GENERALIZED GREEN FUNCTIONS AND UNIPOTENT CLASSES FOR FINITE REDUCTIVE GROUPS, II

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Abstract. This paper is concerned with the problem of the determination of unknown scalars involved in the algorithm of computing the generalized Green functions of reductive groups G over a finite field. In the previous paper, we have treated the case where $G = SL_n$. In this paper, we determine the scalars in the case where G is a classical group Sp_{2n} or SO_N for arbitrary characteristic.

0. Introduction

This paper is a sequel to [S2]. Our aim is to remove an ambiguity from the algorithm of computing generalized Green functions of reductive groups due to Lusztig. Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ with Frobenius map F. Let p be the characteristic of $\mathbf{F}_q$. In [S2], we have treated the case where $G = SL_n$. In this paper we consider the case where $G = Sp_{2n}$ or SO_N for arbitrary p. The case where $G = \mathrm{Spin}_N$ will be treated in a separate paper.

In [S1] it was shown, in the case of Sp_{2n} or SO_N with $p \neq 2$, that there exists a representative in C^F for each unipotent class C, called a distinguished element there (in this paper we call it a split element) which behaves well with respect to the computation of Green functions. Our result in this paper shows that the split elements behave well for any type of generalized Green functions. We also show, in the case where $p = 2$, that such a good representative (called a split element) exists for $G = Sp_{2n}$ or SO_{2n}. This was not known even for the case of Green functions.

The main ingredient for the proof is a variant of the restriction theorem ([L1]) for the generalized Springer correspondence. The restriction theorem is a powerful tool for determining the generalized Springer correspondence, and it was used in [LS], [Sp2] very effectively. We extend this theorem so that it involves the information on the Frobenius action. In [S2], we have investigated the Frobenius action on the cohomology group $H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}})$. But this requires a precise information on the geometry of $\mathcal{P}_u$ related to the local system $\dot{\mathcal{E}}$. In the case of classical groups, one can avoid to deal with $\mathcal{P}_u$ by considering the restriction theorem as above.

1. A variant of the restriction theorem

1.1. We follow the notation in Section 1 in [S2]. In particular, G is a connected reductive group over a finite field $\mathbf{F}_q$, with Frobenius map F. Let k be an algebraic closure of $\mathbf{F}_q$ and p the characteristic of k. Let $P = LU_P$ be a parabolic subgroup of G, with a Levi subgroup L, and let $\mathcal{E}$ be a cuspidal local system on a unipotent class C in L. As in (1.2.2) in [S2], one can define a perverse sheaf K on G associated to the triple $(L, C, \mathcal{E})$. Then K is a semisimple perverse sheaf with $\operatorname{End} K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}]$, where $\mathcal{W} = N_G(L)/L$ is a Coxeter group. Thus K is decomposed as

$$K = \bigoplus_{E \in \mathcal{W}^\wedge} V_E \otimes K_E, \tag{1.1.1}$$

where K_E is a simple perverse sheaf on G such that $V_E = \operatorname{Hom}(K_E, K)$ is an irreducible $\mathcal{W}$-module corresponding to $E \in \mathcal{W}^\wedge$. Put $d = \dim Z_L^0$, where Z_L is the center of L. Let G_{uni} be the unipotent variety of G, and $\mathcal{N}_G$ the set of all the pairs $(C', \mathcal{E}')$, where C' is a unipotent class in G and $\mathcal{E}'$ is a G-equivariant simple local system on C'. Then it is known that $K[-d]|_{G_{\text{uni}}}$ is a semisimple perverse sheaf on G_{uni}, and it is decomposed as

$$K[-d]|_{G_{\text{uni}}} \simeq \bigoplus_{(C', \mathcal{E}') \in \mathcal{N}_G} V_{(C', \mathcal{E}')} \otimes \operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C'], \tag{1.1.2}$$

where $V_{(C', \mathcal{E}')}$ is the multiplicity space for the simple perverse sheaf $\operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C']$ on G_{uni} (cf. [S2, (1.2.4)]). Thus $K_E|_{G_{\text{uni}}}$ coincides with some $\operatorname{IC}(\overline{C}', \mathcal{E}')$ up to shift, and $V_{(C', \mathcal{E}')}$ coincides with V_E. It turns out that all the irreducible $\mathcal{W}$-modules are realized as $V_{(C', \mathcal{E}')}$ for some pair $(C', \mathcal{E}')$. Thus we have an injective map $\mathcal{W}^\wedge \to \mathcal{N}_G$ by $E = V_{(C', \mathcal{E}')} \to (C', \mathcal{E}')$, whose image we denote by $\mathcal{N}_G(C, \mathcal{E})$. Let $\mathcal{M}_G$ be the set of triples $(L, C, \mathcal{E})$ up to G-conjugacy, where L is a Levi subgroup of a parabolic subgroup of G and $\mathcal{E}$ is a cuspidal local system on a unipotent class C of L. The above injective maps form a bijection

$$\coprod_{(L, C, \mathcal{E}) \in \mathcal{M}_G} (N_G(L)/L)^\wedge \to \mathcal{N}_G \tag{1.1.3}$$

which is the so-called generalized Springer correspondence ([L1, 6.5]).

1.2. Let $Q \supset P$ be a parabolic subgroup of G with the Levi subgroup M such that $M \supset L$. Then $\mathcal{W}_1 = N_M(L)/L$ is in a natural way a subgroup of $\mathcal{W}$. Replacing G by M, we have a subset $\mathcal{N}_M(C, \mathcal{E})$ of $\mathcal{N}_M$. For each $(C', \mathcal{E}') \in \mathcal{N}_G(C, \mathcal{E})$ (resp. $(C_1, \mathcal{E}_1) \in \mathcal{N}_M(C, \mathcal{E})$), we denote by E (resp. E_1) the corresponding irreducible representation of $\mathcal{W}$ (resp. $\mathcal{W}_1$) under (1.1.3).

Let $\pi_Q : Q \to M$ be the natural projection. Assume that $(C_1, \mathcal{E}_1) \in \mathcal{N}_M(C, \mathcal{E})$, and that $(C', \mathcal{E}') \in \mathcal{N}_G$. We denote by $f_{C_1, C'} : C_1 U_Q \cap C' \to C_1$ the restriction of π_Q. Then $\mathcal{F} = R^{2d_{C_1, C'}}(f_{C_1, C'})_! \mathcal{E}'$ is a semisimple M-equivariant local system on C_1, where $d_{C_1, C'} = (\dim C' - \dim C_1)/2$. We define an integer $m_{\mathcal{E}_1, \mathcal{E}'}$ to be the multiplicity of $\mathcal{E}_1$ in $\mathcal{F}$. Lusztig proved the following restriction theorem on the generalized Springer correspondence.

Theorem 1.3 (Lusztig [L1, Theorem 8.3])**.** *Under the above setting, $(C', \mathcal{E}') \in \mathcal{N}_G(C, \mathcal{E})$ if and only if $m_{\mathcal{E}_1, \mathcal{E}'} \neq 0$. Moreover in that case we have*

$$m_{\mathcal{E}_1,\mathcal{E}'} = \langle \operatorname{Res} E, E_1 \rangle_{\mathcal{W}_1},$$

where $\langle \, , \, \rangle_{\mathcal{W}_1}$ is the inner product of two representations of $\mathcal{W}_1$ (regarded as characters), and $\operatorname{Res} E$ is the restriction of E on $\mathcal{W}_1$.

1.4. Let $u \in C'$ and $v \in C_1$, and consider the component group $A_G(u)$ and $A_M(v)$. The set of G-equivariant simple local systems on C' is in 1:1 correspondence with the set $A_G(u)^\wedge$ of irreducible characters of $A_G(u)$, and a similar fact holds also for M. As described in [LS], the integer $m_{\mathcal{E}_1,\mathcal{E}'}$ can be interpreted in terms of the representations of $A_G(u)$ and $A_M(v)$, which we explain below. Let $\mathcal{F}_v$ be the stalk of $\mathcal{F}$ at $v \in C_1$. Then we have

$$\mathcal{F}_v \simeq H_c^{2d_{C_1,C'}}(C' \cap vU_Q, \mathcal{E}'). \tag{1.4.1}$$

Let $\pi : \widetilde{C}' = Z_G^0(u)\backslash G \to C', Z_G^0(u)g \mapsto g^{-1}ug$ be the finite covering of C' with group $A_G(u)$. Let $X = (C' \cap vU_Q) \times_{C'} \widetilde{C}'$ be the fibre product of $C' \cap vU_Q$ with $\widetilde{C}'$ over C', and let $\widetilde{\pi} : X \to C' \cap vU_Q$ be the base change of π. Then we have

$$H_c^{2d_{C_1,C'}}(C' \cap vU_Q, \widetilde{\pi}_*\bar{\mathbf{Q}}_l) \simeq H_c^{2d_{C_1,C'}}(X, \bar{\mathbf{Q}}_l),$$

and $A_G(u)$ acts naturally on the right hand side. Now $\widetilde{\pi}_*\bar{\mathbf{Q}}_l$ can be decomposed as $\widetilde{\pi}_*\bar{\mathbf{Q}}_l = \sum_\rho V_\rho \otimes \mathcal{E}_\rho$, where ρ runs over all the irreducible characters of $A_G(u)$. Here $\mathcal{E}_\rho$ is the G-equivariant simple local system on C' corresponding to ρ and V_ρ is the corresponding irreducible representation of $A_G(u)$. It follows that

$$H_c^{2d_{C_1,C'}}(C' \cap vU_Q, \mathcal{E}_\rho) \simeq \big(H_c^{2d_{C_1,C'}}(X, \bar{\mathbf{Q}}_l) \otimes V_{\rho^*}\big)^{A_G(u)},$$

where ρ^* is the dual representation of ρ. On the other hand, the semisimple local system $\mathcal{F}$ can be written as $\mathcal{F} = \sum_{\rho_1} m_{\rho_1}\mathcal{E}_{\rho_1}$, where $\mathcal{E}_{\rho_1}$ is the irreducible local system on C_1 corresponding to $\rho_1 \in A_M(v)^\wedge$ and m_{ρ_1} is the multiplicity of $\mathcal{E}_{\rho_1}$ in $\mathcal{F}$. By taking the stalk at v, we have $\mathcal{F}_v = \sum_{\rho_1} m_{\rho_1}(\mathcal{E}_{\rho_1})_v$. Here $(\mathcal{E}_{\rho_1})_v$ is an irreducible $A_M(v)$-module corresponding to ρ_1. Note that if $\mathcal{E}' = \mathcal{E}_\rho$, and $\mathcal{E}_1 = \mathcal{E}_{\rho_1}$, we have $m_{\mathcal{E}_1,\mathcal{E}'} = m_{\rho_1}$. Now $Z_M(v)$ acts on $C' \cap vU_Q$ by conjugation, and it induces an action of $A_M(v)$ on $H_c^{2d_{C_1,C'}}(C' \cap vU_Q, \mathcal{E}')$. We have

$$\begin{aligned} m_{\mathcal{E}_1,\mathcal{E}'} &= \langle H_c^{2d_{C_1,C'}}(C' \cap vU_Q, \mathcal{E}_\rho), \rho_1 \rangle_{A_M(v)} \\ &= \langle H_c^{2d_{C_1,C'}}(X, \bar{\mathbf{Q}}_l) \otimes V_{\rho^*})^{A_G(u)}, \rho_1 \rangle_{A_M(v)}, \end{aligned}$$

where $\langle \, , \, \rangle_{A_M(v)}$ denotes the inner product of characters of $A_M(v)$.

By Proposition 1.2 in [L1], it is known that $\dim X \le d_{C_1,C'}$. Thus $H_c^{2d_{C_1,C'}}(X, \bar{\mathbf{Q}}_l)$ has a basis corresponding to the set of irreducible components of X of dimension $d_{C_1,C'}$, and the action of $A_G(u)$ on $H_C^{2d_{C_1,C'}}(X, \bar{\mathbf{Q}}_l)$ coincides with the permutation action of $A_G(u)$ on those irreducible components of X. Since $\widetilde{C}' = Z_G^0(u)\backslash G$, we have

$$\begin{aligned} X &= \{(y, Z_G^0(u)g) \in (C' \cap vU_Q) \times \widetilde{C}' \mid y = g^{-1}ug\} \\ &= \{Z_G^0(u)g \mid g^{-1}ug \in vU_Q\} \\ &= Z_G^0(u)\backslash\{g \in G \mid g^{-1}ug \in vU_Q\}. \end{aligned}$$

Put $Y_{u,v} = \{g \in G \mid g^{-1}ug \in vU_Q\}$. Then $Z_G(u) \times Z_M(v)$ acts on $Y_{u,v}$ by $(z, z') : g \mapsto zgz'^{-1}$ for $z \in Z_G(u), z' \in Z_M(v)$, and the projection $Y_{u,v} \to X = Z_G^0(u)\backslash Y_{u,v}$ gives a bijection between the set of irreducible components of X and $Y_{u,v}$, which is compatible with the action of $A_G(u)$ and $A_M(v)$. Note that

$$\begin{aligned} \dim Y_{u,v} &= \dim X + \dim Z_G^0(u) \\ &= d_{C_1,C'} + \dim Z_G^0(u) \\ &= (\dim Z_G(u) + \dim Z_M(v))/2 + \dim U_Q. \end{aligned}$$

Let $X_{u,v}$ be the set of irreducible components of $Y_{u,v}$ of dimension $d_{C_1,C'} + \dim Z_G^0(u)$. It follows from the above discussion, we have

Corollary 1.5 (Lusztig-Spaltenstein [LS, 0.4, (4)]). *Let $\varepsilon_{u,v}$ be the permutation representation of $A_G(u) \times A_M(v)$ on $X_{u,v}$. Then we have*

$$\langle \operatorname{Res} E, E_1 \rangle_{\mathcal{W}_1} = m_{\mathcal{E}_1,\mathcal{E}'} = \langle \varepsilon_{u,v}, \rho \otimes \rho_1^* \rangle_{A_G(u) \times A_M(v)}\,.$$

