Combinatorics of the KR conjecture: Q and T systems as cluster algebras

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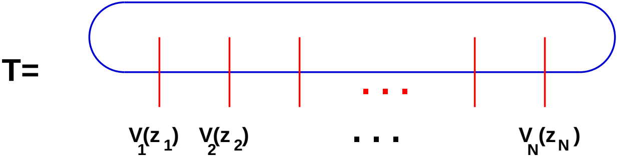
1. Q- and T-systems: Some examples and applications

2. Properties of associated cluster algebras
   - Cluster algebras
   - Formulation of T-systems and Q-systems as cluster algebras

3. KR boundary conditions and polynomiality
   - Application to the HKOTY and KR conjecture
   - Application to the Feigin-Loktev conjecture

4. Next: More about the A_n Q-system

- T-systems [Kirillov-Reshetikhin] were originally formulated as "fusion relations" for the transfer matrix of the (inhomogeneous) Heisenberg spin chain.



- For A_n, if the transfer matrix corresponding to the auxiliary space $V_{i\omega_\alpha}(\zeta)$ (a finite-dimensional Yangian module, or $U_q(\widehat{\mathfrak{sl}})_{n-1}$-module, with "rectangular highest weight") is denoted by $T_{\alpha,i}(\zeta)$ then:

$$T_{\alpha,k+1}(\zeta)T_{\alpha,k-1}(\zeta)+T_{\alpha+1,k}(\zeta)T_{\alpha-1,k}(\zeta)=T_{\alpha,k}(\zeta+1)T_{\alpha,k}(\zeta-1).$$

- This can be considered as a two-step recursion formula in k:

$$T_{\alpha,j;k+1}T_{\alpha,j;k-1}=T_{\alpha,j+1;k}T_{\alpha,j-1;k}-\prod_{\beta\sim\alpha}T_{\beta,j;k},\quad \alpha\in I_r,\ k\in\mathbb{Z}, j\in\mathbb{Z}+c.$$

  with $\beta\sim\alpha$ if $C_{\alpha,\beta}=-1$ (the Cartan matrix). The parameter j takes the place of ζ.
- This formula extends to any simply-laced $\mathfrak{g}$, with KR-modules in Auxiliary space.

- Let Γ a skew-symmetric matrix representing a finite quiver without loops:

$$(\Gamma)_{ij} = k > 0 \quad \Leftrightarrow \quad i \longrightarrow j$$

- Geiss-Leclerc-Schröer (2007) introduced a family $\{T_{\alpha,j;k}\}$ in the context of preprojective algebras, which satisfies the recursion relations:

$$T_{\alpha,j;k+1}T_{\alpha;j;k-1} = T_{\alpha,j+1;k}T_{\alpha,j-1;k} - \prod_{\beta} T_{\beta,j-1;k}^{[\Gamma_{\beta,\alpha}]_+} T_{\beta,j+1;k}^{[\Gamma_{\alpha,\beta}]_+}$$

  where $[m]_+$ is the non-negative part of m.
  The indices α, β take values in the nodes of the quiver graph. The indices j and k take values in $\mathbb{Z}$, modulo boundary conditions (see below).
- This can be taken to be a two-step recursion relation in k for $T_{\alpha,j;k}$.
- The special case of the simply-laced T-system for transfer matrices is when Γ is the signed incidence matrix (pick an orientation on the Dynkin diagram).

- If C is the Cartan matrix of a simply-laced Lie algebra, the Q-system is the combinatorial limit ($\zeta \to \infty$) of the T-system and can be written as:

$$Q_{\alpha;k+1}Q_{\alpha;k-1} = Q_{\alpha;k}^2 - \prod_{\beta} Q_{\beta;k}^{[-C_{\beta,\alpha}]_+}, \quad \alpha \in I_r = \{1, ..., r\},\ k \in \mathbb{Z}$$

r is the rank of the algebra $\mathfrak{g}$.

- In the non simply-laced case,

$$Q_{\alpha;k+1}Q_{\alpha;k-1} = Q_{\alpha;k}^2 - \prod_{\beta \sim \alpha} \mathcal{T}_{\beta;k}^{(\alpha)}$$

where

$$\mathcal{T}_{\beta,k}^{(\alpha)} = \prod_{i=1}^{|C_{\alpha,\beta}|-1} Q_{\beta,\lfloor (t_\beta k+i)/t_\alpha \rfloor}$$

where $t_\alpha = 2$ for the short roots of B_r, C_r and F_4 and $t_2 = 3$ for the short root of G_2. [KR, Kuniba-Nakanishi-Suzuki...]

Boundary conditions!!

