest part shows that

$$s(n) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} s(i).$$

Thus for all $k \geq 1$ we have $s(2k-1) = s(2k)$. Putting $t(k) = 2s(2k) = s(2k-1) + s(2k)$ it can be shown using recursion formulae, that $t(k)$ equals the number of non-squashing partitions of $2k$, ie. the number of "binary" partitions of $2k$.

Answering the Isaacs-Navarro question

- **Theorem:** *Write* $n = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_t}$, *where* $r_1 > r_2 > \ldots > r_t \geq 0$. *Then* $\mu = (2^{r_1}, 2^{r_2}, \ldots, 2^{r_t})$ *is a sd-partition with support* $\mathrm{supp}(\mu) = Irr_{2'}(S_n)$. *Moreover* Θ_μ *satisfies the conditions (i) and (ii) above. Indeed* Θ_μ *vanishes everywhere except on* μ *where it takes the value* $|P/P'|$.
- Facts needed in proof
  • $|P/P'| = z_\mu = 2^{r_1+r_2+\ldots+r_t}$.
  • $|Irr_{2'}(S_n)| = 2^{r_1+r_2+\ldots+r_t}$ (Macdonald, 1971, Bull. London Math. Soc)
  • supp$(\mu) \subseteq Irr_{2'}(S_n)$. (Malle-Navarro-Olsson, 2000, J. Group Theory)
- The last two facts are special cases of more general results utilizes the theory of cores and quotients of partitions.

Another example of existence of signs

In $SL(2, 2^n)$ the 2-Sylow subgroup is self centralizing. It has a unique conjugacy class of involutions and $2^n + 1$ irreducible characters, all of which (with the exception of the Steinberg character) have odd degrees. The involutions are sign elements, so that Θ_t, t involution, vanishes on all elements of odd order. The value on t is 2^n. Thus this is another example of the existence of signs for odd degree irreducible characters such that the signed sum satisfy the conditions mentioned above.

Towards a classification of sign partitions

- **Lemma:** *A sign partition cannot have its smallest part repeated except for the part 1, which may be repeated once.*

  Proof: Suppose that 1 is repeated $m \geq 2$ times then by MN $[n-1,1](\mu) = [m-1,1](1^m) = m-1$. Thus $m = 2$. If $b > 1$ is the smallest part, repeated $m \geq 2$ times then by MN $[n-b,b](\mu) = m$. □

- We have that much more is true:

  Theorem: *A sign partition cannot have repeated parts except for the part 1, which may be repeated once.*

Corollaries

- **Corollary:** *If μ is a sign partition, then the centralizer of elements of cycle type μ is abelian. Short: Centralizers of sign elements in S_n are abelian.*
- There exists a group of order 32 containing a sign element with a non-abelian centralizer. (G. Navarro)
- Which 2-elements in S_n are sign classes?
  Corollary: *Suppose that $n = 2^{r_1} + 2^{r_2} + ... + 2^{r_t}$, where $r_1 > r_2 > ... > r_t \geq 0$. The sign classes of 2-elements in S_n have for n odd (ie. $r_t = 0$) cycle type $(2^{r_1}, 2^{r_2}, ..., 2^{r_t})$. If $n = 4k + 2$ (ie. $r_t = 1$) we have in addition $(2^{r_1}, 2^{r_2}, ..., 2^{r_{t-1}}, 1^2)$. If $n = 8k + 4$ (ie. $r_t = 2$) we have in addition $(2^{r_1}, 2^{r_2}, ..., 2^{r_{t-1}}, 2, 1^2)$.*

Remarks about the proof I

- Assume that the sign partition μ of n has a smallest repeated part $a > 1$ repeated $m > 1$ times. We want to show that we can find a not-too-complicated partition λ such that $|[\lambda](\mu)| \geq m$.
- In fact we show that λ can be chosen such that *all hook lengths of λ outside the first row are $\leq a$.*
- It is not difficult to see that we may assume that a is the largest part of μ.
- Thus $\mu = (a^m, a_2, ..., a_k)$ where $a > a_2 > ... > a_k > 0$. Put $t = a_2 + ... + a_k$, so that $m = ma + t$. We have by the lemma $t > 0$.

