Jack polynomial, random matrix theory, and hyperdeterminant

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Outline

Introduction

Problem

Let $\mathcal{S}$ be a set of $N \times N$ matrices. Assume a probability measure $\mathfrak{M}_{\mathcal{S}}(\mathrm{d}M)$ on $\mathcal{S}$ is given. For a function $F = F(M)$ on $\mathcal{S}$, denote by $\langle F(M) \rangle_{M \in \mathcal{S}}$ the average of F: $\langle F(M) \rangle_{M \in \mathcal{S}} = \int_{\mathcal{S}} F(M) \mathfrak{M}_{\mathcal{S}}(\mathrm{d}M)$.

Problem

Calculate the moment

$$\langle \det(I + xM)^m \rangle_{M \in \mathcal{S}}$$

and its asymptotic behavior as the matrix size N goes to the infinity.

Problem

More generally,

Problem

Calculate the average of a product

$$\langle \det(I + x_1 M) \cdots \det(I + x_m M) \rangle_{M \in \mathcal{S}}$$

and the average of a ratio

$$\left\langle \frac{\det(I + x_1 M) \cdots \det(I + x_m M)}{\det(I + y_1 M) \cdots \det(I + y_l M)} \right\rangle_{M \in \mathcal{S}} .$$

History

Case $\mathcal{S}$ = Classical groups with normalized Haar measures

- Moment [Keating-Snaith (2000)]
- Product [Conrey-Farmer-Keating-Rubinstein-Snaith (2003)]
- Ratio [Conrey-Forrester-Snaith (2005)]

Case $\mathcal{S}$ = Hermitian matrices with Gaussian measures (GOE, GUE, GSE)
Determinantal or Pfaffian expression.
[Brézin-Hikami (2000-)], [Baik-Deift-Strahov (2003)], [Borodin-Strahov (2006)].

Example

Consider $\mathcal{S} = U(n)$, the unitary group.

Theorem [Conrey-Farmer-Keating-Rubinstein-Snaith (2003)]

Let $x_1, \dots, x_{L+K}$ be any complex numbers. Then

$$\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1}) \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M) \right\rangle_{M \in U(n)}$$
$$= (x_1 \cdots x_L)^{-n} \cdot s_{(n^L)}(x_1, \dots, x_{L+K}).$$

Here $s_{(n^L)}(x_1, \dots, x_{L+K})$ is the Schur polynomial with partition $(n^L) = (n, n, \dots, n)$.

Example

In particular, we have [Keating-Snaith (2000)]

$$\langle |\det(I+\xi M)|^{2m}\rangle_{M\in U(n)} = \prod_{j=0}^{n-1} \frac{j!\,(j+2m)!}{(j+m)!^2}, \qquad |\xi| = 1.$$

As a corollary, we obtain an asymptotic behavior

$$\lim_{n\to\infty} \frac{1}{n^{m^2}} \langle |\det(I+\xi M)|^{2m}\rangle_{M\in U(n)} = \prod_{j=0}^{m-1} \frac{j!}{(j+m)!}.$$

This limit value is closely related to the moment of the Riemann zeta function. (**Keating-Snaith conjecture (2000)**).

Our purpose in this talk

We calculate the averages of the product and ratio of characteristic polynomials in a random matrix model, called Dyson's circular β-ensembles.

In order to calculate the characteristic polynomial averages, we employ Jack polynomials and super-Jack polynomials.

Furthermore, we present an expression for the characteristic polynomial average in terms of a hyperdeterminant.

Dyson's circular β-ensembles

Definition of the CβE

Let β be a positive real number. Let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\mathrm{d}z$ be the normalized Haar measure on $\mathbb{T}$:

$$\int_{\mathbb{T}} f(z)\mathrm{d}z = \int_{-\pi}^{\pi} f(e^{i\theta})\frac{\mathrm{d}\theta}{2\pi} \qquad \text{for a Laurent polynomial } f \text{ on } \mathbb{T}.$$

We equip each element $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$ to the probability proportional to

$$\prod_{1 \le i < j \le n} |z_i - z_j|^{\beta}.$$

Then the space $\mathbb{T}^n$ is called Dyson's circular β-ensemble (CβE). The CβE arises from the well-known three random matix models: COE ($\beta = 1$), CUE ($\beta = 2$), CSE ($\beta = 4$).

