GENERALIZED GREEN FUNCTIONS AND UNIPOTENT CLASSES FOR FINITE REDUCTIVE GROUPS, I

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Abstract. The algorithm of computing generalized Green functions of a reductive group G contains some unknown scalars occurring from the $\mathbf{F}_q$-structure of irreducible local systems on unipotent classes of G. In this paper, we determine such scalars in the case where $G = SL_n$ with Frobenius map F of split type or non-split type. In the case where F is of non-split type, we use the theory of graded Hecke algebras due to Lusztig.

0. Introduction

Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ with Frobenius map F. In [L1], Lusztig classified the irreducible characters of finite reductive groups G^F in the case where the center of G is connected. Later in [L5], he extended his results to the disconnected center case. In the course of the classification, in particular in the connected center case, he defined almost characters of G^F, which forms an orthonormal basis of the space $\mathcal{V}(G^F)$ of class functions of G^F different from the basis consisting of irreducible characters. They are defined as explicit linear combinations of irreducible characters, and the transition matrix between these two bases are almost diagonal. So, the determination of the character values of irreducible characters of G^F is equivalent to that of almost characters.

On the other hand, Luszitg founded in [L3] the theory of character sheaves, and showed that the characteristic functions of character sheaves form an orthonormal basis of $\mathcal{V}(G^F)$. He conjectured that those functions coincide, up to scalar, with almost characters (with an appropriate generalization of almost characters if the center is disconnected). Lusztig's conjecture was proved by the author in [S3] in the case where the center is connected. It was also proved for certain groups with disconnected center, i.e., for Sp_{2n} and (under a suitable modification for a disconnected group) O_{2n} with ch $\mathbf{F}_q \neq 2$ by Waldspurger [W], for SL_n by the author [S4] (with ch $\mathbf{F}_q$ not too samll), and independently, for SL_n and SU_n by Bonnafé [B] (with q not too small.)

If Lusztig's conjecture is established, the computation of irreducible characters of G^F is reduced to the computation of characteristic functions of character sheaves, and to the determination of scalars involved in Lusztig's conjecture. In [L3], Lusztig proved that the computation of the characteristic functions of character sheaves are reduced to the computation of generalized Green functions of various reductive subgroups of G^F. Then he showed that there exists a general algorithm of computing generalized Green functions. More precisely, he showed that generalized Green functions can be expressed as an explicit linear combination of various characteristic functions $\chi_{C',\mathcal{E}'}$ of the G-equivariant local system $\mathcal{E}'$ on a unipotent class C' in G. Up to scalar, $\chi_{C',\mathcal{E}'}$ can be easily described in terms of the irreducible character of the component group $A_G(u) = Z_G(u)/Z_G^0(u)$ for $u \in C'^F$ corresponding to $\mathcal{E}'$. However, this scalar depends on the choice of the isomorphism $F^*\mathcal{E} \,\underset{\sim}{\to}\, \mathcal{E}$ for a cuspidal pair $(C, \mathcal{E})$ on a Levi subgroup L of a parabolic subgroup P of G, and on the intersection cohomology complex K induced from $\mathcal{E} \boxtimes \bar{\mathbf{Q}}_l$ on $C \times Z_L^0$ (see (1.2.2)).

The purpose of this paper is to determine these scalars occurring in the computation of generalized Green functions. In the case of Green functions, this problem is equivalent to determining a representative $u \in C'^F$ such that the action of F on the l-adic cohomology group $H^m(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ can be described explicitly, where $\mathcal{B}_u$ is the variety of Borel subgroups of G containing u, and $m/2 = \dim \mathcal{B}_u$. It was shown in [S1], [S2] and [BS] that there exists a unipotent element $u \in C'^F$, in the case where G^F is of split type, and G is not of type E_8, such that F acts on $H^m(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ by a scalar multiplication $q^{m/2}$. Such a unipotent element is called a split element. Even if the remaining cases, the action of F can be described, and by using this, Green functions of exceptional groups (F_4, E_6, E_7 and E_8) were computed explicitly by [S1], [BS] for a good characteristic case. The case G_2 had been computed by Springer [Spr] in an earlier stage. (Green functions of exceptional groups in certain bad characteristic case were computed by Malle [M] by a direct computation).

In the case of generalized Green functions, one has to consider the cohomology group $H_c^m(\mathcal{P}_u, \dot{\mathcal{E}})$, where $\mathcal{P}_u$ is a certain subvariety of parabolic subgroups of G conjugate to P, and $\dot{\mathcal{E}}$ is a local system on $\mathcal{P}_u$ determined from the cuspidal pair $(C, \mathcal{E})$ on a Levi subgroup L of P, and $m/2 = \dim \mathcal{P}_u$. We need to describe the action of F on such cohomology groups. This problem is reduced to the case where G is simply connected, and simple modulo center. In this paper, we discuss the case where $G = SL_n$ with F of split type or non-split type. In the case where F is of split type, the method employed here is to compare the Frobenius action in the case of SL_n with SL_{n-1}, which is a natural generalization of the method in the case of GL_n. In the case of GL_n with F of non-split type, the Frobenius action was determined by investigating the action of F on $H^*(\mathcal{B}, \bar{\mathbf{Q}}_l)$ by making use of the F-equivariant surjective map $\pi_u : H^m(\mathcal{B}, \bar{\mathbf{Q}}_l) \to H^m(\mathcal{B}_u, \bar{\mathbf{Q}}_l)$ induced from the inclusion $\mathcal{B}_u \hookrightarrow \mathcal{B}$, where $\mathcal{B}$ is the Flag variety of G. However, this argument is not generalized to our case. Although we have a counter part $\mathcal{P}_{u_1}$ of $\mathcal{B}$, and a natural map $\pi_u : H_c^m(\mathcal{P}_{u_1}, \dot{\mathcal{E}}) \to H_c^m(\mathcal{P}_u, \dot{\mathcal{E}})$, there does not exist an immersion $\mathcal{P}_u \hookrightarrow \mathcal{P}_{u_1}$, and the surjectivity of π_u is no longer trivial. In order to overcome such difficulties, following the idea of Lusztig, we appeal to the theory of graded Hecke algebra developed in [L7], which makes it possible to compare the Frobenius actions via the isomorphism $H_c^0(\mathcal{P}_{u_1}, \dot{\mathcal{E}}) \simeq H_c^0(\mathcal{P}_u, \dot{\mathcal{E}}) \simeq \bar{\mathbf{Q}}_l$.

The remaining cases where $G \neq SL_n$ will be treated in a subsequent paper.

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1. Preliminaries

1.1. Let G be a connected reductive algebraic group over a field k, where k is an algebraic closure of a finite filed $\mathbf{F}_q$ of characteristic p. Let C be a unipotent conjugacy class in G, and $\mathcal{E}$ an irreducible local system on C which is G-equivariant for the conjugation action. $\mathcal{E}$ is called a cuspidal local system on C if the following condition is satisfied: for any proper parabolic subgroup P of G with Levi decomposition $P = LU_P$ and for any unipotent element $u \in L$, we have $H_c^{\delta}(uU_P \cap C, \mathcal{E}) = 0$, where $\delta = \dim C - \dim$ (class of u in L) (cf. [L2, 2.4]). It is known by [L3, V, 23.1], that if p is almost good then the above condition is equivalent to the condition that $H_c^i(uU_P \cap C, \mathcal{E}) = 0$ for any i (i.e., $\mathcal{E}$ is strongly cuspidal). We also say that $(C, \mathcal{E})$ is a cuspidal pair in G.

Let $\mathcal{N}_G$ be the set of pairs $(C', \mathcal{E}')$ up to G-conjugacy, where C' is a unipotent class in G and $\mathcal{E}'$ is a G-equivariant irreducible local system on C. We also denote by $\mathcal{M}_G$ the set of triples $(L, C, \mathcal{E})$ up to G-conjugacy, where L is a Levi subgroup of a parabolic subgroup of G, and $\mathcal{E}$ is a cuspidal local system on a unipotent class C of L. In [L2, 6.5], Lusztig has shown that there exists a natural bijection

$$\mathcal{N}_G \simeq \coprod_{(L,C,\mathcal{E}) \in \mathcal{M}_G} (N_G(L)/L)^{\wedge}, \tag{1.1.1}$$

which is called the generalized Springer correspondence between unipotent classes and irreducible characters of various Coxeter groups. (For a finite group H, we denote by $H^{\wedge}$ the set of irreducible characters of H). Note that $N_G(L)/L$ is a Coxeter group with standard generators whenever $(L, C, \mathcal{E}) \in \mathcal{M}_G$.

1.2. We describe the generalized Springer correspondence more precisely. Take $(L, C, \mathcal{E}) \in \mathcal{M}_G$. Let Z_L^0 be the connected center of L, and put $\widetilde{C}_{\text{reg}} = C \cdot (Z_L^0)_{\text{reg}} \subset \widetilde{C} = C \cdot Z_L^0$, where

$$(Z_L^0)_{\text{reg}} = \{z \in Z_L^0 \mid Z_G^0(z) = L\}.$$

We define a diagram

$$\widetilde{C} \xleftarrow{\alpha_1} \widehat{Y} \xrightarrow{\beta_1} \widetilde{Y} \xrightarrow{\pi} Y, \tag{1.2.1}$$

where

$$\begin{aligned}
Y &= \bigcup_{x \in G} x\widetilde{C}_{\text{reg}}x^{-1} \subset G, \\
\widetilde{Y} &= \{(g, xL) \in G \times (G/L) \mid x^{-1}gx \in \widetilde{C}_{\text{reg}}\}, \\
\widehat{Y} &= \{(g, x) \in G \times G \mid x^{-1}gx \in \widetilde{C}\},
\end{aligned}$$

and

$$\alpha_1(g, x) = x^{-1}gx, \quad \beta_1(g, x) = (g, xL), \quad \pi(g, xL) = g.$$

Then Y is a smooth, irreducible subvariety of G, and π is a principal covering of Y with group $\mathcal{W} = N_G(L)/L$. There is a canonical local system $\widetilde{\mathcal{E}}$ on $\widetilde{Y}$ satisfying the property that $\beta_1^*\widetilde{\mathcal{E}} = \alpha_1^*(\mathcal{E} \boxtimes \bar{\mathbf{Q}}_l)$, where $\mathcal{E} \boxtimes \bar{\mathbf{Q}}_l$ is the inverse image of $\mathcal{E}$ under the natural map $\widetilde{C} = C \times Z_L^0 \to C$. We define an intersection cohomology complex K by

$$K = \mathrm{IC}(\overline{Y}, \pi_!\widetilde{\mathcal{E}})[\dim Y] \tag{1.2.2}$$

and regard it as a perverse sheaf on G by extending by 0 outside of $\overline{Y}$. Lusztig showed that K is a G-equivariant semisimple perverse sheaf on G, and that $\operatorname{End} K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}]$. It follows that K can be decomposed as

$$K \simeq \bigoplus_{E \in \mathcal{W}^\wedge} V_E \otimes K_E, \tag{1.2.3}$$

where K_E is a simple perverse sheaf on G such that $V_E = \operatorname{Hom}(K_E, K)$ is an irreducible $\mathcal{W}$-module corresponding to $E \in \mathcal{W}^\wedge$.

Let G_{uni} be the unipotent variety of G. Then $K[-d]|_{G_{\mathrm{uni}}}$ turns out to be a G-equivariant semisimple perverse sheaf on G_{uni}, where $d = \dim Z_L^0 = \dim Y - \dim(Y \cap G_{\mathrm{uni}})$. Hence it is decomposed as

$$K[-d]|_{G_{\mathrm{uni}}} = \bigoplus_{(C', \mathcal{E}') \in \mathcal{N}_G} V_{(C', \mathcal{E}')} \otimes \mathrm{IC}(\overline{C}', \mathcal{E}')[\dim C'], \tag{1.2.4}$$

where $V_{(C', \mathcal{E}')}$ is a multiplicity space for the simple perverse sheaf $\mathrm{IC}(\overline{C}', \mathcal{E}')[\dim C']$ on G_{uni}. Comparing (1.2.3) with (1.2.4), we see that for each $E \in \mathcal{W}^\wedge$, there exists a pair $(C', \mathcal{E}') \in \mathcal{N}_G$ such that

$$K_E|_{G_{\mathrm{uni}}} \simeq \mathrm{IC}(\overline{C}', \mathcal{E}')[\dim C' + \dim Z_L^0]. \tag{1.2.5}$$

The correspondence $E \mapsto (C', \mathcal{E}')$ gives a bijection $\coprod_{(L,C,\mathcal{E})} (N_G(L)/L)^\wedge \to \mathcal{N}_G$ in (1.1.1).

1.3. We now consider the $\mathbf{F}_q$-structure on G. So assume that G is defined over $\mathbf{F}_q$ with Frobenius endomorphism $F : G \to G$. Then F acts naturally on the set $\mathcal{N}_G$ and $\mathcal{M}_G$ by $(C', \mathcal{E}') \mapsto (F^{-1}C', F^*\mathcal{E}')$, $(L, C, \mathcal{E}) \mapsto (F^{-1}L, F^{-1}C, F^*\mathcal{E})$, and the map in (1.1.1) is compatible with F-action. Now assume that $(L, C, \mathcal{E}) \in \mathcal{M}_G$ is F-stable. Then we may choose $(L, C, \mathcal{E})$, as a representative of its G-conjugacy class, such that L is an F-stable Levi subgroup of an F-stable parabolic subgroup P of G, with $FC = C, F^*\mathcal{E} \simeq \mathcal{E}$. We choose an isomorphism $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which induces a map of finite order on the stalk of $\mathcal{E}$ at any point of C^F. Since the diagram in (1.2.1), and so the construction of the complex K is compatible with $\mathbf{F}_q$-structure, φ_0 induces a natural isomorphism $\varphi : F^*K \xrightarrow{\sim} K$. We consider the characteristic function $\chi_{K,\varphi}$ of K. The restriction of $\chi_{K,\varphi}$ on G_{uni} gives a G^F-invariant function on G^F_{uni}, which is the generalized Green function $Q^G_{L,C,\mathcal{E},\varphi_0}$ (cf. [L3, II]).

Here F acts naturally on $\mathcal{W}$, which induces a Coxeter group automorphism of degree, say c. We consider the semidirect product $\widetilde{\mathcal{W}} = \mathcal{W} \rtimes (\mathbf{Z}/c\mathbf{Z})$. If an irreducible representation V_E of $\mathcal{W}$ is F-stable, it can be extended to an irreducible representation of $\widetilde{\mathcal{W}}$, in c different ways. Assume that $E \in \mathcal{W}^\wedge$ is F-stable. Then the corresponding $(C', \mathcal{E}') \in \mathcal{N}_G$ is also F-stable, and we have $F^*K_E \xrightarrow{\sim} K_E$. A choice of an isomorphism $\varphi_E : F^*K_E \xrightarrow{\sim} K_E$ induces a bijection $\sigma_E : V_E \to V_E$, which makes V_E into an irreducible $\widetilde{\mathcal{W}}$-module $\widetilde{V}_E$. We choose φ_E so that $\widetilde{V}_E$ turns out to be a preferred extension of V_E (cf. [L3, IV, (17.2)]. By making use of $\varphi_E : F^*K_E \simeq K_E$, we shall define an isomorphism $\psi : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ as follows; By (1.2.5), we have $\mathcal{H}^{a_0}(K_E)|_{C'} = \mathcal{E}'$ for $a_0 = -\dim Z_L^0 - \dim C'$. We define ψ so that $q^{(a_0+r)/2}\psi$ corresponds to the map defined by $\varphi_E : F^*\mathcal{H}^{a_0}(K_E) \xrightarrow{\sim} \mathcal{H}^{a_0}(K_E)$, where

$$r = \dim Y = \dim G - \dim L + \dim(C \times Z_L^0),$$

and so

$$a_0 + r = (\dim G - \dim C') - (\dim L - \dim C). \tag{1.3.1}$$

We define a function Y_j on G^F_{uni} for each $j = (C', \mathcal{E}') \in \mathcal{N}_G^F$ by

$$Y_j(g) = \begin{cases} \mathrm{Tr}\,(\psi, g) & \text{if } g \in C'^F, \\ 0 & \text{if } g \notin C'^F. \end{cases}$$

Then $\{Y_j \mid j \in \mathcal{N}_G^F\}$ gives rise to a basis of the space of G^F-invariant functions on G^F_{uni}. Now the computation of $\chi_{K,\varphi}$ is reduced to the computation of χ_{K_E,φ_E} for each F-stable irreducible character $\mathcal{E}$ of $\mathcal{W}$. We denote χ_{K_E,φ_E} by X_j if E corresponds to $j = (C', \mathcal{E}')$ under the generalized Springer correspondence. In [L3, V], Lusztig gave a general algorithm of expressing X_i as an explicit linear combination of various Y_j. Thus the computation of $\chi_{K,\varphi}$ is reduced to the computation of Y_j.

We shall describe the functions Y_j. Let us choose $u \in C'^F$, and put $A_G(u) = Z_G(u)/Z_G^0(u)$. Then F acts naturally on $A_G(u)$, and the set of G-equivariant simple local systems on C' is in bijective correspondence with the set of F-stable irreducible characters of $A_G(u)$. Let us denote by ρ the irreducible character of $A_G(u)$ corresponding to $\mathcal{E}'$. Let σ be the restriction of F on $A_G(u)$. Then ρ can be extended to an irreducible character of the semidirect group $\widetilde{A}_G(u) = A_G(u) \rtimes \langle\sigma\rangle$. We choose an extension $\widetilde{\rho}$ of ρ. $\mathcal{E}'_u$ has a structure of $A_G(u)$-module affording the character ρ, which is extended to the $\widetilde{A}_G(u)$-module affording $\widetilde{\rho}$. We choose an isomorphism $\psi_0 : F^*\mathcal{E}' \xrightarrow{\sim} \mathcal{E}'$ by the condition that ψ_0 induces an isomorphsim on $\mathcal{E}'_u$ corresponding to the action of σ on $\widetilde{\rho}$.

Since $\mathcal{E}'$ is a simple local system, there exists $\gamma \in \bar{\mathbf{Q}}_l^*$ (depending on the choice of φ_0, u and $\widetilde{\rho}$) such that $\psi = \gamma\psi_0$. We define functions Y_j^0 on the set G^F_{uni} in a similar way as Y_j, but replacing ψ by ψ_0. Then clearly we have $Y_j = \gamma Y_j^0$. We note that the functions Y_j^0 are described in an explicit way as follows. The set of G^F-conjugacy classes in C'^F is in bijective correspondence with the set of F-twisted conjugacy classes in $A_G(u)$. We denote by u_a a representative in the G^F-conjugacy class contained in C'^F corresponding to an F-twisted conjugacy class in $A_G(u)$ containing a. Then we have

$$Y_j^0(g) = \begin{cases} \widetilde\rho(a\sigma) & \text{if } g \text{ is } G^F\text{-conjugate to } u_a, \\ 0 & \text{if } g \notin C'^F. \end{cases}$$

It follows from the above discussion that the computation of generalized Green functions is reduced to the determination of the scalar constant γ for each pair $(C', \mathcal{E}') \in \mathcal{N}_G^F$. Let us choose $v \in C^F$, and let ρ_0 be the F-stable irreducible character of $A_L(v)$ corresponding to $\mathcal{E}$. Then as in the discussion above, the isomorpshims $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is given by choosing an extension $\widetilde\rho_0$ of ρ_0 to the semidirect group $\widetilde A_L(v) = A_L(v) \rtimes \langle\sigma\rangle$. Thus γ is determined by $v, \widetilde\rho_0, u, \widetilde\rho$, which we denote by $\gamma = \gamma(v, \widetilde\rho_0, u, \widetilde\rho)$. The purpose of this paper is to describe the constants $\gamma(v, \widetilde\rho_0, u, \widetilde\rho)$ explicitly.

