Linear quivers, generic extensions and Kashiwara operators

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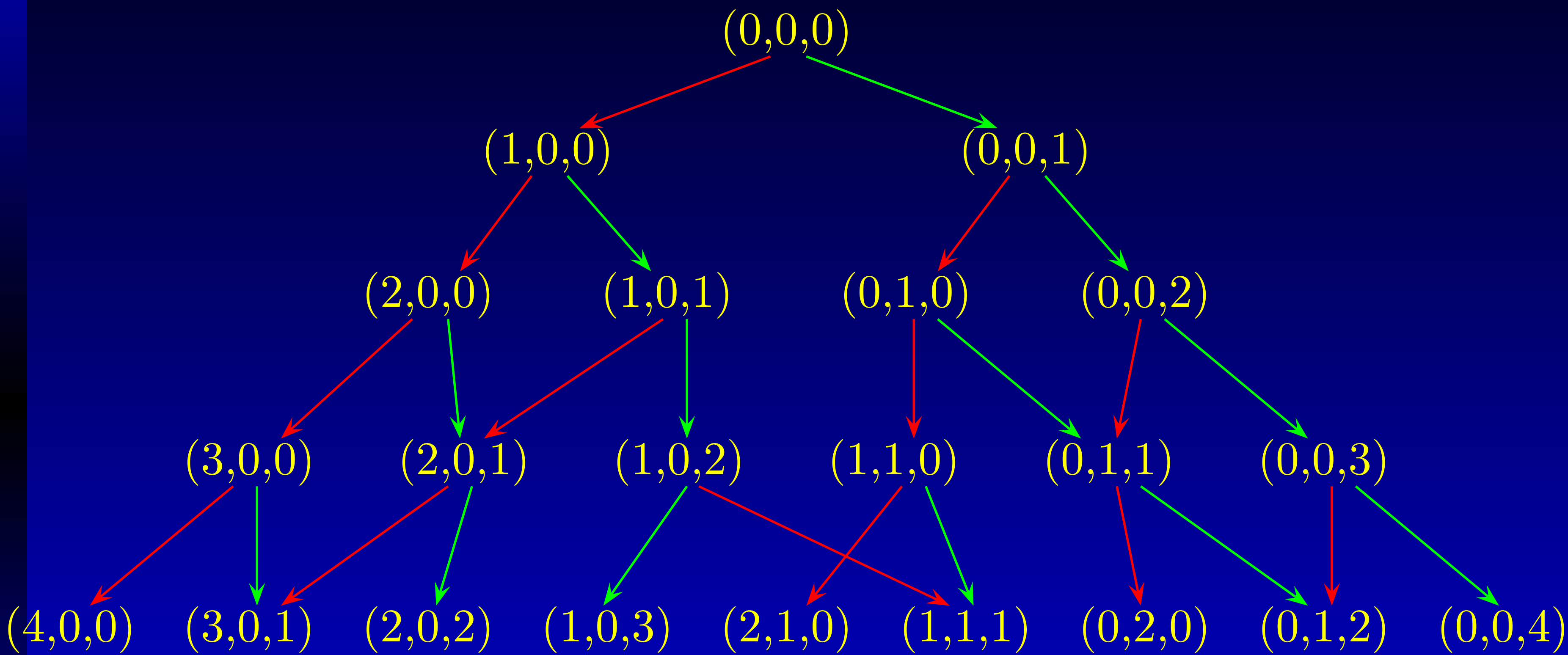
The crystal graph $\mathsf{C}(\mathsf{A}_2)$



κ : $\mathrm{Path}_0(\mathsf{C}) \longrightarrow \Lambda; p \mapsto$ finish vertex

E.g., $\kappa(2^2 12) = (0, 1, 2)$ with $1 =$red and $2 =$green.

The Problem

Let Ω be the set of all words on the alphabet $\{1, 2, \ldots, n\}$. Then we obtain a map

$$\kappa : \Omega \to \Lambda.$$

We shall call it the Kashiwara map.

- representation theoretic interpretation
- better understanding of crystal graphs
- word parametrization of the canonical basis

It seems hard to understand the map.

Quivers and path algebras

- Let $Q = (Q_0, Q_1) = (I, Q_1)$ be a quiver.

  If $\rho \in Q_1$ is an arrow from tail i to head j, we write $h(\rho)$ for j and $t(\rho)$ for i.
- A vertex $i \in I$ is called a sink (resp. source) if there is no arrow ρ with $t(\rho) = i$ (resp. $h(\rho) = i$).
- Let kQ be the path algebra of Q over a field k.

  E.g., if Q is the quiver $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$ then

$$kQ \cong \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

Quiver representations

- A representation $V = (V_i, V_\rho)$ of Q over k consists of

  a set of f. d. k-vector spaces $V_i, i \in I$, and
  a set of linear maps $V_\rho : V_{t(\rho)} \to V_{h(\rho)}, \rho \in Q_1$.

  We may identify V as a (left) kQ-module.
- Call $\mathbf{dim} V := (\dim_k V_1, \ldots, \dim_k V_n)$ the dimension vector of V.
- Let $\mathcal{M}$ be the set of isoclasses of nilpotent representations of Q.

Representation varieties ($k = \bar{k}$)

- Fix $\mathbf{d} = (d_i)_i \in \mathbb{N}^n$ and define the affine space

$$R(\mathbf{d}) := \prod_{\rho \in Q_1} \mathrm{Hom}_k(k^{d_{t(\rho)}}, k^{d_{h(\rho)}}).$$

  Thus, a point $x = (x_\rho)_\rho$ of $R(\mathbf{d})$ determines a representation $V(x)$ of Q.

- The algebraic group $GL(\mathbf{d}) = \prod_{i=1}^n GL_{d_i}(k)$ acts on $R(\mathbf{d})$ by $(g_i)_i \cdot (x_\rho)_\rho = (g_{h(\rho)} x_\rho g_{t(\rho)}^{-1})_\rho$.

- There is a bijection:

$$\{GL(\mathbf{d})\text{-orbits}\} \longleftrightarrow \{\text{isoclasses in } \mathcal{M}_{\mathbf{d}}\}.$$

Dim of orbits and the poset $\mathcal{M}$

- The stabilizer $GL(\mathbf{d})_x = \{g \in GL(\mathbf{d}) \mid gx = x\}$ of x is the group of automorphisms on $M := V(x)$ which is Zariski-open in $\mathrm{End}_{kQ}(M)$ and has dimension equal to $\dim \mathrm{End}_{kQ}(M)$.