CUE ($\beta = 2$)

The circular unitary ensemble (CUE) is nothing but the unitary group $U(n)$ with its normalized Haar measure.
By Weyl's integration formula, the probability density function for eigenvalues $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$ of a CUE matrix M is given by

$$\frac{1}{n!} \prod_{1 \le i < j \le n} |z_i - z_j|^2.$$

Therefore the distribution of the eigenvalues of a CUE matrix is the $\mathrm{C}\beta\mathrm{E}$ with $\beta = 2$.

COE ($\beta = 1$)

Consider the set of symmetric unitary matrices:

$$\mathcal{S}_{\mathrm{COE}}(n) := \{M \in U(n) \mid M \text{ is symmetric}\} \simeq U(n)/O(n).$$

The set $\mathcal{S}_{\mathrm{COE}}(n)$ has a probability measure induced from $U(n)$ and is called the circular orthogonal ensemble (COE). It is well known that the probability density function of eigenvalues $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$ of a COE matrix M is proportional to

$$\prod_{1 \le i < j \le n} |z_i - z_j|.$$

Therefore the distribution of the eigenvalues of a COE matrix is the $\mathrm{C}\beta\mathrm{E}$ with $\beta = 1$.

CSE ($\beta = 4$)

Consider the set of Hermitian unitary quaternion matrices:

$$\mathcal{S}_{\mathrm{CSE}}(n) := \{M \in U(n, \mathbb{H}) \mid M \text{ is Hermitian}\} \simeq U(2n)/Sp(2n).$$

The set $\mathcal{S}_{\mathrm{CSE}}(n)$ has a probability measure induced from $U(2n)$ and is called the circular symplectic ensemble (CSE). It is well known that the probability density function of eigenvalues $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$ of a CSE matrix M is proportional to

$$\prod_{1 \le i < j \le n} |z_i - z_j|^4.$$

Therefore the distribution of the eigenvalues of a CSE matrix is the $\mathrm{C}\beta\mathrm{E}$ with $\beta = 4$.

Characteristic polynomials for general β

For each $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$ and $x \in \mathbb{C}$, we put

$$\Psi(\mathbf{z}; x) = \prod_{j=1}^{n} (1 + x z_j).$$

When $\mathbf{z}$ are eigenvalues of an $n \times n$ COE, CUE, or CSE matrix M, then

$$\Psi(\mathbf{z}; x) = \det(I_n + xM).$$

We call $\Psi(\mathbf{z}; x)$ the characteristic polynomial of $\mathbf{z}$.
For a function $F = F(\mathbf{z})$ on $\mathbb{T}^n$, we define the average of F by

$$\langle F(\mathbf{z}) \rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)} := \frac{\int_{\mathbb{T}^n} F(\mathbf{z}) \prod_{1 \le i < j \le n} |z_i - z_j|^\beta \mathrm{d}z_1 \cdots \mathrm{d}z_n}{\int_{\mathbb{T}^n} \prod_{1 \le i < j \le n} |z_i - z_j|^\beta \mathrm{d}z_1 \cdots \mathrm{d}z_n}.$$

Characteristric polynomial averages

First, we would like to evaluate the following characteristic polynomial averages.
For $x_1, \dots, x_L, y_1, \dots, y_K \in \mathbb{C}$,

$$\left\langle \prod_{l=1}^{L} \Psi(\overline{\mathbf{z}}, x_l) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; y_k) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)} .$$

We will express this average by a Jack polynomial.