- Need to specify **boundary conditions** in order to specify the family of solutions. We need two boundary conditions as this is a two-step recursion relation. A possible choice is (**the KR-point**):
  1. For the T-systems, the boundary conditions are
  $$T_{\alpha,j;0} = 1 \text{ for all } \alpha \in I_r, j \in \mathbb{Z}.$$
  Here $I_r = \{1, ..., r\}$ where r is the rank of the Lie algebra or the number of nodes in the quiver graph.
  2. For the Q-systems,
  $$Q_{\alpha,0} = 1 \quad \text{for all } \alpha \in I_r.$$

- For the second boundary condition in the direction of k we take $T_{\alpha,j;1}$ and $Q_{\alpha,1}$ to be formal variable. Solutions are expressed in terms of these formal variables.
- Under those conditions, it is known that
  1. Solutions of the T-system for the transfer matrices are q-characters of Kirillov-Reshetikhin modules parametrized by the highest weight $i\omega_\alpha$ and spectral parameter ζ [Hernandez].
  2. Solutions of the Q-system are the $U_q(\mathfrak{g})$-characters of the same KR-modules. This is one version of the **Kirillov-Reshetikhin conjecture**. [KR,Nakajima, Hernandez]
  3. For GLS, this is also the natural boundary condition.

KR point

Evaluating the equations

$$T_{\alpha,j;k+1}T_{\alpha,j;k-1} = T_{\alpha,j+1;k}T_{\alpha,j-1;k} - \prod_{\beta\sim\alpha} T_{\beta,j;k},$$

$$T_{\alpha,j;k+1}T_{\alpha;j;k-1} = T_{\alpha,j+1;k}T_{\alpha,j-1;k} - \prod_{\beta} T_{\beta,j-1;k}^{[\Gamma_{\beta,\alpha}]_+} T_{\beta,j+1;k}^{[\Gamma_{\alpha,\beta}]_+},$$

$$Q_{\alpha;k+1}Q_{\alpha;k-1} = Q_{\alpha;k}^2 - \prod_{\beta} \mathcal{T}_{\beta;k}^{(\alpha)},$$

when $k=0$ at the KR-point $T_{\alpha,j;0} = Q_{\alpha,0} = \mathcal{T}_{\beta,0}^{(\alpha)} = 1$, **The RHS=0**.

Another way to say this is: at the KR-point,

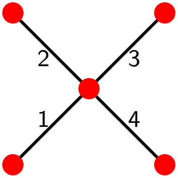
$$T_{\alpha,j;-1} = Q_{\alpha,-1} = 0.$$

A property of Q and T-systems with KR boundary

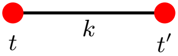
Theorem (Di Francesco, K)

The T-systems and Q-systems introduced above (and generalizations) can be formulated as cluster algebra evolutions. After evaluation of cluster variables at the KR boundary conditions, all cluster variables are polynomials in the formal variables $T_{\alpha,j;1}$ or $Q_{\alpha,1}$.

- A cluster algebra of rank n is an evolution on an n-ary tree $\mathbb{T}_n$ with labeled edges.
- Each node is connected to n edges labeled distinctly. For example, if $n = 4$ we have:



- Each node has associated with it a set:
  - a **cluster variable** $\mathbf{x} = (x_1, ..., x_n)$
  - an **exchange matrix** B: an $n \times n$ skew-symmetric integer matrix.
- A **cluster algebra** is a dynamical evolution of the variables along this tree: If nodes t and t' are connected with an edge labeled by k:



then $(\mathbf{x}, B)_{t'} = \mu_k((\mathbf{x}, B)_t)$.
- The map μ_k is called a **mutation.**

Mutations

- Mutations act as follows on $(\mathbf{x}, B)$:

$$\mu_k : \; x_j \mapsto \begin{cases} \dfrac{\prod_j x_i^{[B_{ij}]_+} + \prod_i x_i^{[-B_{ij}]_+}}{x_j}, & j = k \\ x_j, & j \neq k. \end{cases}$$

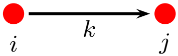
($[m]_+$ is the positive part of m.)

$$\mu_k : B_{ij} \mapsto \begin{cases} -B_{ij} & \text{If } i = k \text{ or } j = k; \\ B_{ij} + \mathrm{sign}(B_{ik})[B_{ik}B_{kj}]_+ & \text{otherwise.} \end{cases}$$

- Cluster mutations on cluster variables are **rational transformations.**
- The kth **column** of B determines the mutation in the direction k.
- The mutations on cluster variables are **subtraction-free expressions!**
- Apart from this last detail, evolutions of cluster variables look like T-system or Q-system evolutions.

Graphical representations of mutations of B

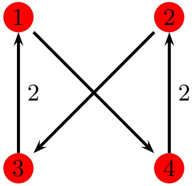
Any skew-symmetric integer matrix corresponds to a quiver graph. If $B_{ij} = k > 0$ then the graph contains the arrow



Example:

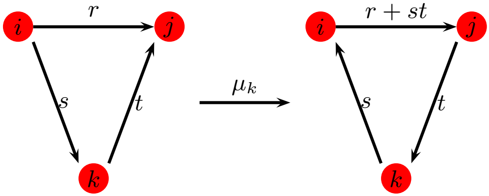
$$B = \begin{pmatrix} 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix}$$

corresponds to the quiver



Then it is possible to illustrate an evolution of B along the cluster tree...