1.4. In order to make the Frobenius action more explicit, we shall consider the following varieties. Put

$$\begin{aligned} \mathcal{P}_u &= \{gP \in G/P \mid g^{-1}ug \in CU_P\}, \\ \widehat{\mathcal{P}}_u &= \{g \in G \mid g^{-1}ug \in CU_P\}, \end{aligned} \tag{1.4.1}$$

and consider the diagram

$$C \xleftarrow{\ \alpha\ } \widehat{\mathcal{P}}_u \xrightarrow{\ \beta\ } \mathcal{P}_u \tag{1.4.2}$$

with

$$\alpha : g \mapsto C\text{-component of } g^{-1}ug \in CU_P, \quad \beta : g \mapsto gP.$$

We define a local system $\dot{\mathcal{E}}$ on $\mathcal{P}_u$ by the property that $\alpha^*\mathcal{E} = \beta^*\dot{\mathcal{E}}$. Then it is known by [L3, 24.2.5] that

$$\mathcal{H}_u^{a_0}(K) \simeq H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}}). \tag{1.4.3}$$

It is also known by [L2, 1.2 (b)] that $\dim \mathcal{P}_u \le (a_0 + r)/2$. Since the left hand side of (1.4.3) is non-zero by (1.2.5), we see that

$$\dim \mathcal{P}_u = (a_0 + r)/2. \tag{1.4.4}$$

Since P is F-stable, $\mathcal{P}_u, \widehat{\mathcal{P}}_u$ are F-stable, and the diagram in (1.4.2) is compatible with Frobenius maps. Moreover, the isomorphism φ_0 induces an isomorphism $\dot\varphi_0 : F^*\dot{\mathcal{E}} \xrightarrow{\sim} \dot{\mathcal{E}}$. This induces a linear map Φ on $V = H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}})$. By (1.4.3), $\mathcal{W}$ acts on V. Also $Z_G(u)$ acts naturally on V, where $Z_G^0(u)$ acts trivially on it. Then it induces an action of $A_G(u)$, which commutes with the action of $\mathcal{W}$. Let ρ be an F-stable irreducible character of $A_G(u)$ corresponding to $\mathcal{E}'$ as in 1.3, and V_ρ the ρ-isotipic part of V. Then Φ leaves V_ρ stable. The previous discussion shows that

V_ρ can be identified with $\widetilde{V}_E \otimes \mathcal{E}'_u$, and $\Phi|_{V_\rho}$ coincides with $\sigma_E \otimes q^{(q_0+r)/2}\psi$. Thus the map ψ can be described by investigating Φ on $H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}})_\rho$.

1.5. We show that the description of the mixed structure $\psi : F^*\mathcal{E}' \to \mathcal{E}'$ on C' is reduced to the case where G is simply connected, almost simple. In fact, let $\pi : G \to G' = G/Z_G^0$ be the natural homomorphism. Then π induces a bijection between $\mathcal{M}_G$ (resp. $\mathcal{N}_G$) and $\mathcal{M}_{G'}$ (resp. $\mathcal{N}_{G'}$) which commutes with their $\mathbf{F}_q$-structures. Hence we may assume that G is semisimple. Let $\widetilde{\pi} : \widetilde{G} \to G$ be the simply connected covering of G. Then $(L, C, \mathcal{E}) \mapsto (\widetilde{\pi}^{-1}(L), C, \widetilde{\pi}^*\mathcal{E})$ gives a bijection between the set $\mathcal{M}_G$ and the subset of $\mathcal{M}_{\widetilde{G}}$ on which $\ker \widetilde{\pi}$ acts trivially. Hence the mixed structure $\varphi_0 : F^*\widetilde{\pi}^*\mathcal{E} \to \widetilde{\pi}^*\mathcal{E}$ for the pair $(C, \widetilde{\pi}^*\mathcal{E})$ on $\widetilde{G}$ determines the mixed structure for the pair $(C, \mathcal{E})$ on G. Similarly, $\widetilde{\pi}$ induces a bijection between the set $\mathcal{N}_G$ and the subset of $\mathcal{N}_{\widetilde{G}}$ on which $\ker \widetilde{\pi}$ acts trivially, and so the mixed structure of the pair $(C', \mathcal{E}')$ on G is determined by the mixed structure of the pair $(C', \widetilde{\pi}^*\mathcal{E}')$ on $\widetilde{G}$. The procedure of determining the mixed structure of $(C', \mathcal{E}')$ from that of $(C, \mathcal{E})$ is parallel for $\widetilde{G}$ and G.

It follows from the above discussion that we may assume G is simply connected, semisimple. Then G is isomorphic to the direct product of simply connected, almost simple groups, with F-action. Now it is easy to see that we are reduced to the case where $G \simeq G_1 \times \cdots \times G_r$, with G_i a copy of G_1, and F acts on G as a cyclic permutation of all the factors. Then G_1 is F^r-stable, and the set $\mathcal{M}_G^F$ is in bijective correspondence with the set $\mathcal{M}_{G_1}^{F^r}$, via the correspondence $(L, C, \mathcal{E}) \leftrightarrow (L_1, C_1, \mathcal{E}_1)$, where

$$
\begin{aligned}
L &= L_1 \times F^{-r+1}(L_1) \times \cdots \times F^{-1}(L_1), \\
C &= C_1 \times F^{-r+1}(C_1) \times \cdots \times F^{-1}(C_1), \\
\mathcal{E} &= \mathcal{E}_1 \boxtimes F^{r-1^*}\mathcal{E}_1 \boxtimes \cdots \boxtimes F^*\mathcal{E}_1.
\end{aligned}
$$

Moreover, $C_1^{F^r} \simeq C^F$ via $v_1 \mapsto v = (v_1, F(v_1), \dots, F^{r-1}(v_1))$. Then $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is determined by $\varphi_1 : F^{r*}\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$ as

$$
(\varphi_0)_v = (\varphi_1)_{v_1} \otimes (\varphi_1)_{F^{r-1}(v_1)} \otimes \cdots \otimes (\varphi_1)_{F(v_1)}
$$

on $\mathcal{E}_v = (\mathcal{E}_1)_{v_1} \otimes (\mathcal{E}_1)_{F^{r-1}(v_1)} \otimes \cdots \otimes (\mathcal{E}_1)_{F(v_1)}$. Similarly, the mixed F-structure of $(C', \mathcal{E}') \in \mathcal{N}_G$ is described by the mixed F^r-structure of $({C'}_1, {\mathcal{E}'}_1) \in \mathcal{N}_{G_1}$.

Thus, the determination of the mixed structure of $(C', \mathcal{E}')$ is reduced to the case where G is an F-stable, simply connected, almost simple group.

1.6. Assume that G is almost simple and simply connected. Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G. We further assume that p is good for G unless G is of type A, and that $p > n$ if $G = SL_n$. Then by [BR], there exists a logarithm map $\log : G \to \mathfrak{g}$ satisfying the following properties; log is an $Ad(G)$-equivariant morphism and $\log(1) = 0, d(\log)_1 : \mathfrak{g} \to \mathfrak{g}$ is the identity map. In particular, for any closed subgroup H of G, $\log(H) \subset \operatorname{Lie} H \subset \mathfrak{g}$. Moreover, $\log|_{G_{\text{uni}}}$ turns out to be an isomorphism $G_{\text{uni}} \to \mathfrak{g}_{\text{nil}}$, where $\mathfrak{g}_{\text{nil}}$ is the nilpotent variety of $\mathfrak{g}$.

Let $\mathcal{L}$ be an irreducible G-local system on a nilpotent orbit $\mathcal{C}$ in $\mathfrak{g}$. The notion of cuspidal local system on $\mathcal{C}$ is defined in a similar way as in the case of groups, i.e., $\mathcal{L}$ is said to be cuspidal or $(\mathcal{C}, \mathcal{L})$ is a cuspidal pair if for any proper parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n}_1$ and any $y \in \mathfrak{p}_1$, we have $H^i_c((y + \mathfrak{n}_1) \cap \mathcal{C}, \mathcal{L}) = 0$ for any i. Then it is easily checked (cf. [L4]) that $\log^*$ gives a bijection between the set of cuspidal pairs in G and the set of cuspidal pairs in $\mathfrak{g}$.

Let $(L, C, \mathcal{E}) \in \mathcal{M}_G$, and $(\mathcal{C}, \mathcal{L})$ the corresponding cuspidal pair in $\mathfrak{l} = \operatorname{Lie} L$, where $C = \log^{-1}(\mathcal{C}), \mathcal{E} = \log^* \mathcal{L}$. We put $\mathfrak{p} = \operatorname{Lie}(P)$ and $\mathfrak{n}_P = \operatorname{Lie} U_P$. Let $\mathcal{C}' = \log(C')$ be a nilpotent orbit in $\mathfrak{g}$. For each $y \in \mathcal{C}'$, put

$$\begin{aligned} \mathcal{P}_y &= \{gP \in G/P \mid \operatorname{Ad}(g)^{-1} y \in \mathcal{C} + \mathfrak{n}_P\}, \\ \widehat{\mathcal{P}}_y &= \{g \in G \mid \operatorname{Ad}(g)^{-1} y \in \mathcal{C} + \mathfrak{n}_P\}. \end{aligned} \tag{1.6.1}$$

Then by using a similar diagram as in (1.4.2), one can define a local system $\dot{\mathcal{L}}$ on $\mathcal{P}_y$. It is easy to see that log gives an isomorphism $\widehat{\mathcal{P}}_u \,\tilde{\to}\, \widehat{\mathcal{P}}_y$ with $y = \log(u)$, and so induces an isomorphism $\mathcal{P}_u \,\tilde{\to}\, \mathcal{P}_y$. Then we have $\log^* \dot{\mathcal{L}} = \dot{\mathcal{E}}$. It follows that we have a canonical isomorphism

$$H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}}) \simeq H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{1.6.2}$$

In the case where G has an $\mathbf{F}_q$-structure with Frobenius map F, $\mathfrak{g}$ has also an action of F, and we may assume that log is F-equivariant. Then the isomorphism (1.6.2) is compatible with $\mathbf{F}_q$-structures. We denote by the same symbol Φ the linear map on $H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}})$ obtained as in the case of $H_c^{a_0+r}(\mathcal{P}_u, \dot{\mathcal{E}})$. Hence the linear map $q^{(a_0+r)/2}\psi$ on $\mathcal{E}_u$ can be described in terms of the Frobenius action Φ on $H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}})_\rho$.

2. Graded Hecke algebras

2.1. The graded Hecke algebra $\mathbf{H}$ was introduced by Lusztig [L7], which is a degenerate version of affine Hecke algebras. In this section, following [L7] we review the definition of $\mathbf{H}$ and their representations on equivariant K-homology groups. In [L7], $\mathbf{H}$ are constructed as algebras over $\mathbf{C}$, but here we regard them as algebras over $\bar{\mathbf{Q}}_l$ so that one can relate them to l-adic cohomology groups.

Let Φ be a root system with a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_m\}$ and W the Weyl group of Φ with corresponding simple reflections $\{s_1, \dots, s_m\}$. We assume that the root lattice $\mathbf{Z}\Phi$ is embedded in a vector space $\mathfrak{h}^*$ over $\bar{\mathbf{Q}}_l$. The action of W on $\mathbf{Z}\Phi$ makes $\mathfrak{h}^*$ into a W-module. (Hence $\mathfrak{h}^*$ has a direct sum decomposition, one summand being W-invariant, the other having Π as a basis). Let $\mathbf{S}$ be the symmetric algebra of $\mathfrak{h}^* \oplus \bar{\mathbf{Q}}_l$. We denote $\mathbf{r} = (0, 1) \in \mathfrak{h}^* \oplus \bar{\mathbf{Q}}_l$, so that $\mathbf{S} = S(\mathfrak{h}^*) \otimes \bar{\mathbf{Q}}_l[\mathbf{r}]$. W acts naturally on $\mathbf{S}$ so that $\mathbf{r}$ is left invariant by W. We denote by $\xi \mapsto {}^w\xi$ the action of W on $\mathcal{S}$. Let $c_1, \dots, c_m$ be integers ≥ 2 such that $c_i = c_j$ whenever s_i and s_j are conjugate in W. Let e be the neutral element of W. Lusztig showed in [L7, Theorem 6.3] that there is a unique structure of associative $\bar{\mathbf{Q}}_l$-algebra on the $\bar{\mathbf{Q}}_l$-vector space $\mathbf{H} = \mathbf{S} \otimes \bar{\mathbf{Q}}_l[W]$ with unit $1 \otimes e$ such that

(i) $\xi \mapsto \xi \otimes e$ is an algebra homomorphism $\mathbf{S} \to \mathbf{H}$,
(ii) $w \mapsto 1 \otimes w$ is an algebra homomorphism $\bar{\mathbf{Q}}_l[W] \to \mathbf{H}$,
(iii) $(\xi \otimes e) \cdot (1 \otimes w) = \xi \otimes w, \quad (\xi \in \mathbf{S}, w \in W)$,
(iv) $(1 \otimes s_i)(\xi \otimes e) - ({}^{s_i}\xi \otimes e)(1 \otimes s_i) = c_i \mathbf{r} \dfrac{\xi - {}^{s_i}\xi}{\alpha_i} \otimes e, \quad (\xi \in \mathbf{S}, 1 \le i \le m)$.

$\mathbf{H}$ is called a graded Hecke algebra attached to W with parameters c_i. It follows from (iv) we see that $\mathbf{r}$ is in the center of $\mathbf{H}$.

2.2. The discussion in [L7] is concerned with algebraic groups over $\mathbf{C}$. Hence the equivariant K-homology is defined for the varieties over $\mathbf{C}$. Since we treat algebraic groups over finite fields, we need to construct the equivariant K-homology based on the l-adic cohomology groups. Fortunately, the basic properties established in section 1 in [L7] work well also for our situation, by a suitable modification. We give some comments below.

Let G be an affine algebraic group over k, and let X be a k-variety on which G acts algebraically. As in [L7], for each integer $m \ge 1$, there exists a smooth irreducible variety $\varGamma$ with free G-action such that $\varGamma \to G\backslash\varGamma$ has a locally trivial principal G-fibration, and that $H^i(\varGamma, \bar{\mathbf{Q}}_l) = 0$ for $i = 1, \dots, m$. (As in [L7, 1.1], we embed G as a closed subgroup of GL_r, and consider the embedding

$$G \subset GL_r \times \{e\} \subset GL_r \times GL_{r'} \subset GL_{r+r'}. \tag{2.2.1}$$

Then $\varGamma = (\{e\} \times GL_{r'})\backslash GL_{r+r'}$ for large r' ($2r' \ge m+2$), with the left action of G on $\varGamma$, satisfies the required condition.) For a G-variety X, we consider ${}_\varGamma X = G\backslash(\varGamma \times X)$ (the quotient by the diagonal action of G). Then for an G-equivariant local system $\mathcal{L}$ on X, there exists a unique local system ${}_\varGamma\mathcal{L}$ on ${}_\varGamma X$ such that $\pi^*({}_\varGamma\mathcal{L}) = p^*\mathcal{L}$, where $\pi : \varGamma \times X \to G\backslash(\varGamma \times X)$ is a natural map, and $p : \varGamma \times X \to X$ is a projection. Then as in [L7], we define

$$H^j_G(X, \mathcal{L}) = H^j({}_\varGamma X, {}_\varGamma\mathcal{L}), \quad H^G_j(X, \mathcal{L}) = H^{2d-j}_c({}_\varGamma X, {}_\varGamma\mathcal{L}^*)^*,$$

where $d = \dim({}_\varGamma X)$, and the upper-script $*$ denotes the dual local system or the dual vector space. (We understand that $H^j_G(X, \mathcal{L}) = H^j(X, \mathcal{L})$ and $H^G_j(X, \mathcal{L}) = H_c^{2\dim X - j}(X, \mathcal{L}^*)^*$ in the case where $G = \{e\}$.) We write them as $H^j_G(X), H^G_j(X)$ if $\mathcal{L}$ is a constant sheaf $\bar{\mathbf{Q}}_l$. Also we write $H^i_c(X)$, $H^i(X)$ instead of $H^i_c(X, \bar{\mathbf{Q}}_l)$, $H^i(X, \bar{\mathbf{Q}}_l)$.

By cup-product, $H^*_G(X) = \bigoplus_j H^j_G(X)$ becomes a graded $\bar{\mathbf{Q}}_l$-algebra with 1, and

$$H^*_G(X, \mathcal{L}) = \bigoplus_j H^j_G(X, \mathcal{L}), \quad H^G_*(X, \mathcal{L}) = \bigoplus_j H^G_j(X, \mathcal{L})$$

become graded $H^*_G(X)$-modules.

We write H^*_G, H^G_* instead of $H^*_G(\text{point}), H^G_*(\text{point})$. Then the map $X \to \text{point}$ defines a $\bar{\mathbf{Q}}_l$-algebra homomorphism $\varepsilon : H^*_G \to H^*_G(X)$ preserving the grading. Via the map ε, $H^*_G(X, \mathcal{L}), H^G_*(X, \mathcal{L})$ can be regarded also as H^*_G-modules.

2.3. Let T be a torus and $X(T)$ be its character group. The arguments in 1.10 in [L7] do not hold in that form. We modify them as follows. In the case where $T \simeq \mathbf{G}_m$ is the one dimensional torus, it can be verified directly by the definition that $H_T^* \simeq \bar{\mathbf{Q}}_l[x]$, a polynomial ring with one variable, with $x \in H_T^2$. Since $H_{G\times G'}^* \simeq H_G^* \otimes H_{G'}^*$, we see that $H_T^* \simeq S(V^*)$, the symmetric algebra of a $\bar{\mathbf{Q}}_l$-vector space $V^* = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(T)$. In particular, we have

$$H_T^{2j} \simeq S^j(V^*), \quad H_T^{2j+1} = 0,$$

and we may identify H_T^2 with V^*. ($S^j(V^*)$ denotes the degree j-part of $S(V^*)$).

