**Lusztig's conjecture for finite special linear groups**

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Abstract. In this paper, we prove Lusztig's conjecture for $G^F = SL_n(\mathbf{F}_q)$, i.e., we show that characteristic functions of character sheaves of $G^F$ coincide with almost characters of $G^F$ up to scalar constants, assuming that $\operatorname{ch}\mathbf{F}_q$ is not too small. We determine these scalars explicitly. Our result gives a method of computing irreducible characters of $G^F$.

## 0. Introduction

Let $G = SL_n$ defined over a finite field $\mathbf{F}_q$ with the standard Frobenius map $F$, and $G^F = SL_n(\mathbf{F}_q)$ the finite special linear group. In [S2], a parametrization of irreducible characters of $G^F$ was given by making use of modified generalized Gelfand-Graev characters. Also the almost characters of $G^F$ was defined, and it was shown that the Shintani descent of irreducible characters of $G^{F^m}$, for sufficiently divisible $m$, coincides with almost characters up to scalar. However as explained in the remark of the last part of [S2], the relationship of our parametrization of irreducible characters and the parametrization in terms of the Harish-Chandra induction was not so clear. Now Lusztig's conjecture is formulated in the form that almost characters coincide with the characteristic functions of character sheaves of $G$ under a suitable parametrization.

In this paper we prove that Lusztig's conjecture holds for $G^F$, assuming that $\operatorname{ch}\mathbf{F}_q$ is not too small so that Lusztig's results ([L7]) for generalized Gelfand-Graev characters are applicable. In the course of the proof, it is shown that almost characters of $G^F$ are parametrized in terms of the twisted induction, which is compatible with the parametrization of $F$-stable irreducible characters of $G^{F^m}$ in terms of the Harish-Chandra induction for sufficiently divisible $m$. Thus giving the relationship between two parametrizations of $F$-stable irreducible characters of $G^{F^m}$ is equivalent to giving the relationship between two parametrizations of almost characters of $G^F$.

We give a complete description of the relationship between those two parametrizations of almost characters, by computing the inner products of the characteristic functions of characters sheaves with various modified generalized Gelfand-Graev characters. Note that the inner product of characteristic functions with generalized Gelfand-Graev characters can be computed by using Lusztig's formula without difficulty. However the computation in the case of modified generalized Gelfand-Graev characters is much more complicated since it involves non-uni potent supports. Through this computation, we can describe the scalar constants appearing in Lusztig's conjecture. Although the expression of such scalars are not so simple, they are explicitly computable. (Here we need a result of Digne-Lehrer-Michel [DLM1] that the fourth root of unity occurring in the Lusztig's theory in [L7] is explicitly determined in the case of $SL_n$).

The computation of irreducible characters of $G^F$ is reduced to the computation of characteristic functions of character sheaves. In turn, the computation of those characteristic functions is reduced to the computation of generalized Green functions. Lusztig's algorithm of computing generalized Green functions contains certain unknown constants. In the case of $SL_n$, we can determine such scalars, which will be discussed in [S3]. Thus our result makes it possible to compute the character table of $SL_n(\mathbf{F}_q)$.

**Some notations.** For a finite group $\Gamma$, we denote by $\operatorname{Irr}\Gamma$ or $\Gamma^\wedge$ the set of irreducible characters of $\Gamma$ over $\bar{\mathbf{Q}}_l$. If $F : \Gamma \to \Gamma$ is an automorphism on $\Gamma$, we denote by $\Gamma/\sim_F$ the set of $F$-twisted conjugacy classes in $\Gamma$, where $x, y \in \Gamma$ is $F$-twisted conjugate if there exists $z \in \Gamma$ such that $y = z^{-1}xF(z)$. In the case where $\Gamma$ is abelian, $\Gamma/\sim_F$ is naturally identified with the largest quotient of $\Gamma$ on which $F$ acts trivially, which we denote by $\Gamma_F$.

For a reductive group $H$, we denote by $Z_H$ the center of $H$, and denote by $Z_H^0$ the identity component of $Z_H$.

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## 1. Parametrization of irreducible characters

**1.1.** Let $k$ be an algebraic closure of a finite field $\mathbf{F}_q$ with $\operatorname{ch} k = p$. In this and next section, we review the parametrization of irreducible characters of $SL_n(\mathbf{F}_q)$ (or more generally, its Levi subgroups) following [S2]. we assume that $p$ is large enough so that the Dynkin-Kostant theory on Lie algebras can be applied. For example, $p \geq 2n$ is enough for $G = SL_n$ (See the remark in 2.1).