1.6. We want to consider a variant of Corollary 1.5 which involves the Frobenius action. Assume that P is F-stable, and that the triple $(L, C, \mathcal{E}) \in \mathcal{M}_G$ is F-stable. We choose $u_0 \in C^F$ and fix an isomorphism $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ so that the induced isomorphism $\mathcal{E}_{u_0} \to \mathcal{E}_{u_0}$ is of finite order. φ_0 induces an isomorphism $\varphi : F^*K \xrightarrow{\sim} K$. For each pair $(C', \mathcal{E}') \in \mathcal{N}_G^F$, we choose $u \in {C'}^F$. We fix an isomorphism $\psi_{\mathcal{E}'} : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ as follows; F acts naturally on $A_G(u)$, and we consider the semidirect product $\widetilde{A}_G(u) = \langle\tau\rangle \ltimes A_G(u)$, where τ is the restriction of F on $A_G(u)$. Since $(C', \mathcal{E}')$ is F-stable, ρ is F-stable. We choose an extension $\widetilde{\rho}$ of ρ to $\widetilde{A}_G(u)$ and fix an isomorphism $\psi_{\mathcal{E}'}$ so that the induced isomorphism $\mathcal{E}'_u \to \mathcal{E}'_u$ corresponds to the action of τ on $\widetilde{\rho}$. Now $\psi_{\mathcal{E}'}$ induces an isomorphism $\widetilde{\psi}_{\mathcal{E}'} : F^* \operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C'] \xrightarrow{\sim} \operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C']$. The isomorphism φ also induces an isomorphism $F^*K[-d]|_{G_{\text{uni}}} \xrightarrow{\sim} K[-d]|_{G_{\text{uni}}}$, which we also denote by φ. Then under the decomposition of (1.1.2), φ induces an isomorphism

$$V_{(C'\mathcal{E}')} \otimes F^* \operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C'] \xrightarrow{\sim} V_{(C'\mathcal{E}')} \otimes \operatorname{IC}(\overline{C}', \mathcal{E}')[\dim C']$$

for each pair $(C', \mathcal{E}') \in \mathcal{N}_G^F$, and one can define a linear isomorphism $\sigma_{(C',\mathcal{E}')}$ on $V_{(C',\mathcal{E}')}$ such that this isomorphism can be written as $\sigma_{(C',\mathcal{E}')} \otimes \widetilde{\psi}_{\mathcal{E}'}$. Now F acts naturally on $\mathcal{W} = N_G(L)/L$, and $\sigma_{(C',\mathcal{E}')}$ becomes $\mathcal{W}$-semilinear, namely we have a relation $\sigma_{(C',\mathcal{E}')}w = F(w)\sigma_{(C',\mathcal{E}')}$ on $V_{(C',\mathcal{E}')}$ for each $w \in \mathcal{W}$. Replacing G by M, we can define a $\mathcal{W}_1$-semilinear map $\sigma_{(C_1,\mathcal{E}_1)}$ on $V_{(C_1,\mathcal{E}_1)}$ for each pair $(C_1, \mathcal{E}_1) \in \mathcal{N}_M^F$. The irreducible $\mathcal{W}$-module $V_{(C',\mathcal{E}')}$ can be written as a $\mathcal{W}_1$-module

$$V_{(C',\mathcal{E}')} = \sum_{E_1 \in \mathcal{W}_1^\wedge} M_{E_1} \otimes E_1, \tag{1.6.1}$$

where M_{E_1} is the multiplicity space of the irreducible $\mathcal{W}_1$-module E_1 and is realized as $M_{E_1} = \operatorname{Hom}_{\mathcal{W}_1}(E_1, V_{(C',\mathcal{E}')})$. Suppose that $E_1 \simeq V_{(C_1,\mathcal{E}_1)}$ under the generalized Springer correspondence for M. If E_1 is F-stable, $(C_1, \mathcal{E}_1) \in \mathcal{N}_M^F$, and we have an isomorphism $\sigma_{(C_1,\mathcal{E}_1)}$ on E_1. One can define a map $\sigma_{\mathcal{E}_1,\mathcal{E}'} : M_{E_1} \to M_{E_1}$ by $f \mapsto \sigma_{(C',\mathcal{E}')} \circ f \circ \sigma^{-1}_{(C_1,\mathcal{E}_1)}$. The linear map $\sigma_{(C',\mathcal{E}')}$ stabilizes the subspace $M_{E_1} \otimes E_1$ and we have

$$\sigma_{(C',\mathcal{E}')}|_{M_{E_1} \otimes E_1} = \sigma_{\mathcal{E}_1,\mathcal{E}'} \otimes \sigma_{(C_1,\mathcal{E}_1)}. \tag{1.6.2}$$

On the other hand, since $F(C') = C', F(C_1) = C_1$, the map $f_{C_1,C'}$ is F-equivariant. Hence $\psi_{\mathcal{E}'} : F^*\mathcal{E}' \,\tilde{\to}\, \mathcal{E}'$ induces an isomorphism $\psi_{C_1,C'} : F^*\mathcal{F} \,\tilde{\to}\, \mathcal{F}$, and so a linear isomorphism $\mathcal{F}_v \to \mathcal{F}_v$ which we denote by the same symbol $\psi_{C_1,C'}$. Now the local system $\mathcal{F}$ on C_1 corresponds to a representation V of $A_M(v)$. V can be decomposed as

$$V = \sum_{\rho_1 \in A_M(v)^\wedge} M_{\rho_1} \otimes \rho_1,$$

where $M_{\rho_1} = \operatorname{Hom}_{A_M(v)}(\rho_1, V)$ is the multiplicity space of the irreducible representation ρ_1. F acts on $A_M(v)$, and as in the case of G we consider the semidirect product $\widetilde{A}_M(v) = \langle \tau \rangle \ltimes A_M(v)$, where τ is the restriction of F on $A_M(v)$. For each $(C_1, \mathcal{E}_1) \in \mathcal{N}_M^F$, we fix an isomorphism $\psi_{\mathcal{E}_1} : F^*\mathcal{E}_1 \,\tilde{\to}\, \mathcal{E}_1$ as in G by using an extension $\widetilde{\rho}_1$ of ρ_1 to $\widetilde{A}_M(v)$. Now $\psi_{C_1,C'}$ stabilizes the subspace $M_{\rho_1} \otimes \rho_1$ for an F-stable $\rho_1 \in A_M(v)^\wedge$, and as in (1.6.2) one can define a linear map $\psi_{\rho_1,\rho}$ on M_{ρ_1} such that

$$\psi_{C_1,C'}|_{M_{\rho_1} \otimes \rho_1} = \psi_{\rho_1,\rho} \otimes \psi_{\mathcal{E}_1}. \tag{1.6.3}$$

The following result gives an F-twisted version of the restriction theorem. The proof is done by chasing the argument in [L1].

Proposition 1.7. *Under the notation as above, we have*

$$\operatorname{Tr}(\sigma_{\mathcal{E}_1,\mathcal{E}'}, M_{E_1}) = q^{-d_{C_1,C'} + \dim U_Q} \operatorname{Tr}(\psi_{\rho_1,\rho}, M_{\rho_1}).$$

1.8. Let $Y_{u,v}$ be as in 1.4. Assume that Q is F-stable. Since u, v are F-stable, $Y_{u,v}$ is F-stable, and so F acts as a permutation on $X_{u,v}$. On the other hand, F acts naturally on $A(u,v) = A_G(u) \times A_M(v)$, and we denote by $\widetilde{A}(u,v)$ the semidirect product group $\langle\tau\rangle \ltimes A(u,v)$. Then the permutation representation $\varepsilon_{u,v}$ is extended to a representation of $\widetilde{A}(u,v)$, which we denote by $\widetilde{\varepsilon}_{u,v}$. Now $\psi_{\mathcal{E}'}$ and $\psi_{\mathcal{E}_1}$ determines an extension of $\rho\otimes\rho_1^*$ to $\widetilde{A}(u,v)$, which we denote by $\widetilde{\rho\otimes\rho_1^*}$. By chasing the argument in 1.4, we see that

$$\mathrm{Tr}\,(\psi_{\rho_1,\rho}, M_{\rho_1}) = \langle\widetilde{\varepsilon}_{u,v}, \widetilde{\rho\otimes\rho_1^*}\rangle_{A(u,v)\tau},$$

where in general

$$\langle V_1, V_2\rangle_{A(u,v)\tau} = |A(u,v)|^{-1}\sum_{a\in A(u,v)}\mathrm{Tr}\,(a\tau, V_1)\,\mathrm{Tr}\,((a\tau)^{-1}, V_2)$$

for representations V_1, V_2 of $\widetilde{A}(u,v)$. Hence combined with Proposition 1.7, we have a variant of Corollary 1.5 involving the Frobenius action.

Corollary 1.9. *Let the notations be as above. Then we have*

$$\mathrm{Tr}\,(\sigma_{\mathcal{E}_1,\mathcal{E}'}, M_{E_1}) = q^{-d_{C_1,C'}+\dim U_Q}\langle\widetilde{\varepsilon}_{u,v}, \widetilde{\rho\otimes\rho_1^*}\rangle_{A(u,v)\tau}\,.$$

1.10. We shall connect the above results to the discussion on generalized Green functions in [S2, Section 1]. Take $j = (C', \mathcal{E}') \in \mathcal{N}_G^F$ and let $\psi_0 = \psi_{\mathcal{E}'} : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ be defined in 1.6. ψ_0 determines the G^F-invariant function Y_j^0 on the set G^F_{uni} as in [S2, 1.3]. On the other hand, let $\widetilde{\mathcal{W}} = \mathcal{W}\rtimes\langle c\rangle$ be the semidirect product, where c is a Coxeter group automorphsim on $\mathcal{W}$ induced from the action of F. In the decomposition in (1.1.1), one can define an isomorphism $\varphi_E : F^*K_E \xrightarrow{\sim} K_E$ so that the induced map $\sigma'_E : V_E \to V_E$ makes the irreducible $\mathcal{W}$-module V_E the preferred extension to $\widetilde{\mathcal{W}}$ (cf. [L2, IV, (17.2)]). Put

$$\begin{aligned} a_0 &= -\dim Z_L^0 - \dim C', \\ r &= \dim G - \dim L + \dim\,(C\times Z_L^0). \end{aligned}$$

The we have

$$a_0 + r = (\dim G - \dim C') - (\dim L - \dim C).$$

We have $\mathcal{H}^{a_0}(K_E)|_{C'} = \mathcal{E}'$ and we define $\psi : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ so that $q^{(a_0+r)/2}\psi$ coincides with the map defined by $\varphi_E : F^*\mathcal{H}^{a_0}(K_E) \xrightarrow{\sim} K_E$. The function Y_j is defined as the characteristic function of $\mathcal{E}'$ through ψ, extended by 0 to the function on G^F_{uni} (see [S2, 1.3]). Since $\mathcal{E}'$ is a simple local system, there exists $\gamma\in\bar{\mathbf{Q}}_l^*$ such that $\psi = \gamma\psi_0$, and so $Y_j = \gamma Y_j^0$. Our main objective is the determination of this scalar γ. Note that the determination of γ is equivalent to the determination of the map $\sigma_{(C',\mathcal{E}')}$. In this paper, we determine γ by investigating the map $\sigma_{(C',\mathcal{E}')}$. The following fact is easily verified.

Lemma 1.11. *Suppose that $q^{-(a_0+r)/2}\sigma_{(C',\mathcal{E}')}$ makes the $\mathcal{W}$-module $V_{(C',\mathcal{E}')}$ the preferred extension to $\widetilde{\mathcal{W}}$. Then we have $\gamma = 1$.*

2. Unipotent classes of classical groups

2.1. Let G be a connected classical group defined over $\mathbf{F}_q$. We consider the following type of groups G.

(I) $G = Sp_{2n},\ \ p \ne 2$,

(II) $G = SO_{2n+1},\ \ p \ne 2$,

(III) $G = SO^{\pm}_{2n},\ \ p \ne 2$.

(IV) $G = Sp_{2n},\ \ p = 2$,

(V) $G = SO^{\pm}_{2n},\ \ p = 2$.

These groups are realized as a group of transformations preserving the various forms. Let V be a vector space over k with $\dim V = N$. Assume that $p \ne 2$. Then Sp_N (resp. O_N) is the subgroup of $GL(V)$ leaving f invariant, where f is an alternating form (resp. a symmetric bilinear form) on V and $N = 2n$ in the case of Sp. SO_N is the connected component of O_N, and $SO^{\pm}_{2n}$ corresponds to two $\mathbf{F}_q$-forms of f, one is split, the other is non-split.

Assume that $p = 2$. Then Sp_{2n} is the subgroup of $GL(V)$ with $N = 2n$ leaving an alternating form (= a symmetric bilinear form) f invariant. The quadratic form Q on V is defined by the property that the map $V\times V \to k, (x,y) \mapsto Q(x+y)-Q(x)-Q(y)$ gives rise to a non-singular bilinear form, which we may take the alternating form f. Let O_{2n} be the subgroup of $GL(V)$ leaving Q invariant. Then we have $O_{2n} \subset Sp_{2n}$, and let SO_{2n} be the connected component of O_{2n}. It is known by [D] that there exists two $\mathbf{F}_q$-forms of Q as follows. We regard Q as the quadratic form on $V_0 = \mathbf{F}_q^{2n}$. Then there exists a basis of V_0 such that, for $x = (x_1, \dots, x_{2n}) \in V_0$ with respect to this basis, $Q(x)$ can be expressed as

$$Q(x) = x_1x_{n+1} + \cdots + x_nx_{2n}, \tag{2.1.1}$$

$$Q(x) = x_1x_{n+1} + \cdots x_{n-1}x_{2n-1} + \alpha x_n^2 + x_nx_{2n} + \alpha x_{2n}^2, \tag{2.1.2}$$

where $\alpha \in \mathbf{F}_q$ is an element such that $\alpha X^2 + X + \alpha$ is an irreducible polynomial in $\mathbf{F}_q[X]$. We denote by O^+_{2n} (resp. O^-_{2n}) the group O_{2n} associated to the form in (2.1.1) (resp. (2.1.2)), and let $SO^{\pm}_{2n}$ be the connected component of $O^{\pm}_{2n}$.

2.2. We shall describe the unipotent classes in G. As is well-known, in the case where $p \ne 2$, the unipotent classes of G are described by unipotent classes in $GL(V)$ which are parametrized by partitions of N through Jordan normal form. Let $\widetilde{C}_\lambda$ be the unipotent class in $GL(V)$ corresponding to a partition λ of N. We write λ as $\lambda = (\lambda_1 \le \lambda_2 \le \cdots \le \lambda_r)$ with $\sum_{i=1}^r \lambda_i = N$ or $\lambda = (1^{c_1}, 2^{c_2}, \dots)$, where $r = l(\lambda)$ is called the length of λ. Assume that $G = Sp_{2n}$. Then $C_\lambda = \widetilde{C}_\lambda \cap G$ is non-empty if and only if c_i is even for odd i, and in that case C_λ is a single conjugacy class in G. While for $\widetilde{G} = O_N$, $C_\lambda = \widetilde{C}_\lambda \cap \widetilde{G}$ is non-empty if and only if c_i is even for even i, and in that case C_λ form a single class in $\widetilde{G}$. Now C_λ is already contained in $G = SO_N$, and so gives a unipotent class in G in almost all cases. The exceptions are the cases where λ satisfies the condition ; $c_i = 0$ if i is even, and c_i is even for all odd i. In that case, C_λ is divided into two classes C'_λ and C''_λ in G.

2.3. In the case where $p = 2$, the parametrization of unipotent classes is more complicated. We shall describe it following Spaltenstein [Sp1, 2.6]. First assume that $G = Sp_{2n}$ with $p = 2$, and let f be the associated alternating form. Then the unipotent classes in G are parametrized by a pair (λ, ε), where λ is a partition of $2n$ such that c_i is even for odd i, and ε is an assignment $\varepsilon : i \mapsto \varepsilon_i \in \{0, 1\}$ for even i such that $c_i \neq 0$. Here $\varepsilon_i = 1$ if c_i is odd, and $\varepsilon_i = 0$ or 1 if c_i is even. The correspondence with unipotent classes are given as follows. Let u be a unipotent element in G. Then as an element in $GL(V)$, u is parametrized by a partition λ of $2n$, which satisfies a similar condition as in the case of $p \neq 2$. Now take even i such that c_i is even non-zero. We define a function h_i on $\operatorname{Ker}(u-1)^i$ by $h_i(x) = f((u-1)^{i-1}x, x)$. Then we put

$$\varepsilon_i = \begin{cases} 0 & \text{if } h_i \equiv 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

The pair (λ, ε) is the one corresponding to the unipotent class in G containing u. We denote by $C_{\lambda,\varepsilon}$ the unipotent class in G corresponding to (λ, ε). For a convenience sake, we extend ε to the function on $\mathbb{N}$ by $\varepsilon : i \mapsto \varepsilon_i$, where $\varepsilon_i = \omega$ for i not appeared above (ω is a symbol not contained in $\{0, 1\}$).

Next assume that $G = SO_{2n}$ with $p = 2$. We have $\widetilde{G} = O_{2n} \subset Sp_{2n}$. Let $\widetilde{C}_{\lambda,\varepsilon}$ be the unipotent class in Sp_{2n} corresponding to (λ, ε). Then $C_{\lambda,\varepsilon} = \widetilde{C}_{\lambda,\varepsilon} \cap \widetilde{G}$ is a unipotent class in $\widetilde{G}$. Thus unipotent classes in $\widetilde{G}$ are 1:1 correspondence with unipotent classes in Sp_{2n}. Now $C_{\lambda,\varepsilon}$ is contained in G if and only if $l(\lambda)$ is even. Assume that $l(\lambda)$ is even. Then $C_{\lambda,\varepsilon}$ forms a single unipotent class in G except for the case where $c_i = 0$ for all odd i, and $\varepsilon_i = 0$ for all even i such that $c_i \neq 0$ (here c_i is even for even i). In the latter case, $C_{\lambda,\varepsilon}$ splits into two classes $C'_{\lambda,\varepsilon}$ and $C''_{\lambda,\varepsilon}$ in G.