For $\chi \in X(T)$, let k_χ be the T-module k with the T-action by $(t, z) \mapsto \chi(t)z$. Let $i : \{0\} \hookrightarrow k, \pi : k \to \{0\}$ be the obvious maps. Then π^* is an isomorphism, and the composition

$$H_*^T(\{0\}) \xrightarrow{i_!} H_*^T(k_\chi) \xrightarrow{(\pi^*)^{-1}} H_*^T(\{0\})$$

is H_T^*-linear of degree 2. Since $H_*^T(\{0\}) \simeq H_T^*$ as H_T^*-modules, $(\pi^*)^{-1} \circ i_!$ is given by multiplication by an element $c(\chi) \in H_T^2$ (cf. [L7, 1.10]). The map $c : X(T) \to H_T^2 = V^*, \chi \mapsto c(\chi)$ gives an injective group homomorphism.

Assume that G is an algebraic group such that G^0 is a torus T. Then $W = G/G^0$ acts naturally on H_T^*, preserving the grading (see [L7, 1.9]). W acts also on $X(T)$, and we have

(2.3.1) The map $c : X(T) \to H_T^2 = V^*$ is W-equivariant.

In fact, take $\varGamma$ on which G acts freely. Then, for a representative $\dot{w} \in G$ of $w \in W$, the map $\varGamma \times k_\chi \to \varGamma \times k_{w(\chi)}, (g, x) \mapsto (\dot{w}g, x)$ induces a map $f_w : T\backslash(\varGamma \times k_\chi) \to T\backslash(\varGamma \times k_{w(\chi)})$, which makes the following diagram commutative.

$$\begin{array}{ccccc}
H_*^T(\{0\}) & \xrightarrow{i_!} & H_*^T(k_\chi) & \xrightarrow{(\pi^*)^{-1}} & H_*^T(\{0\}) \\
{\scriptstyle w}\downarrow & & \downarrow{\scriptstyle (f_w^*)^{-1}} & & \downarrow{\scriptstyle w} \\
H_*^T(\{0\}) & \xrightarrow{i_!} & H_*^T(k_{w(\chi)}) & \xrightarrow{(\pi^*)^{-1}} & H_*^T(\{0\}).
\end{array}$$

(2.3.1) follows from this.

It follows from (2.3.1) that we have

$$H_T^* \simeq S(\bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(T)) \tag{2.3.2}$$

as graded W-modules.

We don't know whether the counter part of 1.11 in [L7] holds in our setting. However, the following related fact holds.

Lemma 2.4. *Assume that G is a connected algebraic group. Let G_r be a maximal reductive subgroup of G, and T a maximal torus of G_r. Let $W = N_{G_r}(T)/T$ be the Weyl group of G_r. Then W acts naturally on H_T^*, and the natural map $H_G^* \to H_T^*$ (cf. [L7, 1.4 (g)]) induced from the inclusion $T \hookrightarrow G$ gives an isomorphism*

$$H_G^* \xrightarrow{\sim} (H_T^*)^W.$$

Proof. By [L7, 1.4 (h)], we know that $H^*_G \xrightarrow{\sim} H^*_{G_r}$. Hence it is enough to show the lemma in the case where G is reductive. Assume that $G = G_r$. Let m be a large integer and let $\varGamma$ be an irreducible, smooth variety with a free G-action such that $H^i(\varGamma) = 0$ for $1 \le i \le m$. We consider the map $f : T\backslash\varGamma \to G\backslash\varGamma$, which is a locally trivial fibration with fibre isomorphic to $T\backslash G$. We have a spectral sequence

$$H^p(G\backslash\varGamma, R^q f_*\bar{\mathbf{Q}}_l) \Rightarrow H^{p+q}(T\backslash\varGamma). \tag{2.4.1}$$

The map f is W-equivariant with respect to the trivial action of W on $G\backslash\varGamma$, and the left action of W on $T\backslash\varGamma$, and so $R^q f_*\bar{\mathbf{Q}}_l$ has a structure of W-sheaf, which induces an action of W on $H^p(G\backslash\varGamma, R^q f_*\bar{\mathbf{Q}}_l)$. W acts naturally on $H^{p+q}(T\backslash\varGamma)$, and by taking the W-invariant parts in (2.4.1), we have a spectral sequence

$$H^p(G\backslash\varGamma, R^q f_*\bar{\mathbf{Q}}_l)^W \Rightarrow H^{p+q}(T\backslash\varGamma)^W. \tag{2.4.2}$$

Since f is a locally trivial fibration, $R^q f_*\bar{\mathbf{Q}}_l$ is a local system with fibre $H^q(T\backslash\varGamma)$. We may assume that $\varGamma = (\{e\} \times GL_{r'})\backslash GL_{r+r'}$ as in 2.2. Then f is $GL_{r+r'}$-equivariant, and so $R^q f_*\bar{\mathbf{Q}}_l$ is a $GL_{r+r'}$-local system on the space $G\backslash\varGamma$ (with respect to the right action of $GL_{r+r'}$). Now $GL_{r+r'}$ acts transitively on $G\backslash\varGamma$ with a stabilizer of a point isomorphic to $G \times GL_{r'}$. Since G is connected, we see that $R^q f_*\bar{\mathbf{Q}}_l$ is a constant sheaf $H^q(T\backslash\varGamma)$. It follows that

$$H^p(G\backslash\varGamma, R^q f_*\bar{\mathbf{Q}}_l) \simeq H^p(G\backslash\varGamma) \otimes H^q(T\backslash G)$$

and we have

$$H^p(G\backslash\varGamma, R^q f_*\bar{\mathbf{Q}}_l)^W \simeq H^p(G\backslash\varGamma) \otimes H^q(T\backslash G)^W$$

since W acts trivially on $H^p(G\backslash\varGamma)$. It is known that $H^*(T\backslash G)$ is a graded regular W-module, and

$$H^q(T\backslash G)^W = \begin{cases} \bar{\mathbf{Q}}_l & \text{if } q = 0, \\ 0 & \text{otherwise .} \end{cases}$$

Hence the spectral sequence (2.4.2) collapses, and we have

$$H^p(G\backslash\varGamma) \simeq H^p(T\backslash\varGamma)^W. \tag{2.4.3}$$

This isomorphism is induced from the natural map $H^p(G\backslash\varGamma) \to H^p(T\backslash\varGamma)$. Since $H^p_G = H^p(G\backslash\varGamma)$, and $H^p_T = H^p(T\backslash\varGamma)$ by definition, the lemma follows from (2.4.3). □

For later discussion, we note the following.

Corollary 2.5. *Assume that G is connected reductive, and let T, W be as before. Let L be a Levi subgroup of a parabolic subgroup of G containing T. Assume further that L contains a cuspidal pair as in 1.1. Put $\mathcal{W} = N_G(Z_L^0)/L = N_G(L)/L$. Then the image of the natural map $H^*_G \to H^*_{Z_L^0}$ coincides with $(H^*_{Z_L^0})^{\mathcal{W}}$.*

Proof. The inclusions $Z_L^0 \hookrightarrow T \hookrightarrow G$ induces the maps $H^*_G \to H^*_T \to H^*_{Z_L^0}$. Put $V^* = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(T)$, $V_1^* = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(Z_L^0)$. Then by (2.3.2), the map $H^*_T \to H^*_{Z_L^0}$ is nothing but the natural map $\varphi : S(V^*) \to S(V_1^*)$ obtained from the restriction map $X(T) \to X(Z_L^0)$. Now $W, \mathcal{W}$ acts naturally on $S(V^*), S(V_1^*)$, respectively. Since $\mathcal{W} \simeq N_W(W_L)/W_L$, φ induces a map $\widetilde{\varphi} : S(V^*)^W \to S(V_1^*)^{\mathcal{W}}$. By [L7, Proposition 2.6], Z_L^0 coincides with a maximal torus of a certain connected reductive subgroup H of G, and $\mathcal{W}$ is regarded as the Weyl group of H. Thus in view of Lemma 2.4, it is enough to show that $\widetilde{\varphi}$ is surjective. This is equivalent to the fact that $V_1/\mathcal{W} \to V/W$ is a closed embedding, where V is the dual space of V^* which is identified with the Lie algebra of the torus $T_{\bar{\mathbf{Q}}_l}$ over $\bar{\mathbf{Q}}_l$, and similarly for V_1. But by using the classification of the triple $(L, C, \mathcal{E}) \in \mathcal{M}_G$, it is checked that $V_1/\mathcal{W} \to V/W$ is a closed embedding. Thus the corollary follows. □

2.6. 1.12 (a), (b) in [L7] were deduced by using 1.11 there. Here we show the corresponding facts by using 2.3 as follows.

(2.6.1) Let G be an algebraic group such that G^0 is a central torus in G. Then we have

$$H_G^* \simeq H_{G^0}^*.$$

In fact, by [L7, 1.9 (a)], we have

$$H_G^* \simeq (H_{G^0}^*)^{G/G^0}.$$

But $H_{G^0}^* \simeq S(V^*)$ with $V^* = H_{G^0}^2$, and the action of G/G^0 on $S(V^*)$ is determined by the action of G/G^0 on $X(G^0)$ by (2.3.2). By our assumption, G/G^0 acts trivially on $X(G^0)$, and so on $S(V^*)$. This implies that $H_G^* \simeq S(V^*) \simeq H_{G^0}^*$, and (2.6.1) follows.

(2.6.2) In the same setting as above, let E be an irreducible representation of G/G^0 over $\bar{\mathbf{Q}}_l$. Then we have

$$H_*^G(\text{point}, E \otimes E^*) \simeq H_*^{G^0}.$$

The proof is similar to [L7, 1.12 (b)], by making use of (2.6.1).

2.7. We return to the setting in 1.1, and consider a connected reductive algebraic group G, and its Lie algebra $\mathfrak{g}$. We further assume that G is almost simple, simply connected. Let $\mathbf{G}_m$ be the multiplicative group of k. Then G acts on $\mathfrak{g}$ by the adjoint action, and $G \times \mathbf{G}_m$ acts on $\mathfrak{g}$ by $(g_1, t) : x \mapsto t^{-2} \operatorname{Ad}(g_1)x$. For $x \in \mathfrak{g}$, we denote by $Z_G(x)$ the stabilizer of x in G, and by $M_G(x)$ the stabilizer of x in $G \times \mathbf{G}_m$. Hence

$$M_G(x) = \{(g_1, t) \in G \times \mathbf{G}_m \mid Ad(g_1)x = t^2 x\}.$$

We assume that p is large enough so that Jacobson-Morozov's theorem and Dynkin-Kostant theory hold for $\mathfrak{g}$, (e.g., $p > 3(h-1)$, where h is the Coxeter number of W, [C, 5.5]). Then, for each nilpotent element $y \in \mathfrak{g}$, there exists a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$, and elements $y^-, h \in \mathfrak{g}$ such that

$$y = \phi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y^- = \phi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \phi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we have $[h, y] = 2y, [h, y^-] = -2y^-, [y, y^-] = h$. Moreover, we have a decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the i-eigenspace of $\operatorname{ad} h : \mathfrak{g} \to \mathfrak{g}$. In particular, note that $y \in \mathfrak{g}_2, y^- \in \mathfrak{g}_{-2}$. One can define an algebra homomorphism $\rho' : \mathbf{G}_m \to \operatorname{Aut} \mathfrak{g}$ by $\rho'(t)z = t^i z$ for $z \in \mathfrak{g}_i$. Since the identity component of $\operatorname{Aut} \mathfrak{g}$ coincides with $\operatorname{ad} G = G/Z_G$, $\rho'(\mathbf{G}_m)$ is a one-dimensional torus in $\operatorname{ad} G$. By taking the identity component of $\pi^{-1}(\rho'(\mathbf{G}_m))$ for $\pi : G \to \operatorname{ad} G$, one obtains a one parameter subgroup $\rho : \mathbf{G}_m \to G$ such that $\rho' = \pi \circ \rho$.

We put

$$
\begin{aligned}
Z_G(\phi) &= Z_G(y) \cap Z_G(y^-),\\
M_G(\phi) &= \{(g_1, t) \in G \times \mathbf{G}_m \mid \operatorname{Ad}(g_1)y = t^2 y, \operatorname{Ad}(g_1)y^- = t^{-2}y^-\}.
\end{aligned}
$$

It is known that $Z_G(\phi)$ is a maximal reductive subgroup of $Z_G(y)$. It is easy to check that $(g_1, t) \mapsto (g_1\rho(t), t)$ gives isomorphisms of algebraic groups

$$
Z_G(y) \times \mathbf{G}_m \xrightarrow{\sim} M_G(y), \qquad Z_G(\phi) \times \mathbf{G}_m \xrightarrow{\sim} M_G(\phi). \tag{2.7.1}
$$

Hence $M_G(\phi)$ is also a maximal reductive subgroup of $M_G(y)$. It also follows from (2.7.1) that the embedding $Z_G(y) \hookrightarrow M_G(y)$ by $g_1 \mapsto (g_1, 1)$ induces an isomorphism

$$
Z_G(y)/Z_G^0(y) \xrightarrow{\sim} M_G(y)/M_G^0(y).
$$

This implies that the G-orbit of $x \in \mathfrak{g}$ is also a $G \times \mathbf{G}_m$-orbit, and a G-local system on a nilpotent G-orbit in $\mathfrak{g}$ is automatically a $G \times \mathbf{G}_m$-local system. In later discussions, we use the notation $M(y), M^0(y)$, etc. instead of $M_G(y), M_G^0(y)$, etc. by omitting the subscript G if there is no fear of confusion.

2.8. Under the setting in 1.1, let $\mathfrak{p}, \mathfrak{l}, \mathfrak{n}_P$ be the Lie algebras of P, L, U_P so that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_P$. Let $\mathfrak{z}$ the Lie algebra of Z_L^0. We assume that $(L, C, \mathcal{E}) \in \mathcal{M}_G$, and let $(\mathcal{C}, \mathcal{L})$ be the corresponding cuspidal pair on $\mathfrak{l}$ (cf. 1.6.) Let

$$
\dot{\mathfrak{g}} = \{(x, gP) \in \mathfrak{g} \times G/P \mid \operatorname{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{z} + \mathfrak{n}_P\}, \tag{2.8.1}
$$

and $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}$ be the first projection. $G \times \mathbf{G}_m$ acts on $\dot{\mathfrak{g}}$ by $(g_1, t) : (x, gP) \mapsto (t^{-2} \operatorname{Ad}(g_1)x, g_1 gP)$, and π is $G \times \mathbf{G}_m$-equivariant. We consider the diagram

$$
\mathcal{C} \xleftarrow{\ \alpha\ } \widehat{\mathfrak{g}} = \{(x, g) \in \mathfrak{g} \times G \mid \operatorname{Ad}(g^{-1})x \in \mathcal{C} + \mathfrak{z} + \mathfrak{n}_P\} \xrightarrow{\ \beta\ } \dot{\mathfrak{g}},
$$

where $\alpha(x, g) = \operatorname{pr}_{\mathcal{C}}(\operatorname{Ad}(g^{-1})x), \beta(x, g) = (x, gP)$. Here α, β are $G \times \mathbf{G}_m$-equivariant with respect to the action of $G \times \mathbf{G}_m$ on $\mathcal{C}$ given by $(g_1, t) : x \mapsto t^{-2}x$, and the action of it on the middle term given by $(g_1, t) : (x, g) \mapsto (t^{-2} \operatorname{Ad}(g_1)x, g_1 g)$. Since $\mathcal{L}$ is an L-local system, there exists a unique local system $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}$ such that $\alpha^*\mathcal{L} = \beta^*\dot{\mathcal{L}}$. By 2.7, $\mathcal{L}$ is $L \times \mathbf{G}_m$-equivariant, and so is $G \times \mathbf{G}_m$-equivariant with respect to the above action. Hence $\dot{\mathcal{L}}$ turns out to be $G \times \mathbf{G}_m$-equivariant.

Let $\dot{\mathcal{L}}^*$ be the dual local system of $\dot{\mathcal{L}}$, and consider $K = \pi_!(\dot{\mathcal{L}}^*)$. Then it is shown in [L7, 3.4] that $K[\delta]$ is a $G \times \mathbf{G}_m$-equivariant perverse sheaf on $\mathfrak{g}$ with a canonical $\mathcal{W}$ action, where $\delta = \dim(\mathfrak{g}/\mathfrak{l}) + \dim(\mathcal{C} + \mathfrak{z})$.

Let X be an algebraic variety with a given morphism $m : X \to \mathfrak{g}$. We consider the fibre product $\dot{X} = X \times_{\mathfrak{g}} \dot{\mathfrak{g}}$ with the cartesian diagram

$$\begin{array}{ccc} \dot{X} & \xrightarrow{\dot{m}} & \dot{\mathfrak{g}} \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ X & \xrightarrow{m} & \mathfrak{g} \end{array} \tag{2.8.2}$$

Then m^*K is a complex with $\mathcal{W}$-action, and it induces a natural $\mathcal{W}$-action on the cohomologies

$$\mathbb{H}^j_c(X, m^*K) \simeq \mathbb{H}^j_c(X, \pi'_!\dot{m}^*\dot{\mathcal{L}}^*) \simeq H^j_c(\dot{X}, \dot{m}^*\dot{\mathcal{L}}^*). \tag{2.8.3}$$

We further assume that X is a G'-variety, where G' is a connected closed subgroup of $G \times \mathbf{G}_m$, and that m is compatible with G'-actions. If we choose a smooth irreducible variety $\varGamma$ with a free G'-action as in 2.2, the cartesian diagram (2.8.2) is lifted to the cartesian diagram

$$\begin{array}{ccc} {}_\varGamma\dot{X} & \xrightarrow{{}_\varGamma\dot{m}} & {}_\varGamma\dot{\mathfrak{g}} \\ {\scriptstyle {}_\varGamma\pi'}\downarrow & & {\scriptstyle {}_\varGamma\pi}\downarrow \\ {}_\varGamma X & \xrightarrow{{}_\varGamma m} & {}_\varGamma\mathfrak{g} \end{array}$$

As in 2.2, we have a local system ${}_\varGamma\dot{\mathcal{L}}^*$ on ${}_\varGamma\dot{\mathfrak{g}}$, and a perverse sheaf (up to shift) ${}_\varGamma K$ on ${}_\varGamma\mathfrak{g}$ which inherits a $\mathcal{W}$-action from K. Since ${}_\varGamma K = ({}_\varGamma\pi)_!({}_\varGamma\dot{\mathcal{L}}^*)$, as in (2.8.3) we have natural $\mathcal{W}$-actions on cohomologies

$$\mathbb{H}^j_c({}_\varGamma X, ({}_\varGamma m)^*({}_\varGamma K)) \simeq \mathbb{H}^j_c({}_\varGamma X, ({}_\varGamma\pi')_!({}_\varGamma\dot{m})^*{}_\varGamma\dot{\mathcal{L}}^*) \simeq H^j_c({}_\varGamma\dot{X}, ({}_\varGamma\dot{m})^*{}_\varGamma\dot{\mathcal{L}}^*).$$

Hence we have an action of $\mathcal{W}$ on the equivariant homology

$$H^{G'}_j(\dot{X}, \dot{\mathcal{L}}) = H^{2d-j}_c({}_\varGamma\dot{X}, {}_\varGamma\dot{\mathcal{L}}^*)^*,$$

where $d = \dim({}_\varGamma\dot{X})$. (Here we write $\dot{m}^*\dot{\mathcal{L}}^*$, $({}_\varGamma\dot{m})^*{}_\varGamma\dot{\mathcal{L}}^*$, etc. as $\dot{\mathcal{L}}^*$, ${}_\varGamma\dot{\mathcal{L}}^*$, etc. by abbreviation.)