Let $\widetilde{G} = GL_{n_1} \times \cdots \times GL_{n_r}$. We regard $\widetilde{G}$ as a subgroup of $GL_n(k)$ with $n = \sum n_i$ in a natural way, and put $G = \widetilde{G} \cap SL_n$. Thus $G$ is a Levi subgroup of a parabolic subgroup of $SL_n$. We consider a Frobenius map $F$ on $\widetilde{G}$ of the form $F = \phi F_0$, where $F_0$ is the split Frobenius map on $\widetilde{G}$ with respect to the $\mathbf{F}_q$-structure, and $\phi$ is a permutation of the factors in $\widetilde{G}$.

Put $\widetilde{G}_i = GL_{n_i}$. We choose an $F_0$-split maximal torus $\widetilde{T}_i$ in $\widetilde{G}_i$ so that $\widetilde{T} = \widetilde{T}_1 \times \cdots \times \widetilde{T}_r$ is an $F$-stable maximal torus in $\widetilde{G}$, which is maximally split with respect to $F$. Let $W = N_{\widetilde G}(\widetilde T)/\widetilde T$ be the Weyl group of $\widetilde G$. Then $W \simeq W_1 \times \cdots \times W_r$ where $W_i = N_{\widetilde G_i}(\widetilde T_i)/\widetilde T_i$ is the Weyl group of $\widetilde G_i$.

Let $\widetilde G^* \simeq \widetilde G_1^* \times \cdots \times \widetilde G_r^*$ be the dual group of $\widetilde G$ over $\mathbf{F}_q$, and $G^*$ the dual group of $G$. We denote also by $F$ the corresponding Frobenius actions on $\widetilde G^*$ and $G^*$. The natural inclusion map $G \hookrightarrow \widetilde G$ induces a map $\pi : \widetilde G^* \to G^*$, which is identified with the projection $\widetilde G^* \to \widetilde G^*/Z_1$, where $Z_1$ is the center of $GL_n$ under the identification of $\widetilde G^*$ with the subgroup of $GL_n^* = GL_n$. Then $Z_{\widetilde G^*}$ and $Z_{G^*}$ are connected, and $Z_{G^*} = Z_{\widetilde G^*}/Z_1$. We have natural inclusions and projections

$$\begin{aligned} &\widetilde G \hookleftarrow G \hookleftarrow G_{\mathrm{der}} = \widetilde G_{\mathrm{der}}, \\ &\widetilde G^* \to G^* \to G^*/Z_{G^*} = \widetilde G^*/Z_{\widetilde G^*}, \end{aligned}$$

where the dual group of $G_{\mathrm{der}}$ (resp. $\widetilde G_{\mathrm{der}}$) is identified with $G^*/Z_{G^*}$ (resp. $\widetilde G^*/Z_{\widetilde G^*}$).

We note that, for a connected reductive group $H$ defined over $\mathbf{F}_q$ with Frobenius map $F$, there exists an isomorphism of abelian groups

$$f : (Z^0_{H^*})^F \xrightarrow{\sim} (H^F/H^F_{\mathrm{der}})^\wedge, \tag{1.1.1}$$

where $H^F_{\mathrm{der}}$ means $(H_{\mathrm{der}})^F$. Returning to the original setting, let $S = \widetilde T \cap \widetilde G_{\mathrm{der}}$ be an $F$-stable maximal torus of $\widetilde G_{\mathrm{der}}$. We have $\widetilde G = Z_{\widetilde G}\widetilde G_{\mathrm{der}}$, $\widetilde T = Z_{\widetilde G}S$, and $Z_{\widetilde G} \cap \widetilde G_{\mathrm{der}} = Z_{\widetilde G} \cap S$ is finite. It follows that $\widetilde G^F/\widetilde G^F_{\mathrm{der}} \simeq \widetilde T^F/S^F$ and we have a natural inclusion map $Z^F_{\widetilde G}S^F/S^F \hookrightarrow \widetilde T^F/S^F$, which induces a surjective map $(\widetilde G^F/\widetilde G^F_{\mathrm{der}})^\wedge \to (Z^F_{\widetilde G}S^F/S^F)^\wedge$. Then we have the following lemma.