  Hence the orbit $\mathcal{O}_M$ of M has dimension
  $$\dim \mathcal{O}_M = \dim GL(\mathbf{d}) - \dim \mathrm{End}_{kQ}(M).$$
- For two representations M, N of Q, define $[N] \leqslant [M]$ (or simply $N \leqslant M$) iff $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_M$, the closure of $\mathcal{O}_M$.
- This order is opposite to the degeneration order which is independent of k.

Dynkin quivers

- If the underlying graph of Q is a Dynkin graph, then, by Gabriel's Theorem, there is a bijection
$$\mathrm{ind}(Q) \longleftrightarrow \Phi^+(Q).$$

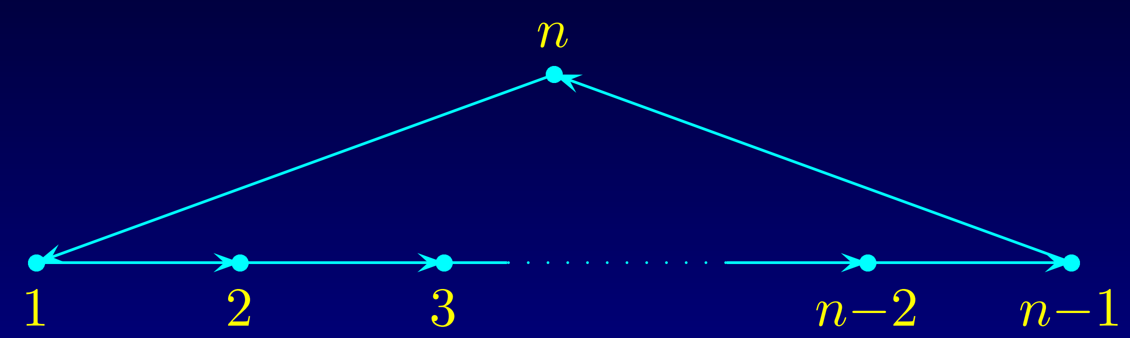
- This induces a bijection from Λ to $\mathcal{M}$, where
$$\Lambda = \{\lambda : \Phi^+ \longrightarrow \mathbb{N}\},$$
sending every $\lambda \in \Lambda$ to
$$M(\lambda) = M_k(\lambda) := \bigoplus_{\alpha\in\Phi^+} \lambda(\alpha) M_k(\alpha).$$

- Thus, we obtain a poset $(\Lambda, \leqslant)$.

Cyclic quivers

- If Q is the cyclic quiver

Δ_n :



then, for each integer $l \geqslant 1$, there is a unique (up to isomorphism) indecomposable nilpotent representation $S_i[l]$ of length l with top S_i.

- Thus, we obtain a bijection from Λ to $\mathcal{M}$, where $\Lambda = \{\lambda : I \times \mathbb{Z}_+ \longrightarrow \mathbb{N} \mid \text{supp}(\lambda) \text{ is finite}\}$,

sending every $\lambda \in \Lambda$ to

$$M(\lambda) := \bigoplus_{i \in I, l \geqslant 1} \lambda_{i,l} S_i[l].$$

Generic extensions

From now on, we assume that
Q is a Dynkin or cyclic quiver.

- Given $[M], [N] \in \mathcal{M}$, consider the extensions

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

  of M by N.

  Reineke proved that the one with $\dim \mathcal{O}_E$ maximal is a unique (up to isomorphism).

- We call E the generic extension of M by N, denoted by $M * N$.

The monoid $\mathcal{M}$

- Define operation $[M] * [N] = [M * N]$ on $\mathcal{M}$.

  It is associative.

  Thus, $(\mathcal{M}, *)$ is a monoid with identity $1 = [0]$.

- If Q is a Dynkin quiver, then $\mathcal{M}$ is generated by simples $[S_i]$, $i \in I$.

- If Q is a cyclic quiver, then the simples $[S_i]$, $i \in I$, generate a proper submonoid $\mathcal{M}_c$, which consists of aperiodic modules.

The generic extension map

- Let Ω be the set of all words on the alphabet $I = \{1, 2, \ldots, n\}$.

  For $w = i_1 i_2 \cdots i_m \in \Omega$, let $\wp(w) \in \Lambda$ be the element defined by
  $$[S_{i_1}] * \cdots * [S_{i_m}] = [M(\wp(w))].$$
  Thus, we obtain a monoid homomorphism
  $$\wp : \Omega \longrightarrow \Lambda, \; w \longmapsto \wp(w),$$
- Call $\wp$ the generic extension map.
- If Q is a Dynkin quiver, then $\wp$ is surjective.
- If Q is a cyclic quiver, then $\wp$ is not surjective with $\mathrm{Im}\,\wp = \Lambda^a$, the set of all aperiodic elements.

The generic extension graph G

- For each $i \in I$, there is a map
$$\sigma_i : \Lambda \longrightarrow \Lambda;\ \lambda \longmapsto \sigma_i\lambda$$
defined by $M(\sigma_i\lambda) \cong S_i * M(\lambda)$.

  Clearly, for each $w = i_1 i_2 \dots i_m$, we have
$$\wp(w) = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_m}(0),$$
where 0 is the zero function.

- The generic extension graph associated to Q is the directed graph G with
$$\text{vertices: } \lambda \in \Lambda,$$
$$\text{arrows: } \lambda \xrightarrow{i} \mu,$$
where $\lambda, \mu \in \Lambda$ and $\sigma_i\lambda = \mu$ for some $i \in I$.

Hall polynomials

- Ringel proved that, for $\lambda, \mu_1, \ldots, \mu_m$ in Λ, there is a polynomial $\varphi^{\lambda}_{\mu_1,\ldots,\mu_m}(T) \in \mathbb{Z}[T]$ such that

$$\varphi^{\lambda}_{\mu_1,\ldots,\mu_m}(q_k) = F^{M_k(\lambda)}_{M_k(\mu_1),\ldots,M_k(\mu_m)}$$

  for any finite field k of q_k elements

  Here $F^{M}_{N_1,\ldots,N_m}$ denotes the number of filtrations

$$M = M_0 \supset M_1 \supset \cdots \supset M_{m-1} \supset M_m = 0$$

  such that $M_{s-1}/M_s \cong N_s$ for all $1 \leqslant s \leqslant m$.
- These polynomials are called Hall polynomials.