2.9. We fix an element $x_0 \in \mathcal{C}$ and a Lie algebra homomorphism $\phi_0 : \mathfrak{sl}_2 \to \mathfrak{l}$ such that $\phi_0\binom{01}{00} = x_0$. As in [L7, 2.3 (b)], we have

$$Z^0_L(\phi_0) = Z^0_L. \tag{2.9.1}$$

It follows that $Z^0_L(\phi_0)$ is central in $Z_L(\phi)$. Hence by (2.7.1), we see that

(2.9.2) $M^0_L(\phi_0) \simeq Z^0_L \times \mathbf{G}_m$, and $M^0_L(\phi_0)$ is contained in the center of $M_L(\phi)$.

Put $\mathfrak{h}^* = \bar{\mathbf{Q}}_l \otimes_{\mathbb{Z}} X(Z_L^0)$. The $\mathfrak{h}^*$ is a $\bar{\mathbf{Q}}_l$-space of $\dim_{\bar{\mathbf{Q}}_l} \mathfrak{h}^* = \dim_k \mathfrak{z}$, on which $\mathcal{W}$ acts naturally. We define a symmetric algebra $\mathbf{S}$ over $\bar{\mathbf{Q}}_l$ by

$$\mathbf{S} = S(\mathfrak{h}^* \oplus \bar{\mathbf{Q}}_l) = S(\mathfrak{h}^*) \otimes \bar{\mathbf{Q}}_l[\mathbf{r}],$$

where $\bar{\mathbf{Q}}_l[\mathbf{r}]$ is the polynomial ring with an indeterminate $\mathbf{r}$ corresponding to $(0,1) \in \mathfrak{h}^* \oplus \bar{\mathbf{Q}}_l$. We now consider the equivariant cohomology $H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}})$. As in [L7, Proposition 4.2], we have an isomorphism

$$H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}}) \simeq \mathbf{S} \tag{2.9.3}$$

as graded algebras. In particular, $H^j_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}}) = 0$ for odd j. For the proof, the argument in [L7] implies that

$$H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}}) \simeq H^*_{M_L(\phi_0)}.$$

Then by using (2.6.1) and (2.9.2), combined with (2.3.2), we have

$$H^*_{M_L(\phi_0)} \simeq H^*_{M_L^0(\phi_0)} \simeq H^*_{Z_L^0\times\mathbf{G}_m} = \mathbf{S}.$$

Hence (2.9.3) follows.

Let $\widetilde{X}$ be a G'-variety (G' is a connected closed subgroup of $G \times \mathbf{G}_m$), with a given G'-equivariant morphism $\widetilde{m} : \widetilde{X} \to \dot{\mathfrak{g}}$. $\widetilde{m}^*\dot{\mathcal{L}}$ is a G'-local system on $\widetilde{X}$, which we denote by $\dot{\mathcal{L}}$ by abbreviation. Now $\widetilde{m}^*$ induces an algebra homomorphism $H^*_{G'}(\dot{\mathfrak{g}}) \to H^*_{G'}(\widetilde{X})$. By combining the natural homomorphism $H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}}) \to H^*_{G'}(\dot{\mathfrak{g}})$ (cf. [L7, 1.4 (g)]), we have a homomorphism $H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}}) \to H^*_{G'}(\widetilde{X})$. Since $H^{G'}_*(\widetilde{X}, \dot{\mathcal{L}})$ is a $H^*_{G'}(\widetilde{X})$-module by 2.2, $H^{G'}_*(X, \dot{\mathcal{L}})$ has a structure of a left $H^*_{G\times\mathbf{G}_m}(\dot{\mathfrak{g}})$-module. Thus by (2.9.3), $H^{G'}_*(\widetilde{X}, \dot{\mathcal{L}})$ turns out to be an $\mathbf{S}$-module.

2.10. Let $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}$ be as in 2.8. Then for each $y \in \mathfrak{g}_{\mathrm{nil}}$, $\pi^{-1}(y)$ coincides with $\mathcal{P}_y$ in (1.6.1). The variety $X = \{y\}$ is invariant under the action of $M^0(y) \subset G \times \mathbf{G}_m$. Let G' be a connected closed subgroup of $M^0(y)$. By applying 2.8 to the inclusion $m : X \hookrightarrow \mathfrak{g}$ together with $\dot{X} = \mathcal{P}_y$, we see that $H^{G'}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ has a natural $\mathcal{W}$-action. By applying 2.9 for $\widetilde{X} = \dot{X}$, $H^{G'}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ has a natural $\mathbf{S}$-action. It also has a structure of $H^*_{G'}$-module by 2.2.

We consider the graded Hecke algebra $\mathbf{H} = \mathbf{S} \otimes \bar{\mathbf{Q}}_l[\mathcal{W}]$ as defined in 2.1, where $\mathbf{S}$ is as in 2.9, with a natural action of the Coxeter group $\mathcal{W}$. Lusztig proved the following theorem.

Theorem 2.11 (Lusztig [L7, Theorem 8.13])**.** *There is a unique* $\mathbf{H}$*-module structure on* $H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ *such that the actions of* $\mathbf{S}$ *and* $\mathcal{W}$ *are given as in 2.10. (The integers* c_i *are determined according to the cuspidal pair* $(\mathcal{C}, \mathcal{L})$*. See [L7, 2.13] for explicit values for* c_i*). Moreover, the* $\mathbf{H}$*-module structure commutes with the* $H^*_{M^0(y)}$*-module structure on* $H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}})$*.*

Remark 2.12. The arguments used in [L7] to prove the theorem are valid also for our setting in almost all cases, by taking 2.3 $\sim$ 2.7 into account. We give further comments on the discrepancies of the arguments.

(a) In [L7, 4.3], the property of the image $H^*_{G\times \mathbf{G}_m} \to H^*_{M^0_L(\phi_0)}$ is used. For this we appeal to Corollary 2.5.

(b) In the proof of Proposition 7.2 in [L7], a property of simply connected space is used, which is not valid in the positive characteristic case. As in 7.1, we consider a connected algebraic group M, and an M-variety X, M-equivariant local system $\mathcal{E}$ on X. Let $\varGamma$ be an irreducible, smooth variety with a free M-action as before. Let $f : M\backslash(\varGamma \times X) \to M\backslash\varGamma$ be the locally trivial fibration. We consider the Leray-Serre spectral sequence

$$H^p_c(M\backslash\varGamma, R^qf_!({}_\varGamma\mathcal{E}^*)) \Rightarrow H^{p+q}_c(M\backslash(\varGamma\times X), {}_\varGamma\mathcal{E}^*).$$

We show that

$$E_2^{p,q} = H^p_c(M\backslash\varGamma, R^qf_!({}_\varGamma\mathcal{E}^*)) = H^p_c(M\backslash\varGamma)\otimes H^q_c(X, \mathcal{E}^*). \tag{2.12.1}$$

(In [L7], this is obtained as a consequence of the fact that $M\backslash\varGamma$ can be chosen to be simply connected). We consider the cartesian diagram

$$\begin{array}{ccc} \varGamma\times X & \xrightarrow{\pi} & M\backslash(\varGamma\times X) \\ {\scriptstyle\widetilde f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \varGamma & \xrightarrow{\widetilde\pi} & M\backslash\varGamma. \end{array}$$

Now ${}_\varGamma\mathcal{E}^*$ on $M\backslash(\varGamma\times X)$ satisfies the property that $\bar{\mathbf{Q}}_l \boxtimes \mathcal{E}^* = \pi^*({}_\varGamma\mathcal{E}^*)$. By the base change theorem, we have $\widetilde\pi^*R^qf_!({}_\varGamma\mathcal{E}^*) \simeq R^q\widetilde f_!\pi^*({}_\varGamma\mathcal{E}^*)$. It is easy to see that $R^q\widetilde f_!(\bar{\mathbf{Q}}_l\boxtimes\mathcal{E}^*)$ is an M-equivariant constant sheaf, and $R^qf_!({}_\varGamma\mathcal{E}^*)$ is obtained from it as the unique quotient. Thus, $R^qf_!({}_\varGamma\mathcal{E}^*)$ is also a constant sheaf with the stalk $H^q_c(X,\mathcal{E}^*)$. This implies (2.12.1).

Once this is established, the other parts in the proof of Proposition 7.2 work without change.

2.13. We return to the setting in 2.10. Let $T(y)$ be a maximal torus of $M^0(y)$ and $W(y)$ the Weyl group of a maximal reductive subgroup of $M^0(y)$ with respect to $T(y)$. Then by (2.3.2) and Lemma 2.4, $H^*_{M^0(y)}$ can be identified with $S(V^*)^{W(y)}$, where $V^* = \bar{\mathbf{Q}}_l\otimes_{\mathbf{Z}} X(T(y))$. Hence $H^*_{M^0(y)}$ may be regarded as the coordinate ring of an affine algebraic variety (over $\bar{\mathbf{Q}}_l$) $V_1 = V/W(y)$, where V is the dual space of V^*. Then for each $v\in V_1$, one obtains an algebra homomorphism $H^*_{M^0(y)}\to\bar{\mathbf{Q}}_l$, $f\mapsto f(v)$. We denote the thus obtained $H^*_{M^0(y)}$-module $\bar{\mathbf{Q}}_l$ by $(\bar{\mathbf{Q}}_l)_v$. It is known by [L7, 8.6] that $H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ is a finitely generated projective $H^*_{M^0(y)}$-module. It follows that $H^{M^0(y)}_*(\mathcal{P}_y,\dot{\mathcal{L}})$ may be regarded as a space of sections of algebraic vector bundle E over V_1, where the fibre of E at $v \in V_1$ is given by

$$E_v = (\bar{\mathbf{Q}}_l)_v \otimes_{H^*_{M^0(y)}} H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{2.13.1}$$

Put $\overline{M}(y) = M(y)/M^0(y)$. Then the finite group $\overline{M}(y)$ acts on $H^*_{M^0(y)}$ as a $\bar{\mathbf{Q}}_l$-algebra automorphism, and acts on $H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ compatible with the action of $H^*_{M^0(y)}$. Also this action of $\overline{M}(y)$ on $H^{M^0(y)}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ commutes with the action of $\mathbf{H}$. The action of $\overline{M}(y)$ on $H^*_{M^0(y)}$ induces an action of $\overline{M}(y)$ on V_1, and E turns out to be an $\overline{M}(y)$-equivariant vector bundle over V_1. For each $v \in V_1$, we denote by $\overline{M}(y,v)$ the stabilizer of v in $\overline{M}(y)$. Then $\overline{M}(y,v)$ acts naturally on E_v.

Let $\overline{M}(y,v)^\wedge$ be the set of irreducible representations of $\overline{M}(y,v)$ up to isomorphisms. For each $\rho \in \overline{M}(y,v)^\wedge$, put $E_{v,\rho} = (\rho^* \otimes E_v)^{\overline{M}(y,v)}$, where ρ^* is the dual representation of ρ. Then $E_{v,\rho}$ is an $\mathbf{H}$-module, and E_v is decomposed as

$$E_v = \bigoplus_{\rho \in \overline{M}(y,v)^\wedge} \rho \otimes E_{v,\rho}.$$

The action of $M(y)$ on $\mathcal{P}_y, \dot{\mathcal{L}}, \dot{\mathcal{L}}^*$ induces an action of $\overline{M}(y)$ on $H^*_c(\mathcal{P}_y, \dot{\mathcal{L}})$, $H^*_c(\mathcal{P}_y, \dot{\mathcal{L}}^*)$, hence on $H^{\{e\}}_*(\mathcal{P}_y, \dot{\mathcal{L}}) = H^*_c(\mathcal{P}_y, \dot{\mathcal{L}}^*)^*$. It is known by [L7, 8.10] that $E_{v,\rho} \neq 0$ if and only if ρ occurs in the restriction of $\overline{M}(y)$-module $H^{\{e\}}_*(\mathcal{P}_y, \dot{\mathcal{L}})$ to $\overline{M}(y,v)$. The $\mathbf{H}$-modules $E_{v,\rho}$ are called standard modules.

Remarks 2.14. (i) Standard modules $E_{v,\rho}$ are parametrized in [L7] (i.e., in the setting that G and $\mathfrak{g}$ are defined over $\mathbf{C}$) as $E_{h,r_0,\rho}$ in terms of the pair $(h, r_0) \in \mathfrak{g} \oplus \mathbf{C}$ such that $[h, y] = 2r_0 y$ with h semisimple. This is also possible in our situation, though we cannot use the Lie algebra $\mathfrak{g}$ over k. Since p is good, we have corresponding objects $G_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}$, and the parametrization of nilpotent orbits and the structure of $\overline{M}(y)$ are the same for $\mathfrak{g}_{\mathbf{C}}$ also. If we consider the maximal torus $T(y)_{\mathbf{C}}$ in $M(y)_{\mathbf{C}}$ corresponding to $T(y)$ in $M(y)$, the space V^* may be identified (under a choice of an isomorphism $\bar{\mathbf{Q}}_l \simeq \mathbf{C}$) with the dual of the Cartan subalgebra $\mathfrak{h}(y)_{\mathbf{C}}$ of a maximal reductive subalgebra $\mathfrak{m}(y)_{\mathbf{C},r}$ of $\mathfrak{m}(y)_{\mathbf{C}} = \operatorname{Lie} M(y)_{\mathbf{C}}$ with the action of $W(y)$. Then the action of $\overline{M}(y)$ on $S(V^*)^{W(y)}$ coincides with the action of $\overline{M}(y)$ on $S(\mathfrak{h}(y)^*_{\mathbf{C}})^{W(y)} \simeq S(\mathfrak{m}(y)^*_{\mathbf{C},r})^{M^0(y)_{\mathbf{C}}}$. Here

$$\mathfrak{m}(y)_{\mathbf{C}} = \operatorname{Lie} M^0(y)_{\mathbf{C}} = \{(x, r_0) \in \mathfrak{g}_{\mathbf{C}} \oplus \mathbf{C} \mid [x, y] = 2r_0 y\}.$$

Moreover, the action of $\overline{M}(y)$ on $S(\mathfrak{m}(y)^*_{\mathbf{C},r})$ is induced from the action of $M(y)_{\mathbf{C}}$, $(g_1, t) : (x, r_0) \mapsto (t^{-2} \operatorname{Ad}(g_1)x, t^{-2} r_0)$. Hence V_1 is identified with the set of semisimple $M^0(y)_{\mathbf{C}}$-orbits on $\mathfrak{m}(y)_{\mathbf{C}}$. This implies, in our case, that $E_{v,\rho}$ may be expressed as $E_{h,r_0,\rho}$, and $\overline{M}(y,v)$ as $\overline{M}(y,h,r_0)$, if (h, r_0) is a semisimple orbit in $\mathfrak{g}_{\mathbf{C}} \oplus \mathbf{C}$ corresponding to $v \in V_1$.

(ii) Standard modules play a crucial role in the representation theory of $\mathbf{H}$. The structure of $\mathbf{H}$-module $E_{v,\rho}$ was studied throughly in [L8], [L9]. However, the result in [L7] is enough for our purpose.

In view of the above remarks, the following result of Lusztig can be applied to our setting.

Theorem 2.15 ([L7, Theorem 8.17]). *Let $(h, r_0) \in \mathfrak{g}_{\mathbf{C}} \oplus \mathbf{C}$ be a semisimple element such that $r_0 \neq 0$. Then*

(i) *Let $Y_{(h,r_0)} = \{x \in \mathfrak{g}_{\mathbf{C}} \mid [h, x] = 2r_0 x\}$. Then $Y_{(h,r_0)}$ consists of nilpotent elements, and $Z_{G_{\mathbf{C}}}(h)$ acts (by the adjoint action) on $Y_{(h,r_0)}$ with finitely many orbits.*

(ii) *Let y be an element in the unique open dense orbit in $Y_{(h,r_0)}$. Then $(h, r_0) \in \mathfrak{m}(y)_{\mathbf{C}}$. Let $\rho \in \overline{M}(y, h, r_0)^{\wedge}$ be such that $E_{h,r_0,\rho} \neq \{0\}$. Then $E_{h,r_0,\rho}$ is a simple $\mathbf{H}$-module.*

2.16. Here we summarize the properties connecting the equivariant homology with the ordinary cohomology. Let M be a connected algebraic group, X an M-variety and $\mathcal{E}$ an M-equivariant local system on X. We consider $H_*^M(X, \mathcal{E})$. For each i, we define F^i as the H_M^*-submodule of $H_*^M(X, \mathcal{E})$ generated by $\bigoplus_{j \le i} H_j^M(X, \mathcal{E})$. Then F^i gives a filtration $F^0 \subseteq F^1 \subseteq \cdots$ and $F^i = 0$ for $i < 0$. Put $\Pi_i = H_i^M(X, \mathcal{E})/H_i^M(X, \mathcal{E}) \cap F^{i-1}$. We have a natural injection $\Pi_i \to F^i/F^{i-1}$ as $\bar{\mathbf{Q}}_l$-spaces. Since F^i/F^{i-1} is an H_M^*-module, this is extended to an H_M^*-linear map

$$H_M^* \otimes_{\bar{\mathbf{Q}}_l} \Pi_i \to F^i/F^{i-1}. \tag{2.16.1}$$

The natural homomorphism $H_i^M(X, \mathcal{E}) \to H_i^{\{e\}}(X, \mathcal{E})$ is zero on $H_i^M(X, \mathcal{E}) \cap F^{i-1}$, and it factors through a $\bar{\mathbf{Q}}_l$-linear map

$$\Pi_i \to H_i^{\{e\}}(X, \mathcal{E}). \tag{2.16.2}$$

Lusztig showed in [L7, 7.2] that the maps (2.16.1) and (2.16.2) are isomorphisms whenever $H_c^{\mathrm{odd}}(X, \mathcal{E}) = 0$, and in that case we obtain an isomorphism

$$H_M^* \otimes_{\bar{\mathbf{Q}}_l} H_i^{\{e\}}(X, \mathcal{E}) \xrightarrow{\sim} F^i/F^{i-1}. \tag{2.16.3}$$