**Lemma 1.2.** *Assume that $n_1 = \cdots = n_r = d$, and put $(Z_{\widetilde G^*})_d = \{z \in Z_{\widetilde G^*} \mid z^d = 1\}$. Then there exists an isomorphism $f_0 : Z^F_{\widetilde G^*}/(Z_{\widetilde G^*})^F_d \to (\widetilde Z^F S^F/S^F)^\wedge$ which makes the following diagram commutative.*

$$\begin{array}{ccc} Z^F_{\widetilde G^*} & \xrightarrow[f]{} & (\widetilde G^F/\widetilde G^F_{\mathrm{der}})^\wedge \\ \downarrow & & \downarrow \\ Z^F_{\widetilde G^*}/(Z_{\widetilde G^*})^F_d & \xrightarrow[f_0]{} & (Z^F_{\widetilde G}S^F/S^F)^\wedge, \end{array} \tag{1.2.1}$$

*where the vertical maps are natural surjections.*

*Proof.* By the isomorphism $f$, the subgroup $(Z_{\widetilde G^*})^F_d$ of $Z^F_{\widetilde G^*}$ is mapped onto the subgroup $A = \{\theta \in (\widetilde T^F/S^F)^\wedge \mid \theta^d = 1\}$ of $(\widetilde T^F/S^F)^\wedge$. We want to show that

$$A = \{\theta \in (\widetilde T^F/S^F)^\wedge \mid \theta|_{Z^F_{\widetilde G}S^F/S^F} = 1\}. \tag{1.2.2}$$

Let $B$ be the right hand side of (1.2.2). Note that $\widetilde T = Z_{\widetilde G}S$, and $Z_{\widetilde G} \cap S$ consists of elements $z$ such that $z^d = 1$ since $\widetilde G_{\mathrm{der}} \simeq SL_d \times \cdots \times SL_d$. It follows that for $zv \in (Z_{\widetilde G}S)^F$ with $z \in Z_{\widetilde G}, v \in S$, we have $z^d \in Z^F_{\widetilde G}, v^d \in S^F$. Hence for $\theta \in B$, we have $\theta^d(zv) = \theta(z^d v^d) = 1$. This implies that $\theta \in A$, and we have $B \subseteq A$.

Here $|A| = |(Z_{\widetilde G^*})^F_d|$. On the other hand, $|B| = |\widetilde T^F|/|Z^F_{\widetilde G}S^F| = |Z^F_{\widetilde G}\cap S^F|$ since $|\widetilde T^F| = |Z^F_{\widetilde G}||S^F|$ (cf. [C, Prop. 3.3.7]). Under the identification $Z^F_{\widetilde G^*}\simeq Z^F_{\widetilde G}$, we have $(Z_{\widetilde G^*})^F_d\simeq Z^F_{\widetilde G}\cap S^F$. It follows that $A = B$, and (1.2.2) follows.

Now we have natural isomorphisms

$$Z^F_{\widetilde G^*}/(Z_{\widetilde G^*})^F_d\simeq (\widetilde T^F/S^F)^\wedge/A\simeq (\widetilde T^F/S^F)^\wedge/B\simeq (Z^F_{\widetilde G}S^F/S^F)^\wedge,$$

which makes the diagram (1.2.1) commutative. This proves the lemma. □

**1.3.** We fix a dual torus $\widetilde T^*$ of $\widetilde T$ over $\mathbf F_q$ in $\widetilde G^*$. Then the Weyl group $N_{\widetilde G^*}(\widetilde T^*)/\widetilde T^*$ may be identified with $W$. For any semisimple element $\dot s\in\widetilde T^*$ such that the conjugacy class $\{\dot s\}$ of $\dot s$ in $\widetilde G^*$ is $F$-stable, put

$$\begin{aligned} W_{\dot s} &= \{w\in W\mid w(\dot s)=\dot s\}\\ Z_{\dot s} &= \{w\in W\mid Fw(\dot s)=\dot s\}.\end{aligned}$$

We fix an $F$-stable Borel subgroup $\widetilde B$ of $\widetilde G$ containing $\widetilde T$. Let $\varSigma$ (resp. $\varSigma^+$) be a root system (resp. positive root system) with respect to the pair $(\widetilde B,\widetilde T)$. Then $Z_{\dot s}$ may be written as $Z_{\dot s}=w_1W_{\dot s}$ for some $w_1\in W$. The element $w_1$ is determined uniquely by the condition that $w_1$ maps $\varSigma^+_{\dot s}$ into $\varSigma^+$, where $\varSigma_{\dot s}$ is the subroot system of $\varSigma$ corresponding to $\dot s$.