Jack polynomials

Monomial symmetric polynomials

We review the definition of Jack polynomials.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we denote by $\ell(\lambda)$ the length of λ.
We deal with partitions such that $\ell(\lambda) \le n$.

Let $x_1, \dots, x_n$ be variables. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$m_\lambda(x_1, \dots, x_n) := \sum_{\mu \in \mathfrak{S}_n \lambda} x_1^{\mu_1} \cdots x_n^{\mu_n} \qquad \in \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n},$$

where $\mathfrak{S}_n \lambda = \{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) \in (\mathbb{Z}_+)^n \mid \sigma \in \mathfrak{S}_n\}$.

Scalar product

Let $\alpha > 0$. Define a scalar product on $\mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ by

$$\langle m_\lambda, m_\mu \rangle_\Delta = \frac{1}{n!} \int_{\mathbb{T}^n} m_\lambda(z_1, \dots, z_n) \overline{m_\mu(z_1, \dots, z_n)} \times \Delta(z_1, \dots, z_n; \alpha) \mathrm{d}z_1 \cdots \mathrm{d}z_n$$

with

$$\Delta(z_1, \dots, z_n; \alpha) = \prod_{1 \le i < j \le n} |z_i - z_j|^{2/\alpha}.$$

Theorem & Definition (Jack polynomials)

There exists a unique family of symmetric polynomials $P_\lambda^{(\alpha)} = P_\lambda^{(\alpha)}(x_1, \dots, x_n)$ satisfying

- $P_\lambda^{(\alpha)} = m_\lambda + \sum_{\mu:\mu<\lambda} u_{\lambda\mu}^{(\alpha)} m_\mu, \qquad u_{\lambda\mu}^{(\alpha)} \in \mathbb{Q}(\alpha)$,
- $\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \rangle_\Delta = 0, \qquad \lambda \neq \mu.$ (orthogonality)

Here $<$ stands for the dominance order:

$$\lambda \le \mu \quad \overset{\text{def}}{\Leftrightarrow}$$
$$|\lambda| = |\mu| \quad \text{and} \quad \lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i \quad (\forall i \ge 1).$$

Special cases of Jack polynomials

- $\alpha = 1$. $P_\lambda^{(1)}$ is the Schur polynomial s_λ.
- $\alpha = 2$. $P_\lambda^{(2)}$ is the zonal spherical function of the Gelfand pair $(U(n), O(n))$.
- $\alpha = 1/2$. $P_\lambda^{(1/2)}$ is the zonal spherical function of the Gelfand pair $(U(2n), Sp(2n))$.

Theorem 1: Product average

Theorem 1 [M, 2007]

For each $\beta > 0$, we have

$$\left\langle \prod_{l=1}^{L} \Psi(\overline{\mathbf{z}}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k}) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)} = (x_1 \cdots x_L)^{-n} \cdot P_{(n^L)}^{(\beta/2)}(x_1, \dots, x_{L+K}).$$

Here $\overline{\mathbf{z}} = (\overline{z_1}, \dots, \overline{z_n}) = (z_1^{-1}, \dots, z_n^{-1}) \in \mathbb{T}^n$. In particular, for any ξ such that $|\xi| = 1$,

$$\left\langle |\Psi(\mathbf{z}; \xi)|^{2k} \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)} = \prod_{i=0}^{k-1} \frac{\Gamma(\frac{2}{\beta}(i+1))\Gamma(n + \frac{2}{\beta}(k+i+1))}{\Gamma(\frac{2}{\beta}(k+i+1))\Gamma(n + \frac{2}{\beta}(i+1))}.$$

($\rightarrow$ the $\beta = 2$ case = Theorem by Conrey et. al Link)