We now consider the case where $X = \mathcal{P}_y$, $\mathcal{E} = \dot{\mathcal{L}}$ and $M = M^0(y)$. It is known that $H_c^{\mathrm{odd}}(\mathcal{P}_y, \dot{\mathcal{L}}) = 0$ by [L3, V, 24.8], and so the previous argument can be applied. We consider E_v as in (2.13.1) and $H_{M^0(y)}^*$-module $(\bar{\mathbf{Q}}_l)_v$. We define an $\bar{\mathbf{Q}}_l$-space F_v^i by $F_v^i = (\bar{\mathbf{Q}}_l)_v \otimes_{H^*_{M^0(y)}} F^i$. Then F_v^i is naturally identified with a quotient of $\bigoplus_{j \le i} H_j^{M^0(y)}(\mathcal{P}_y, \dot{\mathcal{L}})$. We denote by $f_i : F_v^{i-1} \to F_v^i$ the natural map induced from $F^{i-1} \hookrightarrow F^i$. It follows from (2.16.3) we have an exact sequence of $\bar{\mathbf{Q}}_l$-spaces

$$F_v^{i-1} \xrightarrow{f_i} F_v^i \longrightarrow H_i^{\{e\}}(\mathcal{P}_y, \dot{\mathcal{L}}) \longrightarrow 0. \tag{2.16.4}$$

In particular, we have

$$F_v^0 \simeq H_0^{\{e\}}(\mathcal{P}_y, \dot{\mathcal{L}}). \tag{2.16.5}$$

2.17. We consider the $\mathbf{F}_q$-structure on the equivariant homology. Assume that G and X are defined over $\mathbf{F}_q$ with Frobenius map F, and G acts on X over $\mathbf{F}_q$. Let $\mathcal{E}$ be an G-equivariant local system on X such that $F^*\mathcal{E} \simeq \mathcal{E}$. We fix an isomorphism $\varphi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. Then φ induces natural linear isomorphisms on $H_*^G(X, \mathcal{E}), H_G^*(X, \mathcal{E})$, etc. In fact, one can choose a G-variety $\varGamma$ so that $\varGamma$ is defined over $\mathbf{F}_q$. (We may assume that G is an F-stable closed subgroup of some GL_r. The case where GL_r has a split $\mathbf{F}_q$-structure, the construction of $\varGamma$ in 2.2 works well. If GL_r is of non-split type, we choose $F = \sigma_0 F_0$, where F_0 is a split Frobenius, and σ_0 is an automorphism of GL_r defined by $\sigma_0(g) = {}^t g^{-1}$. By choosing similar Frobenius maps for $GL_{r'}$ and $GL_{r+r'}$, the inclusions in (2.2.1) are F-equivariant. Hence $\varGamma = \{e\} \times GL_{r'} \backslash GL_{r+r'}$ is defined over $\mathbf{F}_q$.) Then the maps $\pi : \varGamma \times X \to {}_{\varGamma}X$, $p : \varGamma \times X \to X$ are defined over $\mathbf{F}_q$. Hence ${}_{\varGamma}\mathcal{E}$ inherits an $\mathbf{F}_q$-structure of $\mathcal{E}$, which induces a linear map on $H_G^j(X, \mathcal{E}) = H^j({}_{\varGamma}X, {}_{\varGamma}\mathcal{E})$. The thus obtained linear map is independent of the choice of $\varGamma$. In fact, if $\varGamma'$ is another choice, we have an isomorphism $H^j({}_{\varGamma}X, {}_{\varGamma}\mathcal{E}) \xrightarrow{\sim} H^j_{\varGamma \times \varGamma'}({}_{\varGamma\times\varGamma'}X, {}_{\varGamma\times\varGamma'}\mathcal{E})$, etc. as in [L7, 1.1], which are compatible with the induced F-actions on them.

3. $G = SL_n$ with F of split type

3.1. In this section, we assume that p is arbitrary, and consider $G = SL_n$ with the standard Frobenius map F on G, i.e., for $g = (g_{ij}) \in G$, $F(g) = (g_{ij}^q)$. Let $V = k^n$ with the standard basis $e_1, \dots, e_n$ and we identify SL_n with $SL(V)$.

Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of G, and we denote by F the corresponding Frobenius map on $\mathfrak{g}$. The unipotent classes in G and nilpotent orbits in $\mathfrak{g}$ are parametrized by partitions of n, via Jordan normal form. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of n, and let C_λ (resp. $\mathcal{C}_\lambda$) be the corresponding unipotent class in G (resp. nilpotent orbit in $\mathfrak{g}$). Each $\mathcal{C}_\lambda$ is F-stable, and we construct a specific nilpotent transformation $y = y_\lambda \in \mathcal{C}_\lambda^F$ by defining a basis $\{y^a f_j \mid 1 \le j \le r, 0 \le a < \lambda_j\}$ of V obtained from the standard basis as follows;

$$y^a f_j = e_i \qquad \text{with} \qquad i = \lambda_1 + \cdots \lambda_{j-1} + a. \tag{3.1.1}$$

Then $u_\lambda = y_\lambda + 1 \in C_\lambda^F$. The element $y_\lambda \in \mathcal{C}_\lambda^F$ (resp. $u_\lambda \in C_\lambda^F$) is called the split element corresponding to λ.

3.2. By [L2], [LS], the generalized Springer correspondence for the case where $G = SL_n$ is described as follows. Let n' be the largest divisor of n which is prime to p. Then the center Z_G is a cyclic group of order n'. For a divisor d of n', consider a Levi subgroup L of P of the type $A_{d-1} \times \cdots \times A_{d-1}$ (n/d-factors). Let C be the regular unipotent class in L. Then for $v \in C$, $A_L(v) = Z_L/Z_L^0 \simeq \mathbf{Z}/d\mathbf{Z}$. Let $\mathcal{E}$ be an L-equivariant local system on C corresponding to a character ρ_0 of $A_L(v)$ of order d. Then $(C, \mathcal{E})$ is a cuspidal pair on L, and any cuspidal pair on a Levi subgroup of a parabolic subgroup of G is obtained in this way. Hence for a Levi subgroup L determined by d, there exist exactly $\varphi(d)$ cuspidal pairs in L, where φ is the Euler function.

Let K be as in (1.2.2) with respect to the cuspidal pair $(C, \mathcal{E})$ on L. Let C' be a unipotent class in G corresponding to a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$. Then for $u \in C'$, $A_G(u)$ is a cyclic group of order n'_λ, where n'_λ is the greatest common divisor of $n', \lambda_1, \lambda_2, \dots, \lambda_r$. Let $\mathcal{E}'$ be the local system on C' corresponding to $\rho \in A_G(u)^\wedge$. The condition for C' such that $\mathrm{IC}(\overline{C'}, \mathcal{E}')$ is a component of K (up to shift) is that each λ_i is divisible by d. In this case n'_λ is divisible by d, and we have a surjective homomorphism $A_G(u) \to A_L(v)$ which factors through the natural maps $Z_G \to A_G(u)$ and $Z_G \to A_L(v)$. Let $\rho \in A_G(u)^\wedge$ be the character obtained as the pull back of $\rho_0 \in A_L(v)^\wedge$. Then $\mathrm{IC}(\overline{C'}, \mathcal{E}')$ is the unique component in K whose support is $\overline{C}'$.

Now $\mathcal{W} = N_G(L)/L$ is isomorphic to the symmetric group $S_{n/d}$. The irreducible character $E = E_\mu \in S_{n/d}^\wedge$ corresponding to $(C', \mathcal{E}')$ under the generalized Springer correspondence is given by $\mu = (\lambda_1/d, \lambda_2/d, \dots)$.

3.3. We fix an F-stable Borel subgroup B of G and an F-stable maximal torus contained in B, where B (resp. T) is the subgroup of G consisting of upper triangular matrices (resp. diagonal matrices). We fix d as in 3.2, and put $t = n/d$. Let $P = LU_P$ be the parabolic subgroup of G containing B, where L is the Levi subgroup of P containing T of type $A_{d-1} \times \cdots \times A_{d-1}$, ($t$-times). Hence P, L and U_P are all F-stable. Let $(C, \mathcal{E})$ be the cuspidal pair in L corresponding to $\rho_0 \in A_L(v)^\wedge$ as in 3.2, and $(\mathcal{C}, \mathcal{L})$ the corresponding objects in $\mathfrak{l}$. The unipotent class C in L can be identified with $C_1 \times \cdots \times C_t$ in $SL_d \times \cdots \times SL_d$ with C_i regular unipotent in SL_d. We choose $v = v_0 \in C^F$ so that v_0 is a product of split elements in C_i^F, and let $y_0 = v_0 - 1$ the corresponding element in $\mathcal{C}^F$. Let $\widetilde{A}_L(v_0)$ be as in 1.3. Since $A_L(v_0)$ is abelian, $\rho_0 \in A_L(v)^\wedge$ is linear. We choose an extension $\widetilde{\rho}_0$ so that $\widetilde{\rho}_0(\sigma) = 1$. This corresponds to an isomorphism $\varphi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which induces the identity map on the stalk $\mathcal{E}_{v_0}$.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n such that all the λ_i are divisible by d, and $u = u_\lambda$ the split unipotent element in G^F. As in the case of $(C, \mathcal{E})$, we choose an extension $\widetilde{\rho}$ of $\rho \in A_G(u)^\wedge$ corresponding to $\mathcal{E}'$ by the condition that $\widetilde{\rho}(\sigma) = 1$, and consider $\gamma = \gamma(v, \widetilde{\rho}_0, u, \widetilde{\rho})$ as in 1.3. Passing to the Lie algebra situation, we consider $y = y_\lambda \in \mathfrak{g}^F$ and $y_0 \in \mathcal{C}^F$. Under this setting, we write γ as $\gamma = \gamma(y_0, \widetilde{\rho}_0, y, \widetilde{\rho})$. We consider the subvariety $\mathcal{P}_y$ of G/P as given in (1.6.1). As in 1.6, the map φ_0 induces a linear isomorphism Φ on $H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}})$. We have

Theorem 3.4. *Assume that p is arbitrary, and let $G = SL_n$ with the standard Frobenius map F. Then Φ acts on $H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}}) = H_c^{a_0+r}(\mathcal{P}_y, \dot{\mathcal{L}})_\rho$ as $q^{(a_0+r)/2}$ times identity. In particular, we have $\gamma(y_0, \widetilde{\rho}_0, y, \widetilde{\rho}) = 1$.*

3.5. The remainder of this section is devoted to the proof of Theorem 3.4. Since the second statement easily follows from the first one, we concentrate to the proof of the first statement. First we note that $\mathcal{P}_y$ may be identified with the set $\mathcal{F}_y$ of partial flags

$$D = (V_d \subset V_{2d} \subset \cdots \subset V_{(t-1)d}),$$

such that D is y-stable and that y induces a regular nilpotent transformation on $V_{id}/V_{(i-1)d}$ for each $i \geq 1$. (Here V_j denotes a subspace of V with $\dim V_j = j$).

Let $\mathcal{G}_y$ be the set of d-dimensional subspaces V_d of V such that V_d is y-stable and that y acts as a regular nilpotent transformation on V_d. We have a natural surjective map $p : \mathcal{F}_y \to \mathcal{G}_y$ by $p(D) = V_d$. Then $\mathcal{G}_y$ is identified with the variety $\mathbf{P}(\operatorname{Ker} y^d) - \mathbf{P}(\operatorname{Ker} y^{d-1})$; for each $v \in \operatorname{Ker} y^d - \operatorname{Ker} y^{d-1}$, the space spanned by $v, yv, \dots y^{d-1}v$ gives an element in $\mathcal{G}_y$. We have a filtration of $\mathcal{G}_y$

$$\mathcal{G}_y = \mathcal{G}_0 \supset \mathcal{G}_1 \supset \cdots ,$$

where $\mathcal{G}_i - \mathcal{G}_{i+1} \simeq \mathbf{A}^{s-i}$ with $\dim \mathcal{G}_y = s = d(\dim \operatorname{Ker} y) - 1$. Here $\mathcal{G}_i$ is defined by $\mathbf{P}(U_i) - \mathbf{P}(\operatorname{Ker} y^{d-1})$ for a certain subspace U_i of $\operatorname{Ker} y^d$ containing $\operatorname{Ker} y^{d-1}$ such that $\operatorname{Ker} y^d = U_0 \supset U_1 \supset \cdots$. Let us choose a non-zero vector $w_i \in U_i - U_{i+1}$ for each i. We can choose some e_j as w_i. As in the case of $\mathcal{B}_u$ for GL_n, one can define a map $f^{(i)} : \mathbf{A}^{s-i} \to Z_{\widetilde{G}}(y), v \mapsto f_v^{(i)}$ such that $f_v^{(i)} \cdot w_i = v$ for $v \in U_i - U_{i-1}$, under the identification $\mathbf{P}(U_i) - \mathbf{P}(U_{i-1}) \simeq \mathbf{A}^{s-i}$. (Here $\widetilde{G}$ denotes GL_n). Let $V_d^{(i)}$ be the element in $\mathcal{G}_y$ corresponding to w_i. Then y induces a nilpotent transformation $\overline{y}$ on $\overline{V} = V/V_d^{(i)}$, which corresponds to a partition λ' of $n-d$ obtained from λ by replacing some λ_j by $\lambda_j - d$. Moreover, $p^{-1}(V_d^{(i)})$ is isomorphic to $\mathcal{F}_{\overline{y}}$, the corresponding variety for $SL(\overline{V})$, under the correspondence

$$D = (V_d^{(i)} \subset V_{2d} \subset \cdots \subset V_{(t-1)d}) \mapsto \overline{D} = (\overline{V}_{2d} \subset \cdots \subset \overline{V}_{(t-1)d})$$

with $\overline{V}_{jd} = V_{jd}/V_d^{(i)}$. As in the case of GL_n, by using the map $f^{(i)} : \mathbb{A}^{s-i} \to Z_{\tilde{G}}(y)$, we have an isomorphism

$$p^{-1}(V_d^{(i)}) \times (\mathcal{G}_i - \mathcal{G}_{i+1}) \simeq p^{-1}(\mathcal{G}_i - \mathcal{G}_{i+1}), \qquad (D, v) \mapsto f_v^{(i)} \cdot D. \tag{3.5.1}$$

Note that $\mathcal{F}_y$ and $\mathcal{G}_y$ have natural $\mathbf{F}_q$-structures inherited from G/P. Then $\mathcal{G}_i$ are all F-stable, and the isomorphism in (3.5.1) is F-equivariant.

3.6. Let Q be the maximal parabolic subgroup of G containing P of type $A_{n-d-1} \times A_{d-1}$. Let $\mathcal{G}$ be the set of subspaces of dimension d in V. Then $\mathcal{G}$ may be identified with G/Q and $\mathcal{G}_y$ is a locally closed subvariety of $\mathcal{G}$. The map $p : \mathcal{F}_y \to \mathcal{G}_y$ is obtained from the map $G/P \to G/Q$ by restricting it to $\mathcal{P}_y$, which we also denote by p. Now, $V_d^{(i)} \in \mathcal{G}_y$ corresponds to $gQ \in G/Q$ for some $g = g_i \in G$ and $p^{-1}(V_d^{(i)})$ may be identified with $p^{-1}(gQ)$, where

$$p^{-1}(gQ) = \{xP \in gQ/P \mid \operatorname{Ad}(x)^{-1} y \in \mathcal{C} + \mathfrak{n}_P\}.$$

We may choose g so that $gP \in \mathcal{P}_y$.

We note that Q/P is isomorphic to M/P_M, where M is the subgroup of G isomorphic to SL_{n-d}, and is isogeneous to a component of the Levi subgroup of Q containing T. Then $P_M = P \cap M$ is the parabolic subgroup of M of type $A_{d-1} \times \cdots \times A_{d-1}$, ($t-1$ factors), and $L_M = L \cap M$ is the Levi subgroup of P_M. The regular nilpotent orbit $\mathcal{C}$ in $\mathfrak{l}$ can be written as $\mathcal{C} = \mathcal{C}_M \times \mathcal{C}_t$, where $\mathcal{C}_M$ is the regular nilpotent orbit in $\operatorname{Lie} L_M = \mathfrak{l}_M$ and $\mathcal{C}_t$ is the regular nilpotent orbit in the

t-th component of $\mathfrak{l}$. Since $\operatorname{Ad}(g)^{-1}y \in \mathcal{C} + \mathfrak{n}_P$, one can write $\operatorname{Ad}(g)^{-1}y = y' + z'$ with $y' \in \mathfrak{m}$ and $z' \in \mathcal{C}_t + \mathfrak{n}_Q$. (Here $\mathfrak{m} = \operatorname{Lie} M$ and $\mathfrak{n}_Q = \operatorname{Lie} U_Q$). Set

$$\begin{aligned}\mathcal{P}^M_{y'} &= \{xP_M \in M/P_M \mid \operatorname{Ad}(x)^{-1}y' \in \mathcal{C}_M + \mathfrak{n}_{P_M}\},\\ \hat{\mathcal{P}}^M_{y'} &= \{x \in M \mid \operatorname{Ad}(x)^{-1}y' \in \mathcal{C}_M + \mathfrak{n}_{P_M}\}.\end{aligned}$$

We note that

(3.6.1) The map $xP_M \mapsto gxP$ gives an isomorphism $\mathcal{P}^M_{y'} \simeq p^{-1}(gQ)$.

In fact, since M normalizes C_tU_Q, we have $\operatorname{Ad}(x)^{-1}z' \in \mathcal{C}_t + \mathfrak{n}_Q$. Then the condition $\operatorname{Ad}(gx)^{-1}y \in \mathcal{C} + \mathfrak{n}_P$ is equivalent to the condition $\operatorname{Ad}(x)^{-1}y' \in \mathcal{C}_M + \mathfrak{n}_{P_M}$. (3.6.1) follows from this.

By (3.6.1), one can define an injective map $\iota : \mathcal{P}^M_{y'} \to \mathcal{P}_y$. Similarly, $\widehat{\mathcal{P}}^M_{y'}$ is isomorphic to the set $\{x' \in gM \mid \operatorname{Ad}(x')^{-1}y \in \mathcal{C} + \mathfrak{n}_P\}$ which is a subset of $\widehat{\mathcal{P}}_y$. Hence we have an injective map $\hat{\iota} : \widehat{\mathcal{P}}^M_{y'} \to \widehat{\mathcal{P}}_y$. Now it is easy to see that the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{\alpha} & \widehat{\mathcal{P}}_y & \xrightarrow{\beta} & \mathcal{P}_y \\ {\scriptstyle\iota'}\uparrow & & \uparrow{\scriptstyle\hat{\iota}} & & \uparrow{\scriptstyle\iota} \\ \mathcal{C}_M & \xleftarrow{\alpha'} & \widehat{\mathcal{P}}^M_{y'} & \xrightarrow{\beta'} & \mathcal{P}^M_{y'}. \end{array} \tag{3.6.2}$$

Here the left vertical map is an injection $\iota' : \mathcal{C}_M \to \mathcal{C}, x \mapsto (x, y'')$, where y'' is the projection of $z' \in \mathcal{C}_t + \mathfrak{n}_Q$ to $\mathcal{C}_t$, i.e., the projection of $\operatorname{Ad}(g)^{-1}y \in \mathcal{C} + \mathfrak{n}_P$ on $\mathcal{C}_t$. The horizontal maps α', β' are defined in a similar way as α and β by replacing G by M.