Mutations on the quiver



Cluster algebras with coefficients

- T-systems and Q-systems, as evolutions in the "time" direction k, are **not** subtraction free expressions. To fix this detail we can either
  (1) renormalize T's and Q's (possible for simple Lie algebras) or
  (2) introduce coefficients.
- **Coefficients:** Pick m new variables $(x_{n+1}, ..., x_{n+m}) = (q_1, ..., q_m)$ and add m rows to the mutation matrix B representing their connectivity to the cluster variables. **These variables do not evolve**, so we do not bother with adding m columns to B, and the extended matrix $\widetilde{B}$ is now rectangular.
- The evolution of the cluster variables $(x_1, ..., x_n)$ and $\widetilde{B}$ are given by the same formulas, with B replaced by $\widetilde{B}$.

Theorem (Kedem)

Let C be the Cartan matrix for a simply-laced Lie algebra. The set of equations

$$Q_{\alpha,k+1} = Q_{\alpha,k-1}^{-1}\left(Q_{\alpha,k}^2 + q_\alpha \prod_\beta Q_{\beta,k}^{[-C_{\beta,\alpha}]_+}\right)$$

for all $\alpha \in I_r$ and $k \in \mathbb{Z}$ are all cluster mutations in the cluster algebra defined by the "seed"

$$\mathbf{x} = (Q_{1,0}, ..., Q_{r,0}; Q_{1,1}, ..., Q_{r,1}; q_1, ..., q_r), \quad \widetilde{B} = \begin{pmatrix} 0 & -C \\ C & 0 \\ -I & I \end{pmatrix}$$

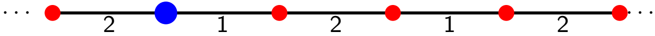
Remarks:

• The coefficients q_α do not mutate. When $q_\alpha = -1$, this is the Q-system.

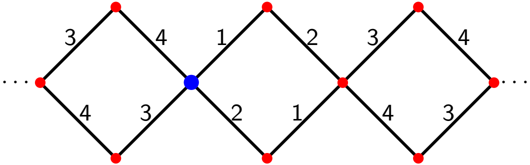
• The full cluster algebra defined by this seed contains variables which are do not satisfy this equation.

• The subgraph of the cluster tree consisting of Qs is a bipartite "strip" in the full cluster tree.

- A mutation of a cluster seed $\mathbf{x}$ will represent a Q-system evolution only when the variables connected by the incidence matrix are at the "right" point.
- $\mathfrak{sl}_2$ there is only one choice:

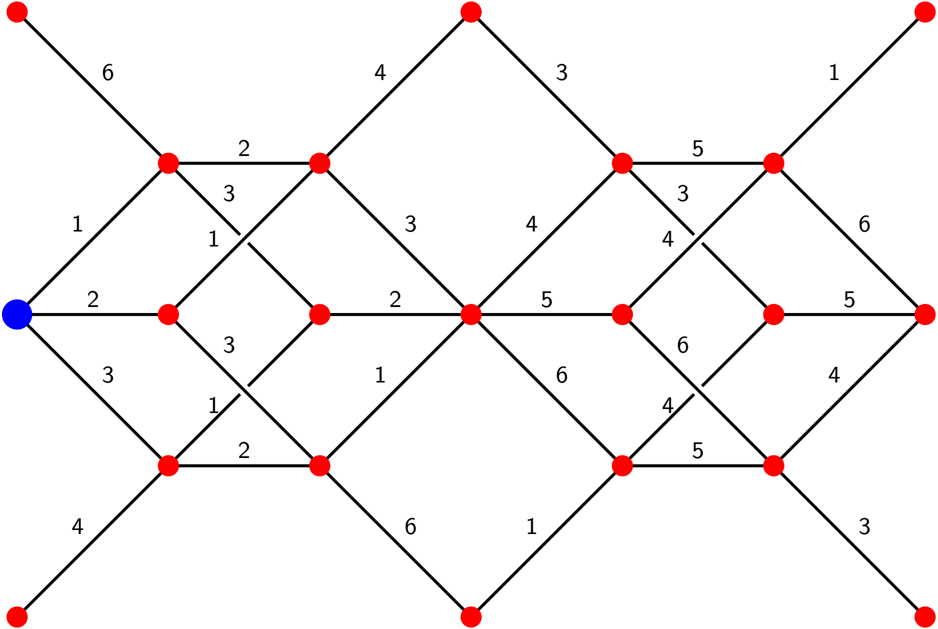


- For $\mathfrak{sl}_3$, the subgraph is as follows:



- For $\mathfrak{sl}_4$ it is more complicated:

Cluster graph for $\mathfrak{sl}_4$



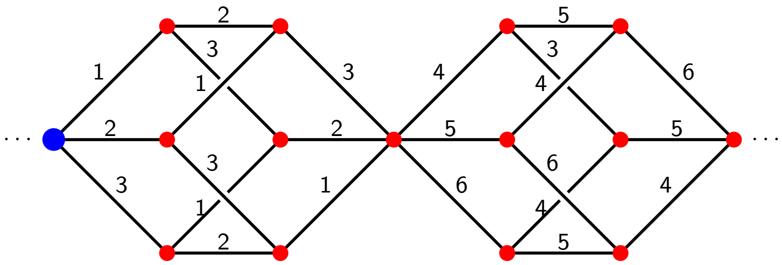
A subgraph containing the full Q-system

Lemma

All Q-system equations and all the variables $\{Q_{\alpha,k}\}$ are obtained in the following subgraph:

1. *From the seed $\mathbf{x}_0 = (Q_{\alpha,0}; Q_{\alpha,1})_{\alpha \in I_r}$ act with any sequence of distinct mutations from the set $(\mu_\alpha)_{\alpha \in I_r}$.* **OR**, *with any sequence from the set $(\mu_\alpha)_{\alpha \in r+I_r}$. All nodes are reached by Q-system evolutions.*
2. *From the node $\prod_{\alpha \in I_r} \mu_\alpha(\mathbf{x}_0)$, act with any sequence of distinct mutations from the set $(\mu_\alpha)_{\alpha \in r+I_r}$. From the node $\prod_{\alpha \in r+I_r} \mu_\alpha(\mathbf{x}_0)$, act with any sequence of distinct mutations from the set $(\mu_\alpha)_{\alpha \in I_r}$.*
3. *Repeat by periodicity.*

For example, for $\mathfrak{sl}_4$, it is sufficient to consider the subgraph:



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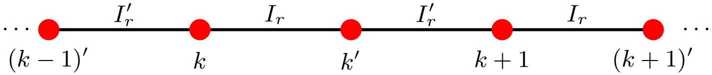
The bipartite cluster graph

- For mutations μ_i, μ_j which commute $B_{ij} = 0$. We can define a compound mutation $\mu_I = \mu_{i_1} \circ \cdots \circ \mu_{i_m}$ if all mutations in the sequence commute.
- For the simply-laced Lie algebras there is a shorthand for the subgraph described above: The bipartite graph:



where I_r and $I'_r = \{r+1, ..., 2r\}$.

- All Q-system evolutions for the simply-laced algebras are encoded in this graph. The union of the cluster seeds at all the (unprimed) nodes is the full set $\{Q_{\alpha,k} : \ \alpha \in I_r, k \in \mathbb{Z}\}$.
- The cluster seed at node k is $(Q_{\alpha,2k}; Q_{\alpha,2k+2})_{\alpha \in I_r}$. The exchange matrix at each unprimed node is $\widetilde{B}$.

Theorem

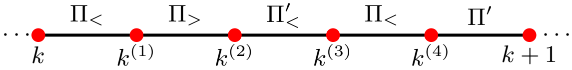
Let C be the Cartan matrix for a non simply-laced Lie algebra. The set of equations

$$Q_{\alpha,k+1} = Q_{\alpha,k-1}^{-1}\left(Q_{\alpha,k}^2 + q_\alpha \prod_\beta \mathcal{T}_{\beta,k}^{(\alpha)}\right)$$

for all $\alpha \in I_r$ and $k \in \mathbb{Z}$ are all cluster mutations in the cluster algebra defined by the "seed"

$$\mathbf{x} = (Q_{1,0}, ..., Q_{r,0}; Q_{1,1}, ..., Q_{r,1}; q_1, ..., q_r), \quad \widetilde{B} = \begin{pmatrix} C^t - C & -C^t \\ C & 0 \\ -I & I \end{pmatrix}$$

Here, the graphs are not bipartite: For B_r, C_r, F_4,



Where $\Pi_<$ is the set indices of short roots, $\Pi'_>$ is the set of indices of long roots shifted by r, etc.

Theorem

The system of equations of the form

$$T_{\alpha,j;k+1}T_{\alpha,j;k-1} = \prod_{j'} T_{\alpha,j';k}^{[C_{\alpha,\alpha}^{j',j}]_+} + q_\alpha \prod_{\alpha,j'} T_{\beta,j';k}^{[-C_{\beta,\alpha}^{j',j}]_+}.$$

is a subset of the mutations of the cluster algebra with seed (for some fixed $k \in \mathbb{Z}$)

$$\mathbf{x} = (\mathbf{T}_{2k}, \mathbf{T}_{2k+1}; q_1, ..., q_r\}, \quad \widetilde{B} = \begin{pmatrix} 0 & -C \\ C & 0 \\ -E & E \end{pmatrix}$$

where $\mathbf{T}_k = (T_{\alpha,j;k})_{\alpha \in I_r, j \in \mathbb{Z}}$, and $E_{q_\alpha;\beta,j} = \delta_{\alpha,\beta}$, provided that the matrix $C = P - A$ (where P and A have non-negative entries) has the properties:

1. *C is symmetric (the simply-laced case);*
2. *A commutes with P.*
3. $\sum_k P_{\alpha,\beta}^{i,j} = 2\delta_{\alpha,\beta}$.