2.4. Let G be as in 2.1. For a convenience sake, we introduce a function ε on $\mathbb{N}$ also in the case of $p \neq 2$. Assume that $p \neq 2$. In the case of $G = Sp_{2n}$, we put $\varepsilon_i = 1$ if i is even and $c_i \neq 0$, and put $\varepsilon_i = \omega$ otherwise. In the case of O_N, we put $\varepsilon_i = 1$ if i is odd and $c_i \neq 0$, and put $\varepsilon_i = \omega$ otherwise. Under this convention, we denote the class C_λ in Sp_{2n} or SO_N by $C_{\lambda,\varepsilon}$. For $u \in G$, let $A_G(u)$ be the component group of $Z_G(u)$ as before. In the case of $\widetilde{G} = O_N$, we also consider $A_{\widetilde{G}}(u) = Z_{\widetilde{G}}(u)/Z^0_{\widetilde{G}}(u)$ for $u \in \widetilde{G}$. Following [Sp1, 2.9], we shall describe the structure of $A_G(u)$ and $A_{\widetilde{G}}(u)$.

Assume that $G = Sp_{2n}$ with $p \neq 2$. Take $u \in C_{\lambda,\varepsilon}$. We consider the generator a_i corresponding to each λ_i. Then $A_G(u)$ is an abelian group generated by a_i such that $\varepsilon(\lambda_i) = 1$ under the condition that $a_i^2 = 1$ and that $a_i = a_j$ if $\lambda_i = \lambda_j$.

Next assume that $\widetilde{G} = O_N$ with $p \neq 2$. Take $u \in C_{\lambda,\varepsilon}$. Then $A_{\widetilde{G}}(u)$ is an abelian group generated by a_i, exactly by the same condition as the case of Sp_{2n}. Now $A_G(u)$ is the subgroup of $A_{\widetilde{G}}(u)$ of index 2 generated by a_ia_j for each $i \neq j$.

Next assume that $G = Sp_{2n}$ with $p = 2$. Take $u \in C_{\lambda,\varepsilon}$. Again we consider the generators a_i corresponding to λ_i. Then $A_G(u)$ is an abelian group generated by a_i such that $\varepsilon(\lambda_i) \neq 0$ under the condition that $a_i^2 = 1$ and that $a_i = a_j$ if $\lambda_i = \lambda_j$ or if $\lambda_i = \lambda_j + 1$ or if λ_i is even and $\lambda_i = \lambda_j + 2$.

Finally assume that $\widetilde{G} = O_{2n}$ with $p = 2$, and $G = SO_{2n}$. Take $u \in C_{\lambda,\varepsilon}$. Then $A_{\widetilde{G}}(u)$ is an abelian group generated by a_i, exactly by the same condition as the case of Sp_{2n} with $p = 2$. $A_G(u)$ is the subgroup of $A_{\widetilde{G}}(u)$ of index 2 generated by a_ia_j for $i \neq j$.

2.5. In what follows, we shall construct a normal form of unipotent elements in G^F. As a preliminary for this, we consider the case where $G = SO_{2n}$ with $p = 2$. So assume given a vector space V over $\mathbf{F}_q$ of dimension $N = 2n$ with a basis $e_1, \dots, e_N$, endowed with an alternating form f. We define an element $v \in GL(V)$ by $(v-1)e_j = e_{j-1}$ (with a convention $e_0 = 0$), and assume that v leaves f invariant. We consider the following condition on f.

(2.5.1) $$f(e_1, e_N) = 1,$$

(2.5.2) $$f(e_i, e_N) = 0 \quad \text{for } i = n+1, \dots, N,$$

(2.5.3) $$f(e_i, e_k) + f(e_{i+1}, e_k) + f(e_i, e_{k+1}) = 0 \quad \text{for } 0 \le i, k \le N-1.$$

Note that (2.5.3) is equivalent to the condition that v leaves f invariant. Also note that the conditions (2.5.1) $\sim$ (2.5.3) determines the alternating form f invariant by v uniquely. In fact, it follows from (2.5.2) and (2.5.3) that

(2.5.4) $$f(e_i, e_j) = 0 \quad \text{for } n+1 \le i, j \le N.$$

Also it follows from (2.5.1) and (2.5.3), we have

(2.5.5) $$f(e_i, e_j) = \begin{cases} 0 & \text{if } i + j \le N, \\ 1 & \text{if } i + j = N + 1. \end{cases}$$

Hence it is enough to show that $f(e_i, e_j)$ is determined for $1 \le i \le n$ and $n+1 \le j \le N$. By (2.5.5) we have $f(e_n, e_{n+1}) = 1$. Since $f(e_{n+1}, e_j) = 0$ for $j \ge n+1$, we have $f(e_n, e_j) = 1$ for $j \ge n+1$ by (2.5.3). Then $f(e_i, e_j) = f(e_i, e_{j-1}) + f(e_{i+1}, e_{j-1})$ is determined for $1 \le i \le n$ by induction on j $(n+1 \le j \le N)$.

We consider a quadratic form Q such that $Q(x+y) - Q(x) - Q(y) = f(x, y)$, which is left invariant by v. We have the following lemma.

Lemma 2.6. *Let the notations be as above. Assume that $Q(e_N) = 0$. Then Q is determined uniquely, which is non-degenerate of split type.*

Proof. Since Q is invariant by v, it is known by [Sp1, 6.10] that $Q(e_i) = f_(e_i, e_{i+1})$ for $i = 1, \dots, N-1$. Hence Q is determined uniquely by f and by the condition $Q(e_N) = 0$. It is easy to see that this Q actually gives rise to a quadratic form invariant by v. In order to show that Q is non-degenerate of split type, it is enough to see that there exists a basis $e'_1, \dots, e'_N$ of V satisfying the property

(2.6.1) $$Q(e'_i) = 0 \quad \text{for } i = 1, \dots, N,$$

$$f(e'_i, e'_j) = \begin{cases} 1 & \text{if } i + j = N + 1, \\ 0 & \text{otherwise.} \end{cases}$$

We show (2.6.1). We consider the square matrix $A = (f(e_i, e_{N-j+1}))_{1 \le i,j \le n}$ of degree n. By (2.5.5) A is a lower unitriangular matrix. For $k = 0, 1, \dots,$ we denote by A_k the principal minor matrix of A of degree 2^k. We have $A_0 = (1)$. We show that

(2.6.2) The matrix A_k has the following property; for k such that $2^{k+1} \le n$, we have

$$A_{k+1} = \begin{pmatrix} A_k & 0 \\ A_k & A_k \end{pmatrix}.$$

If $2^k < n < 2^{k+1}$, A is of the form

$$A = \begin{pmatrix} A_k & 0 \\ A'_k & A''_k \end{pmatrix},$$

where A'_k is the minor matrix of A_k of type $(n - 2^k, 2^k)$ consisting of the first $(n - 2^k)$-rows and all the columns, and A''_k is the principal minor matrix of A_k of degree $n - 2^k$.

In fact, assume that $2^{k+1} \le n$. By induction we may assume that (2.6.2) holds for $k-1$. Put $A = (a_{ij})$ with $a_{ij} = f(e_i, e_{N-j+1})$. Then by (2.5.3), we have

$$a_{i,j} = a_{i-1,j} + a_{i-1,j-1}. \tag{2.6.3}$$

By induction, we see that the last row of A_k is of the form $(1, \dots, 1)$. Hence (2.6.3) implies that the (2^k+1)-th row of A_{k+1} is of the form $(1, 0, \dots, 0, 1, 0, \dots, 0)$ (1 appears in the first and the (2^k+1)-th coordinates), which coincides with the first row of the matrix (A_k, A_k). Since the (2^k+2)-th row of A_{k+1} is determined by (2^k+1)-th row by (2.6.3), and so on, we see that the minor matrix of A_{k+1} of type $(2^k, 2^{k+1})$, consisting of the last 2^k-rows and all the columns, coincides with (A_k, A_k). (Note that since the last column of A_k is of the form ${}^t(0, \dots, 0, 1)$, the interaction between two A_k does not occur in this computation). Thus (2.6.2) holds for the case where $2^{k+1} \le n$. The case where $2^k < n < 2^{k+1}$ is dealt similarly.

For j such that $2^{a-1} < j \le 2^a$, we define a marked matrix $A^{(j)}$ as follows. In the matrix A_a, the (j,j) entry is contained in a minor matrix $A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, where the (j,j)-entry corresponds to the $(1,1)$-entry (resp. $(2,2)$-entry) of A_1 if j is odd (resp. even). We define a marked matrix $A_a^{(j)}$ by replacing the minor matrix A_1 in A_a by $A_1^\bullet$, where

$$A_1^\bullet = \begin{pmatrix} 1^\bullet & 0 \\ 1^\bullet & 1^\bullet \end{pmatrix} \quad \text{or} \quad A_1^\bullet = \begin{pmatrix} 1 & 0 \\ 1 & 1^\bullet \end{pmatrix}$$

according as (j,j) corresponds to $(1,1)$ or $(2,2)$ in A_1. In each of the matrices the marks $\bullet$ are attached to some entries in A_a. For example, for $2 < j \le 2^2$, $A_2^{(j)}$ is given as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1^\bullet & 0 \\ 1 & 1 & 1^\bullet & 1^\bullet \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1^\bullet \end{pmatrix}$$

according to the case where $j = 3$ or $j = 4$. For $k > a$, we define $A_k^{(j)}$ inductively as in (2.6.2) by replacing A_k by $A_k^{(j)}$, starting from $A_a^{(j)}$. Then we define the matrix $A^{(j)}$ for $2^k \le n < 2^{k+1}$ by replacing A'_k, A''_k by $(A_k^{(j)})', (A_k^{(j)})''$ which is defined similarly.

By a direct observation, we have

(2.6.4) The matrix $A^{(j)}$ has the following properties.

(i) In each row, the number of marked 1 is even except the j-th row, where the number is 1.
(ii) In each column containing the marked 1's, the entries except the marked 1 are all zero.

We now define, for $j = 1, \dots, n$, the vector e'_{N-j+1} by

$$e'_{N-j+1} = \sum_k e_{N-k+1},$$

where the sum is taken over $1 \le k \le n$ such that k-th column in $A^{(j)}$ contains a marked 1. It follows from (2.6.4) that we have

$$f(e_i, e'_{N-j+1}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \tag{2.6.5}$$

for $1 \le i, j \le n$.

We now consider the values of Q. Since Q satisfies the relation $Q(e_i) = f(e_i, e_{i+1})$, it follows, by (2.5.4) and (2.5.5) together with our assumption that $Q(e_N) = 0$, that

$$Q(e_i) = \begin{cases} 0 & \text{if } i \ne n, \\ 1 & \text{if } i = n. \end{cases} \tag{2.6.6}$$

Note that we have $e'_{n+1} = e_{n+1}$ by the previous computation. Put $e'_n = e_n + e_{n+1}$. Then we have

$$\begin{aligned} Q(e'_n) &= Q(e_n) + Q(e_{n+1}) + f(e_n, e_{n+1}) = 0, \\ f(e'_n, e'_{n+1}) &= f(e_n + e_{n+1}, e_n) = 1. \end{aligned} \tag{2.6.7}$$

Now put $e'_i = e_i$ for $i = 1, \dots, n-1$. Then by (2.5.4) and (2.5.5), together with (2.6.5) $\sim$ (2.6.7), we see that the basis $\{e'_1, \dots, e'_N\}$ satisfies the relation (2.6.1). The lemma is proved. □

2.7. Let G be as in 2.1. We assume that G^F is of split type. For each F-stable unipotent class C in G, we shall construct a normal form u, called a split element, in C^F. The G^F-conjugacy class of u is called the split class in C^F. In the case where $p = 2$, we construct u following [Sp1, II, 6.19]. First consider the case where

$G = Sp_{2n}$ with $p = 2$. Take a unipotent class $C_{\lambda,\varepsilon}$ of G. For each $j \ge 1$, put

$$n_j = \begin{cases} \lambda_j & \text{if } \varepsilon(\lambda_j) = 1, \\ \lambda_j + 1 & \text{if } \varepsilon(\lambda_j) = \varepsilon, \\ \lambda_j + 2 & \text{if } \varepsilon(\lambda_j) = 0. \end{cases} \tag{2.7.1}$$

We consider the vector space V_j over $\mathbf{F}_q$ of dimension n_j with basis $e^j_1, \dots, e^j_{n_j}$. Assume that a non-degenerate alternating form (= a symmetric bilinear form) f_j on V_j is given. Let $H(V_j)$ be the subgroup of $GL(V_j)$ consisting of $g \in GL(V_j)$ which leaves the form f_j invariant. Put $H = Sp_{n_j}(k)$. Then $H(V_j)$ is regarded as a subgroup H^F of H under the natural $\mathbf{F}_q$-structure F on H.

One can construct $v_j \in H(V_j)$ such that $(v_j - 1)e^j_i = e^j_{i-1}$ for $i = 1, \dots, n_j$ (under the convention that $e^j_0 = 0$) with respect to the alternating form f_j satisfying the property as given in 2.5 (with $f = f_j, N = n_j$).

We also note that

(2.7.2) The image of $v_j \in Z_H(v_j)$ to $A_H(v_j)$ gives a generator $\bar{a}_j$ of $A_H(v_j)$, where $A_H(v_j)$ is of order 1 or 2.

For each $h = \lambda_j$, we shall construct a vector space M_j over $\mathbf{F}_q$ with an alternating form f_j, and $u_j \in H(M_j)$ as follows.

(a) $\varepsilon(h) = 1$. In this case, h is even and c_h is odd or even. We put $M_j = V_j$ and $u_j = v_j$. Thus $u_j \in H(M_j)$ with respect to $f^0_j = f_j$ on M_j. Since $f^0_j(e^j_{n_j}, e^j_1) = 1$, we see that the function $x \mapsto f^0_j((u_j - 1)^{h-1}x, x)$ is non-trivial on $\operatorname{Ker}(u_j - 1)^h$.

(b) $\varepsilon(h) = \omega$. In this case h is odd and c_h is even. Assume that $\lambda_j = \lambda_{j-1} = h$, and let (V_j, f_j) and (V_{j-1}, f_{j-1}) be as before. Note that $\dim V_j = \dim V_{j-1} = n_j = h+1$ by (2.7.1). We define an alternating form f' on $V_j \oplus V_{j-1}$ by the condition that $f'|_{V_j} = f_j$, $f'|_{V_{j-1}} = f_{j-1}$ and that $V_j \perp V_{j-1}$. Let L be a line in $V_j \oplus V_{j-1}$ generated by $e^j_1 + e^{j-1}_1$. Then L is an isotropic line by (2.5.5) and we put $M_j = L^\perp/L$. We have $\dim M_j = 2h$. Since $v_j(e^j_1) = e^j_1, v_{j-1}(e^{j-1}_1) = e^{j-1}_1$, we see that $v_j + v_{j-1}$ fixes L, and so it induces a linear transformation on M_j, which we denote by u_j. The form f' induces an alternating form f^0_j. We have $u_j \in H(M_j)$.