Let $\mathcal{L}$ and $\dot{\mathcal{L}}$ be local systems on $\mathcal{C}$ and $\mathcal{P}_y$, respectively, as in 1.6. We denote by $\mathcal{L}_M$ and $\dot{\mathcal{L}}_M$ similar objects for $\mathcal{C}_M$ and $\mathcal{P}^M_{y'}$ as $\mathcal{L}, \dot{\mathcal{L}}$ for $\mathcal{C}$ and $\mathcal{P}_y$. Then $\mathcal{L}_M$ coincides with $(\iota')^*\mathcal{L}$. This implies, by (3.6.2), that

$$\iota^*\dot{\mathcal{L}} = \dot{\mathcal{L}}_M. \tag{3.6.3}$$

Put

$$\begin{aligned}Y_y &= \{xQ \in G/Q \mid \operatorname{Ad}(x)^{-1}y \in \mathfrak{m} + \mathcal{C}_t + \mathfrak{n}_Q\},\\ \hat{Y}_y &= \{x \in G \mid \operatorname{Ad}(x)^{-1}y \in \mathfrak{m} + \mathcal{C}_t + \mathfrak{n}_Q\}.\end{aligned}$$

Then Y_y is isomorphic to $\mathcal{G}_y$. We consider the subset Y_i of Y_y corresponding to $\mathcal{G}_i$. Since $\mathcal{G}_i - \mathcal{G}_{i+1}$ coincides with the set $\{f^{(i)}_v \cdot w_i \mid v \in \mathbf{A}^{s-i}\}$, one can write as

$Y_i - Y_{i+1} = \{f_v^{(i)} g_i Q \mid v \in \mathbf{A}^{s-i}\}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{C}_t & \xleftarrow{\ \widetilde{\alpha}\ } & \widehat{Y}_y & \xrightarrow{\ \widetilde{\beta}\ } & Y_y \\ \uparrow & & \uparrow & & \uparrow \\ \{y_i''\} & \longleftarrow & f^{(i)}(\mathbb{A}^{s-i})g_i & \longrightarrow & Y_i - Y_{i+1}. \end{array} \tag{3.6.4}$$

Here $y_i'' = y''$ is as in (3.6.2), and $\tilde{\alpha}(x)$ is the $\mathcal{C}_t$-component of $\operatorname{Ad}(x)^{-1}y \in \mathfrak{m}+\mathcal{C}_t+\mathfrak{n}_Q$, $\tilde{\beta}(x) = xQ$. All the vertical maps are natural inclusions and the lower horizontal arrows are the restrictions of upper ones. Note that the right lower horizontal map is an isomorphism since $Y_i - Y_{i+1} \simeq \mathbf{A}^{s-i}$.

Let $\mathcal{L}_t$ be the cuspidal local system on $\mathcal{C}_t$. Then we have a local system $\dot{\mathcal{L}}_t$ on Y_y by the condition that $\widetilde{\alpha}^*\mathcal{L}_t = \widetilde{\beta}^*\dot{\mathcal{L}}_t$. Since $\mathcal{L}_t$ is a local system of rank 1, it follows from (3.6.4) that

(3.6.5) The restriction of $\dot{\mathcal{L}}_t$ on $Y_i - Y_{i+1}$ is the constant sheaf $\bar{\mathbf{Q}}_l$.

We now consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \xleftarrow{\ \alpha\ } & \widehat{\mathcal{P}}_y & \xrightarrow{\ \beta\ } & \mathcal{P}_y \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{C}_M \times \{y_i''\} & \xleftarrow{\ \alpha''\ } & \beta^{-1}(p^{-1}(Y_i - Y_{i+1})) & \xrightarrow{\ \beta''\ } & p^{-1}(Y_i - Y_{i+1}). \end{array} \tag{3.6.6}$$

Here all the vertical maps are natural inclusions, and the horizontal maps α'' and β'' are the restrictions of α and β. By (3.5.1), we have

$$\begin{aligned} p^{-1}(Y_i - Y_{i+1}) &\simeq \mathcal{P}^M_{y'} \times (Y_i - Y_{i+1}), \\ \beta^{-1}(p^{-1}(Y_i - Y_{i+1})) &\simeq \widehat{\mathcal{P}}^M_{y'} \times f^{(i)}(\mathbb{A}^{s-i})g_i, \end{aligned} \tag{3.6.7}$$

and under the above isomorphisms, the maps α'', β'' are given as

$$\alpha''(x, f_v^{(i)} g_i) = (\alpha'(x), y_i''), \qquad \beta''(x, f_v^{(i)} g_i)) = (\beta'(x), v)$$

for $x \in \mathcal{P}^M_{y'}, v \in Y_i - Y_{i+1} \simeq \mathbf{A}^{s-i}$.

Now the restriction of $\mathcal{L}$ to $C_M \times \{y_i''\}$ is a local system $\mathcal{L}_M \boxtimes \bar{\mathbf{Q}}_l$. Hence by making use of (3.6.6) and (3.6.7), we have

(3.6.8) Under the isomorphism $p^{-1}(Y_i - Y_{i+1}) \simeq \mathcal{P}^M_{y'} \times (Y_i - Y_{i+1})$, the restriction of $\dot{\mathcal{L}}$ on $p^{-1}(Y_i - Y_{i+1})$ coincides with $\dot{\mathcal{L}}_M \boxtimes \bar{\mathbf{Q}}_l$.

It follows from (3.6.8) that we have an isomorphism

$$H_c^k(p^{-1}(Y_i - Y_{i+1}), \dot{\mathcal{L}}) \simeq H_c^{k'}(\mathcal{P}^M_{y'}, \dot{\mathcal{L}}_M), \tag{3.6.9}$$

where $k \equiv k'$ (mod 2). Then using the locally trivial filtration of $\mathcal{P}_y = p^{-1}(Y_0) \supset p^{-1}(Y_1) \supset \cdots$, and by induction on the rank of G, we see that

$$H_c^{\text{odd}}(p^{-1}(Y_i), \dot{\mathcal{L}}) = 0 \tag{3.6.10}$$

for any $i \geq 0$.

3.7. We are now ready to prove Theorem 3.4. Put $m = a_0 + r$. First we note the following.

(3.7.1) $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}}) = H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})_\rho$, and the map Φ acts on $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$ as a scalar multiplication.

In fact, it follows from section 2 that $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$ has a natural structure of $\mathcal{W} \times A_G(y)$-module, which is compatible with the isomorphisms (1.4.3) and (1.6.2). Hence by the generalized Springer correspondence, it is decomposed as

$$H_c^m(\mathcal{P}_y, \dot{\mathcal{L}}) \simeq \bigoplus_{\rho' \in A_G(y)^\wedge} V_{y,\rho} \otimes \rho',$$

where $V_{y,\rho'}$ is an irreducible $\mathcal{W}$-module whenever it is non-zero. Now the explicit description of the generalized Springer correspondence in the case of SL_n (see. 3.2) shows that ρ is the unique character such that $V_{y,\rho} \neq 0$. Hence $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}}) = H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})_\rho$. Since $A_G(y)$ is abelian, $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$ is an irreducible $\mathcal{W}$-module. It is easy to see that the map Φ on $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$ commutes with the action of $\mathcal{W}$. Hence Φ is a scalar multiplication, and so (3.7.1) holds.

Note that in the discussion of 3.5 and 3.6, $\mathcal{G}_y, Y_y$, etc. have natural $\mathbf{F}_q$-structures. We may choose the filtration of $\mathcal{G}_y$ and Y_y compatible with the $\mathbf{F}_q$-structure, i.e., all the Y_i and $\widehat{Y}_y$ are F-stable. Then all the diagrams and formulas there hold with $\mathbf{F}_q$-structure. We consider the top piece $Y_0 - Y_1$ of the filtration of Y_y. In this case, we may choose $g = g_0 = 1$ in the discussion in 3.6, and so y is decomposed as $y = y' + z'$ with $y' \in \mathfrak{m}$ and $z' \in \mathcal{C}_t + \mathfrak{n}_Q$. Hence y' (resp. y'') is the projection of y on $\mathfrak{m}$ (resp. on $\mathcal{C}_t$). Since y is a split element, y', y'' are also split. Let

$$m' = (\dim M - \dim \mathcal{C}_{y'}) - (\dim L_M - \dim \mathcal{C}_M),$$

where $\mathcal{C}_{y'}$ is the nilpotent orbit in $\mathfrak{m}$ containing y'. Since $\mathcal{C}$ is the regular nilpotent orbit in $\mathfrak{l}$, we see easily that $m = 2\dim \mathcal{B}_y$, where $\mathcal{B}_y$ is the variety of Borel subgroups whose Lie algebra contains y. Similarly, we have $m' = 2\dim \mathcal{B}^M_{y'}$. Then by using the locally trivial filtration of $\mathcal{B}_y$, we see that

$$m - m' = 2d \dim \operatorname{Ker} y = 2s. \tag{3.7.2}$$

In fact, assume that $y = y_\lambda$ with $\lambda = (\lambda_1, \dots, \lambda_k)$. By using the locally trivial filtration arising from the maximal parabolic subgroup P_1 of G with Levi subgroup L_1 of type A_{n-2}, one obtains that $\dim \mathcal{B}_y - \dim \mathcal{B}^{L_1}_{y_1} = \dim \operatorname{Ker} y$, where y_1 is a nilpotent element in $\operatorname{Lie} L_1$ of type $\lambda' = (\lambda_1, \dots, \lambda_k - 1)$. In the same way, one can find a nilpotent element $y_2 \in \operatorname{Lie} L_2$ with type $(\lambda_1, \dots, \lambda_k - 2)$ such that $\dim \mathcal{B}_{y_1}^{L_1} - \dim \mathcal{B}_{y_2}^{L_2} = \dim \operatorname{Ker} y$, where L_2 is a Levi subgroup of the maximal parabolic subgroup P_2 of L_1. Repeating this procedure, one can find similar formulas for $L_1 \supset L_2 \supset \cdots \supset L_d$ with $\mathcal{B}_{y_d}^{L_d} = \mathcal{B}_{y'}^{M}$. (3.7.2) follows from this.

Since $Y_0 - Y_1 \simeq \mathbf{A}^s$, we have an isomorphism with $\mathbf{F}_q$-structures

$$H_c^m(p^{-1}(Y_0 - Y_1), \dot{\mathcal{L}}) \simeq H_c^{m'}(\mathcal{P}_{y'}^M, \dot{\mathcal{L}}_M)[s] \tag{3.7.3}$$

as a special case of (3.6.9), where $[s]$ is the Tate twist. (The compatibility of the Frobenius actions comes from the discussion in 3.6 by noticing that y'' is a split element in $\mathcal{C}_t$.) Let Φ_M be the map on $H_c^{m'}(\mathcal{P}_{y'}^M, \dot{\mathcal{L}}_M)$ defined in a similar way as Φ. By induction on the rank of G, we may assume that Φ_M acts on $H_c^{m'}(\mathcal{P}_{y'}^M, \dot{\mathcal{L}}_M)$ as a scalar multiplication by $q^{m'/2}$. Then by (3.7.3), Φ acts on $H_c^m(p^{-1}(Y_0 - Y_1), \dot{\mathcal{L}})$ as a scalar multiplication by $q^{m/2}$. Now by using the cohomology long exact sequence with respect to the closed immersion $p^{-1}(Y_1) \subset p^{-1}(Y_0) = \mathcal{P}_u$, together with (3.6.10), we see that the natural map

$$H_c^m(p^{-1}(Y_0 - Y_1), \dot{\mathcal{L}}) \longrightarrow H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$$

is injective. This proves the theorem since Φ acts on $H_c^m(\mathcal{P}_y, \dot{\mathcal{L}})$ by a scalar multiplication by (3.7.1).

4. $G = SL_n$ with F of non-split type

4.1. In this section, we assume that $G = SL_n$ is as in section 3, and that p is large enough so that the argument in section 2 can be applied (e.g., $p > 3(n-1)$). Let $F = \sigma F_0$ be the twisted Frobenius map on G, where F_0 is the standard Frobenius map over $\mathbf{F}_q$ as in 3.1, and σ is the graph automorphism on G of order 2. Here we take $\sigma : G \to G$ defined by $\sigma(g) = w_0{}^t g^{-1} w_0^{-1}$ for $g \in G$ (w_0 is the permutation matrix in GL_n corresponding to the longest element in S_n, and ${}^t g$ means the transpose of the matrix $g = (g_{ij})$). Then B and T in 3.3 are F and F_0-stable.

Unipotent classes in G are all F-stable. In order to describe elements in C^F for each unipotent class C, we introduce a sesqui-linear form as follows. Let $V \simeq k^n$ be as in 3.1, and V_0 the $\mathbf{F}_{q^2}$-subspace of V generated by $\{e_i\}$. We define a sesqui-linear form $\langle\ ,\ \rangle$ on V_0 by $\langle \sum_i a_i e_i, \sum_j b_j e_j \rangle = \sum_i a_i b_{n-i}^q$. Then it is easy to see that for $g \in G^{F_0^2}$, $g \in G^F$ if and only if $\langle gv, gw \rangle = \langle v, w \rangle$ for any $v, w \in V_0$. Let $\mathfrak{g}$ be the Lie algebra of G, on which F acts naturally. Then for $x \in \mathfrak{g}^{F_0^2}$, $x \in \mathfrak{g}^F$ if and only if $\langle xv, w \rangle + \langle v, xw \rangle = 0$ for any $v, w \in V_0$.

4.2. For a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, we shall construct a nilpotent element $y_\lambda \in \mathfrak{g}^F$. First we note that there exist basis vectors

$$f_j^{(i)} \quad (1 \le i \le r, 1 \le j \le \lambda_i)$$

of V_0 satisfying the property that

$$\langle f_j^{(i)}, f_k^{(i')}\rangle = \begin{cases} 1 & \text{if } i = i', j+k = \lambda_i + 1 \text{ and } j \ne k, \\ \pm 1 & \text{if } i = i', j+k = \lambda_i + 1 \text{ and } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, we can choose $f_j^{(i)} = e_k$ for some k if λ_i is even. If λ_i is odd, put $\lambda_i = 2t_i + 1$. Then $f_j^{(i)}$ is of the form e_k if $j \ne t_i + 1$, and we can choose $f_{t_i+1}^{(i)}$ from one of the vectors $e_k \pm \frac{1}{2} e_{n-k+1}$ with $2k \ne n+1$ (note that $p > 2$), or e_l with $n = 2l+1$.

Put $t_i = [\lambda_i/2]$ ([ ] is the Gauss symbol) for each λ_i. We now define a nilpotent transformation $y_\lambda \in \mathfrak{g}^{F^2}$ on V_0 by

$$y_\lambda f_j^{(i)} = \begin{cases} f_{j+1}^{(i)} & \text{if } 1 \le j \le t_i - 1, \\ \varepsilon_i f_{j+1}^{(i)} & \text{if } j = t_i, \\ -f_{j+1}^{(i)} & \text{if } t_i + 1 \le j \le \lambda_i - 1, \\ 0 & \text{if } j = \lambda_i, \end{cases}$$

where $\varepsilon_i = 1$ if λ_i is even, and $\varepsilon_i = \langle f_{j+1}^{(i)}, f_{j+1}^{(i)}\rangle$ if λ_i is odd. Then

$$\{y_\lambda^j f_1^{(i)} \mid 1 \le i \le r, 0 \le j \le \lambda_i - 1\} \tag{4.2.1}$$

gives a basis of V_0 satisfying the relation

$$\langle y_\lambda^j f_1^{(i)}, y_\lambda^{\lambda_i - j + 1} f_1^{(i)}\rangle = (-1)^j a_i, \quad (a_i = \pm 1) \tag{4.2.2}$$

and $\langle y_\lambda^j f_1^{(i)}, y_\lambda^k f_1^{(i')}\rangle = 0$ for all other pairs. It follows from this that $y_\lambda \in \mathfrak{g}^F$.

Let d be as in 3.2, and assume that $d \ge 2$. We assume that the partition λ satisfies the condition that all the parts λ_i are divisible by d. We shall construct a nilpotent element $y_1 \in \mathfrak{g}^F$ of type $\nu = (d, \dots, d)$ associated to y_λ. We define a map y_1 on V_0 by

$$y_1 f_j^{(i)} = \begin{cases} 0 & \text{if } j \equiv 0 \pmod{d}, \\ y_\lambda f_j^{(i)} & \text{otherwise.} \end{cases} \tag{4.2.3}$$

Then in view of (4.2.2), it is easy to check that y_1 leaves the form $\langle\, , \rangle$ invariant, and we have $y_1 \in \mathfrak{g}^F$.

4.3. Let L be a Levi subgroup of the standard parabolic subgroup P of G of type $A_{d-1} \times \cdots \times A_{d-1}$ ($t = n/d$-factors). (Here P and L are as in 3.1 with respect to F_0, P is σ-stable, σ permutes the i-th factor and $(t-i+1)$-th factor, etc.) Thus P and L are F-stable. Let $\mathcal{C}$ be the regular nilpotent orbit in $\mathfrak{l}$. We choose a representative $y_0 \in \mathcal{C}^F$ in the following way; we define a basis $\{e_j^{(i)} \mid 1 \le i \le t, 1 \le j \le d\}$ of V_0 by $e_j^{(i)} = e_{(i-1)d+j}$. Then in the case where t is even, or t is odd and

$i \neq (t+1)/2$, we define

$$y_0 e_j^{(i)} = \begin{cases} e_{j+1}^{(i)} & \text{if } 1 \le i \le [t/2], j \neq d, \\ -e_{j+1}^{(i)} & \text{if } t - [t/2] + 1 \le i \le t, j \neq d, \\ 0 & \text{if } j = d \end{cases}$$

If t is odd and $i = (t+1)/2$, let V_1 be the subspace of V spanned by $e_j^{(i)}$ with $1 \le j \le d$. We define $y_0|_{V_1}$ as a regular nilpotent element $y_\lambda \in \mathfrak{sl}_d^F$ as in 4.2.

Let $(\mathcal{C}, \mathcal{L})$ be the cuspidal pair in $\mathfrak{l}$ corresponding to an F-stable character ρ_0 of $A_L(y_0)$. We have a natural homomorphism $A_L(y_0) \to A_G(y_0)$. Since $A_G(y_0)$ is a cyclic group of order d, this gives an isomorphism compatible with F-action. Thus ρ_0 is regarded as an F-stable character of $A_G(y_0)$. Since y_0 and y_1 are conjugate under G, there exists $c_1 \in A_G(y_0)$ (up to F-conjugacy) such that y_1 is obtained from y_0 by twisting by c_1. Since ρ_0 is F-stable, the value $\rho_0(c_1)$ is well-defined. This value is determined by y_1, hence by y_λ, which we denote by η_λ. Let $\gamma(y_0, \widetilde{\rho}_0, y_\lambda, \widetilde{\rho})$ be the scalar defined by choosing the extensions $\widetilde{\rho}_0, \widetilde{\rho}$ in a similar way as the case of split F (cf. 3.3). Put $m = a_0 + r$ as before. We have the following theorem.