In the cases above, P is the "shift" matrix $P_{\alpha,\beta}^{i,j} = \delta_{\alpha,\beta}(\delta_{i,j+1} + \delta_{i,j-1})$. The evolution subgraph is the bipartite graph.

Theorem (Fomin-Zelevinsky)

The Laurent phenomenon: The cluster variables at any node of a cluster algebra are Laurent polynomials in the cluster variables at any other node in the cluster algebra.

This is remarkable because the evolution is a rational function.

Corollary (DFK)

Polynomiality for KR boundary conditions: *Let t be a node with cluster variables $\mathbf{x} = (a_1, ..., a_n; b_1, ..., b_m) = (\mathbf{a}, \mathbf{b})$ and exchange matrix B such that $B_{ij} = 0$ for $i, j \leq n$. Moreover, we assume that $N_i(\mathbf{b}) := a_i \mu_i(a_i)$ vanishes when evaluated at the point $\mathbf{b} = \mathbf{b}_0$ for all i. (The numerator is not a function of a_i because $B_{ij} = 0$ if $i, j \leq n$). Then all cluster variables in the cluster algebra are polynomials of $\mathbf{a}$ when evaluated at the point $\mathbf{b}_0$.*

In the case of KR boundary conditions, we have **polynomiality**.
The case above holds for all the systems above at the KR point (with $q_\alpha = -1$).

Proof for rank 2

Lemma (Rank 2 case of polynomiality)

Suppose we have a cluster variable (a, b) *at node* t*, such that*

$$a' = \mu_2(a) := N(b)/a$$

with $N(b_0) = 0$ *for some* b_0*, then any other cluster variable* $z(a, b_0)$ *is a polynomial in* a*.*

Proof.

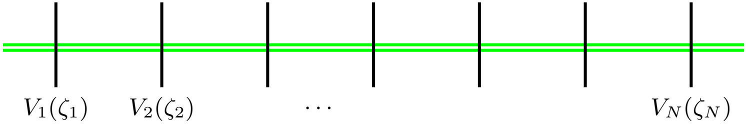
$$\begin{aligned} z(a,b) &= b^{-m} \textstyle\sum_n P_n(b) a^n && \text{(z is a Laurent polynomial)} \\ &= b^{-m} \textstyle\sum_n P_n(b) N(b)^n (a')^{-n} && \text{(from exchange relation)} \end{aligned}$$

Here, $P_n(b)$ is a polynomial in b and so is $N(b)$. By the Laurent phenomenon, if $n < 0$, $P_n(b)$ is divisible by $N(b)^n$. Therefore, if $n < 0$, $P_n(b_0) = 0$. Therefore, $z(a, b_0)$ is a polynomial in a. $\square$

This generalizes easily to higher rank.

Application to combinatorial KR conjecture

- The Heisenberg spin chain (periodic boundary conditions)



Problem: Find the eigenvectors and eigenvalues of the transfer matrix or Hamiltonian, and prove that you have all of them [Bethe 1931, Kirillov-Reshetikhin 80's, many others].

- $V_i(\zeta_i)$ are "special modules" of $Y(\mathfrak{g})$ or $U_q(\widehat{\mathfrak{g}})$ [KR-modules defined by Chari; applied by Kuniba/Nakanishi].
- The Hilbert space is

$$\mathcal{H} = V_1(\zeta_1) \otimes \cdots \otimes V_N(\zeta_N) \underset{\mathfrak{g}}{\simeq} \bigoplus_{\lambda \in \Lambda_+} V(\lambda)^{\oplus \mathcal{M}_{\{V_i\},\lambda}}.$$

- **Completeness problem:** KR conjectured that (1) Bethe vectors enumerated by "rigged configuraitons" and (2) The set of Bethe vectors is complete:

$$|\{\text{rigged configs}(\{V_i\}, \lambda)\}| = \mathcal{M}_{\{V_i\},\lambda}$$