By (2.5.1) and (2.5.5), $L^\perp$ has a basis

$$e^j_n + e^{j-1}_n, e^j_{n-1}, e^{j-1}_{n-1}, \dots, e^j_2, e^{j-1}_2, e^j_1 + e^{j-1}_1.$$

Hence$L^\perp/L$ has a basis

$$\bar{e}^j_n + \bar{e}^{j-1}_n, \bar{e}^j_{n-1}, \bar{e}^{j-1}_{n-1}, \dots, \bar{e}^j_2, \bar{e}^{j-1}_2, \bar{e}^j_1 = \bar{e}^{j-1}_1,$$

where $\bar{e}^j_i, \bar{e}^{j-1}_i$ denote the image of e^j_i, e^{j-1}_i on $(V_j \oplus V_{j-1})/L$.

(c) $\varepsilon(h) = 0$. In this case, h is even and c_h is even. Assume that $\lambda_j = \lambda_{j-1}$, and consider the vector spaces V_j and V_{j-1} as before. By (2.7.1), we have $n_j = \dim V_j = \dim V_{j-1} = h + 2$. We consider the alternating form f' on $V_j \oplus V_{j-1}$ as before. Let

N be the subspace of $V_j \oplus V_{j-1}$ spanned by $e_1^j + e_1^{j-1}$ and $e_2^j + e_2^{j-1}$. Then N is an isotropic subspace of $V_j \oplus V_{j-1}$ of dimension 2, and we put $M_j = N^\perp/N$. We have $\dim M_j = 2h$. The alternating form f' induces an alternating form f_j^0 on M_j. Now $v_j + v_{j-1}$ stabilized N, and so induces a linear transformation on M_j which we denote by u_j. We see that $u_j \in H(M_j)$.

By (2.5.1) and (2.5.5), $N^\perp$ has a basis

$$e_n^j + e_n^{j-1}, e_{n-1}^j + e_{n-1}^{j-1}, e_{n-2}^j, e_{n-2}^{j-1}, \dots, e_1^j, e_1^{j-1}.$$

Hence $N^\perp/N$ has a basis

$$\bar{e}_n^j + \bar{e}_n^{j-1}, \bar{e}_{n-1}^j + \bar{e}_{n-1}^{j-1}, \bar{e}_{n-2}^j, \bar{e}_{n-2}^{j-1}, \dots, \bar{e}_3^j, \bar{e}_3^{j-1}, \bar{e}_2^j = \bar{e}_2^{j-1}, \bar{e}_1^j = \bar{e}_1^{j-1},$$

where $\bar{e}_i^j, \bar{e}_i^{j-1}$ denotes the image of e_i^j on $(V_j \oplus V_{j-1})/N$. The action of u_j on this basis is easily described, and by using the formulas in 2.5, one can check that $f_j^0((u_j-1)^{h-1}x, x) = 0$ for all $x \in \operatorname{Ker}(u_j-1)^h$. For example,

$$f_j^0(\bar{e}_n^j + \bar{e}_n^{j-1}, \bar{e}_3^j + \bar{e}_3^{j-1}) = 2f_j(e_n^j, e_3^j) = 0,$$

and the other cases are dealt similarly.

We now define a vector space $\bar{V}$ as $\bar{V} = \bigoplus_j M_j$ so that $\dim \bar{V} = 2n$, and let $f = \bigoplus_j f_j^0$ be the alternating form obtained from f_j^0. Put $\bar{u} = \prod_j u_j \in H(\bar{V})$. It follows from the previous construction, we have

(2.7.3) $H(\bar{V})$ can be identified with G^F, and under this isomorphism, the element $u \in G^F$ corresponding to $\bar{u}$ gives an element in $C_{\lambda,\varepsilon}^F$. We call u a split element in $C_{\lambda,\mathcal{E}}^F$.

The structure of the group $A_G(u)$ is also described as follows (cf. [Sp1, II, 6.19]) Take λ_j such that $\varepsilon(\lambda_j) \neq 0$. We denote by $\bar{a}_j$ an automorphism of $\bigoplus_{k\ge1} V_k$ defined by $v \mapsto v_j(v)$ for $v \in V_j$, and $v \mapsto v$ for $v \in V_k$ such that $k \neq j$. Then $\bar{a}_j$ induces an automorphism on $\bar{V}$ commuting with $\bar{u}$, which we denote also by $\bar{a}_j$. It is checked that $\bar{a}_j \in H(\bar{V})$, and so this gives an element of $Z_G(u)$. The image of $\bar{a}_j$ on $A_G(u)$ coincides with the generator a_j stated in 1.4. In particular, we see that F acts trivially on $A_G(u)$. This implies, since $A_G(u)$ is abelian, that

(2.7.4) For any $u' \in C_{\lambda,\varepsilon}^F$, F acts trivially on $A_G(u')$.

2.8. Next we consider the case where $G = SO_{2n}$ with $p = 2$. Take a unipotent class $C_{\lambda,\varepsilon}$ of G, and let n_j be as in (2.7.1). We consider the vector space V_j over $\mathbf{F}_q$ with basis $e_1^j, \dots, e_{n_j}^j$ and a unipotent element $v_j \in GL(V_j)$ as in 2.7. By Lemma 2.6, one can construct a split quadratic form Q_j on V_j which is invariant by v_j. Let $H(V_j)$ be the subgroup of $GL(V_j)$ consisting of g which leaves Q_j invariant. Let $\widetilde{H} = O_{n_j}(k)$. Since Q_j is of split type, $H(V_j)$ can be identified with the subgroup $\widetilde{H}^F$ of $\widetilde{H}$, where F is a split Frobenius map. Then by a similar argument as in 2.7, we obtain $u_j \in H(M_j)$ for each case (a), (b) or (c), where $H(M_j)$ is the group of invariants with respect to the induced quadratic from Q_j^0. Note that the explicit computation in the proof of Lemma 2.6 shows that Q_j^0 is of split type. As in 2.7, we define a vector space $\bar V = \bigoplus_j M_j$ and $Q = \bigoplus_j Q_j^0$, and put $\bar u = \prod_j u_j \in H(\bar V)$. We have

(2.8.1) Let $\widetilde G = O_{2n}^+$. Then $H(\bar V)$ can be identified with $\widetilde G^F$ with split Frobenius map F, and under this isomorphism, the element $u \in \widetilde G^F$ corresponding to $\bar u$ gives an element in $\widetilde C_{\lambda,\varepsilon}^F$. In the ordinary case $u \in C_{\lambda,\varepsilon}^F$. In the exceptional case we have $u \in (C'_{\lambda,\varepsilon})^F$ and $\sigma(u) \in (C''_{\lambda,\varepsilon})^F$, where σ is the graph automorphism on SO_{2n}. We call u and $\sigma(u)$ the split elements in $\widetilde C_\lambda^F$.

2.9. Next we consider the case where $G = Sp_{2n}$ or SO_N with $p \ne 2$. We assume that F is a split Frobenius map. Let $\mathfrak g$ be the Lie algebra of G. Since the unipotent classes in G are in bijection with the nilpotent orbits in $\mathfrak g$ with $\mathbf F_q$-structure, we consider the normal form of nilpotent orbits instead of unipotent classes. Let $\mathcal O_{\lambda,\varepsilon}$ be the nilpotent orbit in $\mathfrak g$ corresponding to the unipotent class $C_{\lambda,\varepsilon}$ in G. For each λ_j, we construct a vector space M_j over $\mathbf F_q$ and a nilpotent transformation X_j on M_j as follows.

(a) $\varepsilon(\lambda_j) = 1$. We consider a vector space M_j of dimension $h = \lambda_j$ with basis $e_1^j, \dots, e_h^j$. We define a non-degenerate alternating form (resp. a symmetric bilinear form) f_j on M_j in the case where $G = Sp_{2n}$ (resp. SO_N) by

(2.9.1) $$f_j(e_{h-i+1}^j, e_i^j) = (-1)^{\delta_j - i} \quad \text{for } i = 1, \dots, h,$$

where
$$\delta_j = \begin{cases} \lambda_j/2 + j & \text{if } G = Sp_{2n}, \\ (\lambda_j - 1)/2 + j & \text{if } G = SO_N. \end{cases}$$
We put the value of f_j zero for any other pair of the basis. We define a nilpotent transformation X_j on M_j by $X_j(e_i^j) = e_{i-1}^j$ (under the convention that $e_0 = 0$). Then $X_j \in \mathfrak h(M_j)$, where $\mathfrak h(M_j)$ is the subalgebra of $\mathfrak{gl}(M_j)$ consisting of X such that $f_j(Xx, y) + f_j(x, Xy) = 0$ for $x, y \in M_j$.

(b) $\varepsilon(\lambda_j) = \omega$. In this case c_h is even for $\lambda_j = h$. We assume that $\lambda_j = \lambda_{j-1}$. We consider a vector space M_j of dimension $2h = 2\lambda_j$ with basis $e_1^j, \dots, e_h^j, e_1^{j-1}, \dots, e_h^{j-1}$. We define an alternating form (resp. a symmetric bilinear form) f_j on M_j in the case where $G = Sp_{2n}$ (resp. $G = SO_N$) by

(2.9.2) $$f_j(e_{h-i+1}^j, e_i^{j-1}) = \varepsilon f_j(e_i^{j-1}, e_{h-i+1}^j) = (-1)^{i-1} \quad \text{for } i = 1, \dots, h,$$

where $\varepsilon = -1$ (resp. $\varepsilon = 1$) if $G = Sp_{2n}$ (resp. $G = SO_N$). We put the values of f_j zero for any other pair of the basis. We define a nilpotent transformation X_j on M_j by $X_j e_i^j = e_{i-1}^j, X_j e_i^{j-1} = e_{i-1}^{j-1}$ (we put $e_0^j = e_0^{j-1} = 0$ as before). Then $X_j \in \mathfrak h(M_j)$.

We define a vector space $\bar V$ by $\bar V = \bigoplus_j M_j$ so that $\dim \bar V = N$, and let $f = \sum_j f_j$ be an alternating form (resp. a symmetric bilinear form) on $\bar V$ obtained from f_j. Put $\bar X = \bigoplus_j X_j \in \mathfrak h(\bar V)$. Then it is known by [SS], (2.9.3) $\mathfrak{h}(\bar{V})$ can be identified with $\mathfrak{g}^F$. Under this correspondence $\bar{X}$ gives an element $X \in \mathcal{O}^F_{\lambda,\varepsilon}$, which we call a split element in $\mathcal{O}^F_{\lambda,\varepsilon}$. X also determines the G^F-class in $C^F_{\lambda,\varepsilon}$, which we call the split class in $C^F_{\lambda,\varepsilon}$.

2.10. We consider the case where $G = SO_{2n}$ with non-split Frobenius F (for arbitrary p). Let F_0 be the split Frobenius map. Then one can write $F = F_0\sigma$ with the graph automorphism σ. We may choose $s \in \widetilde{G} \setminus G$ such that $\sigma = \operatorname{ad} s$, and fix it for all. Let $u \in G^{F_0}$ be a split element in $C_{\lambda,\varepsilon}$. Since $A_{\widetilde{G}}(u) \neq A_G(u)$, there exists $a \in A_{\widetilde{G}}(u) \setminus A_G(u)$. Let $\dot{a} \in Z_{\widetilde{G}}(u)$ be a representative of a. Since $[\widetilde{G} : G] = 2$, there exists $g \in G$ such that $\dot{a} = gs$. It follows that ${}^{gF}u = u$. Now there exists $\alpha \in G$ such that $\alpha^{-1}F(\alpha) = g$, and we have $u' = \alpha u \alpha^{-1} \in C^F_{\lambda,\varepsilon}$. It is easy to see that the G^F-conjugacy class of u' is uniquely determined by $\dot{a} \in Z_{\widetilde{G}}(u)$. For the exceptional case, we have $u' \in (C'_{\lambda,\varepsilon})^F$ and $\sigma(u') \in (C''_{\lambda,\varepsilon})^F$. In what follows, we fix a split element $u' \in C^F_{\lambda,\varepsilon}$ as follows.

(2.10.1) Let $G = SO^-_{2n}$. Let u be the split element in $C^{F_0}_{\lambda,\varepsilon}$ or $(C'_{\lambda,\varepsilon})^{F_0}$. We choose $a = a_i \in A_{\widetilde{G}}(u)$ such that $\varepsilon(\lambda_i) = 1$ (resp. $\varepsilon(\lambda_i) \neq 0$) in the case where $p \neq 2$ (resp. $p = 2$) and that λ_i is minimal under this condition. Take a representative $\dot{a}_i \in Z_{\widetilde{G}}(u)$ as in 2.7, and define $u' \in C^F_{\lambda,\varepsilon}$ by using $\dot{a}_i$. (In the exceptional case, define $u' \in (C'_{\lambda,\varepsilon})^F$ and $\sigma(u') \in (C''_{\lambda,\varepsilon})^F$.) We call u' the split element in $C^F_{\lambda,\varepsilon}$.

Under the notation above, $aF_0 = gF$ acts trivially on $A_G(u)$ by (2.7.4). It follows that F acts trivially on $A_G(u')$. Since $A_G(u')$ is abelian, we have

(2.10.2) The statement (2.7.4) holds also for the case where F is of non-split type.

Remark 2.11. The definition of the split elements for Sp_{2n} or SO_N (for $p \neq 2$) in 2.9, 2.10 is essentially the same as the one used in [S1, 3.3, 3.7] (where it is called distinguished elements). Note that in the case where $q \equiv 1 \pmod 4$, δ_j can be removed in the formula (2.9.1) by a suitable base change. Also note that the definition of f_j involves the case where $\varepsilon(h) = 1$ and c_h is even, which is necessary for later discussions, though these cases are ignored in [S1].

In the case of non-split groups with $p \neq 2$, our definition of split elements is not the same as in [S1, 3.7], where it is defined by using $a = a_i$ corresponding to λ_i of maximal length instead of minimal length. This is not essential, but the definition here is more convenient since it produces preferred extensions of $\mathcal{W}$-modules as will be seen in Theorem 4.3. (The elements defined in [S1, 3.7] do not necessarily produce them).

3. Generalized Springer correspondence

3.1. We review here the generalized Springer correspondence for classical groups following [L1] and [LS]. In what follows, we denote by W_n the Weyl group of type C_n, and by W_n' the Weyl group of type D_n. Throughout the whole cases, for a given $(L, C, \mathcal{E}) \in \mathcal{M}_G$, the cuspidal pair $(C, \mathcal{E})$ is uniquely determined. So, we just describe L which has a cuspidal pair.