Theorem 4.4. *Assume that p is large enough so that Dynkin-Kostant theory can be applied. Let w_0 be the longest element in $\mathcal{W}$. Then Φw_0 acts on $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}) = H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})_\rho$ as a scalar multiplication by $\eta_\lambda(-q)^{m/2}$. In particular, $\gamma(y_0, \widetilde{\rho}_0, y_\lambda, \widetilde{\rho}) = \eta_\lambda(-1)^{m/2}$.*

4.5. The remainder of this section is devoted to the proof of the theorem. If we notice that the preferred extension $\widetilde{V}_E$ of V_E is given by defining the action of $\sigma \in \widetilde{\mathcal{W}}$ by the action of $w_0 \in \mathcal{W}$, the second statement follows easily from the first one. So we concentrate the proof of the first statement. For y_1 of type $(d, \dots, d)$, we construct a $\mathfrak{sl}_2$-triple $\{y_1, y_1^-, h_1\}$ as follows. On each Jordan block, y_1 can be expressed as a matrix of degree d with respect to the basis in (4.2.1) as

$$Y = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & 0 & \\ & & 1 & 0 \end{pmatrix}.$$

We define matrices Y^-, H of degree d by

$$Y^- = \begin{pmatrix} 0 & 1\cdot(d-1) & & & \\ & 0 & 2(d-2) & & \\ & & 0 & \ddots & \\ & & & \ddots & (d-1)\cdot 1 \\ & & & & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1-d & & & \\ & 3-d & & \\ & & \ddots & \\ & & & d-1 \end{pmatrix}.$$

Then $[H, Y] = 2Y, [H, Y^-] = -2Y, [Y, Y^-] = H$. Thus by combining these matrices for all the Jordan blocks, one obtains $y_1^-, h_1 \in \mathfrak{g}$ satisfying the property that $[h_1, y_1] = 2y_1, [h_1, y_1^-] = -2y_1^-, [y_1, y_1^-] = h_1$ as asserted. It follows from the construction, we see easily that $y_1^-, h_1 \in \mathfrak{g}^F$.

We define a transversal slice $\varSigma$ with respect to the orbit through y_1 in $\mathfrak{g}$ by $\varSigma = y_1 + Z_{\mathfrak{g}}(y_1^-)$. Hence $\varSigma$ is F-stable. We have the following lemma.

Lemma 4.6. *Let y_λ be as in 4.2. Then we have $y_\lambda \in \varSigma^F$.*

Proof. We write y_λ as $y_\lambda = y_1 + y$. It is enough to show that $y \in Z_{\mathfrak{g}}(y_1^-)$. Now y is a nilpotent transformation on V_0 determined by the condition that

$$y : y_\lambda^j f_1^{(i)} \mapsto y_\lambda^{j+1} f_1^{(i)}$$

for $j \equiv 0 \pmod d$, and it maps all other $y_\lambda^j f_1^{(i)}$ to 0. Since y_1^- maps $y_\lambda^j f_1^{(i)}$ to $y_\lambda^{j-1} f_1^{(i)}$ up to scalar if $j \not\equiv 1 \pmod d$, and to 0 if $j \equiv 1 \pmod d$. we see easily check that $y_1^- \circ y = y \circ y_1^- = 0$ on V_0. Hence $y \in Z_{\mathfrak{g}}(y_1^-)$. □

4.7. By using the $\mathfrak{sl}_2$-triple $\{y_1, y_1^-, h_1\}$, one can define a Lie algebra homomorphism $\phi_1 : \mathfrak{sl}_2 \to \mathfrak{g}$ as in 2.7. The construction of $\mathfrak{sl}_2$-triple given in 4.5 works well for y_λ in general, and one gets the $\mathfrak{sl}_2$-triple containing y_λ. We denote by ϕ_λ the homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ obtained from it. Thus $Z_G(\phi), M_G(\phi)$, are defined as in 2.7 for $\phi = \phi_1, \phi_\lambda$.

Let $\pi : \dot{\mathfrak{g}} \to \mathfrak{g}$ be as in 2.8. Then $\mathcal{P}_y \subset \dot{\mathfrak{g}}$, and the local system $\dot{\mathcal{L}}$ on $\mathcal{P}_y$ can be extended to a local system on $\dot{\mathfrak{g}}$ (cf. 2.8), which we denote also by $\dot{\mathcal{L}}$ as in 2.8. Put $K_1 = \pi_!\dot{\mathcal{L}}$. K_1 is essentially the same as $K = \pi_!\dot{\mathcal{L}}^*$ in 2.8, and so $K_1[\delta]$ is a perverse sheaf on $\mathfrak{g}$ with a canonical $\mathcal{W}$-action, where δ is as in 2.8. By making use of the transversal slice $\varSigma$, we show the following proposition.

Proposition 4.8. *There exist natural maps of $\mathcal{W}$-modules, which make the following diagram commutative.*

$$\begin{array}{ccc} \mathbb{H}^i(\mathfrak{g}, K_1) & \xrightarrow{\pi_1} & H_c^i(\mathcal{P}_{y_1}, \dot{\mathcal{L}}) \\ \Big\downarrow{\scriptstyle \pi_\lambda} & \swarrow{\scriptstyle \xi_\lambda} & \\ H_c^i(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}) & & \end{array}$$

Moreover, the map ξ_λ is equivariant with respect to the actions of Φ on both cohomologies.

Proof. By the inclusion $\{y_\lambda\} \subset \Sigma \subset \mathfrak{g}$, we have the canonical maps

$$
\begin{array}{ccc}
\mathbb{H}^i(\mathfrak{g}, K_1) & \longrightarrow & \mathbb{H}^i(\Sigma, K_1) \\
\downarrow & \swarrow & \\
\mathcal{H}^i_{y_\lambda}(K_1) & &
\end{array}
\tag{4.8.1}
$$

Since K_1 is a $\mathcal{W}$-complex with respect to the trivial action of $\mathcal{W}$ on $\mathfrak{g}$, the above maps are $\mathcal{W}$-equivariant. Since $K_1 = \pi_!\dot{\mathcal{L}}$, we have

$$\mathcal{H}^i_{y_\lambda}(K_1) \simeq H^i_c(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$$

by the proper base change theorem. On the other hand, $K_1[\delta]$ is a perverse sheaf on $\mathfrak{g}$. Since the morphism $G \times \Sigma \to \mathfrak{g}$ is smooth with all fibres of pure dimension equal to $\dim Z_G(y_1)$, by a similar argument as in [L6, 3.2], $K_1[\dim \Sigma]|_\Sigma$ is a perverse sheaf on Σ. Σ is stable under the $\mathbf{G}_m$-action ($t : x \mapsto t^{i-2}x$ for each $x \in \mathfrak{g}_i$ with respect to the grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ associated to $\phi_1 : \mathfrak{sl}_2 \to \mathfrak{g}$), and contracts to $y_1 \in \Sigma$. Since K_1 is $\mathbf{G}_m$-equivariant, the canonical map $\mathbb{H}^i(\Sigma, K_1) \to \mathcal{H}^i_{y_1}(K_1)$ gives rise to an isomorphism

$$\mathbb{H}^i(\Sigma, K_1) \simeq \mathcal{H}^i_{y_1}(K_1) \simeq H^i_c(\mathcal{P}_{y_1}, \dot{\mathcal{L}}).$$

The proposition follows from this. □

For the special case where $i = 0$, we have the following more precise result.

Lemma 4.9. *The maps π_1, π_λ in Proposition 4.8 give isomorphisms*

$$\mathbb{H}^0(\mathfrak{g}, K_1) \simeq H^0_c(\mathcal{P}_y, \dot{\mathcal{L}}) \simeq \Gamma(\mathcal{C}, \mathcal{L})$$

for $y = y_1, y_\lambda$. In particular, $\xi^0_\lambda : H^0_c(\mathcal{P}_{y_1}, \dot{\mathcal{L}}) \to H^0_c(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ is an isomorphism.

Proof. We consider the following commutative diagram

$$
\begin{array}{ccccccc}
\overline{\mathcal{C}} & \xleftarrow{\overline{\alpha}} & \widehat{\dot{\mathfrak{g}}}' & \xrightarrow{\overline{\beta}} & \dot{\mathfrak{g}}' & \xrightarrow{\overline{\pi}} & \mathfrak{g} \\
\bar{j}\uparrow & & \hat{j}\uparrow & & \uparrow j & & \uparrow \mathrm{id} \\
\mathcal{C} & \xleftarrow{\alpha} & \widehat{\dot{\mathfrak{g}}} & \xrightarrow{\beta} & \dot{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g}
\end{array}
\tag{4.9.1}
$$

where the lower horizontal maps are as in 2.8, and

$$\dot{\mathfrak{g}}' = \{(x, gP) \in \mathfrak{g} \times G/P \mid \mathrm{Ad}(g^{-1})x \in \overline{\mathcal{C}} + \mathfrak{z} + \mathfrak{n}_P\},$$
$$\widehat{\dot{\mathfrak{g}}}' = \{(x, g) \in \mathfrak{g} \times G \mid \mathrm{Ad}(g^{-1})x \in \overline{\mathcal{C}} + \mathfrak{z} + \mathfrak{n}_P\}$$

and $\overline{\alpha}, \overline{\beta}$ and $\overline{\pi}$ are maps defined in a similar way as α, β and π. $\hat{j}, j$ are open immersions, and $\overline{\pi}$ is proper. Now the local system $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}$ is determined by the condition that $\alpha^*\mathcal{L} = \beta^*\dot{\mathcal{L}}$. Since the square in the left hand side in (4.9.1) is cartesian, $\widehat{j}_!(\alpha^*\mathcal{L}) \simeq \overline{\alpha}^*(\bar{j}_!\mathcal{L})$. The middle square is also cartesian, and we have

$$\overline{\beta}^*(j_!\dot{\mathcal{L}}) \simeq \widehat{j}_!(\beta^*\dot{\mathcal{L}}) \simeq \widehat{j}_!(\alpha^*\mathcal{L}) \simeq \overline{\alpha}^*(j_!\mathcal{L}).$$

By the definition of the direct image with compact support, we have $\pi_!\dot{\mathcal{L}} = \overline{\pi}_*(j_!\dot{\mathcal{L}})$. Then

$$\begin{aligned}\mathbb{H}^0(\mathfrak{g}, K_1) &= \mathbb{H}^0(\mathfrak{g}, \pi_!\dot{\mathcal{L}}) \\ &= \mathbb{H}^0(\mathfrak{g}, \overline{\pi}_*(j_!\dot{\mathcal{L}})) \\ &\simeq H^0(\dot{\mathfrak{g}}', j_!\dot{\mathcal{L}}).\end{aligned}$$

Similarly, by using the open immersion

$$j : \mathcal{P}_y \hookrightarrow \overline{\mathcal{P}}_y = \{gP \in G/P \mid \operatorname{Ad}(g^{-1})y \in \overline{\mathcal{C}} + \mathfrak{n}_P\},$$

we see that $H^0_c(\mathcal{P}_y, \dot{\mathcal{L}}) \simeq H^0(\overline{\mathcal{P}}_y, j_!\dot{\mathcal{L}})$ for $y = y_1, y_\lambda$. It follows that the maps π_1, π_λ in Proposition 4.8 for $i = 0$ are nothing but the restriction $\varGamma(\dot{\mathfrak{g}}', j_!\dot{\mathcal{L}}) \to \varGamma(\overline{\mathcal{P}}_y, j_!\dot{\mathcal{L}})$ of the global section of the sheaf $j_!\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}'$ for $y = y_1, y_\lambda$. But since $\overline{\beta}^*(j_!\dot{\mathcal{L}}) \simeq \overline{\alpha}^*(\bar{j}_!\mathcal{L})$, we have

$$\varGamma(\dot{\mathfrak{g}}', j_!\dot{\mathcal{L}}) \simeq \varGamma(\overline{\mathcal{C}}, \bar{j}_!\mathcal{L}) \simeq \varGamma(\overline{\mathcal{P}}_y, j_!\dot{\mathcal{L}}).$$

Finally, we note that $\bar{j}_!\mathcal{L} \simeq \bar{j}_*\mathcal{L}$ since $\mathcal{L}$ is the cuspidal local system and so is clean ([L7, 2.2]). Hence

$$\varGamma(\overline{\mathcal{C}}, \bar{j}_!\mathcal{L}) \simeq \varGamma(\overline{\mathcal{C}}, \bar{j}_*\mathcal{L}) \simeq \varGamma(\mathcal{C}, \mathcal{L})$$

as asserted. □

4.10. Let $\phi_0 : \mathfrak{sl}_2 \to \mathfrak{l} \subset \mathfrak{g}$ be the Lie algebra homomorphism such that $\phi_0\binom{01}{00} = y_0$ constructed as in 4.3. Thus ϕ_0 is F-equivariant with respect to the twisted Frobenius action on $\mathfrak{sl}_2$. Put $G_0 = Z^0_G(\phi_0)$ and $T_0 = Z^0_L(\phi_0)$. Then G_0 and T_0 are F-stable. It is checked that G_0 is isomorphic to SL_t, and F acts as a twisted Frobenius endomorphism on SL_t. By (2.9.1) we have $T_0 \simeq Z^0_L$, and under the identification $G_0 \simeq SL_t$, T_0 coincides with a maximally split maximal torus of SL_t, and $\mathcal{W} = N_G(Z^0_L)/L$ is naturally isomorphic to the Weyl group of G_0 with respect to T_0.

F acts naturally on $\mathcal{W} \simeq S_t$, as a conjugation by $w_0 \in \mathcal{W}$, where w_0 is the longest element in $\mathcal{W}$. Thus Fw_0 acts trivially on $\mathcal{W}$. By 2.17, F acts naturally on $H^*_{T_0} = \bigoplus_i H^{2i}_{T_0} \simeq S(\mathfrak{h}^*)$, where $\mathfrak{h}^* = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(T_0)$. $\mathcal{W}$ also acts on $H^*_{T_0}$, which coincides with the action of $\mathcal{W}$ on $S(\mathfrak{h}^*)$ induced from the action of $\mathcal{W}$ on $X(T_0)$ (cf. (2.3.2)). We have the following lemma.

Lemma 4.11. *Fw_0 acts on $H^{2i}_{T_0}$ as a scalar multiplication by $(-q)^i$.*

Proof. Fw_0 commutes with the graded algebra structure of $H^*_{T_0}$. Since $H^*_{T_0}$ is generated by $H^2_{T_0}$, it is enough to show that Fw_0 acts on $H^2_{T_0}$ as a scalar multiplication by $-q$. We show this by modifying the arguments used in the proof of Lemma 2.4. Let $\varGamma$ be as in the proof of Lemma 2.4 (with respect to G_0). We consider the locally trivial fibration $f : T_0\backslash\varGamma \to G_0\backslash\varGamma$. We may assume that $\varGamma$ is defined over $\mathbf{F}_q$, and f is F-equivariant. We consider the spectral sequence

$$H^p(G_0\backslash\varGamma, R^qf_*\bar{\mathbf{Q}}_l) \Rightarrow H^{p+q}(T_0\backslash\varGamma), \tag{4.11.1}$$

which have natural actions of $\mathcal{W}$ (cf. 2.4) and of F. Let θ be the reflection representation of $\mathcal{W}$. Then (4.11.1) implies a spectral sequence

$$H^p(G_0\backslash\varGamma, R^qf_*\bar{\mathbf{Q}}_l)_\theta \Rightarrow H^{p+q}(T_0\backslash\varGamma)_\theta,$$

where X_θ denotes the θ-isotypic subspace for a $\mathcal{W}$-module X. As in 2.4, we have

$$H^p(G_0\backslash\varGamma, R^qf_*\bar{\mathbf{Q}}_l) \simeq H^p(G_0\backslash\varGamma)\otimes H^q(T_0\backslash G_0),$$

and so

$$H^p(G_0\backslash\varGamma, R^qf_*\bar{\mathbf{Q}}_l)_\theta \simeq H^p(G_0\backslash\varGamma)\otimes H^q(T_0\backslash G_0)_\theta$$

since $\mathcal{W}$ acts trivially on $H^p(G_0\backslash\varGamma)$. Now it is known that $\bigoplus_i H^{2i}(T_0\backslash G_0)$ is a graded regular representation of $\mathcal{W}$, and that

$$H^q(T_0\backslash G_0)_\theta = \begin{cases} H^q(T_0\backslash G_0) & \text{if } q = 2, \\ 0 & \text{if } q < 2. \end{cases}$$

Since $H^*(T_0\backslash\varGamma) = H^*_{T_0} \simeq S(\mathfrak{h}^*)$, we have $H^2(T_0\backslash\varGamma)_\theta = H^2(T_0\backslash\varGamma)$. Moreover, $H^0(G_0\backslash\varGamma) = H^0_G = \bar{\mathbf{Q}}_l$ be Lemma 2.4. It follows that

$$H^2(T_0\backslash\varGamma) \simeq H^2(T_0\backslash G_0).$$

This isomorphism is compatible with the actions of F and $\mathcal{W}$. It is well-known that Fw_0 acts as a scalar multiplication $-q$ on $H^2(T_0\backslash G_0) = H^2(B_0\backslash G_0)$, where B_0 is the F-stable Borel subgroup of G_0 containing T_0. Hence Fw_0 acts similarly on $H^2_{T_0} = H^2(T_0\backslash\varGamma)$. This proves the lemma. □

4.12. We consider the equivariant homology $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$, where $M^0(y_\lambda) = M^0_G(y_\lambda)$. By results in Section 2, the graded Hecke algebra $\mathbf{H} = \mathbf{S}\otimes\bar{\mathbf{Q}}_l[\mathcal{W}]$ acts on $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$, where $\mathbf{S} = S(\mathfrak{h}^*)\otimes\bar{\mathbf{Q}}_l[\mathbf{r}]$ as defined in 2.9 with $S(\mathfrak{h}^*)$ in 4.10. We shall construct a standard $\mathbf{H}$-module $E_{v,\rho'}$ obtained from $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$ for a certain pair (v, ρ'). Let y be the nilpotent element in $\mathfrak{g}_{\mathbf{C}}$ corresponding to $y_\lambda \in \mathfrak{g}$. We choose $y^-, h_0 \in \mathfrak{g}_{\mathbf{C}}$ such that $\{y, y^-, h_0\}$ forms an $\mathfrak{sl}_2$-triple. Put $h = h_0, r_0 = 1$. Then $(h, r_0) \in \mathfrak{m}(y)_{\mathbf{C}}$ with h semisimple. We denote by v an element in $H^*_{M^0(y_\lambda)}$ corresponding to the $M^0(y)$-orbit of (h, r_0). Let ρ be the irreducible character of $A_G(y_\lambda)$ as in 4.3. Since $A_G(y_\lambda) \simeq \overline{M}(y_\lambda) \simeq \overline{M}(y)$, one can regard ρ as a character of $\overline{M}(y)$. Let ρ^* be the dual representation of ρ.