Put $T=\widetilde T\cap G$ and $B=\widetilde B\cap G$. Then $T$ is the maximally split maximal torus of $G$, and $B$ is an $F$-stable Borel subgroup of $G$ containing $T$. We identify $W$ with the Weyl group $N_G(T)/T$ of $G$.

**1.4.** The element $w_1$ induces an automorphism $\gamma : W_{\dot s}\to W_{\dot s}$ by $\gamma(w) = F(w_1ww_1^{-1})$. Let $W_{\dot s}\langle\gamma\rangle$ be the semidirect product of $W_{\dot s}$ with the cyclic group $\langle\gamma\rangle$ generated by $\gamma$. We denote by $(W^\wedge_{\dot s})^\gamma$ the set of $\gamma$-stable irreducible characters of $W_{\dot s}$. Each $E\in(W^\wedge_{\dot s})^\gamma$ is extendable to an irreducible character of $W_{\dot s}\langle\gamma\rangle$. We fix the preferred extension $\widetilde E$ of $E$ (see [L3, 17.2]).

Let $\widetilde T_w$ be an $F$-stable maximal torus of $\widetilde G$ obtained from $\widetilde T$ by twisting by $w\in W$. Then to any $w\in Z_{\dot s}$, one can attach an irreducible character $\theta_w$ of $\widetilde T^{Fw}\simeq\widetilde T^F_w$ (see e.g., [S2, 2]). Let $R^{\widetilde G}_{\widetilde T_w}(\theta_w)$ be the Deligne-Lusztig character of $\widetilde G^F$ associated to $\theta_w\in(\widetilde T^F_w)^\wedge$. For each $\widetilde E\in(W^\wedge_{\dot s})^\gamma$, put

$$\widetilde\rho_{\dot s,E} = (-1)^{l(w_1)}|W_{\dot s}|^{-1}\sum_{w\in W_{\dot s}}\mathrm{Tr}\,(\gamma w,\widetilde E)R^{\widetilde G}_{\widetilde T_{w_1w}}(\theta_{w_1w}).$$

Then $\widetilde\rho_{\dot s,E}$ gives rise to an irreducible character of $\widetilde G^F$. The set $\mathrm{Irr}\,\widetilde G^F$ is decomposed as

$$\mathrm{Irr}\,\widetilde G^F = \coprod_{\{\dot s\}}\mathcal E(\widetilde G^F,\{\dot s\}),$$

where $\{\dot{s}\}$ runs over all the $F$-stable semisimple classes in $\widetilde{G}^*$. The Lusztig series $\mathcal{E}(\widetilde{G}^F, \{\dot{s}\})$ associated to the $F$-stable class $\{\dot{s}\}$ is given as

$$\mathcal{E}(\widetilde{G}^F, \{\dot{s}\}) = \{\widetilde{\rho}_{\dot{s},E} \mid E \in (W_{\dot{s}}^{\wedge})^{\gamma}\}.$$

**1.5.** We now describe the irreducible characters of $G^F$ following [S2]. Let $\pi : \widetilde{G}^* \to G^*$ be as in 1.1. Let $T^* = \pi(\widetilde{T}^*)$ be the maximal torus of $G^*$. Then $W = N_{\widetilde{G}^*}(\widetilde{T}^*)/\widetilde{T}^*$ is naturally identified with $N_{G^*}(T^*)/T^*$. As in the case of $\widetilde{G}^F$, the set $\operatorname{Irr} G^F$ is partitioned as

$$\operatorname{Irr} G^F = \coprod_{\{s\}} \mathcal{E}(G^F, \{s\}),$$

where $\{s\}$ runs over all the $F$-stable semisimple classes in $G^*$. We fix $s \in T^*$ for a given $F$-stable class $\{s\} \subset G^*$. There exists $\dot{s} \in \widetilde{T}^*$ such that $\pi(\dot{s}) = s$ and that the class $\{\dot{s}\}$ is $F$-stable. One can find $w_1 \in Z_{\dot{s}}$ and an isomorphism $\gamma = Fw_1 : W_{\dot{s}} \to W_{\dot{s}}$ as in 1.4.