- Define the **binomial coefficient**

$$\binom{m+p}{m} = \frac{(p+m)(p+m-1)\cdots(p+1)}{m!}, \quad p \in \mathbb{Z}$$

- Define $\mathbf{n} = (n_{\alpha,m})_{\alpha\in\{1,\ldots,r\},m\in\mathbb{Z}_+}$ with $n_{\alpha,m}$ the number of modules $V_i(\zeta_i)$ of KR type with highest weight $m\omega_\alpha$.
- for $\mathfrak{g}$ simply-laced, the number of rigged configurations is $\lim_{k\to\infty}$ of

$$M_{\mathbf{n},\lambda} = \sum_{\substack{\mathbf{m}\in\mathbb{Z}_+^{r\times k} \\ p_{\alpha,i}\geq 0 \\ p_{\alpha,k}=\ell_\alpha}} \prod_{i=1}^{k} \binom{p_{\alpha,i}(\mathbf{n},\mathbf{m}) + m_{\alpha,i}}{m_{\alpha,i}}$$

with

$$p_{\alpha,i} = \sum_j \min(i,j) n_{\alpha,j} - \sum_\beta \sum_j C_{\alpha,\beta} \min(i,j) m_{\beta,j}, \quad \lambda = \sum_\alpha \ell_\alpha \omega_\alpha$$

- **Combinatorial KR conjecture is**

$$\mathcal{M}_{\mathbf{n},\lambda} := \dim\left(\mathrm{Hom}_{\mathfrak{g}}\left(V_1(\zeta_1)\otimes\cdots\otimes V_N(\zeta_N),\ V_\ell\right)\right) = \lim_{k\to\infty} M_{\mathbf{n},\lambda}$$

Proved by KR for A_n; Proof for special cases [Kirillov, Schilling, Shimozono, Okado] where the crystal picture is available by using the KKR bijection.

The HKOTY conjecture

- In general, it can be proved that [Bethe, KR, HKOTY]

$$\lim_{k\to\infty} N_{\mathbf{n},\lambda} = \lim_{k\to\infty} \sum_{\substack{\mathbf{m}\in\mathbb{Z}_+^{r\times k}\\ p_{\alpha,i}\in\mathbb{Z}\\ p_{\alpha,k}=\ell_\alpha}} \prod_{i=1}^{k} \binom{p_{\alpha,i}(\mathbf{n},\mathbf{m})+m_{\alpha,i}}{m_{\alpha,i}} = \mathrm{Hom}_{\mathfrak{g}}\left(\prod_{\alpha,i} \mathrm{KR}_{\alpha,i}^{\otimes n_{\alpha,i}}, V(\lambda)\right).$$

provided that Solutions of the Q-system are characters of KR-modules. (proved by [KR,Nakajima,Hernandez])

- The **HKOTY** $M = N$ **conjecture** is that

$$M_{\mathbf{n},\lambda} = \sum_{p_{\alpha,i}\geq 0} \prod_{i=1}^{k} \binom{p_{\alpha,i}(\mathbf{n},\mathbf{m})+m_{\alpha,i}}{m_{\alpha,i}} = \sum_{p_{\alpha,i}\in\mathbb{Z}} \prod_{i=1}^{k} \binom{p_{\alpha,i}(\mathbf{n},\mathbf{m})+m_{\alpha,i}}{m_{\alpha,i}} = N_{\mathbf{n},\lambda}$$

Theorem (Di Francesco-K, 07)

The HKOTY identity is true for all simple Lie algebras, provided that the solutions of the Q-system are polynomials in the initial variables $Q_{\alpha,1}$ after evaluation at the boundary condition $Q_{\alpha,0}=1$.

Proof that $M = N$ for $\mathfrak{g} = \mathfrak{sl}_2$:

1. The HKOTY identity has the form :

$$\sum_{\mathbf{m}:\text{Restrictions}} f(\mathbf{m}) = \sum_{\mathbf{m}} f(\mathbf{m})$$

2. Construct a generating function in the formal variables $\{t, u_1, u_2, ..., u_k\}$:

$$Z_{\mathbf{n},\ell}(t;\mathbf{u}) = \sum_{\mathbf{m}\in\mathbb{Z}^k} t^{-q} \prod_{i=1}^{k} \binom{m_i + p_i + q}{m_i} u_i^{p_i+q}, \quad q = \ell - \sum_i i(n_i - 2m_i)$$

Notice: No restrictions on the summation!

3. The constant term in t is the term with $q = 0$, which is one of the restrictions on our sum.
4. $N_{\mathbf{n},\ell} =$ The constant term of $Z_{\mathbf{n},\ell}(t;1,...,1)$ in t.
5. $M_{\mathbf{n},\ell} =$ The constant term in t of **only the positive powers in** $\{u_i\}$ of $Z_{\mathbf{n},\ell}(t;\mathbf{u})$, evaluated at $u_1 = ... = u_k = 1$.
6. We must show these constant terms are equal.