Ringel-Hall algebras

Let $\mathscr{Z} = \mathbb{Z}[v, v^{-1}]$.

- The (twisted generic) Ringel-Hall algebra $\mathcal{H}_v(Q)$ of Q is the free $\mathscr{Z}$-module having basis

$$\{u_\lambda = u_{[M(\lambda)]} \mid \lambda \in \Lambda\}$$

and satisfying the mult'n rules

$$u_\mu u_\nu = v^{\varepsilon(\mu,\nu)} \sum_{\lambda \in \Lambda} \varphi^\lambda_{\mu,\nu}(v^2) u_\lambda.$$

- Here

$$\varepsilon(\mu, \nu) = \dim{}_k \mathrm{Hom}_{kQ}(M(\mu), M(\nu)) - \dim{}_k \mathrm{Ext}^1_{kQ}(M(\mu), M(\nu))$$

is the Euler form associated to the quiver Q.

Quantum enveloping algebras

- Let $\mathbf{U}$ be the QEA over $\mathbb{Q}(v)$ associated to Q with generators $E_i, F_i, K_i^{\pm}$, $i \in I$.
- Let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by E_i.
- The Lusztig integral form U^+ is the $\mathscr{Z}$-subalgebra of $\mathbf{U}^+$ generated by divided powers $E_i^{(m)}$ $(i \in I, m \geqslant 1)$.
- (Ringel) U^+ is isomorphic to the Ringel-Hall algebra $\mathcal{H}$ (resp. $\mathcal{C}$) via $E_i^{(m)} \mapsto u_i^{(m)}$, if Q is Dynkin (resp. cyclic).

PBW type bases

Let $\tilde{u}_\lambda = v^{-\dim M(\lambda)+\dim \mathrm{End}(M(\lambda))} u_\lambda \in U^+$.

- If Q is a Dynkin quiver, then $\{E_\lambda = \tilde{u}_\lambda\}_{\lambda \in \Lambda}$ is a PBW type basis, which

  (1) coincides with Lusztig's one defined by braid group actions;

  (2) is used in the construction of the canonical basis.

- If Q is a cyclic quiver, a similar basis $\{E_\lambda\}_{\lambda \in \Lambda^a}$ can be constructed in the form (D-D-Xiao) $E_\lambda = \tilde{u}_\lambda + (\text{lin comb of } \tilde{u}_\mu, \mu \notin \Lambda^a)$.

- Lin-Xiao-Zhang, (Hubery), Beck–Nakajima

The SMB property

Theorem (Deng-Du)

For each $w = i_1 i_2 \cdots i_m \in \Omega$, define a monomial

$$E_w = E_{i_1} E_{i_2} \cdots E_{i_m} \in \mathbf{U}^+.$$

Let Q be a Dynkin quiver (resp. a cyclic quiver). For each $\lambda \in \Lambda$ (resp. Λ^a), choose an arbitrary word $w_\lambda \in \wp^{-1}(\lambda)$. Then

(1) the set $\{E_{w_\lambda} \mid \lambda \in \Lambda \text{ (resp. } \Lambda^a)\}$ is a $\mathbb{Q}(v)$-basis of $\mathbf{U}^+$;

(2) if all w_λ are *distinguished*, then $\{E^{(w_\lambda)} \mid \lambda \in \Lambda \text{ (resp. } \Lambda^a)\}$ is a $\mathscr{Z}$-basis of U^+.

Canonical bases (Lusztig)

- For each $\lambda \in \Lambda$, there is a unique element
$$\mathfrak{b}_\lambda = E_\lambda + \sum_{\mu<\lambda} p_{\lambda,\mu} E_\mu \in U^+$$
with all $p_{\lambda,\mu} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $\bar{\mathfrak{b}}_\lambda = \mathfrak{b}_\lambda$,
where $^-$ is the $\mathbb{Z}$-algebra involution
$^- : U^+ \longrightarrow U^+;\ E_i^{(m)} \longmapsto E_i^{(m)},\ v \longmapsto v^{-1}.$
- If Q is a Dynkin quiver, then $\mathbf{B} = \{\mathfrak{b}_\lambda \mid \lambda \in \Lambda\}$ is a $\mathscr{Z}$-basis of U^+.
- If Q is a cyclic quiver, then $\mathbf{B} = \{\mathfrak{b}_\pi \mid \pi \in \Lambda^a\}$ is a $\mathscr{Z}$-basis of U^+

Kashiwara operators

- Each $x \in \mathbf{U}^+$ can be written uniquely in the form

$$x = \sum_{m \geqslant 0} E_i^{(m)} x_m$$

where $x_m \in \mathbf{U}^+$ satisfy $F_i x_m - x_m F_i \in K_i \mathbf{U}^+$ and $x_m = 0$ for $m \gg 0$.

- The Kashiwara operator

$$\tilde{E}_i : \mathbf{U}^+ \to \mathbf{U}^+$$

is defined by

$$\tilde{E}_i(x) = \sum_{m \geqslant 0} E_i^{(m+1)} x_m.$$

The Kashiwara map κ

Let
$\mathscr{A}_\infty = \{f(v) \in \mathbb{Q}(v) \mid f(v^{-1})$ regular at $v = 0\}$.