Put $W_s = \{w \in W \mid w(s) = s\}$. Then $W_{\dot{s}}$ is naturally regarded as a subgroup of $W_s$, and we have $W_s \simeq W_{\dot{s}} \rtimes \Omega_s$, where $\Omega_s$ is a cyclic group isomorphic to $Z_{G^*}(s)/Z^0_{G^*}(s)$. $W_{\dot{s}}$ is characterized as the largest reflection subgroup of $W_s$, and sometimes we denote it by $W_s^0$. Let $F' = F\dot{w}_1$, where $\dot{w}_1$ is the representative of $w_1$ in $N_{\widetilde{G}^*}(\widetilde{T}^*)$. Then $\pi$ is $F'$-equivariant, and $s \in (T^*)^{F'}$. So $F'$ acts naturally on $W_s$, leaving $W_{\dot{s}}$ and $\Omega_s$ invariant. We consider the set $\Omega_s/\!\sim_{F'}$ of $F'$-twisted classes in $\Omega_s$. Since $\Omega_s$ is abelian, $\Omega_s/\!\sim_{F'}$ is identified with $(\Omega_s)_{F'}$, the largest quotient on which $F'$ acts trivially. For each $x \in (\Omega_s)_{F'}$, one can choose $\dot{s}_x \in (\widetilde{T}^*)^{xF'}$ such that $\pi(\dot{s}_x) = s$, and obtain an isomorphism $\gamma_x = xF' : W_{\dot{s}} \to W_{\dot{s}}$. To each $\gamma_x$-stable irreducible character $E$ of $W_{\dot{s}}$, one can attach the irreducible character $\widetilde{\rho}_{\dot{s}_x,E}$ of $\widetilde{G}^F$ as before. We denote by $\mathcal{T}_{\dot{s}_x,E}$ the set of irreducible characters of $G^F$ occurring in the restriction of $\widetilde{\rho}_{\dot{s}_x,E}$ on $G^F$. Then by [L5], we can decompose $\mathcal{E}(G^F, \{s\})$ as

$$\mathcal{E}(G^F, \{s\}) = \coprod_{(x,E)} \mathcal{T}_{\dot{s}_x,E}, \tag{1.5.1}$$

where the pair $(x, E)$ runs over all $x \in (\Omega_s)_{F'}$ and $E \in (W_{\dot{s}}^{\wedge})^{\gamma_x}/\Omega_s^{F'}$ (the set of $\Omega_s^{F'}$-orbits in $(W_{\dot{s}}^{\wedge})^{\gamma_x}$).

**1.6.** Following [S2], we shall modify the partition in (1.5.1). For $E \in W_{\dot{s}}^{\wedge}$, let $\Omega_{s,E}$ be the stabilizer of $E$ in $\Omega_s$. (In [S2], the notation $\Omega_s(E)$ is used instead of $\Omega_{s,E}$). If the $\Omega_s$-orbit of $E$ in $W_{\dot{s}}^{\wedge}$ is $F'$-stable, then $\Omega_{s,E}$ is $F'$-stable, and one can consider the largest quotient $(\Omega_{s,E})_{F'}$ as before. If we put $\widetilde{\Omega}_{s,E} = \{x \in \Omega_s \mid {}^{xF'}E = E\}$, then $\widetilde{\Omega}_{s,E} \neq \emptyset$, and one can write it as $\widetilde{\Omega}_{s,E} = \Omega_{s,E}a_E$ for some $a_E \in \Omega_s$. It follows that $\widetilde{\Omega}_{s,E}/\!\sim_{F'}$ can be identified with the set $(\Omega_{s,E}/\!\sim_{F'})a_E$ and with $(\Omega_{s,E})_{F'}a_E$. We denote this set by $(\widetilde{\Omega}_{s,E})_{F'}$. By (4.4.2) in [S2], we have the following natural bijection

$$\coprod_{E\in (W_{\dot s}^\wedge/\Omega_s)^{F'}} (\widetilde{\Omega}_{s,E})_{F'} \simeq \coprod_{x\in(\Omega_s)_{F'}} (W_{\dot s}^\wedge)^{\gamma_x}/\Omega_s^{F'}, \tag{1.6.1}$$

where $(W_{\dot s}^\wedge/\Omega_s)^{F'}$ denotes the set of $F'$-stable $\Omega_s$-orbits in $W_{\dot s}^\wedge$. (In [S2], $\Omega_{s,E}$ is used instead of $\widetilde{\Omega}_{s,E}$. This is justified since we have a bijection $\widetilde{\Omega}_{s,E} \simeq \Omega_{s,E}$. However this bijection depends on the choice of $a_E$, and so the form as in (1.6.1) is more convenient for our later purpose.)