Properties of the generating function $Z(t;\mathbf{u})$

- **Factorization:** The generating function Z has nice properties: It factorizes

$$Z_{l,\mathbf{n}} = \frac{\mathcal{Q}_1(t)\mathcal{Q}_k(t;\mathbf{u})^{l+1}}{\mathcal{Q}_{k+1}(t;\mathbf{u})^{l+1}} \prod_{j=1}^{k} \frac{\mathcal{Q}_j(t;\mathbf{u})^{n_j}}{u_j}$$

in terms of functions $\mathcal{Q}_m(t;\mathbf{u})$ which satisfy **deformed Q-system:**

$$\mathcal{Q}_{m+1}(t;\mathbf{u})\mathcal{Q}_{m-1}(t;\mathbf{u}) = \frac{\mathcal{Q}_m(t;\mathbf{u})^2 - 1}{u_m}, \quad \mathcal{Q}_0 = 1, \mathcal{Q}_1 = t, \quad (m > 1).$$

Lemma (Translation property)

The deformed Q-system is equivalent to the system

$$\mathcal{Q}_{k+1}(t,\mathbf{u}) = \mathcal{Q}_k(t',\mathbf{u}'),\ t' = \mathcal{Q}_2,\ u_1' = \mathcal{Q}_1 u_2,\ u_j' = u_{j+1}$$

with appropriate boundary conditions. (This is used to prove the factorization formula by induction from the definition. The function $\mathcal{Q}_2(t,u) = (t^2-1)/u$.)

- When $\{\mathbf{u} = (1, ..., 1)\}$ the deformed Q-system reduces to the Q-system for $\mathfrak{sl}_2$

$$Q_{m+1}Q_{m-1} = Q_m^2 - 1, \; Q_0 = 1, Q_1 = t,$$

  and solutions are **Chebyshev polynomials** of the second kind.
  Therefore, $Q_m(t; 1, ..., 1)$ are all polynomials in t.
- The proof is a direct calculation using
  1. The translation recursion of $\mathcal{Q}(t, \mathbf{u})$;
  2. The factorization of the generating function $Z_{\ell,\mathbf{n}}(t, \mathbf{u})$;
  3. **Solutions of the Q-system are polynomials in t.**

Lemma (DFK)

The constant term in t of $Z_{\ell,\mathbf{n}}(t, \mathbf{u})$ when $u_j = 1$ for all j has no contribution from negative powers of u_j, $j = 1, ..., k$.

- This implies the proof of HKOTY's identity $M = N$ for $\mathfrak{sl}_2$.

General case

- For higher rank algebras, there is always a **factorization** of the generating function in terms of solutions of a **deformed Q-system.** The proof that $M = N$ always depends only on the fact that:

Lemma

The solutions $\{Q_{\alpha,m}\}_{\alpha\in[1,\dots,r]}$ of the recursion relation known as the "Q-system" for any $\mathfrak{g}$ are **polynomials** *in the initial data $\{Q_{\alpha,1}\}_\alpha$.*

- This follows from the formulation as a cluster algebra and the polynomiality of solutions with KR-boundary conditions. (A purely combinatorial argument)
- It also follows from representation theoretical arguments:

Theorem (Kirillov-Reshetikhin, Nakajima, Hernandez)

solutions of the Q-system with $m > 0$ where $Q_{\alpha,0} = 1$ are characters of KR-modules of $U_q(\widehat{\mathfrak{g}})$ restricted to $U_q(\mathfrak{g})$.

- The trivial and fundamental KR-modules generate the Groethendieck group finite-dimensional representations, hence all characters of KR-modules are polynomials in $t_\alpha = \{Q_{\alpha,1}\}_\alpha$. (Natural generalization of Chebyshev polynomials.)

Application to proof of Feigin-Loktev conjecture

- Feigin and Loktev defined a grading on the tensor product of finite-dimensional $\mathfrak{g}$-modules:

$$V_1 \otimes \cdots \otimes V_N \simeq \bigoplus_{\lambda} V_{\lambda}^{\oplus M_{\{V_i\},\lambda}}$$

$$\Longrightarrow V_1 \star \cdots \star V_N(\zeta_1, ..., \zeta_N) \simeq \bigoplus_{\lambda} \bigoplus_{n} V_{\lambda}^{\oplus M_{\{V_i\},\lambda}[n]}$$

- **Example:** For $\mathfrak{g} = \mathfrak{sl}_2$ and V the fundamental representation, $V^{\otimes 3} \simeq V_3 \oplus V_1^{\oplus 2}$ whereas

$$V \star V \star V \simeq V_3[0] \oplus V_1[1] \oplus V_1[2]$$

- The LHS is a $\mathfrak{g}[t]$-module, and the decomposition is into $\mathfrak{g}$-modules.
- The definition depends on some "localization parameters" of the $\mathfrak{g}[t]$-modules V_i. The FL-conjecture is

Conjecture (Feigin-Loktev)

The multiplicities $M_{\{V_i\},\lambda}[n]$ do not depend on the localization parameters, for "sufficiently nice" modules V_i.