Under the notation in Remark 2.14 and Theorem 2.15, we note that

(4.12.1) Let v be as above. Then $E_{v,\rho'}$ is a simple $\mathbf{H}$-module, where ρ' is the restriction of ρ^* on $\overline{M}(y,v)$.

It is enough to show that (h, r_0) satisfies the property in Theorem 2.15. By Dynkin-Kostant theory, y is contained in the open dense orbit in $Y_{(h,r_0)} = \mathfrak{g}_2$ (the graded space with respect to h) under the action of $Z_{G_{\mathbf{C}}}(h)$. It remains to show that ρ' occurs in $H_*^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$. But this is clear since $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}) = H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})_\rho$. Thus (4.12.1) holds.

4.13. The $\mathbf{F}_q$-structure $\varphi_0 : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ induces a linear isomorphism Φ on $H_c^*(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$. φ_0 also induces a linear isomorphism Ψ on $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$ satisfying the following property; by [L6, 7.2, (d)], there exists a $\bar{\mathbf{Q}}_l$-linear isomorphism

$$\bar{\mathbf{Q}}_l \otimes_{H^*_{M^0(y_\lambda)}} H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*) \to H_*^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*), \tag{4.13.1}$$

where $\bar{\mathbf{Q}}_l$ is regarded as an $H^*_{M^0(y_\lambda)}$-module via the canonical map $H^*_{M^0(y_\lambda)} \to H^*_{\{e\}} = \bar{\mathbf{Q}}_l$. F acts naturally on $H^*_{M^0(y_\lambda)}$ and on $H^*_{\{e\}}$, and the last map is F-equivariant with respect to the trivial action on $\bar{\mathbf{Q}}_l$. Thus Ψ induces a linear map $\overline{\Psi}$ on the left hand side of (4.13.1). The $\bar{\mathbf{Q}}_l$-linear map in (4.13.1) is compatible with $\overline{\Psi}$ and the map Φ^* on $H_*^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*) = H_c^*(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})^*$, where Φ^* is the transposed inverse of Φ.

Note that $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ is an irreducible $\mathcal{W}$-module. Since Φw_0 commutes with all the elements in $\mathcal{W}$, we see that Φw_0 acts on $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ as a scalar multiplication. Then we have the following lemma.

Lemma 4.14. *Assume that Φw_0 acts on $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ by a scalar multiplication by ζ. Then Φw_0 acts on $H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ by a scalar multiplication by $\zeta(-q^{-1})^{m/2}$.*

Proof. Let $v = (h, r_0)$ be as in (4.12.1). Let $\gamma_v : H^*_{M^0(y_\lambda)} \to \bar{\mathbf{Q}}_l$ be the algebra homomorphism corresponding to v (cf. 2.13). Since $M^0(y_\lambda)$ is F-stable, F acts naturally on $H^*_{M^0(y_\lambda)}$ such that $\Psi(mx) = F(m)\Psi(x)$ for $m \in H^*_{M^0(y_\lambda)}$ and $x \in H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$. Since $M_G^0(y_\lambda) \simeq Z_G^0(y_\lambda) \times \mathbf{G}_m$, we have $H^*_{M^0(y_\lambda)} \simeq S(\mathfrak{h}_1^*)^{W_1} \otimes \bar{\mathbf{Q}}_l[\mathbf{r}]$, where W_1 is the Weyl group of a reductive group $Z_G^0(\phi_\lambda)$ and $\mathfrak{h}_1^* = \bar{\mathbf{Q}}_l \otimes_{\mathbf{Z}} X(T_1)$ with a maximally split maximal torus T_1 of $Z_G^0(\phi_\lambda)$. We note that

(4.14.1) The maximal ideal $\operatorname{Ker}\gamma_v$ in $H^*_{M^0(y_\lambda)}$ is generated by homogeneous polynomials.

In fact, by the previous argument, we may replace $H^*_{M^0(y_\lambda)}$ by $S(\mathfrak{m}(y)^*_{\mathbf{C},r})^{M^0(y)}$, and v by $(h, r_0) \in \mathfrak{m}(y)_{\mathbf{C},r}$. It is enough to show that if a polynomial function f on $\mathfrak{m}(y)_{\mathbf{C},r}$ which is invariant under the action of $M^0(y)$ vanishes on (h, r_0), then its homogeneous parts also vanish at (h, r_0). But the $\mathbf{G}_m$-action on $\mathfrak{m}(y)_{\mathbf{C}}$ implies that $t : (h, r_0) \mapsto (t^{-2}h, t^{-2}r_0)$. Since f is invariant under $M^0(y)$, we see that f vanishes also on $(t^{-2}h, t^{-2}r_0)$ for any $t \in \mathbf{C}^*$. It follows that each homogeneous part of f also vanishes at (h, r_0) as asserted.

Next we note that

(4.14.2) The maximal ideal $\operatorname{Ker}\gamma_v$ is F-stable.

Let w_1 be the longest element in W_1. As in Lemma 4.11, Fw_1 acts on $S(\mathfrak{h}_1^*)_i$ (the i-th homogeneous part) as a scalar multiplication by $(-q)^i$. Hence F acts on $S(\mathfrak{h}_1^*)_i^{W_1}$ by a scalar multiplication by $(-q)^i$. Also, F acts on $\bar{\mathbf{Q}}_l[\mathbf{r}]_i$ as a scalar multiplication by q^i. Since $\operatorname{Ker}\gamma_v$ is a homogeneous ideal, F stabilizes $\operatorname{Ker}\gamma_v$. Hence (4.14.2) holds.

Now $E_{v,\rho'}$ is obtained as the quotient of $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ by the $\mathbf{H}$-submodule $I_v = \operatorname{Ker}\gamma_v \cdot H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$. Since $\operatorname{Ker}\gamma_v$ is F-stable, we see that I_v is Ψ-stable. Thus Ψ induces a linear map on $E_{v,\rho'}$. We consider the filtration $F^0 \subseteq F^1 \subseteq \cdots$ of $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$ as in 2.16. Then each F^i, as well as its ρ'-isotypic part $F^i_{\rho'}$, is Ψ-stable. Then $(F^i_{\rho'})_v$ is also Ψ-stable since it is the quotient of $F^i_{\rho'}$ by $F^i_{\rho'} \cap I_v$. By (2.16.5) and by our assumption, Ψw_0 acts on the non-zero space $F^0_v = (F^0_{\rho'})_v$ as a scalar multiplication by ζ^{-1}. This implies that Ψw_0 acts on $H_0^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$ modulo I_v by ζ^{-1}. On the other hand, since $E_{v,\rho'}$ is a simple $\mathbf{H}$-module, $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ is generated by $H_0^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ mod I_v as an $\mathbf{H}$-module. Since $\mathbf{r}$ acts as a scalar multiplication by r_0 on $E_{v,\rho'}$, the action of $\mathbf{H}$ on $E_{v,\rho'}$ is given by the action of $S(\mathfrak{h}^*) = H^*_{T_0}$ and of $\mathcal{W}$. Note that $\Psi w_0(\xi x) = Fw_0(\xi)\Psi w_0(x)$ for $\xi \in \mathbf{S}, x \in H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$. The action of $\mathcal{W}$ preserves the grading of $H_*^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$, and Ψw_0 commutes with $\mathcal{W}$. It follows, by Lemma 4.11 that Ψw_0 acts on $H_m^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ modulo I_v as a scalar multiplication by $\zeta^{-1}(-q)^{m/2}$. Let f_m be the map $F_v^{m-1} \to F_v^m$ as in 2.16, which is $\overline{M}(y_\lambda, v)$-equivariant. Since $(F_v^m)_{\rho'}/(\operatorname{Im} f_m)_{\rho'}$ is regarded as a natural quotient of $H_m^{M^0(y_\lambda)}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ modulo I_v, Ψw_0 acts on $(F_v^m)_{\rho'}/(\operatorname{Im} f_m)_{\rho'}$ as $\zeta^{-1}(-q)^{m/2}$. Since $H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)$ is isomorphic to $F_v^m/\operatorname{Im} f_m$ by (2.16.4), we see that Ψw_0 acts on $H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$ by a scalar multiplication by $\zeta^{-1}(-q)^{m/2}$, which coincides with the action of $\Phi^* w_0$ on it. We claim that $H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*) = H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)_{\rho'}$. In fact,

$$H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*) = H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})^* = \mathit{\Gamma}(\mathcal{C}, \mathcal{L})^*$$

by Lemma 4.9. $A_L(y_0)$ acts on $\mathit{\Gamma}(\mathcal{C}, \mathcal{L})$ by the character ρ_0. Since ρ is the pull back of ρ_0 under the map $A_G(y_\lambda) \to A_L(y_0)$ (cf. 3.2), the action of $\overline{M}(y_\lambda) = A_G(y_\lambda)$ on $H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ is via ρ_0. Hence $H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}) = H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})_\rho$ and the claim follows.

Thus Φw_0 acts on $H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}) = H_m^{\{e\}}(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}}^*)^*$ by a scalar multiplication by $\zeta(-q^{-1})^{m/2}$ as asserted. □

4.15. We are now ready to prove Theorem 4.4. First we note that $\mathcal{W}$ acts trivially on $H_c^0(\mathcal{P}_y, \dot{\mathcal{L}})$ for any $y = y_\nu$ such that all the parts of ν are divisible by d. In fact, if y_ν is regular nilpotent, $a_0 + r = 0$ by (1.3.1) since $\mathcal{C}$ is also a regular nilpotent class in L. It follows, by the generalized Springer correspondence (see 3.2), that $H_c^0(\mathcal{P}_{y_\nu}, \dot{\mathcal{L}})$ is the irreducible $\mathcal{W}$-module corresponding to the unit representation. Thus by Lemma 4.9, $H_c^0(\mathcal{P}_y, \dot{\mathcal{L}})$ is also a trivial $\mathcal{W}$-module for any y.

Now assume that Φw_0 acts on $H_c^m(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ by a scalar multiplication by ζ. Then by Lemma 4.14, Φw_0 acts on $H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ by a scalar multiplication by $\zeta(-q^{-1})^{m/2}$. Since the map $H_c^0(\mathcal{P}_{y_1}, \dot{\mathcal{L}}) \to H_c^0(\mathcal{P}_{y_\lambda}, \dot{\mathcal{L}})$ is Φw_0-equivariant isomorphism by Lemma

4.9 (and Proposition 4.8), we see that Φw_0 acts on $H_c^0(\mathcal{P}_{y_1}, \dot{\mathcal{L}})$ by a scalar multiplication by $\zeta(-q^{-1})^{m/2}$. Since w_0 acts trivially on it, we see that

(4.15.1) Φ acts on $H_c^0(\mathcal{P}_{y_1}, \dot{\mathcal{L}})$ by $\zeta(-q^{-1})^{m/2}$.

On the other hand, by a similar argument as in the proof of Lemma 4.9, the natural map

$$H_c^0(\mathcal{P}_{y_0}, \dot{\mathcal{L}}) \simeq \varGamma(\mathcal{C}, \mathcal{L}) \to \mathcal{L}_{y_0}$$

gives an isomorphism. This isomorphism is compatible with the action of Φ and of φ_0. It follows that Φ acts on $H_c^0(\mathcal{P}_{y_0}, \dot{\mathcal{L}})$ as an identity map. Since y_1 is in the G-orbit of y_0, $H_c^0(\mathcal{P}_{y_0}, \dot{\mathcal{L}}) \simeq H_c^0(\mathcal{P}_{y_1}, \dot{\mathcal{L}})$. As discussed in the proof of Lemma 4.14, $A_G(y_0)$ acts on $H_c^0(\mathcal{P}_{y_0}, \dot{\mathcal{L}})$ via ρ_0. We also note that $A_L(y_0) \simeq A_G(y_0)$. Since y_1 is G^F-conjugate to y_{c_1}, an element twisted by $c_1 \in A_G(y_0)$ from y_0, we see that

(4.15.2) Φ acts on $H_c^0(\mathcal{P}_{y_1}, \dot{\mathcal{L}})$ by a scalar multiplication by $\rho_0(c_1) = \eta_\lambda$.

Comparing (4.15.1) and (4.15.2), we see that $\zeta = \eta_\lambda(-q)^{m/2}$. This proves the theorem.

4.16. In order to apply Theorem 4.4, we need to know $c_1 \in A_G(y_0)$ such that $y_1 = (y_0)_{c_1}$. For a given y_0, we shall choose a specific y_1 and y_λ, and determnie c_1 explicitly. Put $\lambda' = (\lambda'_1, \dots, \lambda'_r)$ with $\lambda'_i = \lambda_i/d$. Hence λ' is a partition of t. Let $\{e_j^{(i)}\}$ be the basis of V_0 as in 4.3. Put $d' = [d/2]$. Let us define a subspace W_0 of V_0 and define y_0 by

$$W_0 = \begin{cases} \langle e_{d'+1}^{(i)} \mid 1 \le i \le t \rangle & \text{if } d \text{ is odd}, \\ \{0\} & \text{if } d \text{ is even}. \end{cases}$$

Also we define subspaces W_1, W_2 of V_0 by

$$\begin{aligned} W_1 &= \langle e_j^{(i)} \mid 1 \le j \le d', 1 \le i \le t \rangle, \\ W_2 &= \langle e_j^{(i)} \mid d - d' + 1 \le j \le d, 1 \le i \le t \rangle. \end{aligned}$$

Clearly we have $V_0 = W_1 \oplus W_0 \oplus W_2$. We define a new basis $\{h_j^{(i)} \mid 1 \le j \le d, 1 \le i \le t\}$ of V_0 satisfying the following conditions.

(i) $h_j^{(i)} = e_j^{(i)}$ if $e_j^{(i)} \in W_1$.
(ii) The set $\{h_j^{(i)} \mid d - d' + 1 \le j \le d, 1 \le i \le t\}$ coincides with the set of the basis $\{e_j^{(i)}\}$ of W_2.
(iii) Let z be the number of i such that λ'_i is odd. Then

$$\begin{aligned} \langle h_j^{(2i-1)}, h_{d-j+1}^{(2i)} \rangle &= 1 \quad \text{for } 1 \le i \le (t-z)/2, 1 \le j \le d, \\ \langle h_j^{(i)}, h_{d-j+1}^{(i)} \rangle &= 1 \quad \text{for } (t-z)/2 + 1 \le i \le t, 1 \le j \le d', \\ \langle h_{d'+1}^{(i)}, h_{d'+1}^{(i)} \rangle &= \pm 1 \quad \text{for } (t-z)/2 + 1 \le i \le t \text{ if } d\text{: odd}, \\ \langle h_j^{(i)}, h_{j'}^{(i')} \rangle &= 0. \quad \text{for all other cases.} \end{aligned}$$

(iv) $\{h^{(i)}_{d'+1}\}$ gives a basis of W_0 in the case where d is odd.

The conditions (i) $\sim$ (iv) determines $\{h^{(i)}_j\}$ uniquely except the vectors contained in W_0. We note that one can choose the basis $\{h^{(i)}_j\}$ of W_0 so that the transition matrix between $\{e^{(i)}_j\}$ and $\{h^{(i)}_j\}$ has the determinant ± 1 (see the construction of $f^{(i)}_j$ in 4.2).

We define a nilpotent transformation $y'_1 \in \mathfrak{g}^F$ as in 4.2 replacing $f^{(i)}_j$ by $h^{(i)}_j$. Then it is easy to construct $y'_\lambda \in \mathfrak{g}^F$ such that y'_1 is obtained from y'_λ by a similar procedure as y_1 is obtained from y_λ. Clearly, y'_1 (resp. y'_λ) is conjugate to y_1 (resp. y_λ). The argument in the proof of Theorem 4.4 works well for y'_1, y'_λ. Thus we consider y'_1 and y'_λ, and write them as y_1, y_λ. We shall describe $c_1 \in A_G(y_0)$ such that $y_1 = (y_0)_{c_1}$. It follows from the construction that there exists $g \in \widetilde{G} = GL_n$ such that $\mathrm{Ad}(g)y_0 = y_1$, where g stabilizes the subspaces W_0, W_1, W_2. More precisely, g acts trivially on W_1, and gives a permutation matrix with respect to the basis $\{e^{(i)}_j\}$ up to ± 1 on W_2. Thus by our choice of the basis $\{h^{(i)}_j\}$, we have $\det g = \pm 1$. Let us take $\alpha \in \overline{\mathbf{F}}_q$ such that $\alpha^n = \pm 1$ (if $\det g = 1$, we choose $\alpha = 1$). We note that $\mathrm{Ker}\, y_0 \subset W_2$, and that g stabilizes $\mathrm{Ker}\, y_0$. We denote by g_0 the restriction of g on $\mathrm{Ker}\, y_0$. Then we have

Lemma 4.17. *Let the notations be as before. Then we have* $y_1 = (y_0)_{c_1}$, *where* $c_1 \in A_G(y_0)$ *is given, under the identification* $A_G(y_0) \simeq \{x \in \overline{\mathbf{F}}^*_q \mid x^d = 1\}$, *by*

$$c_1 = \alpha^{t(1-q)} \det g_0.$$

Proof. Let $\phi_0 : \mathfrak{sl}_2 \to \mathfrak{g}$ be as in 4.10, and we consider the group $Z_G(\phi_0)$. Then $A_G(y_0) \simeq Z_G(\phi_0)/Z^0_G(\phi_0)$. We have $Z_G(\phi_0) \simeq \{x \in GL_t \mid \det x^d = 1\}$, where the element $g_1 \in Z_G(\phi_0)$ corresponding to x is given as follows; g_1 acts on the subspace V_j of V_0 spanned by $\{e^{(i)}_j \mid 1 \le i \le t\}$, for a fixed j, as $x \in GL_t$. Now if we can find $g_1 \in G$ such that $Ad(g_1)y_0 = y_1$, then $g_1^{-1}F(g_1) \in A_G(y_0)$, and it leaves $\mathrm{Ker}\, y_0$ invariant. Moreover, the determinant of the restriction of $g_1^{-1}F(g_1)$ gives rise to the corresponding element in $A_G(y_0) \simeq \{x \in \overline{\mathbf{F}}^*_q \mid x^d = 1\}$.

Now in our situation, if we put $g_1 = \alpha^{-1}g$, we have $g_1 \in G$ and $\mathrm{Ad}(g_1)y_0 = y_1$. Then $g_1^{-1}F(g_1) = \alpha^{1-q}g^{-1}F(g)$. On the other hand, since $F(g) = w_0({}^tg^{-1})w_0^{-1}$, $F(g)$ also stabilizes the subspaces W_0, W_1, W_2. Moreover, it acts on W_2 trivially, and on W_1 as a permutaiton of the basis $\{e^{(i)}_j\}$ up to sign. It follows that $g^{-1}F(g)$ acts on the space $\mathrm{Ker}\, y_0$ as g_0^{-1}. Thus $g_1^{-1}F(g_1)$ acts on $\mathrm{Ker}\, y_0$ as a map $\alpha^{1-q}g_0^{-1}$, and we have $\det(\alpha^{1-q}g_0^{-1}) = \alpha^{t(1-q)}\det g_0$ as asserted. □

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