Let $E \in (W_{\dot s}^\wedge/\Omega_s)^{F'}$, i.e., the $\Omega_s$-orbit of $E$ is $F'$-stable. Then for each $y \in (\widetilde{\Omega}_{s,E})_{F'}$, one can associate the pair $(x, E')$, where $x \in (\Omega_s)_{F'}$ and $E' \in (W_{\dot s}^\wedge)^{\gamma_x}/\Omega_s^{F'}$, by (1.6.1). We denote by $\mathcal{T}_{s,E}$ the union of various $\mathcal{T}_{\dot s_x,E'}$ where $(x, E')$ runs over all the pairs in the image of $(\Omega_{s,E})_{F'}$ under the bijection in (1.6.1). Thus we can rewrite (1.5.1) as

$$\mathcal{E}(G^F, \{s\}) = \coprod_{E\in (W_{\dot s}^\wedge/\Omega_s)^{F'}} \mathcal{T}_{s,E}. \tag{1.6.2}$$

For a pair $(s, E)$ with $E \in (W_{\dot s}^\wedge/\Omega_s)^{F'}$, put

$$\overline{\mathcal{M}}_{s,E} = (\Omega_{s,E}^{F'})^\wedge \times (\widetilde{\Omega}_{s,E})_{F'}, \tag{1.6.3}$$

where $\Omega_{s,E}^{F'}$ is the $F'$-fixed subgroup of $\Omega_{s,E}$.

It is known by [L5] that $\mathcal{T}_{\dot s_x,E}$ is in bijection with the set $(\Omega_{s,E}^{xF'})^\wedge = (\Omega_{s,E}^{F'})^\wedge$. Hence by (1.5.1), $\mathcal{E}(G^F,\{s\})$ is parametrized by various $(\Omega_{s,E}^{F'})^\wedge$. However, this parametrization of $\mathcal{T}_{\dot s_x,E}$ is not canonical. It depends on the choice of an irreducible character $\rho_0 \in \operatorname{Irr} G^F$ occurring in the decomposition of $\widetilde{\rho}_{\dot s_x,E}$. In [S2], a bijective correspondence $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$ was constructed by making use of generalized Gelfand-Graev characters. This bijection is determined uniquely once we fix a representative $u \in C^F$ for each $F$-stable unipotent class $C$ in $G$. Thus we have a parametrization of $\mathcal{E}(G^F,\{s\})$ as

$$\mathcal{E}(G^F, \{s\}) = \coprod_{E\in (W_{\dot s}^\wedge/\Omega_s)^{F'}} \overline{\mathcal{M}}_{s,E}. \tag{1.6.4}$$

In the next section, we shall explain this parametrization in details.

## 2. Generalized Gelfand-Graev characters

**2.1.** In order to explain the bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$, we shall review results on generalized Gelfand-Graev characters following [S2]. (Although no restriction on $p$ was assumed in [S2], this must be changed. In fact Kawanaka's construction of generalized Gelfand-Graev characters of $GL_n$ or $SL_n$ requires no assumption on $p$. However, our construction ([S2, 2.3]) relies on the Dynkin-Kostant theory, which requires that $p$ is not too small). We also prove a character formula for modified generalized Gelfand-Graev characters, which will play an essential role in later sections.