- Let $\mathscr{L}$ be the $\mathscr{A}_\infty$-submodule of $\mathbf{U}^+$ generated by
$$e_w = \tilde{E}_{i_1}\tilde{E}_{i_2}\cdots\tilde{E}_{i_m}\cdot 1$$
for all words $w = i_1 i_2 \cdots i_m \in \Omega$.
- Kashiwara tells us:
$$B = \{e_w + v^{-1}\mathscr{L} \mid q \in \Omega\}$$
is a $\mathbb{Q}$-basis for $\mathscr{L}/v^{-1}\mathscr{L}$, from which the global crystal basis is constructed.
- Thus, for each word $w = i_1 i_2 \ldots i_m$, there is a unique $\kappa(w) \in \Lambda$ s. t.
$$\tilde{E}_{i_1}\tilde{E}_{i_2}\cdots\tilde{E}_{i_m}\cdot 1 \equiv \mathfrak{b}_{\kappa(w)} \pmod{v^{-1}\mathscr{L}}.$$

The crystal graph C

- For each $i \in I$, there is a map
$$\tau_i : \Lambda \longrightarrow \Lambda; \ \lambda \longmapsto \tau_i\lambda \text{ define by}$$
$$\tilde{E}_i(\mathfrak{b}_\lambda) \equiv \mathfrak{b}_{\tau_i\lambda} \pmod{v^{-1}\mathscr{L}}.$$
- We have
$\kappa(w) = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_m}(0), \forall w = i_1 i_2 \ldots i_m.$
- The crystal graph associated to Q is the directed graph C with
$$\text{vertices: } \lambda \in \Lambda,$$
$$\text{arrows: } \lambda \xrightarrow{i} \mu,$$
where $\lambda, \mu \in \Lambda$ and $\tau_i\lambda = \mu$ for some $i \in I$.

Word parametrization

- The maps $\wp$ and κ are different (e.g., not both are monoid homom), but can be used to parameterize the canonical (or crystal) bases.

  For each $\lambda \in \Lambda$, choose
  $$y_\lambda \in \wp^{-1}(\lambda) \text{ and } w_\lambda \in \kappa^{-1}(\lambda).$$

  Both sets $\{y_\lambda\}_{\lambda\in\Lambda}$ and $\{w_\lambda\}_{\lambda\in\Lambda}$ give two parametrizations of the canonical basis.

Question: Can the two word parametrizations be made the same?

In other words, can we prove that $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) \neq \emptyset$ for all $\lambda \in \Lambda$?

A comparison of σ_i and τ_i

Let Q be a Dynkin quiver.

- (Lusztig) If i is a sink of Q, then
$$\tau_i = \sigma_i : \Lambda \to \Lambda.$$

- In general, write $E_i E_\lambda = \sum_\mu f_{i,\lambda;\mu} E_\mu$. Then

  (1) $f_{i,\lambda;\mu} \neq 0 \Longrightarrow M(\mu) \leqslant M(\sigma_i \lambda)$.

  (2) (Reineke) $\quad \tau_i \lambda = \mu \iff f_{i,\lambda;\mu} \neq 0$
  and $\deg^+ f_{i,\lambda;\mu} = a_i(\lambda) \geqslant a_i(\mu) - 1$ ($Q \neq E_8$),

  where $\quad a_i(\lambda) := \max_\mu \deg^+ f_{i,\lambda;\mu}$.

- Thus, combining (1) and (2) and ... yields $\kappa(w) \leqslant \wp(w)$ for all w.

The linear quiver case

For $\lambda = (\lambda_{s,t}) \in \Lambda$, let

$$m_{\sigma} = \max\{j \mid \lambda_{i+1,j} \neq 0\},$$
$$m_{\tau} = \min\{k| \mid s_{ik} = \max_j s_{ij}\},$$

where $s_{ij} = \sum_{l \geqslant j} \lambda_{i,l} - \sum_{l \geqslant j+1} \lambda_{i+1,l}$.

Then we have

$$(\sigma_i \lambda)_{s,t} = \begin{cases} \lambda_{s,t} + 1 & \text{if } (s,t) = (i, m_{\sigma}), \\ \lambda_{s,t} - 1 & \text{if } (s,t) = (i+1, m_{\sigma}), \\ \lambda_{s,t} & \text{otherwise.} \end{cases}$$

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Then we have

$$(\tau_i \lambda)_{s,t} = \begin{cases} \lambda_{s,t} + 1 & \text{if } (s,t) = (i, m_{\tau}), \\ \lambda_{s,t} - 1 & \text{if } (s,t) = (i+1, m_{\tau}), \\ \lambda_{s,t} & \text{otherwise.} \end{cases}$$

Example

- Let Q be the quiver 

  Then 2 is a sink and so $\sigma_2 = \tau_2$.
- Write $\lambda \in \Lambda$ as the triple (a, b, c), if $a = \lambda(\alpha_1)$, $b = \lambda(\alpha_1 + \alpha_2)$, and $c = \lambda(\alpha_2)$.
- Thus, we have

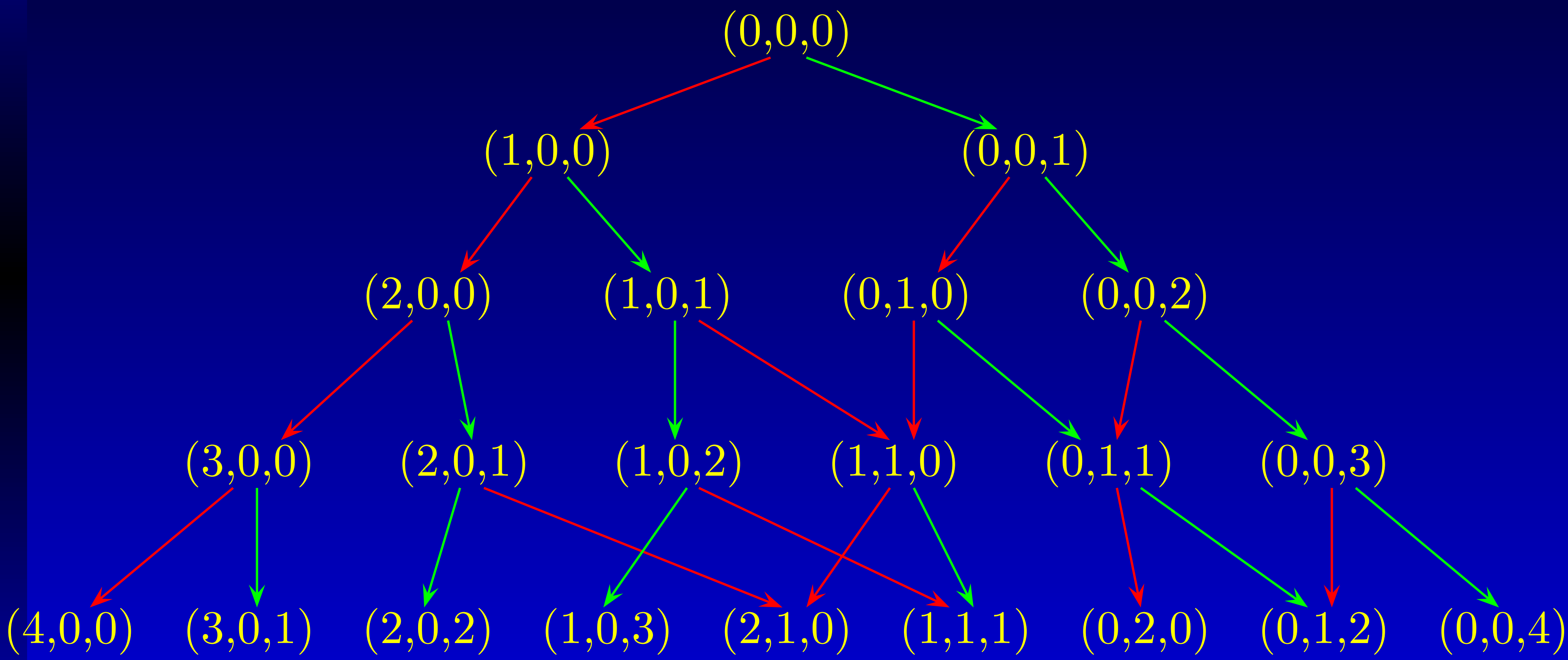
$$\sigma_1(a, b, c) = \begin{cases} (a+1, b, c) & \text{if } c = 0, \\ (a, b+1, c-1) & \text{if } c \geqslant 1 \end{cases}$$