Remarks about the proof II

- Let for $0 \leq i \leq m$ μ_i be μ with i parts equal to a removed. Thus $\mu_0 = \mu$ and $\mu_m = \mu^* = (a_2, ..., a_k)$.
- The partition $(n-a, 1^a)$ has precisely two hooks of length a (since multiplicity of a is ≥ 2. The MN-formula shows $[n-a, 1^a](\mu) = (-1)^{a-1}[n-a](\mu_1) + [n-2a, 1^a](\mu_1) = (-1)^{a-1} + [n-2a, 1^a](\mu_1)$. Inductively we get $[n-a, 1^a](\mu) = (m-1)(-1)^{a-1} + [t, 1^a](\mu_{m-1})$. Need to understand the last term $[t, 1^a](\mu_{m-1})$.
- If $t \leq a$, then $[t, 1^a]$ has only one hook of length a and we get by MN that $[t, 1^a](\mu_{m-1}) = (-1)^{a-1}[t](\mu_m) = (-1)^{a-1}$ and thus $[n-a, 1^a](\mu) = m(-1)^{a-1}$. Thus $[n-a, 1^a]$ may be chosen as the desired λ.

Remarks about the proof III

- We are left with the case $a < t$. This is divided into the subcases $t < 2a$ and $t \geq 2a$, which demand similar arguments. We consider only the first subcase.
- If $t < 2a$ then $t - a < a$. There are exactly a partitions of t obtained by adding an a-hook to the partition $(t - a)$. Suppose that κ_i is obtained by adding a hook with leg length i to $(t - a)$.
- Since $t < 2a$ each κ_i has only one hook of length a. Removing it we get $(t - a)$. Note that $\kappa_0 = (t)$.
- By the theorem, which was a consequence of MN, the generalized character $\sum_{i=0}^{a-1}(-1)^i\kappa_i$ takes the value 0 on μ^*, since μ^* has no part divisible by a.

Remarks about the proof IV

- Choose an $j > 0$ such that $(-1)^j[\kappa_j](\mu^*) \geq 0$. (Clearly, the $(-1)^j[\kappa_j](\mu^*)$ cannot all be < 0, since the contribution from $[\kappa_0] = [t]$ is equal to 1 and $a \geq 4$.)
- Put $\lambda^* = \kappa_j$ so that
$$(-1)^j[\lambda^*](\mu^*) \geq 0.$$
- If λ is obtained from λ^* by adding ma to its largest part, then the largest part of λ is at least $n - a$ so that trivially all hook lengths outside the first row are $\leq a$. We can then show that $|[\lambda](\mu)| \geq m$.
- This is done by a calculation analogous to above. We get
$$[\lambda](\mu) = [\lambda^*](\mu^*) + m(-1)^j = (-1)^j((-1)^j[\lambda^*](\mu^*) + m).$$
This has absolute value $\geq m$, so that μ is not a sign class.

What is next?

It would seem that eventually sign partitions have to behave much like *sd*-partitions. In most cases the following seems to hold:

If $\mu' = (a_2, ..., a_k)$ is a sign partition of t, $a > a_2$ and $\mu = (a, a_2, ..., a_k)$ then μ is a sign partition if and only if $a > t$.

Example: (4,3,2,1) is a (non-*sd*) sign partition of 10, but (a,4,3,2,1) is not a sign partition for $a = 5, ..., 10$.

However the partitions $(1, 1)$ and $(a, a-1, 1), a \geq 2$ provide (the only) counterexamples to the above statement in the case $a = t$. There may be only finitely many counterexamples in the case where $a < t$.

An open question is the following:

Is it true that if $\mu = (a_1, a_2, ..., a_k)$ is a sign partition, then also $\mu' = (a_2, ..., a_k)$ is an sign partition?