Let $\mathfrak{g}$ be the Lie algebra of $G$ with Frobenius map $F = \phi F_0$. Let $G_{\mathrm{uni}}$ (resp. $\mathfrak{g}_{\mathrm{nil}}$) be the set of unipotent elements in $G$ (resp. nilpotent elements in $\mathfrak{g}$). We have a bijection $\log : G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}, v \mapsto v - 1$. Let $\mathcal{O}$ be an $F$-stable nilpotent orbit in $\mathfrak{g}$, and choose a representative $N \in \mathfrak{g}^F$. Correspondingly, we consider an $F$-stable unipotent class $C$ containing $u = \log^{-1} N \in C^F$. By Dynkin-Kostant theory, there exists a natural grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ associated to $N$. Let $\mathfrak{u}_i = \oplus_{j\geq i}\mathfrak{g}_j$ for $j \geq 1$. Then $\mathfrak{u}_i$ is a nilpotent subalgebra of $\mathfrak{g}$, and there exists a connected unipotent subgroup $U_i$ of $G$ defined over $\mathbf{F}_q$ such that $\log(U_i) = \mathfrak{u}_i$. Also one can find a parabolic subgroup $P = P_N$ and its Levi subgroup $L = L_N$ such that $P = LU_1$, where $L$ is an $F$-stable Levi subgroup of $P$ with $\mathrm{Lie}(L) = \mathfrak{g}_0$ and $U_1$ is the unipotent radical of $P$. Moreover, we have $N \in \mathfrak{g}_2$.

Let $N^* \in \mathfrak{g}_{-2}^F$ be the element such that $\{N, N^*, H\}$ gives a TDS triple for some semisimple element $H \in \mathfrak{g}_0$. We define a linear map $\lambda : \mathfrak{u}_1 \to k$ by $\lambda(x) = \langle N^*, x\rangle$ , where$\langle\, ,\rangle$ is a fixed $G$-invariant non-degenerate bilinear form on $\mathfrak{g}$. It is known that the map $(x, y) \mapsto \lambda([x, y])$ defines a symplectic form on $\mathfrak{g}_1$, and according to [S2, 2.3] one can find an $F$-stable Lagrangian subspace $\mathfrak{s}$ of $\mathfrak{g}_1$ satisfying the following properties. Put $\mathfrak{u} = \mathfrak{s} + \mathfrak{u}_2$ ($\mathfrak{u} = \mathfrak{u}_{1.5}$ in the notation in [S2]). Then $\mathfrak{u}$ is a subalgebra of $\mathfrak{u}_1$, and we obtain an $F$-stable closed subgroup $U$ of $U_1$ such that $\log(U) = \mathfrak{u}$. $U$ is a normal subgroup of $U_1$. Moreover, $U$ is stable by the conjugation action of $L$. (Note that the last property does not hold for a general Lagrangian subspace (see [S2, 2.6]).) Now the map $\lambda \circ \log : U \to k$ turns out to be an $F$-stable homomorphism from $U$ to $k$. Thus we obtain a linear character $\varLambda_N$ on $U^F$ by $\varLambda_N = \psi \circ \lambda \circ \log$, where $\psi : \mathbf{F}_q \to \bar{\mathbf{Q}}_l^*$ is a fixed non-trivial additive character of $\mathbf{F}_q$. The generalized Gelfand-Graev character $\varGamma_N$ on $G^F$ associated to $N$ is defined as $\varGamma_N = \mathrm{Ind}_{U^F}^{G^F} \varLambda_N$.

Following [K3], we construct modified generalized Gelfand-Graev characters. Let $A_\lambda = Z_L(\lambda)/Z_L^0(\lambda)$ for $\lambda : \mathfrak{u} \to k$. Then by [S2, 2.7] we have

$$A_\lambda \simeq A_G(N) = Z_G(N)/Z_G^0(N). \tag{2.1.1}$$

Since the latter group is abelian, $A_\lambda$ is an abelian group. Moreover, we have a surjective map $Z_G \to A_\lambda$. Put

$$\overline{\mathcal{M}} = (A_\lambda)_F \times (A_\lambda^F)^\wedge. \tag{2.1.2}$$

For each $(c, \xi) \in \overline{\mathcal{M}}$, one can construct a character $\varGamma_{c,\xi}$ on $G^F$ as follows; for $c \in A_\lambda$, we choose a representative $\dot{c} \in Z_L(\lambda)$. Then we find $\alpha_c \in L$ such that $\alpha_c^{-1}F(\alpha_c) = \dot{c}$. Let us define a linear map $\lambda_c : \mathfrak{u} \to k$ by $\lambda_c = \lambda \circ \mathrm{Ad}\, \alpha_c^{-1}$, where Ad is the adjoint action of $L$ on $\mathfrak{u}$. We define a linear character $\varLambda_c$ on $U^F$ by $\varLambda_c = \psi \circ \lambda_c \circ \log$. If we notice that $Z_L(\lambda_c)^F$ coincides with $Z_{L^F}(\varLambda_c)$, the linear character $\varLambda_c$ can be extended to the character of $Z_L(\lambda_c)^F U^F$ so that it is trivial on $Z_L(\lambda_c)^F$, which we denote by the same symbol $\varLambda_c$. On the other hand, by the isomorphism