and

$$\tau_1(a, b, c) = \begin{cases} (a+1, b, c) & \text{if } a \geqslant c, \\ (a, b+1, c-1) & \text{if } a < c. \end{cases}$$

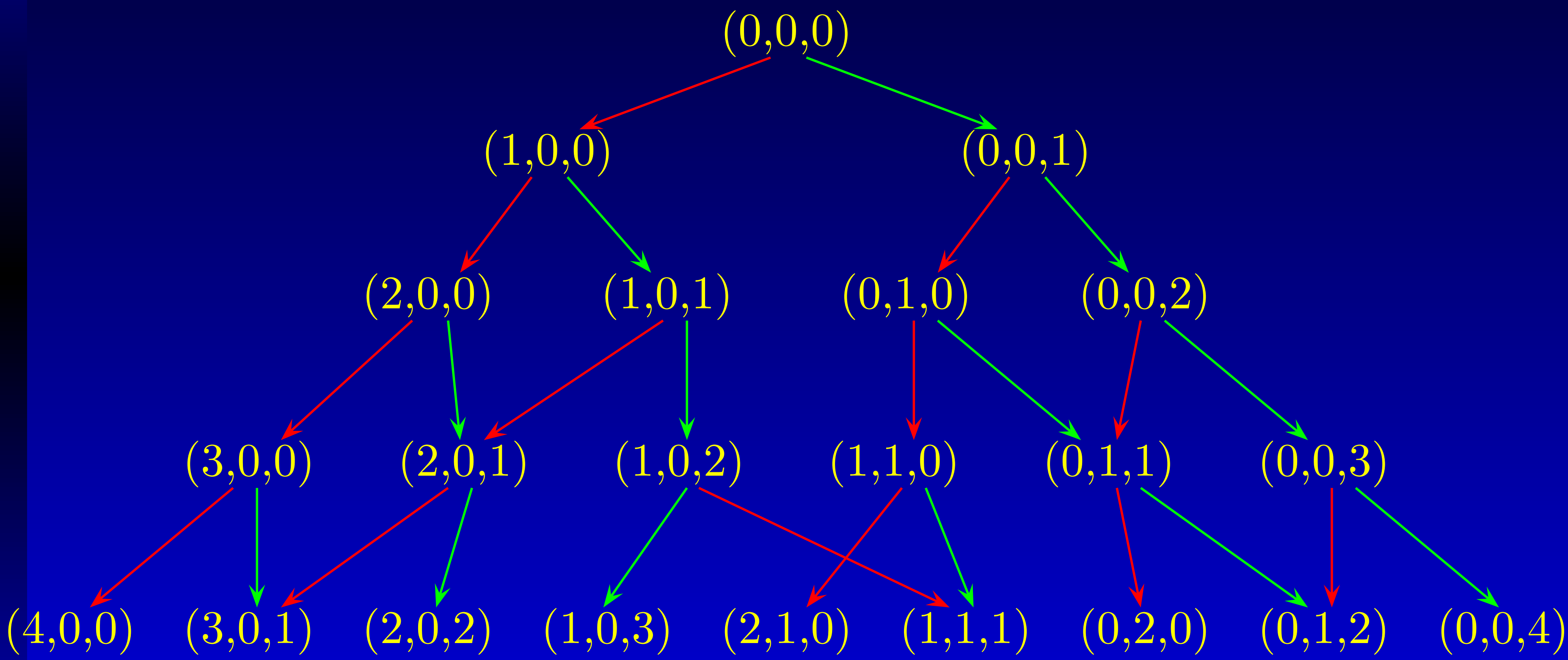
The graph $\mathsf{G}(\mathsf{A}_2)$ and $\mathsf{C}(\mathsf{A}_2)$

The graph $\mathsf{G}(\mathsf{A}_2)$ with $\sigma_1 =$ red, $\sigma_2 =$ green.



The graph $\mathsf{G}(\mathsf{A}_2)$ and $\mathsf{C}(\mathsf{A}_2)$

The graph $\mathsf{C}(\mathsf{A}_2)$ with $\tau_1 =$ red, $\tau_2 =$ green.



Good modules

- $\Phi^+(Q) = \{(i,j) \mid 1 \leqslant i \leqslant j \leqslant n\}$.
- Let $M_{i,j}$ be the indecomposable whose top and socle are isomorphic to S_i and S_j, respectively.
- $\lambda = (\lambda_{s,t}) \in \Lambda$ is said to be good at i if there exists $j > i$ such that

$$M(\lambda) \cong \left(\bigoplus_{s=j}^{n} \lambda_{i,s} M_{i,s} \right) \oplus \lambda_{i+1,j} M_{i+1,j} \oplus N,$$

where the top of N contains no S_i or S_{i+1}.

Lemma

If λ is good at i, then $\sigma_i \lambda = \tau_i \lambda$.

Theorem (D-D-Z)

Let Q be a linear quiver.