Example of Theorem 1

For $x, y \in \mathbb{C}$ $(x \neq 0)$, we have

$$\begin{aligned} & \left\langle \Psi(\mathbf{z}^{-1}; x^{-1}) \cdot \Psi(\mathbf{z}; y) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)} \\ = & x^{-n} P_{(n)}^{(\beta/2)}(x, y) \\ = & \frac{n!}{(2/\beta)_n} \sum_{j=0}^{n} \frac{(2/\beta)_j (2/\beta)_{n-j}}{j!\,(n-j)!} x^{-j} y^j. \end{aligned}$$

Here $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

Proof of Theorem 1

$$
\begin{aligned}
&\prod_{l=1}^{L} \Psi(\mathbf{z}^{-1}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k}) \\
=&\prod_{l=1}^{L} x_l^{-n} \cdot (z_1 \cdots z_n)^{-L} \cdot \prod_{k=1}^{L+K} \prod_{j=1}^{n} (1 + x_k z_j) \\
=&\prod_{l=1}^{L} x_l^{-n} \cdot \overline{P_{(L^n)}^{(2/\beta)}(z_1, \dots, z_n)} \\
&\quad \times \sum_{\lambda} P_{\lambda}^{(\beta/2)}(x_1, \dots, x_{L+K}) P_{\lambda'}^{(2/\beta)}(z_1, \dots, z_n).
\end{aligned}
$$

Here λ' is the conjugate partition (i.e. the transposed Young diagram) of λ.

Proof of Theorem 1

The average over the $\mathrm{C}\beta\mathrm{E}$ is

$$\left\langle \prod_{l=1}^{L} \Psi(\mathbf{z}^{-1}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k}) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)}$$

$$= \prod_{l=1}^{L} x_l^{-n} \cdot \sum_{\lambda} P_{\lambda}^{(\beta/2)}(x_1, \dots, x_{L+K})$$

$$\times \left[\frac{\langle P_{\lambda'}^{(2/\beta)}, P_{(L^n)}^{(2/\beta)} \rangle_{\Delta}}{\langle 1, 1 \rangle_{\Delta}} \right]_{\alpha \mapsto 2/\beta}$$

$$= \prod_{l=1}^{L} x_l^{-n} \cdot P_{(n^L)}^{(\beta/2)}(x_1, \dots, x_{L+K}).$$

Average over COE

Denote by $\langle F(M) \rangle_{M \in \mathrm{COE}(n)}$ the average of F over the $\mathrm{COE}(n)$ (Link).

Corollary

$$\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1}) \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M) \right\rangle_{M \in \mathrm{COE}(n)}$$

$$= \left\langle \prod_{i=1}^{L} \Psi(\mathbf{z}^{-1}; x_i^{-1}) \cdot \prod_{i=1}^{K} \Psi(\mathbf{z}; x_{L+i}) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)|_{\beta=1}}$$

$$= \prod_{i=1}^{L} x_i^{-n} \cdot P_{(n^L)}^{(1/2)}(x_1, \dots, x_{L+K}).$$

Average over CSE

Denote by $\langle F(M) \rangle_{M \in \mathrm{CSE}(n)}$ the average of F over the $\mathrm{CSE}(n)$ (▸ Link).

Corollary

$$\left\langle \prod_{i=1}^{L} \det(I + x_i^{-1} M^{-1}) \cdot \prod_{i=1}^{K} \det(I + x_{L+i} M) \right\rangle_{M \in \mathrm{CSE}(n)}$$

$$= \left\langle \prod_{i=1}^{L} \Psi(\mathbf{z}^{-1}; x_i^{-1}) \cdot \prod_{i=1}^{K} \Psi(\mathbf{z}; x_{L+i}) \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)|_{\beta=4}}$$

$$= \prod_{i=1}^{L} x_i^{-n} \cdot P^{(2)}_{(n^L)}(x_1, \dots, x_{L+K}).$$

Super-Jack polynomials

Jack Q-polynomials

In order to evaluate the ratio average of characteristic polynomials, we introduce the super-Jack polynomials.