(a) Let $G = Sp_{2n}$ with $p \neq 2$. Then $(L, C, \mathcal{E}) \in \mathcal{M}_G$ if and only if L is of type C_m for some m of the form $m = \frac{1}{2}d(d-1)$ with $d \geq 1$. We have $N_G(L)/L \simeq W_{n-\frac{1}{2}d(d-1)}$. Since the set $\{d(d-1) \mid d \geq 1\}$ coincides with the set $\{d(d-1) \mid d \in \mathbf{Z}, d : \text{odd}\}$, the generalized Springer correspondence (1.1.3) is given by a bijection

$$\mathcal{N}_G \leftrightarrow \coprod_{\substack{d \in \mathbf{Z} \\ d \text{ odd}}} (W_{n-\frac{1}{2}d(d-1)})^\wedge.$$

(b) Let $G = SO_N$ with $p \neq 2$. Then $(L, C, \mathcal{E}) \in \mathcal{M}_G$ if and only if L is of type B_m (resp. D_m) for some m such that $m = \frac{1}{2}(d^2 - 1)$ (resp. $m = \frac{1}{2}d^2$) and that $d \equiv N \pmod 2$ in the case where N is odd (resp. N is even) with $m \geq 0$. We have $N_G(L)/L \simeq W_{(N-d^2)/2}$ if $m \geq 1$, and $N_G(L)/L \simeq W_n$ (resp. W_n') if $m = 0$, namely if L is a maximal torus T, in the case where N is odd (resp. N is even). Thus the generalized Springer correspondence is given by a bijection

$$\mathcal{N}_G \leftrightarrow \coprod_{\substack{d \geq 1 \\ d \text{ odd}}} (W_{(N-d^2)/2})^\wedge \qquad (N \text{ odd}),$$

$$\mathcal{N}_G \leftrightarrow W_n' \coprod \Big(\coprod_{\substack{d > 0 \\ d \text{ even}}} (W_{(N-d^2)/2})^\wedge \Big) \qquad (N \text{ even}).$$

(c) Let $G = Sp_{2n}$ with $p = 2$. Then $(L, C, \mathcal{E}) \in \mathcal{M}_G$ if and only if L is of type C_m for some m of the form $m = d(d-1)$ with $d \geq 1$. We have $N_G(L)/L \simeq W_{n-d(d-1)}$. Hence as in the case (a), the generalized Springer correspondence is given by

$$\mathcal{N}_G \leftrightarrow \coprod_{\substack{d \in \mathbf{Z} \\ d \text{ odd}}} (W_{n-d(d-1)})^\wedge.$$

(d) Let $G = SO_{2n}$ with $p = 2$. Then $(L, C, \mathcal{E}) \in \mathcal{M}_G$ if and only if L is of type D_m for some m of the form $m = d^2$ with $d \geq 0$, even. We have $N_G(L)/L \simeq W_{n-d^2}$ if $d \geq 1$ and $N_G(L)/L \simeq W_n'$ if $d = 0$, namely if L is a maximal torus of G. Hence the generalized Springer correspondence is given by

$$\mathcal{N}_G \leftrightarrow W_n' \coprod \Big(\coprod_{\substack{d > 0 \\ d \text{ even}}} (W_{n-d^2})^\wedge \Big).$$

3.2. The generalized Springer correspondence for classical groups is described in terms of symbols. We review the notion of symbols following [L1], [LS]. Let $r, s \in \mathbf{Z}_{\ge 1}$, $d \in \mathbf{Z}$. For each integer $n \ge 1$ let $\widetilde{X}^{r,s}_{n,d}$ be the set of all ordered pairs (A, B) where $A = \{a_1, \dots, a_{m+d}\}$, $B = \{b_1, \dots, b_m\}$ (for some m) are subsets of $\mathbf{Z}_{\ge 0}$ satisfying the following conditions.

$$
\begin{aligned}
a_i - a_{i-1} &\ge r + s \quad (1 < i \le m + d),\\
b_i - b_{i-1} &\ge r + s \quad (1 < i \le m),\\
b_1 &\ge s,
\end{aligned}
\tag{3.2.1}
$$

$$\sum a_i + \sum b_i = n + (r + s)(m + [d/2])(m + d - [d/2]) - r(m + [d/2]).$$

(In the case where $r + s = 0$, A or B contains elements with multiplicities. In that case, we regard it as a sequence of integers.)

Note that if $r = s = 1$ and d is odd the fourth condition is written as

$$\sum a_i + \sum b_i = n + \frac{1}{2}(2m + d)(2m + d - 1), \tag{3.2.2}$$

and if $r = 2, s = 0$ it is written as

$$\sum a_i + \sum b_i = n + \frac{1}{2}((2m + d - 1)^2 - 1). \tag{3.2.3}$$

Let $X^{r,s}_{n,d}$ be the set of equivalence classes on $\widetilde{X}^{r,s}_{n,d}$ for the equivalence relation generated by

$$(A, B) \sim \big(\{0\} \cup (A + r + s), \{s\} \cup (B + r + s)\big),$$

where $A + r + s$ denotes the set $\{a_1 + (r + s), \dots, a_{m+d} + (r + s)\}$ and so on for B. We put

$$X^{r,s}_n = \coprod_{d \text{ odd}} X^{r,s}_{n,d}.$$

An element in $X^{r,s}_{n,d}$ is called an (r, s)-symbol of rank n and defect d, which we also denote by (A, B).

We consider the special case where $s = 0$. In that case, $(A, B) \mapsto (B, A)$ defines a bijection from $X^{r,0}_{n,d}$ to $X^{r,0}_{n,-d}$ and so induces an involution of each of the following sets,

$$X^{r,0}_{n,\text{ even}} = \coprod_{d \text{ even}} X^{r,0}_{n,d}, \quad X^{r,0}_{n,\text{ odd}} = \coprod_{d \text{ odd}} X^{r,0}_{n,d}.$$

Let $Y^r_{n,\text{ odd}}$ (resp. $Y^r_{n,\text{ even}}$) be the set obtained as the quotient of $X^{r,0}_{n,\text{ odd}}$ (resp. $X^{r,0}_{n,\text{ even}}$) by this involution, with the convention that the symbol invariant by the involution, i.e., the symbol (A, A) which we call the degenerate symbol, is counted twice. For $d \ge 0$, the image of $X^{r,0}_{n,d}$ in $Y^r_{n,\text{ odd}}$ or $Y^r_{n,\text{ even}}$ is denoted by $Y^r_{n,d}$. One can regard the element in $Y^r_{n,d}$ as a symbol (A, B) in $X^{r,0}_{n,d}$ considered as an unordered pair.

3.3. The set $W_n^\wedge$ is parametrized by ordered pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. For a fixed $d \in \mathbf{Z}$, one can express the partitions α, β as $\alpha : \alpha_1 \le \alpha_2 \le \cdots \le \alpha_{m+d}$, $\beta : \beta_1 \le \beta_2 \le \cdots \le \beta_m$ for a suitable m, by allowing 0 in the entries. Then $(A, B) \in \widetilde{X}^{0,0}_{n,d}$ for $A = \{\alpha_1, \dots, \alpha_{m+d}\}$, $B = \{\beta_1, \dots, \beta_m\}$, and this induces a well-defined bijection between $W_n^\wedge$ and $X^{0,0}_{n,d}$. The same map induces a bijection between $W_n^\wedge$ and $Y^0_{n,d}$ if $d \ge 1$. On the other hand, the set $(W'_n)^\wedge$ is parametrized by unordered pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$, under the convention that (α, α) is counted twice. Thus in a similar way as above, we have a natural bijection between $(W'_n)^\wedge$ and $Y^0_{n,0}$.

For a given r, s, d, we define a symbol $\Lambda^{r,s}_d = (A, B)$ as follows.

$$\begin{cases} A = \{0, (r+s), \dots, (d-1)(r+s)\}, \quad B = \emptyset & \text{if } d > 0, \\ A = \emptyset, \quad B = \{s, s + (r+s), \dots, s + (-d-1)(r+s)\} & \text{if } d < 0, \\ A = \emptyset, \quad B = \emptyset & \text{if } d = 0. \end{cases}$$

It is easy to see that $\Lambda^{r,s}_d \in X^{r,s}_{n_0,d}$ with $n_0 = (r+s)[d/2](d - [d/2]) - s[d/2]$, and that the set $X^{r,s}_{n_0,d}$ consists of a unique element $\Lambda^{r,s}_d$. In the case where $s = 0$, let Λ^r_d be the image of $\Lambda^{r,0}_d$ under the map $X^{r,0}_{n_0,d} \to Y^r_{n_0,d}$. For each $d \in \mathbf{Z}$, one can define a map

$$X^{0,0}_{n-n_0,d} \to X^{r,s}_{n,d}, \quad \Lambda \mapsto \Lambda + \Lambda^{r,s}_d, \tag{3.3.1}$$

which gives a bijection $X^{r,s}_{n-n_0,d} \simeq X^{r,s}_{n,d}$. (Note, in general, that the sum of two symbols Λ, Λ' with the same defect is defined by choosing representatives $\Lambda = (A, B), \Lambda' = (A', B')$ such that $|A| + |B| = |A'| + |B'|$, namely of the same shape, and then by adding entry-wise.) Similarly for each $d \ge 0$, one can define a bijection

$$Y^0_{n-n_0,d} \to Y^r_{n,d}, \quad \Lambda \mapsto \Lambda + \Lambda^r_d. \tag{3.3.2}$$

Combining (3.3.1) with the bijection $W^\wedge_{n-n_0} \simeq X^{0,0}_{n-n_0,d}$ above, we have a bijection

$$W^\wedge_{n-n_0} \to X^{r,s}_{n,d}. \tag{3.3.3}$$

Similarly, combining (3.3.2) with the bijections $W^\wedge_{n-n_0} \simeq Y^0_{n-n_0,d}$ for $d > 0$ and $(W'_{n-n_0})^\wedge \simeq Y^0_{n-n_0,0}$, we have bijections

$$\begin{aligned} W^\wedge_{n-n_0} &\to Y^r_{n,d} \quad (d > 0), \\ (W'_{n-n_0})^\wedge &\to Y^r_{n,d} \quad (d = 0). \end{aligned} \tag{3.3.4}$$

3.4. A symbol $(A, B) \in X^{r,s}_{n,d}$ is said to be distinguished if $d = 0$ or 1, and

$$\begin{aligned} &a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_m \le b_m && (d = 0), \\ &a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_m \le b_m \le a_{m+1} && (d = 1). \end{aligned}$$

A symbol $(A, B) \in Y^r_{n,d}$ for $d \geq 0$ is said to be distinguished if it is an image of a distinguished symbol in $X^{r,0}_{n,d}$.

Assume that $r \geq 1$. The two symbols $(A, B), (A', B') \in X^{r,s}_n$ are said to be similar if $A \cup B = A' \cup B', A \cap B = A' \cap B'$, namely under some shift, $A \cup B$ coincides with $A' \cup B'$ with multiplicities. This defines an equivalence relation on the set $X^{r,s}_n$, and an equivalence class is called a similarity class in $X^{r,s}_n$. A similarity class in $Y^r_{n,\text{ even}}$ or $Y^r_{n,\text{ odd}}$ is defined in a similar way. A similarity class in $Y^r_{n,\text{ even}}$ containing (A, A) is called a degenerate class, which consists of two copies of (A, A). It is easy to see that each (non-degenerate) similarity class contains a unique distinguished symbol.

It is known by [L1], [LS] that a similarity class in $X^{r,s}_n$, $Y^r_{n,\text{ even}}$ or $Y^r_{n,\text{ odd}}$ is in a natural way regarded as a vector space over $\mathbf{F}_2$ as follows. Let $\Lambda = (A, B)$ be a distinguished symbol in a similarity class $\mathcal{C}$ in $X^{r,s}_n, Y^r_{n,\text{ even}}$ or $Y^r_{n,\text{ odd}}$. We assume that $A \neq B$ if $(A, B) \in Y^r_{n,\text{ even}}$, and put $S = (A \cup B) \backslash (A \cap B)$. Then $S \neq \emptyset$ and it is written as $S = \{c_1, c_2, \dots, c_t\}$ in an increasing order. A non-empty subset $I = \{c_i, c_{i+1}, \dots, c_j\}$ of S is called an interval if $c_{k+1} - c_k < r + s$ for $i \leq k < j$ and it is maximal with respect to this condition. We say that c_i is the tail of I. An interval is called an initial interval if $c_i < s$. Hence the initial interval exists only in the case where $s > 0$, and in that case, it exists uniquely after some shift. S is a disjoint union of intervals.

Assume that $\mathcal{C} \subset X^{r,s}_n$. Let I be an interval which is not initial with the tail c. If $c \in A$ (resp. $c \in B$), then there exists a unique $(A', B') \in \mathcal{C}$ such that $c \in B$ (resp. $c \in A$) and that $A \cap J = A' \cap J, B \cap J = B' \cap J$ for all other intervals J. This means that (A', B') is obtained from (A, B) by permuting the entries in the interval I. All the symbols in $\mathcal{C}$ are obtained from (A, B) by permuting the entries in certain intervals. Let $\mathcal{I}$ be the set of non-initial intervals in S and $\mathcal{P}(\mathcal{I})$ the set of all subsets of $\mathcal{I}$. The above argument shows that $\mathcal{C}$ is in bijection with the set $\mathcal{P}(\mathcal{I})$, which has a natural structure of $\mathbf{F}_2$ vector space with origin Λ and is denoted by $V^{r,s}_\Lambda$. In the case where $\mathcal{C} \subset Y^r_{n,\text{ even}}$ or $\mathcal{C} \subset Y^r_{n,\text{ odd}}$ ($\mathcal{C}$: non-degenerate), $\mathcal{C}$ is in bijection with the quotient set of $\mathcal{P}(\mathcal{I})$ under the relation $\mathcal{K} \sim \mathcal{I} \backslash \mathcal{K}$ for $\mathcal{K} \in \mathcal{P}(\mathcal{I})$. Hence $\mathcal{C}$ is identified with the $\mathbf{F}_2$ vector space $V^{r,0}_\Lambda / L$, where L is a line generated by $\mathcal{I} \in \mathcal{P}(\mathcal{I})$, which we denote by V^r_Λ.

3.5. Let G be as in 3.1. We associate the sets $X^{1,1}_n, Y^2_{n,\text{ even}}, Y^2_{n,\text{ odd}}$ for $Sp_{2n}, SO_{2n}, SO_{2n+1}$ with $p \neq 2$ and $X^{2,2}_n, Y^4_{n,\text{ even}}$ for Sp_{2n}, SO_{2n} with $p = 2$. Recall n_0 in 3.3.

(a) The case $X^{1,1}_n$. We have $r = s = 1$, and $n_0 = \frac{1}{2}d(d-1)$ for odd d. Hence (3.3.3) implies a bijection

$$X^{1,1}_n \leftrightarrow \coprod_{\substack{d \in \mathbf{Z} \\ d \text{ odd}}} (W_{n - \frac{1}{2}d(d-1)})^\wedge.$$

(b) The case $Y^2_{n,\text{ even}}$ or $Y^2_{n,\text{ odd}}$. We have $r = 2, s = 0$, and $n_0 = \frac{1}{2}d^2$ for even d and $n_0 = \frac{1}{2}(d^2 - 1)$ for odd d. Hence (3.3.4) implies bijections

$$Y^2_{n,\text{ odd}} \leftrightarrow \coprod_{\substack{d\ge 1\\ d\text{ odd}}} (W_{n-\frac{1}{2}(d^2-1)})^\wedge,$$

$$Y^2_{n,\text{ even}} \leftrightarrow (W'_n)^\wedge \coprod \Big(\coprod_{\substack{d>0\\ d\text{ even}}} (W_{n-\frac{1}{2}d^2})^\wedge \Big).$$

(c) The case $X^{2,2}_n$. We have $r = s = 2$, and $n_0 = d(d-1)$ for odd d. Hence (3.3.3) implies a bijection

$$X^{2,2}_n \leftrightarrow \coprod_{\substack{d\in\mathbf{Z}\\ d\text{ odd}}} (W_{n-d(d-1)})^\wedge.$$

(d) The case $Y^4_{n,\text{ even}}$. We have $r = 4, s = 0$, and $n_0 = d^2$ for even d. Hence (3.3.4) implies a bijection

$$Y^4_{n,\text{ even}} \leftrightarrow (W'_n)^\wedge \coprod \Big(\coprod_{\substack{d>0\\ d\text{ even}}} (W_{n-d^2})^\wedge \Big).$$

3.6. Let $X = X^{1,1}_n, Y^2_{n,\text{ odd}}, Y^2_{n,\text{ even}}$ or $X^{2,2}_n, Y^4_{n,\text{ even}}$ according as $G = Sp_{2n}$, SO_{2n+1}, SO_{2n} with $p \ne 2$, or $G = Sp_{2n}, SO_{2n}$ with $p = 2$. In view of the bijections in 3.5 and the discussion in 3.1, the generalized Springer correspondence can be described by giving a bijection between $\mathcal{N}_G$ and X. By [L1], [LS], this bijection is given explicitly in such a way that the set $G_{\text{uni}}/\!\sim$ of unipotent classes in G is in bijection with the set $X/\!\sim$ of similarity classes in X. In what follows, we define a map $\rho : G_{\text{uni}}/\!\sim \to X/\!\sim$ by associating a distinguished symbol $\varLambda = \rho(C) \in X$ for each unipotent class C in G.