Let $\lambda \in \Lambda$, and define, for $1 \leqslant i \leqslant j \leqslant n$,

$$w_{i,j} = \underbrace{i \ldots i}_{\lambda_{ij}} \underbrace{i+1 \ldots i+1}_{\lambda_{ij}} \ldots \ldots \underbrace{j \ldots j}_{\lambda_{ij}},$$

and

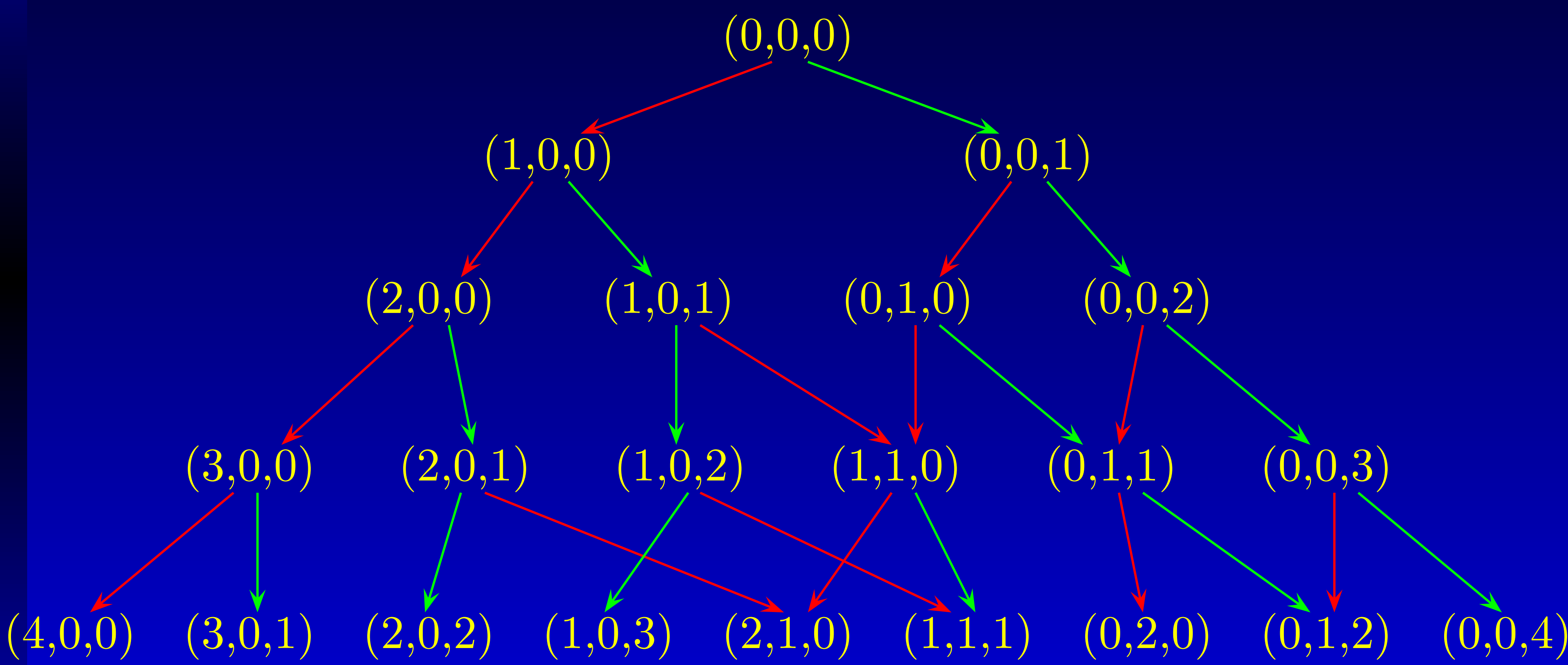
$$w = w_{n,n} w_{n-1,n-1} w_{n-1,n} \cdot \cdots \cdot w_{1,1} w_{1,2} \ldots w_{1,n}.$$

Then $\wp(w) = \lambda = \kappa(w)$. In particular,

$$\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) \neq \emptyset.$$

From the example above

The graph $\mathsf{G}(\mathsf{A}_2)$ with $\sigma_1 =$ red, $\sigma_2 =$ green.