Let $P_\lambda^{(\alpha)}$ be the Jack polynomial. Define $Q_\lambda^{(\alpha)}$ by

$$Q_\lambda^{(\alpha)} = b_\lambda^{(\alpha)} P_\lambda^{(\alpha)}$$

with

$$b_\lambda^{(\alpha)} = \prod_{(i,j)\in\lambda} \frac{\alpha(\lambda_i - j) + \lambda_j' - i + 1}{\alpha(\lambda_i - j) + \lambda_j' - i + \alpha}.$$

For example,

$\frac{2\alpha+2}{3\alpha+1}$	$\frac{\alpha+2}{2\alpha+1}$	$\frac{1}{\alpha}$
$\frac{\alpha+1}{2\alpha}$	$\frac{1}{\alpha}$	

.

Skew Jack polynomials

Define the coefficients $f^{\lambda}_{\mu\nu}(\alpha) \in \mathbb{Q}(\alpha)$ via

$$P^{(\alpha)}_{\mu} P^{(\alpha)}_{\nu} = \sum_{\lambda} f^{\lambda}_{\mu\nu}(\alpha) P^{(\alpha)}_{\lambda}.$$

Define skew Jack polynomials by

$$P^{(\alpha)}_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu}(\alpha) P^{(\alpha)}_{\nu}, \qquad Q^{(\alpha)}_{\lambda/\mu} = \sum_{\nu} f^{\lambda}_{\mu\nu}(\alpha) Q^{(\alpha)}_{\nu}.$$

Super-Jack polynomials

Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be the sequence of (possibly infinite many) variables.
Define super-Jack polynomials

$$\widehat{P}_\lambda^{(\alpha)}(\mathbf{x};\mathbf{y}) = \sum_{\mu,\nu} f_{\mu\nu}^\lambda(\alpha) P_\mu^{(\alpha)}(\mathbf{x}) Q_{\nu'}^{(1/\alpha)}(\mathbf{y}) = \sum_\nu P_{\lambda/\nu}^{(\alpha)}(\mathbf{x}) Q_{\nu'}^{(1/\alpha)}(\mathbf{y}),$$

$$\widehat{Q}_\lambda^{(\alpha)}(\mathbf{x};\mathbf{y}) = \sum_{\mu,\nu} f_{\mu\nu}^\lambda(\alpha) Q_\mu^{(\alpha)}(\mathbf{x}) P_{\nu'}^{(1/\alpha)}(\mathbf{y}) = \sum_\nu Q_{\lambda/\nu}^{(\alpha)}(\mathbf{x}) P_{\nu'}^{(1/\alpha)}(\mathbf{y}).$$

The super-Jack polynomial is first defined in [Kerov-Okounkov-Olshanski (1998)].

Fundamental facts

When $\alpha = 1$, the polynomial $\widehat{P}^{(1)}_\lambda(\mathbf{x};\mathbf{y})$ is the super-Schur polynomial (or sometimes called the Littlewood-Schur polynomial).
In Macdonald's Book, this polynomial is written as $s_\lambda(\mathbf{x}/\mathbf{y})$.

$$\widehat{P}^{(\alpha)}_\lambda(\mathbf{x};\emptyset) = P^{(\alpha)}_\lambda(\mathbf{x}), \qquad \widehat{P}^{(\alpha)}_\lambda(\emptyset;\mathbf{y}) = Q^{(1/\alpha)}_{\lambda'}(\mathbf{y}),$$
$$\widehat{Q}^{(\alpha)}_\lambda(\mathbf{x};\emptyset) = Q^{(\alpha)}_\lambda(\mathbf{x}), \qquad \widehat{Q}^{(\alpha)}_\lambda(\emptyset;\mathbf{y}) = P^{(1/\alpha)}_{\lambda'}(\mathbf{y}).$$

$$\widehat{Q}^{(\alpha)}_\lambda(\mathbf{x};\mathbf{y}) = b^{(\alpha)}_\lambda \widehat{P}^{(\alpha)}_\lambda(\mathbf{x};\mathbf{y}).$$