(a) $G = Sp_{2n}$ with $p \ne 2$. Let C_λ be a unipotent class of G as in 2.2, where λ is a partition of $2n$. We express λ as $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2m}$ for some m, by allowing 0 in the entries if necessary. We divide the sequence $\{\lambda_1, \lambda_2, \dots, \lambda_{2m}\}$ into the union of blocks as follows. If λ_i is even, let $\{\lambda_i\}$ be a block. If $\lambda_i = h$ is odd, the sequence $A_h = \{\lambda_k \mid \lambda_k = h\}$ consisting of even elements, which we write as $\{\lambda_a, \lambda_{a+1}, \dots, \lambda_b\}$ for some $b > a$. Then we divide A_h into a disjoint union of two elements blocks $\{\lambda_a, \lambda_{a+1}\} \cup \{\lambda_{a+2}, \lambda_{a+3}\} \cup \cdots \cup \{\lambda_{b-1}, \lambda_b\}$. We define a sequence $\nu_1, \dots, \nu_{2m}$ as follows. Put

$$\begin{cases} \nu_i = \lambda_i/2 + i & \text{if } \{\lambda_i\} \text{ is a block,} \\ \nu_i = \nu_{i+1} = (\lambda_i + 1)/2 + i & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block,} \end{cases}$$

and put $A = \{0, \nu_2, \nu_4, \dots, \nu_{2m}\}$, $B = \{\nu_1, \nu_3, \dots, \nu_{2m-1}\}$. Then $\varLambda = (A, B)$ gives rise to a distinguished symbol in $X^{1,1}_n$, which is independent of the choice of m, and

$C_\lambda \mapsto \Lambda$ gives the required bijection ρ. Actually, the map ρ was defined in [L1, 11.6]. Although the definition given there is not the same as ours, it is easily checked that they coincide with each other.

(b) $G = SO_N$ with $p \ne 2$. Let C_λ be a unipotent class of G as in 2.2, where λ is a partition of N. We choose M such that $M \equiv N \pmod 2$ and express λ as $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_M$. We divide the sequence $\{\lambda_1, \lambda_2, \dots, \lambda_M\}$ into the union of blocks as follows. If λ_i is odd, let $\{\lambda_i\}$ be a block. If $\lambda_i = h$ is even, the sequence $A_h = \{\lambda_k \mid \lambda_k = h\}$ consists of even elements. As in the case (a), we divide A_h as a disjoint union of two elements blocks, $\{\lambda_a, \lambda_{a+1}\} \cup \{\lambda_{a+2}, \lambda_{a+3}\} \cup \cdots \cup \{\lambda_{b-1}, \lambda_b\}$. We define a sequence $\nu_1, \nu_2, \dots, \nu_M$ as follows. Put

$$\begin{cases} \nu_i = (\lambda_i - 3)/2 + i & \text{if } \{\lambda_i\} \text{ is a block,} \\ \nu_i = \nu_{i+1} = (\lambda_i - 2)/2 + i & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block,} \end{cases}$$

and put $A = \{\nu_1, \nu_3, \dots, \nu_{[(M+1)/2]}\}$, $B = \{\nu_2, \nu_4, \dots, \nu_{[M/2]}\}$. Then $\Lambda = (A, B)$ gives rise to a distinguished symbol in $Y^2_{n,\ \text{odd}}$ or $Y^2_{n,\ \text{even}}$ according as N is odd or even, which is independent of the choice of M. The map $C_\lambda \mapsto \Lambda$ gives the required bijection ρ. The proof follows, as in the case (a), from the discussion in [L1, 11.7].

(c) $G = Sp_{2n}$ with $p = 2$. The map ρ is defined in [LS, 2.1]. The following definition is slightly modified from the original one so as to fit to the case (a). Let $C_{\lambda,\varepsilon}$ be a unipotent class of G as in 2.3, where λ is a partition of $2n$. Here λ is the same as in the case (a), and we express it as $\lambda_1 \le \cdots \le \lambda_{2m}$ for some m. We use the convention that $\varepsilon(0) = 1$. We divide the set $\{\lambda_1, \lambda_2, \dots, \lambda_{2m}\}$ into a disjoint union of blocks as follows. If $\varepsilon(\lambda_i) = 1$, then $\{\lambda_i\}$ is a block. If $\varepsilon(\lambda_i) = 0$ or ω for $\lambda_i = h$, the sequence $A_h = \{\lambda_k \mid \lambda_k = h\}$ has even cardinality, and it is divided into blocks as in the case (a). We define a sequence $\nu_1, \dots, \nu_{2m}$ as follows.

(i) If $\{\lambda_i\}$ is a block, put

$$\nu_i = \lambda_i/2 + 2i.$$

(ii) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\varepsilon(\lambda_i) = \omega$, put

$$\begin{aligned} \nu_i &= (\lambda_i + 1)/2 + 2i, \\ \nu_{i+1} &= \nu_i + 1. \end{aligned}$$

(iii) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\varepsilon(\lambda_i) = 0$, put

$$\begin{aligned} \nu_i &= (\lambda_i + 2)/2 + 2i, \\ \nu_{i+1} &= \nu_i. \end{aligned}$$

Put $A = \{0, \nu_2, \nu_4, \dots, \nu_{2m}\}$, $B = \{\nu_1, \nu_3, \dots, \nu_{2m-1}\}$. Then by [LS, 2.2], $\Lambda = (A, B)$ gives rise to a distinguished symbol in $X^{2,2}_n$, which is independent of the choice m, and $C_{\lambda,\varepsilon} \mapsto \Lambda$ gives the required bijection

(d) $G = SO_{2n}$ with $p = 2$. The map ρ is defined in [LS, 3.1]. Let $C_{\lambda,\varepsilon}$ be a unipotent class in G. λ is the same as in the case (c), and we define the block in the same way as the case (c). However, here we use the convention that $\varepsilon(0) = 0$. Note that in the sequence $\{\lambda_1, \dots, \lambda_{2m}\}$ the multiplicity of 0 is even since the number of non-zero λ_i is even (cf. 2.3). We define a sequence $\nu_1, \dots, \nu_{2m}$ as follows.

(i) If $\{\lambda_i\}$ is a block, put

$$\nu_i = (\lambda_i - 6)/2 + 2i.$$

(ii) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\varepsilon(\lambda_i) = \omega$, put

$$\begin{aligned}\nu_i &= (\lambda_i - 5)/2 + 2i,\\ \nu_{i+1} &= \nu_i + 1.\end{aligned}$$

(iii) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\varepsilon(\lambda_i) = 0$, put

$$\begin{aligned}\nu_i &= (\lambda_i - 4)/2 + 2i,\\ \nu_{i+1} &= \nu_i.\end{aligned}$$

Put $A = \{\nu_1, \nu_3, \dots, \nu_{2m-1}\}$, $B = \{\nu_2, \nu_4, \dots, \nu_{2m}\}$. Then by [LS, 3.2], $\Lambda = (A, B)$ gives rise to a distinguished symbol in $Y^4_{n,\text{ even}}$, which is independent of the choice of m, and $C_{\lambda,\varepsilon} \mapsto \Lambda$ gives the required bijection.

3.7. We return to the setting in the beginning of 3.6, and let $\rho : G_{\text{uni}}/{\sim} \to X/{\sim}$ be the bijection constructed in 3.6. By making use of ρ, we shall construct a bijection $\widetilde{\rho} : \mathcal{N}_G \to X$. For a unipotent class C in G, let $\mathcal{C}$ be the similarity class in X containing the distinguished symbol $\Lambda = \rho(C)$. Take $u \in C$. As discussed in 3.4, $\mathcal{C}$ has a natural structure of $\mathbf{F}_2$-vector space $V^{r,s}_\Lambda$ for $X = X^{1,1}_n$ or $X^{2,2}_n$ with the basis corresponding to the set of intervals. Since $A_G(u)$ is an elementary abelian 2-group, it has a natural structure of $\mathbf{F}_2$-vector space, and so does the dual group $A_G(u)^\wedge$. It was shown in [L1, 11], [LS, 2.2] that $V^{r,s}_\Lambda$ is naturally identified with $A_G(u)^\wedge$, where the set of intervals is in bijection with the set of generators in $A_G(u)$ given in 2.4; if I is an interval corresponding to the generator a_i of $A_G(u)$, we associate the character χ_i of $A_G(u)$ which takes the value -1 for a_i and 1 for other generators. Similar argument also works for the case where $X = Y^2_{n,\text{ odd}}, Y^2_{n,\text{ even}}$ and $Y^4_{n,\text{ even}}$, and by [L1, 11], [LS, 3.2] V^r_Λ is naturally identified with $A_G(u)^\wedge$. Hence the similarity class $\mathcal{C}$ can be identified with $A_G(u)^\wedge$.

On the other hand, for a fixed C, G-equivariant simple local systems on C are parametrized by $A_G(u)^\wedge$. Thus combining with the above argument, we obtain a bijection $\widetilde{\rho} : \mathcal{N}_G \to X$. This bijection describes combinatorially the generalized Springer correspondence, namely,

Theorem 3.8 ([L1, 12.3,13.3], [LS, 2.4, 3.3])**.** *Let G be as in 3.1. Then the composite of $\widetilde{\rho}$ with the bijection in 3.5 gives the generalized Springer correspondence in 3.1.*

4. Mail results

4.1. Let G be as in 2.1 and we apply the argument in Section 1 for G. First consider the case where G is of split type. For each unipotent class C' in G, we choose a split element $u \in C'^F$ described in Section 2, i.e., (2.9.3) for $G = Sp_{2n}$ or

SO_N with $p \neq 2$, (2.7.3) for $G = Sp_{2n}$ with $p = 2$, (2.8.1) for $G = SO_N$ with $p = 2$. For each pair $(C', \mathcal{E}') \in \mathcal{N}_G^F$, we fix an isomorphism $\psi_0 = \psi_{\mathcal{E}'} : F^*\mathcal{E}' \to \mathcal{E}'$ in 1.6 so that the induced isomorphism $\mathcal{E}'_u \to \mathcal{E}'_u$ is identity. (Note that F acts trivially on $A_G(u)$. This is known by [SS] for the case where $p \neq 2$, and follows from (2.7.4) for the case where $p = 2$.) Take $(L, C, \mathcal{E}) \in \mathcal{M}_G^F$. Then L is of the same type as G of split type. Thus we choose the split element $u_0 \in C^F$. We fix an isomorphism $\varphi_0 : F^*\mathcal{E} \to \mathcal{E}$ in 1.6 so that the induced isomorphism $\mathcal{E}_{u_0} \to \mathcal{E}_{u_0}$ is identity. Let $V_{(C',\mathcal{E}')}$ be the irreducible $\mathcal{W}$-module and $\sigma_{(C',\mathcal{E}')}$ be the isomorphism on $V_{(C',\mathcal{E}')}$ given in 1.6. Since F is of split type, F acts trivially on $\mathcal{W}$ and so $\sigma_{(C',\mathcal{E}')}$ commutes with the action of $\mathcal{W}$. It follows that $\sigma_{(C',\mathcal{E}')}$ is a scalar map.

Next consider the case where $G = SO_{2n}$ with $F = F_0\sigma$ of non-split type. We choose the split elements $u' \in C'^F$ and $u'_0 \in C^F$ as in 2.10, and fix $\psi_0 = \psi_{\mathcal{E}'}, \varphi_0$ as above. (Again F acts trivially on $A_G(u')$ by (2.10.2).) Let $(L, C, \mathcal{E}) \in \mathcal{M}_G^F$. If $L \neq T$, $\mathcal{W}$ is a Weyl group of type B and so F acts trivially on $\mathcal{W}$. Hence $\sigma_{(C',\mathcal{E}')}$ is a scalar map. While if $L = T$, $\mathcal{W} = W'_n$ is the Weyl group of type D_n and F acts non-trivially on $\mathcal{W}$. Note that σ acts on W'_n and the semidirect product $W'_n \rtimes \langle\sigma\rangle$ is isomorphic to W_n. Assume that $V_{(C',\mathcal{E}')} = E$ is F-stable. Then E can be extended to an irreducible representation $\widetilde{E}$ of W_n via the map $\sigma_{(C',\mathcal{E}')}$. Since $\sigma^{-1} \circ \sigma_{(C',\mathcal{E}')}$ commutes with the action of $\mathcal{W}$, it acts as a scalar on E. Thus in order to describe the map $\sigma_{(C',\mathcal{E}')}$, we have only to determine this scalar together with the representation $\widetilde{E}$.

We recall the preferred extension $\widetilde{E}$ of W_n due to [L2, IV, 17.2]. An F-stable irreducible representation E of W'_n is parametrized by an unordered pair $(\alpha; \beta)$ of partitions such that $|\alpha| + |\beta| = n$ and that $\alpha \neq \beta$. We write $\alpha : \alpha_1 \leq \cdots \leq \alpha_m$, $\beta : \beta_1 \leq \cdots \leq \beta_m$, and define a symbol Λ_E associated to E by an unordered pair $(\lambda; \mu)$ with $\lambda_i = \alpha_i + i - 1, \mu_j = \beta_j + j - 1$. (This is a different type of symbols from those appeared in Section 3.) Irreducible representations of W_n are parametrized in a similar way, but by using an ordered pair $(\alpha; \beta)$ and its associated symbol $(\lambda; \mu)$. For an F-stable irreducible representation E, there exists two extensions to W_n, which correspond to two symbols $(\lambda; \mu)$ and $(\mu; \lambda)$ for W_n. An extension $\widetilde{E}$ of E is called the preferred extension of E if in the symbol $\Lambda_{\widetilde{E}}$, the smallest number which does not appear in both entries appears in the second entry. For example, $(n; 0)$ is the symbol associated to the unit representation of W'_n, and it is extended to the unit representation $(n; 0)$ or the long sigh representation $(0; n)$ of W_n. In this case, $(n; 0)$ is the preferred extension.

We can state our main results.

Theorem 4.2. *Let $G = Sp_{2n}, SO_N$ with F of split type (N is even if $p = 2$). Then $\sigma_{(C',\mathcal{E}')}$ is $q^{(a_0+r)/2}$ times identity.*

Theorem 4.3. *Let $G = SO_{2n}$ with F of non-split type.*

(i) *Suppose $L \neq T$. Then $\sigma_{(C',\mathcal{E}')}$ is $q^{(a_0+r)/2}$ times identity.*

(ii) *Suppose $L = T$. Then $\sigma_{C',\mathcal{E}'} = q^{(a_0+r)/2}\sigma$, and $W'_n\langle\sigma\rangle$-module $\widetilde{E}$ coincides with the preferred extension of E.*

In view of Lemma 1.11, we have the following corollary.

Corollary 4.4. *Let G be as in 2.1. For each $(C',\mathcal{E}')\in\mathcal{N}_G^F$, choose $\psi_0 : F^*\mathcal{E}'\xrightarrow{\sim}\mathcal{E}'$ by choosing a split element $u\in C'^F$. Let γ be the constant as in 1.10. Then we have $\gamma=1$, namely we have $Y_j=Y_j^0$ for $j=(C',\mathcal{E}')$.*

4.5. We prove the theorem by making use of the restriction formula in Corollary 1.9. We choose a standard parabolic subgroup $Q=MU_Q$ such that the Levi subgroup M of Q is of the same type as G with semisimple rank $n-1$. Take $(L,C,\mathcal{E})\in\mathcal{M}_G^F$ and let P be the F-stable standard parabolic subgroup of G whose Levi subgroup is L. Then we have $P\subset Q$ and $L\subset M$. Take $(C',\mathcal{E}')\in\mathcal{N}_G^F$, and take a split element $u\in C'^F$. Also take $(C_1,\mathcal{E}_1)\in\mathcal{N}_M^F$, and choose a split element $v\in C_1^F$. Let ρ (resp. ρ_1) be the irreducible representation of $A_G(u)$ (resp. $A_M(v)$) corresponding to $\mathcal{E}'$ (resp. $\mathcal{E}_1$). Since F acts trivially on $A_G(u)$ and $A_M(v)$, the extension $\widetilde{\rho\otimes\rho_1^*}$ to $\widetilde{A}(u,v)$ in 1.8 is just the trivial extension of $\rho\otimes\rho_1^*$. Then we have the following lemma.