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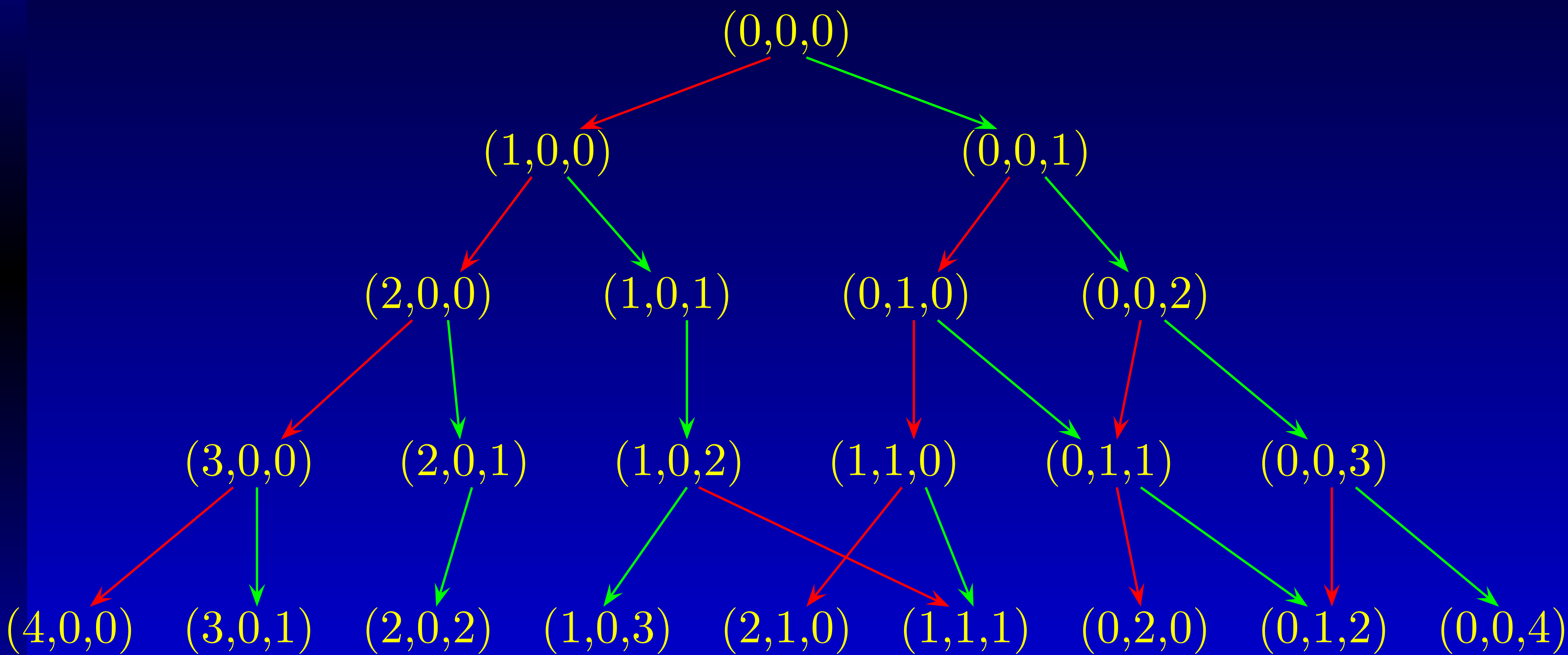
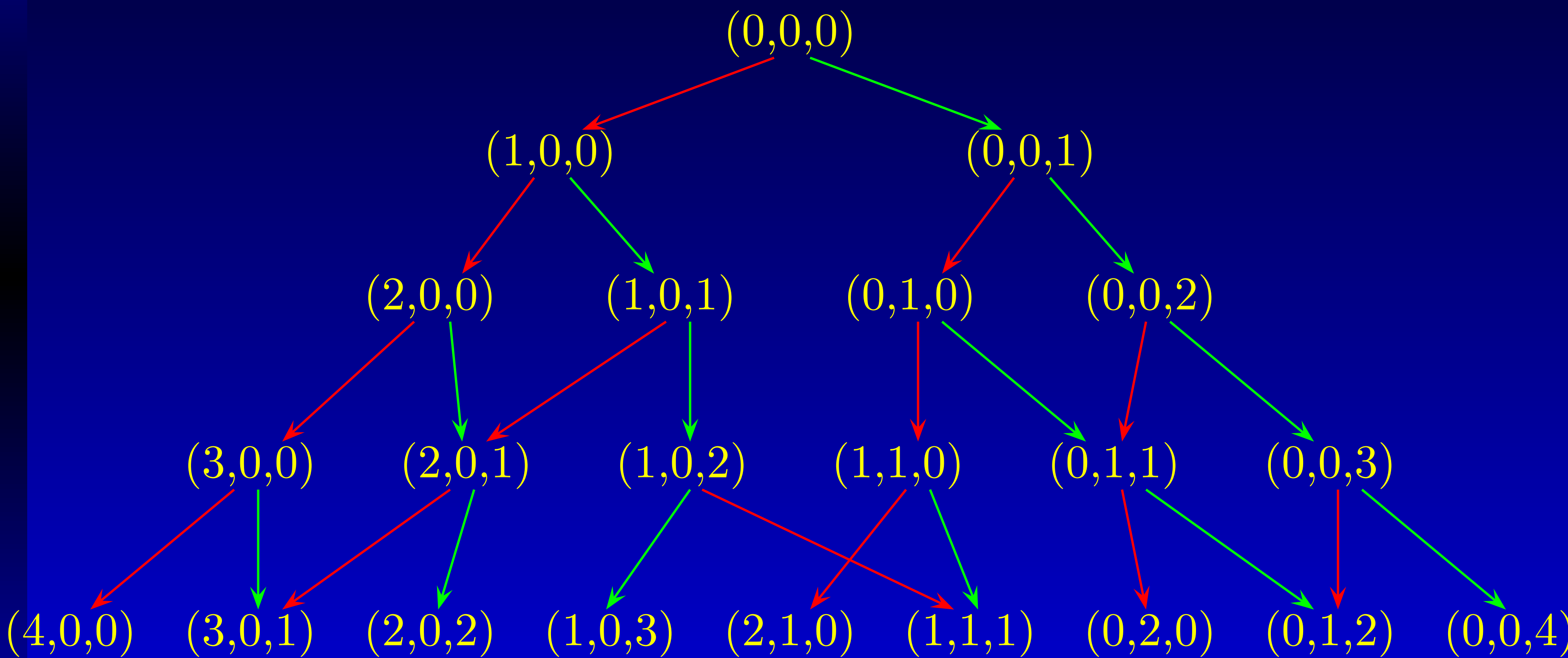
The graph $G(A_2) \cap C(A_2)$:



From the example above

The graph $\mathsf{C}(\mathsf{A}_2)$ with $\tau_1 =$ red, $\tau_2 =$ green.



A Counterexample

However, the theorem fails for cyclic quivers.

Let Q be the cyclic quiver Δ_3.

Let $M(\lambda) = S_1 \oplus S_1[2] \oplus S_1[3] \oplus S_2[2] \oplus S_3[3]$.

Then $\wp^{-1}(\lambda) = \{132^2 1^3 3^2 2^2.\}$

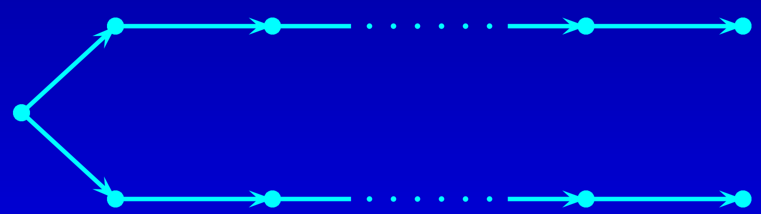
But $\kappa^{-1}(\lambda)$ is given by

$$1^2231213232, 1^22312133^22^2, 1^2231231232,$$
$$1^2231231 32^2, 1^22321^23232, 1^22321^23^22^2,$$
$$1^2232131232, 1^2232131 32^2, 1^2321213232,$$
$$1^23212133^22^2, 1^2321231232, 1^23212313 2^2.$$

Hence we have $\wp^{-1}(\lambda) \cap \kappa^{-1}(\lambda) = \emptyset$.