- Define $M_{\{V_i\},\lambda}(q) = \sum_n q^n M_{\{V_i\},\lambda}[n]$. Then

Theorem (Ardonne,K)

1. *For any simple Lie algebra $\mathfrak{g}$, if V_i are any sequence of $\mathfrak{g}[t]$-modules of Kirillov-Reshetikhin type [Chari], then $M_{\{V_i\},\lambda}(q)$ is bounded from above by HKOTY's graded M-sum formula.*
2. *$M_{\{V_i\},\lambda}(1)$ is bounded from below by the dimension of the multiplicity space of the tensor product of finite-dimensional modules.*
3. *If $M_{\mathbf{n},\lambda}(1)$ is equal to the dimension of the multiplicity space (the combinatorial KR-conjecture) then the equality holds in the first statement.*

- The proof of the $M = N$ conjecture therefore completes the proof of the FL-conjecture in this case, because the explicit formula for the multiplicity does not depend on the localization parameters.

Feigin-Loktev fusion product

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}^*_{\mathbf{n}}\,,\ V(\lambda)\right)\right|$$

Chari's $\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\ V(\lambda)\right)\right|$$

Restricted Sum

$$M_{\mathbf{n},\lambda}$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\,,\ V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

Unrestricted Sum

Feigin-Loktev
fusion product

$$|\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}_{\mathbf{n}}^{*}\ ,\ V(\lambda)\right)|$$

Chari's
$\mathfrak{g}[t]$-modules

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = |\mathrm{Hom}_{\mathfrak{g}}\left(\mathcal{F}_{\mathbf{n}}^{*}\ ,\ V(\lambda)\right)| = M_{\mathbf{n},\lambda}$$

Restricted
Sum

$$\left|\mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = \left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right|$$

$$M_{\mathbf{n},\lambda} \neq N_{\mathbf{n},\lambda}$$

$$\left|\mathrm{Hom}_{U_q(\mathfrak{g})}\left(\bigotimes_{\alpha,m} KR_{\alpha,m}^{\otimes n_{\alpha,m}}\ ,\ V(\lambda)\right)\right| = N_{\mathbf{n},\lambda}$$

$U_q(\widehat{\mathfrak{g}})$-modules

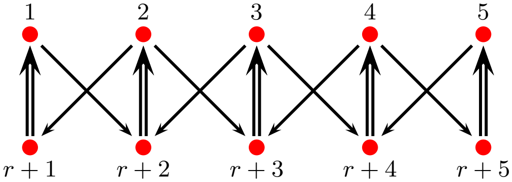
Unrestricted
Sum

- The Q-system is a very special cluster algebra – integrable. This allows us to prove properties of this cluster algebra.
- Instead of considering the cluster algebra with coefficients, we can renormalize any Q-system corresponding to a simple Lie algebra, and write the A_n-system as

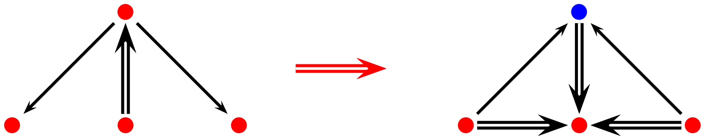
$$R_{\alpha,k+1}R_{\alpha,k-1} = R_{\alpha,k}^2 + R_{\alpha-1,k}R_{\alpha+1,k}$$

with $R_{0,k} = R_{r+1,k} = 1$ for all k, $R_{\alpha,k} = \epsilon_\alpha Q_{\alpha,k}$.

- The exchange matrix B corresponds to the quiver (A_5):



- Q-system evolutions for A_r are those which occur at nodes on the quiver graph such that
  1. There is only one incoming double arrow from the node;
  2. There are only outgoing single arrows to the node.



Without imposing the KR-boundary condition, there is a conjecture of Fomin and Zelevinsky (for any cluster algebra)

Conjecture

The cluster variables at any node of a cluster graph are Laurent polynomials with **positive coefficients** *of the cluster variables at any other node of the cluster graph.*

Claim: We can prove positivity using the integrability of the system for the cluster variables in the subgraph corresponding to Q-system evolutions (see talk by Di Francesco).

The subgraph can be described as: If the node has a cluster variable which includes the elements $\{Q_{\beta,n} : \ C_{\beta,\alpha} \neq 0\}$, then an evolution along the edge labeled by α (if n is even) or $r+\alpha$ (if n is odd) is a Q-system evolution.

Generalizations

- There are more general Q-systems than the ones considered here. For all of them the polynomiality lemma holds, so there is an HKOTY identity corresponding to them.
- A direct cluster-theoretical and geometric interpretation of this identity is missing.
- We do not have a representation-theoretical interpretation for the other cluster variables (outside the Q-system zone). With KR-boundary conditions, they are virtual characters.
- Proof of positivity outside the "integrable" subgraph corresponding to Q-system evolutions is open.