Lemma 4.6. *Assume that F acts trivially on $\mathcal{W}$ and on $\mathcal{W}_1$. Let $E\in\mathcal{W}^\wedge$ (resp. $E_1\in\mathcal{W}_1^\wedge$) be corresponding to $(C',\mathcal{E}')\in\mathcal{N}_G$ (resp. $(C_1,\mathcal{E}_1)\in\mathcal{N}_M$), and assume that E_1 occurs in the restriction of E to $\mathcal{W}_1$. Suppose that the theorem holds for $\sigma_{(C_1,\mathcal{E}_1)}$. If $X_{u,v}\neq\emptyset$ and F acts trivially on $X_{u,v}$, then the theorem holds for $\sigma_{(C',\mathcal{E}')}$.*

Proof. We follow the notation in Section 1. Since E_1 occurs in the restriction of E to $\mathcal{W}_1$, $M_{E_1}\neq 0$ in (1.6.1). Since $\sigma_{(C',\mathcal{E}')}$ and $\sigma_{(C_1,\mathcal{E}_1)}$ is a scalar map, $\sigma_{\mathcal{E}_1,\mathcal{E}'}$ is also a non-zero scalar map by (1.6.2). Since F acts trivially on $X_{u,v}$, $\widetilde{\varepsilon}_{u,v}$ is the trivial extension of $\varepsilon_{u,v}$ to $\widetilde{A}(u,v)=\langle\tau\rangle\times A(u,v)$. It follows that

$$\langle\widetilde{\varepsilon}_{u,v},\widetilde{\rho\otimes\rho_1^*}\rangle_{A(u,v)\tau}=\langle\varepsilon_{u,v},\rho\otimes\rho_1^*\rangle_{A(u,v)}\,.$$

Then Corollary 1.9 together with Corollary 1.5 implies that

$$\operatorname{Tr}(\sigma_{\mathcal{E}_1,\mathcal{E}'},M_{E_1})=q^{-d_{C_1,C'}+\dim U_Q}\dim M_{E_1},$$

and we see that $\sigma_{\mathcal{E}_1,\mathcal{E}'}$ is a scalar map by $q^{-d_{C_1,C'}+\dim U_Q}$. By our assumption, $\sigma_{(C_1,\mathcal{E}_1)}$ is a scalar map by $q^{(a_0'+r')/2}$, where a_0',r' are as in 1.10 with respect to M. Thus again by (1.6.2), we see that $\sigma_{(C',\mathcal{E}')}$ is a scalar map by $q^{(a_0+r)/2}$. □

4.7. In view of Lemma 4.6, it is important to know the Frobenius action on $X_{u,v}$. We note that

(4.7.1) $Z_G(u)\times Z_M(v)U_Q$ acts transitively on the set $Y_{u,v}$.

In fact, put $\mathcal{Q}_{u,C_1}=\{gQ\in G/Q\mid g^{-1}ug\in C_1U_Q\}$. $\mathcal{Q}_{u,C_1}$ is a locally closed subvariety of G/Q. We have a surjective morphism $Y_{u,v}\to\mathcal{Q}_{u,C_1}, g\mapsto gQ$, which induces an isomorphism $Y_{u,v}/Z_M(v)U_Q\simeq\mathcal{Q}_{u,C_1}$. It is known by [Sp1, II, 6.7] that $Z_G(u)$ acts transitively on $\mathcal{Q}_{u,C_1}$. (4.7.1) follows from this.

Lemma 4.8. *If $Y_{u,v}^F\neq\emptyset$, then F acts trivially on $X_{u,v}$.*

Proof. By (4.7.1), the closure of $Z_G^0(u)gZ_M^0(v)U_Q$ in $Y_{u,v}$ ($g\in Y_{u,v}$) gives an element $x\in X_{u,v}$, and $A_G(u)xA_M(v)$ gives all the irreducible components in $Y_{u,v}$. We can choose $g \in Y_{u,v}^F$. Hence $Z_G^0(u)gZ_M^0(v)U_Q$ is F-stable, and so is x. Since F acts trivially on $A_G(u)$ and $A_M(v)$, F stabilizes all the irreducible components in $Y_{u,v}$. □

4.9. Assume that F is of split type. For each $u \in G^F$, we shall give $v \in M^F$ such that $Y_{u,v}^F \neq \emptyset$. Let V and f be as in 2.1. Note that G/Q can be identified with the subvariety of $\mathbb{P}(V)$ consisting of $\langle x \rangle$ for isotropic vectors x with respect to f ($\langle x \rangle$ denotes the line spanned by x). Under the setting in 2.7, 2.9, we consider the vector space $\bar{V} = \bigoplus M_j$ which is identified with V^F. In the following cases, we can find $g \in G^F$ such that $g^{-1}ug \in Q$ and that $v = \pi(g^{-1}ug)$ is a split element in M ($\pi : Q \to M$ is the natural projection). In particular, we have $g \in Y_{u,v}^F$. In the discussion below, we identify the partitions with the corresponding Young diagrams.

First we consider the case where $G = Sp_{2n}$ or SO_N with $p \neq 2$.

(i) Take M_j such that $\varepsilon(\lambda_j) = 1$. Let $e_1^j, \dots, e_h^j$ be the basis of M_j with $h = \lambda_j$ given in 2.9 (a). The stabilizer of $\langle e_1^j \rangle$ in G is a parabolic subgroup gQg^{-1} with some $g \in G^F$. The nilpotent transformation X_j on M_j induces a map $\bar{X}_j$ on $\bar{M}_j = \langle e_1^j \rangle^{\perp} / \langle e_1^j \rangle$, and one can define a nilpotent element $\bar{X}$ on $\bar{M}_j \oplus \bigoplus_{j' \neq j} M_{j'}$. This determines a unipotent element $\pi(g^{-1}ug) = v$ in M^F which is a split element. In this case v is of type λ', where λ' is obtained from λ by replacing one row h such that $\varepsilon(h) = 1$ by $h - 2$.

(ii) Take M_j such that $\varepsilon(\lambda_j) = 1$ and that c_h is even for $h = \lambda_j$. We choose j such that $\lambda_j = \lambda_{j-1} = h$ and consider $N_j = M_j \oplus M_{j-1}$ with $f_j' = f_j + f_{j-1}$. Thus N_j has a basis $e_1^j, \dots, e_h^j, e_1^{j-1}, \dots, e_h^{j-1}$. By our construction, we have

$$f_j(e_1^j, e_h^j) = -f_{j-1}(e_1^{j-1}, e_h^{j-1}) = \pm 1.$$

We consider $\bar{N}_j = \langle x \rangle^{\perp} / \langle x \rangle$ for $x = e_1^j + e_1^{j-1}$. The nilpotent transformation $X_j + X_{j-1}$ on $M_j \oplus M_{j-1}$ induces a linear map $\bar{X}_j$ on $\bar{N}_j$, and one can define a nilpotent element $\bar{X}$ on $\bar{N}_j \oplus \bigoplus M_{j'}$. This determines a unipotent element $\pi(g^{-1}ug) = v$ in M of type λ', where λ' is obtained from λ by replacing two rows of length h by two rows of length $h - 1$. It is easy to see that v is a split element in M^F.

(iii) Take M_j such that $\varepsilon(\lambda_j) = \omega$. Put $h = \lambda_j$. We choose a basis of M_j as in 2.9 (b). Let $x = e_1^j + e_1^{j-1}$. Then x is an isotropic vector with respect to f_j. Now X_j induces $\bar{X}_j$ on $\bar{M}_j = \langle x \rangle^{\perp} / \langle x \rangle$, and one can define $\bar{X}$ on $\bar{M}_j \oplus \bigoplus_{j' \neq j} M_{j'}$. This determines a unipotent element $v = \pi(g^{-1}ug) \in M^F$. In this case, v is of type λ', where λ' is obtained from λ by replacing two rows of length h by two rows of length $h - 1$. It is easy to see that v is a split element in M^F.

It is known (cf. [S1]) that if $Y_{u,v} \neq \emptyset$, for $u \in C_\lambda, v \in C_{\lambda'}$, then λ' is obtained from λ by the procedure described above. Hence if v is split, it coincides with one of the above cases. It follows that

Proposition 4.10. *Let $G = Sp_{2n}$ or SO_N with $p \neq 2$. Assume that F is of split type. Let $u \in G^F$ and $v \in M^F$ be such that $Y_{u,v} \neq \emptyset$. Assume that u, v are split elements. Then F acts trivially on $X_{u,v}$.*

4.11. Next we consider the case where $G = Sp_{2n}$ with $p = 2$. We keep the setting in 4.9, in particular assume that F is of split type.

(i) Choose M_j as in 2.7 (a), i.e., the case where $\varepsilon(h) = 1$ for $h = \lambda_j$. v_j induces a linear map $\bar{v}_j$ on $\bar{M}_j = \langle e_1^j\rangle^{\perp}/\langle e_1^j\rangle$ and one can define $\bar{v}$ on $\bar{M}_j \oplus \bigoplus_{j' \neq j} M_{j'}$. This determines a unipotent element $\pi(g^{-1}ug) = v \in M$ for some $g \in G^F$. v is contained in the class $C_{\lambda',\varepsilon'}$ in M, where λ' is obtained from λ by replacing one row of length h by a row of length $h-2$, and ε' is given by $\varepsilon'(h-2) = 1$ and $\varepsilon'(\lambda'_k) = \varepsilon(\lambda_k)$ if $\lambda'_k \neq h-2$. By our construction of the form f_j in 2.5, we see that v is a split element if $h-2$ does not occur in the row of λ, nor $h-2$ occurs and $\varepsilon(h-2) = 1$.

(ii) Assume that $c_h \geq 2$ and $\varepsilon(h) = 1$ for $h = \lambda_j$, and take $\lambda_j = \lambda_{j-1} = h$. We consider $M_j = V_j \oplus V_{j-1}$ with $f_j = f_j^0 + f_{j-1}^0$, where f_j^0, V_j are as in 2.7. Let $x = e_1^j + e_1^{j-1}$. Put $\bar{M}_j = \langle x\rangle^{\perp}/\langle x\rangle$. Then $v_j + v_{j-1}$ induces a unipotent element $\bar{v}_j$ on $\bar{M}_j$. This determines $\pi(g^{-1}ug) = v$ of M^F as before. This construction is exactly the same as the one in 2.7 (b). Hence v is contained in $C_{\lambda',\varepsilon'}$ in M, where λ' is obtained from λ by removing two rows of length h by two rows of length $h-1$, and $\varepsilon'(h-1) = \omega$, $\varepsilon'(h') = \varepsilon(h')$ for all $h' \neq h$. In this case, v is a split element without any condition.

(iii) Choose M_j as in 2.7 (b), i.e., the case where $\varepsilon(h) = \omega$. The basis of M_j is given in 2.7 (b). Then $\bar{M}_j = \langle \bar{e}_1^j\rangle^{\perp}/\langle \bar{e}_1^j\rangle$ has a basis $\bar{e}_{n-1}^j, \bar{e}_{n-1}^{j-1}, \dots, \bar{e}_2^j, \bar{e}_2^{j-1}$ (we use the same notation for the image on $\bar{M}_j$ as the one in M_j). Thus the induced linear map $\bar{v}_j$ on $\bar{M}_j$ is just a sum of two copies of nilpotent elements as given in 2.7 (a) with respect to the induced form $\bar{f}_j$ on $\bar{M}_j$. It follows that $\bar{v}$ determines a unipotent element $\pi(g^{-1}ug) = v \in M$. v is contained in $C^F_{\lambda',\varepsilon'}$, where λ' is obtained from λ by replacing two rows of length h by two rows of length $h-1$. ε' is given by $\varepsilon'(h-1) = 1$, and is the same as ε for all other $h' \neq h$. In particular, v is a split element if $h-1$ does not occur in the rows of λ nor $\varepsilon(h-1) = 1$.

(iv) Choose M_j as in (iii). Under the notation there, put $x = \bar{e}_2^j + \bar{e}_2^{j-1}$. Then $\bar{M}_j = \langle x\rangle^{\perp}/\langle x\rangle$ has a basis exactly the same as the basis of $N^{\perp}/N$ in 2.7 (c). Thus the induced map $\bar{v}_j$ on $\bar{M}_j$ determines a unipotent element $\pi(g^{-1}ug) = v$ of M in the same way as above. v is a split element in M and is contained in $C_{\lambda',\varepsilon'}$, where λ' is obtained from λ by replacing two rows of length h by two rows of length $h-1$, and $\varepsilon'(h-1) = 0$ if $h-1$ does not occur in λ nor if $\varepsilon(h-1) = 0$.

(v) Choose M_j as in 2.7 (c), i.e., the case where $\varepsilon(h) = 0$. The basis of M_j is given in 2.7 (c). Then $\bar{M}_j = \langle \bar{e}_1^j\rangle^{\perp}/\langle \bar{e}_1^j\rangle$ has a basis $\bar{e}_{n-1}^j + \bar{e}_{n-1}^{j-1}, \bar{e}_{n-2}^j, \bar{e}_{n-2}^{j-1}, \dots, \bar{e}_3^j, \bar{e}_3^{j-1}$, $\bar{e}_2^j = \bar{e}_2^{j-1}$. Thus the induced linear map $\bar{v}_j$ on $\bar{M}_j$ is the same as the case (b) as above with respect to the induced form $\bar{f}_j$. It follows that $\bar{v}$ determines a unipotent element $\pi(g^{-1}ug) = v \in M$. v is contained in $C_{\lambda',\varepsilon'}$ where λ' is obtained from λ by replacing two rows of length h by two rows of length $h-1$, and ε' is given by $\varepsilon'(h-1) = \omega$ if $h-1$ does not occur in the rows of λ. In this case, v is a split element without any condition.

Finally, we consider the case where $G = SO_{2n}$ with $p = 2$. The argument in the case of $G = Sp_{2n}$ with $p = 2$ works well for this case under a suitable modification, since in each case of (II), the induced quadratic form is of the same type as the original one, and so one can check easily that v is a split element.

4.12. We consider the counter part of Proposition 4.10 and 4.11 for the case where $F = F_0\sigma$ is of non-split type. We assume that $G = SO_{2n}$ (p : arbitrary) and $\widetilde G = O_{2n}$. Let $u \in C^{F_0}_{\lambda,\varepsilon}$ and $v \in C^{F_0}_{\lambda',\varepsilon'}$ be split elements as in 4.9 or 4.11, and assume that $Y^{F_0}_{u,v} \ne \emptyset$. Take $g \in Y^{F_0}_{u,v}$. Then $g^{-1}ug \in vU_Q$. By replacing u by $g^{-1}ug$ (an element in the split class) we may assume that $u \in vU_Q$. Let $a = a_i \in Z_{\widetilde G}(u)$ be as in (2.10.1) and $\dot a \in Z_{\widetilde G}(u)$ be its representative. Let $a' \in A_{\widetilde M}(v)$ be defined similar to a. We assume that a and a' are both related to the row of the same length in λ and λ'. By the explicit description of the element $\dot a \in Z_{\widetilde G}(u)$ (see 2.7 $\sim$ 2.10), we see that $\dot a$ normalizes Q. It follows that $\dot a \in Z_{\widetilde M}(v)$, which gives a representative of a'. Let us write $\dot a = xs$ and let $\alpha^{-1}F(\alpha) = x$ be as in (2.10.1). Since we may take $s \in M$, we have $x \in M$, and so $\alpha \in M$ also. Let $u' = \alpha u\alpha^{-1} \in C^F_{\lambda,\varepsilon}$ $v' = \alpha v\alpha^{-1} \in C^F_{\lambda',\varepsilon'}$ be split elements as given in (2.10.1). We have the following lemma.