Remark

- Although the common word parametrization fails for cyclic quivers, it is natural to expect that the same result holds for all Dynkin quivers.
- Using the theory of Frobenius morphisms and module twisting on representations of quivers with automorphisms, we can establish a similar result for type B.
- E.g., consider the quiver:



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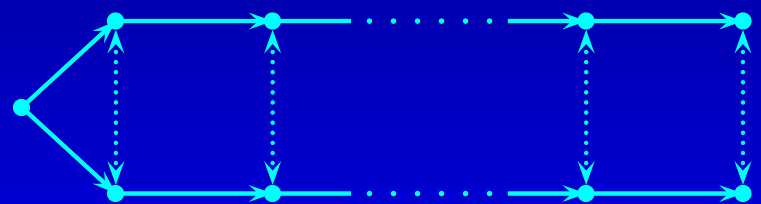

Frobenius maps

Let $\mathbb{F}_q$ be a finite field of q elements and let $k = \overline{\mathbb{F}}_q$ be its algebraic closure.

- A Frobenius map on a vector space over k is an abelian group automorphism $F : V \to V$ satisfying

  (1) $F(\lambda v) = \lambda^q F(v)$ for all $v \in V$ and $\lambda \in k$;

  (2) for any $v \in V$, $F^n(v) = v$ for some $n > 0$.

q-twists of vector spaces

Let f be the field automorphism

$$f : k \longrightarrow k; \lambda \mapsto \lambda^q.$$

- For a k-space V,
  let $V^{(1)} = V \otimes_f k$ with $\lambda v \otimes 1 = v \otimes \lambda^q$.
- We may identify $V^{(1)}$ as V with a twisted scalar multiplication $\lambda \centerdot v = \sqrt[q]{\lambda}\, v$.
- Let $\tau_V : V \longrightarrow V^{(1)}$ be the $\mathbb{F}_q$-linear isomorphism sending v to $v \otimes 1$.
- Clearly, a map $F : V \longrightarrow V$ is a Frobenius map iff $F \circ \tau_V^{-1} : V^{(1)} \longrightarrow V$ is a k-linear isomorphism.

Algebras with Fr. morphisms

- A Frobenius morphism on a k-algebra A (with 1) is a Frobenius map $F = F_A$ on the underlying vector space satisfying $F(ab) = F(a)F(b)$ for all $a, b \in A$.
- If M is an A-module, then we call a Frobenius map F_M on the space M a module Frobenius map (relative to F_A) if $F_M(am) = F(a)F_M(m)$ for all $a \in A$ and $m \in M$.
- In this case, the fixed point space $A^F = \{a \in A \mid F(a) = a\}$ is an $\mathbb{F}_q$-algebra; while M^{F_M} is naturally an A^F-module.

Module twisting

Let A be a f.d. k-algebra with Fr. morphism F.

- Let M be an A-module defined by the k-algebra homomorphism $\pi : A \longrightarrow \mathrm{End}_k(M)$,

  and let F_M be a Frobenius map on M.

  Define an A-module structure on $M^{(1)}$ by
  $$\pi^{[1]}(a) = \tau_M \circ \pi(F^{-1}(a)) \circ \tau_M^{-1}, \forall a \in A.$$
  Denote this module by $M^{[1]}$ and call it the *Frobenius twist* of M.

Module twisting

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- Let M be an A-module defined by the k-algebra homomorphism $\pi : A \longrightarrow \mathrm{End}_k(M)$,

  and let F_M be a Frobenius map on M.

  Define a new A-module structure on M by

  $$\pi^{[F_M]}(a) = F_M \circ \pi(F^{-1}(a)) \circ F_M^{-1}, \forall a \in A.$$

  Denote this module by $M^{[F_M]}$ and call it the F_M-*twist* of M.

Module twisting

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- Let M be an A-module defined by the k-algebra homomorphism $\pi : A \longrightarrow \mathrm{End}_k(M)$,

  and let F_M be a Frobenius map on M.

- We have A-module isomorphism
$$F_M \circ \tau_M^{-1} : M^{[1]} \longrightarrow M^{[F_M]}.$$

F-stable modules

- An A-module is called F-stable if $M \cong M^{[1]}$.
- An A-module is called F-periodic if $M \cong M^{[r]}$ for some $r \geqslant 1$.
- Let $p(M) = p_F(M)$ be the minimal number r satisfying $M \cong M^{[r]}$. We call it the *F-period* of M.

Lemma

$M \cong M^{[r]}$ iff there exists a Fr. map F_M on M such that F_M^r is a module Fr. map (wrt F^r).

- Thus, if M is F-stable, then M^{F_M} is an A^F-module for some Fr. map F_M on M.

Frobenius twist functor

- If $f : M \to N$ is an A-module homomorphism, then the k-linear map $f^{(1)} = f \otimes 1 : M^{(1)} \to N^{(1)}$ becomes an A-module homomorphism $f^{[1]} : M^{[1]} \to N^{[1]}$.
- We obtain a functor $(\ )^{[1]} = (\ )^{[1]}_{A\text{-}\mathbf{mod}} : A\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}$.
- $A\text{-}\mathbf{mod}^F$ whose objects are F-stable A-modules M with a fixed isomorphism $\varphi_M : M^{[1]} \xrightarrow{\sim} M$ and whose morphisms are compatible with the isomorphisms φ_M.

Theorem

There is a cat. equivalence $A^F\text{-}\mathbf{mod} \cong A\text{-}\mathbf{mod}^F$.

Quivers with automorphisms

Let Q be a quiver with automorphism σ. Then

- (Q, σ) gives rise to a valued quiver.
- σ induces a Fr. morphism on $A = kQ$

$$F = F_{Q,\sigma;q} : A \to A;\ \sum_s x_s p_s \mapsto \sum_s x_s^q \sigma(p_s)$$

Theorem

The representation category of an $\mathbb{F}_q$-species is equivalent to A^F-**mod** for some A and F.

Induced automorphisms on G, C

Two observations:

- If M, N are F-stable, then so is $M * N$.

  Thus, the Fr. twist functor induces an automorphism on the generic ext. graph G.

- The structure constants for the Hall algebra is invariant for Fr. twisting.

  Thus, by Reineke's result, the Fr. twist functor induces an automorphism on the crystal graph C.

- We may "fold" these graphs to obtain a common word parametrization for a non-simply case from a simply-laced case.

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