Duality for super-Jack polynomials

Proposition

The super-Jack polynomial has the following duality.

$$\widehat{Q}_{\lambda}^{(\alpha)}(\mathbf{x};\mathbf{y}) = \widehat{P}_{\lambda'}^{(1/\alpha)}(\mathbf{y};\mathbf{x}).$$

Theorem 2: Ratio average

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Theorem 2

Let

$$\mathbf{x} = (x_1, \dots, x_L, x_{L+1}, \dots, x_{L+K}) \in \mathbb{C}^{L+K}, \quad \mathbf{v} = (v_1, \dots, v_T) \in \mathbb{D}^T.$$

Then

$$\left\langle \frac{\prod_{l=1}^{L} \Psi(\overline{\mathbf{z}}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k})}{\prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)}$$

$$=(x_1 \cdots x_L)^{-n} \widehat{P}^{(\beta/2)}_{(n^L)}(\mathbf{x}; \mathbf{v}).$$

($\to$ Theorem 1 Link)

Corollary: Duality for ratio averages

Corollary (Duality between $\beta \longleftrightarrow 4/\beta$)

Let $\beta > 0$ and $\beta' = 4/\beta$.

$$x_1, \dots, x_m, x_{m+1}, \dots, x_{m+K}, v_1, \dots, v_n, v_{n+1}, \dots, v_{n+T} \in \mathbb{D}.$$

Then

$$b^{(\beta/2)}_{(n^m)}(x_1 \dots x_m)^n \left\langle \frac{\prod_{j=1}^{m} \Psi(\overline{\mathbf{z}}; x_j^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{m+k})}{\prod_{t=1}^{n+T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)}$$

$$= (v_1 \dots v_n)^m \left\langle \frac{\prod_{j=1}^{n} \Psi(\overline{\mathbf{w}}; v_j^{-1}) \cdot \prod_{t=1}^{T} \Psi(\mathbf{w}; v_{n+T})}{\prod_{k=1}^{m+K} \Psi(\mathbf{w}; -x_k)^{\beta'/2}} \right\rangle_{\mathbf{w} \in \mathrm{C}\beta'\mathrm{E}(m)}.$$

Corollary: Duality for ratio averages

Roughly speaking, there exists a duality

$$\begin{pmatrix} \beta \\ n \\ m \\ \mathbf{x} \\ \mathbf{v} \end{pmatrix} \longleftrightarrow \begin{pmatrix} 4/\beta \\ m \\ n \\ \mathbf{v} \\ \mathbf{x} \end{pmatrix}$$

Hyperdeterminantal expressions for the ratio average

Definition of hyperdeterminants

For a multi-dimensional array

$$A = (a_{i_1,i_2,\dots,i_{2p}})_{1 \le i_1,i_2,\dots,i_{2p} \le N},$$

we define the hyperdeterminant of A by

$$\det{}^{[2p]}(A) = \frac{1}{N!} \sum_{\sigma_1,\sigma_2,\dots,\sigma_{2p} \in \mathfrak{S}_N} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)\cdots\mathrm{sgn}(\sigma_{2p}) \times \prod_{i=1}^{N} a_{\sigma_1(i),\sigma_2(i),\dots,\sigma_{2p}(i)}.$$

If $p = 1$, this is the ordinary determinant of an $N \times N$ matrix $A = (a_{ij})$. $\det^{[2]}(a_{i,j})_{1 \le i,j \le N} = \frac{1}{N!} \sum_{\sigma_1,\sigma_2 \in \mathfrak{S}_N} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2) \prod_{i=1}^{N} a_{\sigma_1(i),\sigma_2(j)} = det(a_{i,j})_{1 \le i,j \le N}$.

Hyperdeterminantal expression for Jack

When $\alpha = 1/p$, the Jack polynomial of a rectangular-shaped Young diagram is expressed as a hyperdeterminant.