Lemma 4.13. *Let the notations be as above, and u, v (resp. u', v') be split elements with respect to F_0 (resp. F). Then under the assumption in Proposition 4.10 or 4.11, F acts trivially on the set $X_{u',v'}$.*

Proof. Since $\alpha \in M$, $\operatorname{ad}\alpha$ maps $Y_{u,v}$ onto $Y_{u',v'}$, and so induces a bijection between $X_{u,v}$ and $X_{u',v'}$. Since $\dot aF_0 = xF$, $\operatorname{ad}\alpha$ maps $\dot aF_0$-stable elements of $X_{u,v}$ to F-stable elements of $X_{u',v'}$. In view of Proposition 4.10 and 4.11, we may assume that F_0 acts trivially on the set $X_{u,v}$. Hence in order to prove the lemma, it is enough to show that any element in $X_{u,v}$ is stable by $\operatorname{ad}\dot a$. Now an irreducible component of $Y_{u,v}$ is expressed as the closure of $Z^0_G(u)gZ^0_M(v)U_Q$ for some $g \in Y_{u,v}$. By our choice of u, v, we may take $g = 1$. Then $Z^0_G(u)Z_M(v)^0U_Q$ is stable by $\operatorname{ad}\dot a$, and so there exists an irreducible component stable by $\operatorname{ad}\dot a$. Since $A_G(u) \times A_M(v)$ acts transitively on the set $X_{u,v}$, and a commutes with $A_G(u)$ and $A_M(v)$, we conclude that $\operatorname{ad}\dot a$ stabilizes each element in $X_{u,v}$. This proves the lemma. □

4.14. In order to apply Lemma 4.6, we need first to know the condition for $X_{u,v} \ne \emptyset$. By the isomorphism $Y_{u,v}/Z_M(v)U_Q \simeq Q_{u,C_1}$ in 4.7, the elements in $X_{u,v}$ corresponds to the irreducible components of Q_{u,C_1} of dimension $(\dim Z_G(u) - \dim Z_M(v))/2$. The condition for C_1 for the existence of such an irreducible component in Q_{u,C_1} is described in [Sp1, II, 6.7]. By making use of 3.6, it is interpreted in terms of the symbols (cf. [LS, 2.6]);

(4.14.1) Let $\Lambda = \rho_G(u), \Lambda' = \rho_M(v)$ be the distinguished symbols associated to u and v. Then $X_{u,v} \ne \emptyset$ if and only if Λ' is obtained from Λ by decreasing one of the entries of Λ by 1.

Let $E \in \mathcal{W}^\wedge, E_1 \in \mathcal{W}_1^\wedge$ be as in Lemma 4.6. In applying the lemma, we also need to know when E_1 appears in the restriction of E. This is given as follows (cf. [LS, 2.8]).

(4.14.2) Let X be the set of symbols as in 3.6. Let $(A, B) \in X$ corresponding to $E \in \mathcal{W}^\wedge$. Then $E_1 \in \mathcal{W}_1^\wedge$ appears in the restriction of E if and only if the symbol (A', B') corresponding to E_1 is obtained from (A, B) by decreasing one of the entries in A or B by 1. (This holds also for the case of degenerate symbols. If $E, E' \in \mathcal{W}^\wedge$ are corresponding to the degenerate symbol (A, A) and its copy, E and E' have the same restriction on $\mathcal{W}_1$, and its components are parametrized by symbols obtained by decreasing one of the entries in (A, A) by 1.) In particular, if E_1 appears in the restriction of E, we have $X_{u,v} \neq \emptyset$.

4.15. We are now ready to prove Theorem 4.2. So, assume that F is of split type. First consider the case where $G = Sp_{2n}$ or SO_N with $p \neq 2$. By induction on the rank of G, we may assume that the theorem holds for Sp_{2n-2} or SO_{N-2}. Let $(C', \mathcal{E}') \in \mathcal{N}_G$ be corresponding to $E \in \mathcal{W}^\wedge$. Let $E_1 \in \mathcal{W}_1^\wedge$ be an irreducible component of the restriction of E, and $(C_1, \mathcal{E}_1) \in \mathcal{N}_M$ be the corresponding element. We choose the split elements $u \in C'^F$ and $v \in C_1^F$. Then $X_{u,v} \neq \emptyset$ by (4.14.2), and F acts trivially on $X_{u,v}$ by Proposition 4.10. Thus the assertion holds for $\sigma_{(C',\mathcal{E}')}$ by Lemma 4.6, and the theorem follows.

Next we consider the case where $G = Sp_{2n}$ or SO_{2n} with $p = 2$. Since the result in 4.11 is somewhat weaker than Proposition 4.10, we need a more precise argument. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{2m}$ be the sequence of λ as in 3.6, and Λ be the distinguished symbol associated to λ. For a given integer h, let $A_h = \{\lambda_j, \lambda_{j+1}, \dots, \lambda_k\}$ be the subsequence consisting of $\lambda_i = h$. We say that $I = \{a, \dots, b\}$ $(a \le b)$ is a semi-interval if I corresponds to the sequence A_h under the construction of Λ in 3.6 (c), (d). The element a is called the tail of the semi-interval I. They have the following forms. We denote by $I_h^{\varepsilon(h)}$ the semi-interval corresponding to h and $\varepsilon(h)$.

$$(4.15.1) \qquad I_h^{\varepsilon(h)} = \begin{cases} \{a, a+2, a+4, a+6, \dots\} & \text{if } \varepsilon(h) = 1, \\ \{a, a, a+4, a+4, a+8, a+8, \cdots\} & \text{if } \varepsilon(h) = 0, \\ \{a, a+1, a+4, a+5, a+8, a+9, \cdots\} & \text{if } \varepsilon(h) = \omega. \end{cases}$$

For two semi-intervals $I = \{a, \dots, b\}, I' = \{a', \dots, b'\}$ with $b' < a$, the distance of I, I' is defined by $a - b'$. It is easy to see that the distance of two semi-intervals is always ≥ 3. The case where the distance is 3 occurs in the following three cases;

$$(I_h^1, I_{h+2}^1), \quad (I_h^1, I_{h+1}^\omega), \quad (I_h^\omega, I_{h+1}^1).$$

The semi-interval I is a part of an interval unless $I = I_h^0$, and if the distance of I and I' is equal to 3, they are joined to be a part of one big interval. This explains the condition of generators of $A_G(u)$ in 2.4.

By induction, we may assume that the theorem holds for the smaller rank case. Let $E \in \mathcal{W}^\wedge$ corresponding to $(C', \mathcal{E}') \in \mathcal{N}_G$. Let Λ be the distinguished symbol associated to C'. We write $C' = C_{\lambda,\varepsilon}$. Suppose that there exist two semi-intervals $I = \{a, \dots, b\}, I' = \{a', \dots, b'\}$ such that the distance $a - b' \ge 5$. Then a is also a tail of an interval, and it is easy to check that one can decrease a by $a - 1$ in Λ to obtain a new symbol Λ'. This procedure is also valid for a symbol similar to Λ. Moreover if a corresponds to h, and $a - 1$ corresponds to $h' = h - 1$ or $h - 2$ under $C_{\lambda,\varepsilon} \leftrightarrow \Lambda$, then we have $c_{h'} = 0$. Let $C_{\lambda',\varepsilon'} \leftrightarrow \Lambda'$ with $v \in C^F_{\lambda',\varepsilon'}$. There exists $(C_1, \mathcal{E}_1) \in \mathcal{N}_M$ $(C_1 = C_{\lambda',\varepsilon'})$ corresponding to $E_1 \in \mathcal{W}_1^\wedge$ such that E_1 occurs in the restriction of E. By making use of 4.11, we see that $Y^F_{u,v} \neq \emptyset$ for a split element $v \in C_{\lambda',\varepsilon'}$, and so F acts trivially on $X_{u,v}$ by Lemma 4.8. Now Lemma 4.6 can be applied to show that the theorem holds for $\sigma_{(C',\mathcal{E}')}$.

Thus it is enough to consider the case where the distance of I, I' is ≤ 4. Let $I = I_h^\omega$. There are three possibilities for I' with distance ≤ 4, i.e., $I' = I_{h-2}^\omega, I_{h-1}^1$ or I_{h-1}^0. For each case, one can find $(C_1, \mathcal{E}_1) \leftrightarrow E_1$ with a split element $v \in C_1^F = C_{\lambda',\varepsilon'}^F$, to which Lemma 4.6 can be applied. For example consider the case where $I' = I_{h-1}^0$. Then by (4.15.1),

$$I' = \{\dots, a-4, a-4\}, \quad I = \{a, a+1, \dots, \}$$

for some a. It follows that a is the tail of an interval. One can replace I by $J = \{a, a, \dots\}$ which produces a new symbol Λ' corresponding to $C_{\lambda',\varepsilon'}$. This works also for a symbol (A, B) similar to Λ, where $(A, B) \leftrightarrow E$. Now λ' is obtained from λ by replacing two rows of length h by two rows of length $h-1$. Since $\varepsilon(h-1) = 0$, we have $\varepsilon'(h-1) = 0$ and 4.11 can be applied to show that $Y_{u,v}^F \ne \emptyset$ for a split element $v \in C_{\lambda',\varepsilon'}^F$, and we get the assertion in a similar way as above. The other cases are dealt similarly.

Now we may assume that λ consists of even rows. Assume that there exists h such that $\varepsilon(h) = 0$, and consider a semi-interval $I = I_h^0$. There are two possibilities for I' whose distance is ≤ 4, i.e., $I' = I_{h-2}^1$ or I_{h-2}^0. First assume that $I' = I_{h-2}^1$. Then I, I' is written as

$$I' = \{\dots, a-6, a-4\}, \quad I = \{a, a, a+4, a+4, \dots\}$$

by (4.15.1). Then one can replace the tail a of I by $a-1$, which divide I into two semi-intervals

$$J_1 = \{a-1, a\}, \quad J_2 = \{a+4, a+4, \dots\},$$

and $I' \cup J_1$ form a part of some interval. The situation is the same for a symbol (A, B) similar to Λ. This produces a new symbol $\Lambda' \leftrightarrow C_{\lambda',\varepsilon'}$ where λ' is obtained from λ by replacing two rows of length h by two rows of length $h-1$ where $\varepsilon'(h-1) = \omega$. Hence by 4.11, we have $Y_{u,v}^F \ne \emptyset$ for a split element $v \in C_{\lambda',\varepsilon'}$. Another case is dealt similarly.

Finally we may assume that λ consists of even rows h with $\varepsilon(h) = 1$. We consider $I = I_h^1$, $I' = I_{h-2}^1$. Then we have

$$I' = \{\dots, a-5, a-3\}, \quad I = \{a, a+2, a+4, \dots\}.$$

Note that I, I' are a part of a common interval. By replacing a by $a-1$, we have new semi-intervals

$$J' = \{\dots, a-5, a-3, a-1\}, \quad J = \{a+2, a+4, \dots\}.$$

This produces a new symbol $\Lambda' \leftrightarrow C_{\lambda',\varepsilon'}$, where λ' is obtained from λ by replacing one row of length h by one row of length $h-2$. Since $\varepsilon(h-2) = 1$, the argument in 4.11 can be applied, and we get the assertion of the theorem. Theorem 4.2 is now proved.

4.16. We shall prove Theorem 4.3. So assume that F is of non-split type. First consider the case where $L \ne T$. In this case the proof is done almost similar to the proof of Theorem 4.2, by using Lemma 4.13 instead of Proposition 4.10 and 4.11. However, we have to be careful for the choice of v (cf. the condition of a and a' in 4.12) in applying Lemma 4.13. In the case where $p = 2$, this is done along the line in 4.15, by choosing the decreasing number suitably. In the case where $p \neq 2$, Proposition 4.10 cannot be applied directly, and we have to apply a similar argument as in 4.15. But this is easier than the case of $p = 2$; the semi-intervals are of the form $I_h^1 = \{a, a+1, a+2, \dots\}$ or $I_h^\omega = \{a, a, a+2, a+2, \dots\}$, and only I_h^1 gives an interval. The distance of two semi-intervals I, I' is ≥ 2. If the distance is ≥ 3, no interaction occurs for I, I' in decreasing one entry. Then the assertion (i) of the theorem is obtained by considering the following (I, I').

$$(I_h^\omega, I_{h-2}^\omega), \quad (I_h^\omega, I_{h-1}^1), \quad (I_h^1, I_{h-1}^\omega), \quad (I_h^1, I_{h-2}^1).$$

The details are omitted.

Next we consider the case where $L = T$. Hence $\mathcal{W} = W_n'$, and we regard it as a subgroup of $W_n = W_n'\langle\sigma\rangle$. We consider the decomposition of $E = V_{(C',\mathcal{E}')}$ in (1.6.1). Assume that E is F-stable, and let $\widetilde{E}$ be the extension of E by $\sigma_{(C',\mathcal{E}')}$. Let $\widetilde{E}_1$ be the extension of $E_1 = V_{(C_1,\mathcal{E}_1)}$ through $\sigma_{(C_1,\mathcal{E}_1)}$. Note that in our case $\dim M_{E_1} = 1$, and so $\sigma_{\mathcal{E}',\mathcal{E}}$ is a scalar map. By multiplying σ^{-1} on the both side of (1.6.2), one can write

$$\sigma^{-1} \circ \sigma_{(C',\mathcal{E}')}|_{M_{E_1} \otimes E_1} = \sigma_{\mathcal{E}',\mathcal{E}_1} \otimes \sigma^{-1} \circ \sigma_{(C_1,\mathcal{E}_1)}.$$

Since F acts trivially on $A_G(u)$ and $A_M(v)$, the extension $\widetilde{\rho \otimes \rho_1^*}$ is the trivial extension. Thus if F acts trivially on $X_{u',v'}$, then Corollary 1.9 implies that $\sigma_{\mathcal{E}',\mathcal{E}_1}$ is a scalar map by $q^{-d_{C_1,C'} + \dim U_Q}$. Now assume that E corresponds to the symbol Λ and E_1 is an irreducible component of E corresponding to the symbol Λ', where Λ' is obtained from Λ by decreasing an entry by 1, under the condition in 4.12. (Note that Λ is not a degenerate symbol since E is F-stable.) Let $\widetilde{E}$ and $\widetilde{E}_1$ be the preferred extensions of E, E_1, respectively. Then it is easy to check that $\widetilde{E}_1$ occurs in the restriction of $\widetilde{E}$ on W_{n-1}. Thus again by using the arguments in 4.15 (see also the remark for the case where $L \neq T$ with non-split case), thanks to Lemma 4.13, the verification of Theorem 4.3 (ii) is reduced to the case where $n = 2$, namely $G \simeq G_1 \times G_1$, where G_1 is of type A_1, and F acts as a permutation of two factors. This case is checked as follows (cf. [S1, Lemma 3.11]). Since the class C' is F-stable, u' is the product of two regular elements in G_1 or the product of two identity elements in G_1. Let $\mathcal{B}_{u'}$ be the variety of Borel subgroups of G containing u'. Then F acts trivially on the one dimensional W_n'-module $H^{2d_{u'}}(\mathcal{B}_{u'})$, where $d_{u'} = \dim \mathcal{B}_{u'}$. Hence W_n-module $H^{2d_{u'}}(\mathcal{B}_{u'})$ coincides with the identity representation or the long sign representation η (i.e., takes the value $\eta(r_\alpha) = 1$, $\eta(r_\beta) = -1$, where r_α (resp. r_β) is the reflection with respect to the short root α (resp. long root β) of the root system of type B_2.) according to the cases where u' is regular or identity. The corresponding symbol is $(2; 0)$ for the former, and $(12; 01)$ for the latter. The both are preferred extensions, and the theorem follows.

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