Theorem 3 [M, 2008]

Let p, a, b be positive integers.

$$Q^{(1/p)}_{(a^b)} = \frac{b!\,(p!)^b}{(pb)!}\det{}^{[2p]}\left(g^{(1/p)}_{a+i_1+i_2+\cdots+i_p-i_{p+1}-\cdots-i_{2p}}\right)_{1\le i_1,\dots,i_{2p}\le b},$$

where $g^{(\alpha)}_k$ is the one-row Jack Q-polynomial $g^{(\alpha)}_k := Q^{(\alpha)}_{(k)}$.

We call this formula the Jacobi-Trudi-type formula for rectangular Jack polynomials.

Hyperdeterminantal expression for Jack

In the previous theorem, let $\alpha = 1/p = 1$. Then we obtain

$$s_{(a^b)} = \det(h_{a-i+j})_{1 \le i,j \le b},$$

with $h_k = s_{(k)}$.
This is a special case of the Jacobi-Trudi identity

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \le i,j \le n} \qquad n \ge \ell(\lambda).$$

Hyperdeterminantal expression for ratio averages

Theorem 4-1

Let

$$\mathbf{x} = (x_1, \dots, x_L, x_{L+1}, \dots, x_{L+K}) \in \mathbb{C}^{L+K}, \quad \mathbf{v} = (v_1, \dots, v_T) \in \mathbb{D}^T.$$

Suppose that β is an even positive integer: $\beta = 2p$. Then

$$\left\langle \frac{\prod_{l=1}^{L} \Psi(\overline{\mathbf{z}}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k})}{\prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)}$$

$$= \frac{n!\,(p!)^n}{(pn)!} (x_1 \cdots x_L)^{-n} \det{}^{[2p]} \left(\widehat{g}^{(1/p)}_{L+i_1+\cdots+i_p-i_{p+1}-\cdots-i_{2p}}(\mathbf{v}; \mathbf{x}) \right)_{1 \le i_1, \dots, i_{2p} \le n},$$

with $\widehat{g}_k^{(\alpha)} := \widehat{Q}_{(k)}^{(\alpha)}$.

Hyperdeterminantal expression for ratio averages

Theorem 4-2

Let $\mathbf{x}, \mathbf{v}$ as in Theorem 4-1. Suppose that $\beta = 2/p$ for some positive integer p. Then

$$\left\langle \frac{\prod_{l=1}^{L} \Psi(\overline{\mathbf{z}}; x_l^{-1}) \cdot \prod_{k=1}^{K} \Psi(\mathbf{z}; x_{L+k})}{\prod_{t=1}^{T} \Psi(\mathbf{z}; -v_t)^{\beta/2}} \right\rangle_{\mathbf{z} \in \mathrm{C}\beta\mathrm{E}(n)}$$

$$= b_{(L^n)}^{(p)} \frac{L!\,(p!)^L}{(pL)!}$$

$$\times \, (x_1 \cdots x_L)^{-n} \det{}^{[2p]} \left(\widehat{g}^{(1/p)}_{n+i_1+\cdots+i_p-i_{p+1}-\cdots-i_{2p}}(\mathbf{x}; \mathbf{v}) \right)_{1 \le i_1, \ldots, i_{2p} \le L}.$$

Conclusion

Conclusion

- We have evaluated averages of the product and ratio of characteristic polynomials in the $\mathrm{C}\beta\mathrm{E}$.

  Product Average $\to$ Jack polynomials (Theorem 1 Link)
  Ratio Average $\to$ Super-Jack polynomials (Theorem 2 Link)

- As a corollary of Theorem 2, we have obtained a duality for ratio averages between $\beta \leftrightarrow 4/\beta$ (Link).

- Suppose that either $\beta/2$ or $2/\beta$ is an integer. Using a Jacobi-Trudi-type identity for Jack polynomials (Link), we have expressed the ratio average in terms of a hyperdeterminant. (Link, Link).

Thank you for your attention.

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