$$Z_L(\lambda_c)^F/Z_L^0(\lambda_c)^F \simeq Z_L(\lambda)^{\dot{c}F}/Z_L^0(\lambda)^{\dot{c}F} \simeq A_\lambda^{\dot{c}F} = A_\lambda^F,$$

the linear character $\xi \in (A_\lambda^F)^\wedge$ determines a linear character $\xi^\natural$ on $Z_L(\lambda_c)^F$ which is trivial on $Z_L^0(\lambda_c)^F$.

Let $\widetilde{P}$ and $\widetilde{L}$ be the parabolic subgroup of $\widetilde{G}$ and its Levi subgroup associated to $N \in \widetilde{\mathfrak{g}} = \operatorname{Lie} \widetilde{G}$. Then we have $P = \widetilde{P} \cap G$ and $L = \widetilde{L} \cap G$, and so $Z_L(\lambda) \subset Z_{\widetilde{L}}(\lambda)$. Let us take a linear character $\theta$ of $Z_L(\lambda)^F$ of the following type.

(2.1.3) $\theta$ is the restriction to $Z_L(\lambda)^F$ of a linear character of $Z_{\widetilde{L}}(\lambda)^F$ which is trivial on $(Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^F$.

Since $\dot{c} \in Z_L(\lambda)$, we have $(Z_{\widetilde{L}}(\lambda)/Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^F \simeq (Z_{\widetilde{L}}(\lambda)/Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^{\dot{c}F}$. It follows that $\theta$ is regarded as a linear character of $Z_L(\lambda)^{\dot{c}F}$, and it determines a linear character of $Z_L(\lambda_c)^F$ via the isomorphism $\operatorname{ad} \alpha^{-1} : Z_L(\lambda_c)^F \simeq Z_L(\lambda)^{\dot{c}F}$, which we denote also by $\theta$. Then $\theta\xi^{\natural}$ gives rise to a character on $Z_L(\lambda_c)^F U^F$ under the surjective homomorphism $Z_L(\lambda_c)^F U^F \to Z_L(\lambda_c)^F$, which we denote also by $\theta\xi^{\natural}$. Under this setting, we define a modified generalized Gelfand-Graev character $\varGamma_{c,\xi,\theta}$ by

$$\varGamma_{c,\xi,\theta} = \operatorname{Ind}_{Z_L(\lambda_c)^F U^F}^{G^F} \big(\theta\xi^{\natural} \otimes \varLambda_c\big). \tag{2.1.4}$$

In the case where $\theta = 1$, we simply write $\varGamma_{c,\xi,\theta}$ as $\varGamma_{c,\xi}$.

For later use, we also define a generalized Gelfand-Graev character $\varGamma_c$ associated to $c \in A_\lambda$ by $\varGamma_c = \operatorname{Ind}_{U^F}^{G^F} \varLambda_c$. Under the isomorphism $A_\lambda \simeq A_G(N)$ (2.1.1), one can construct a nilpotent element $N_c \in \mathfrak{g}_2^F$ twisted by $c$. $\varGamma_c$ is nothing but the generalized Gelfand-Graev character $\varGamma_{N_c}$ associated to $N_c$. We remark that $\varGamma_{c,\xi}$ occurs as a direct summand of $\varGamma_c$.

**2.2.** We choose $m$ large enough so that $F^m$ acts trivially on $A_\lambda$. Replacing $F$ by $F^m$, we have a modified generalized Gelfand-Graev character $\varGamma^{(m)}_{c,\xi,\theta}$ on $G^{F^m}$. Now the parameter set $\overline{\mathcal{M}}$ is replaced by $A_\lambda \times A_\lambda^\wedge$. We denote by $\mathcal{M}$ the subset of $A_\lambda \times A_\lambda^\wedge$ defined by

$$\mathcal{M} = A_\lambda^F \times (A_\lambda^\wedge)^F,$$

where $(A_\lambda^\wedge)^F$ is the set of $F$-stable irreducible characters of $A_\lambda$. We now construct, for a certain linear character $\theta$ of $Z_L(\lambda_c)^{F^m}$, and for each $(c,\xi) \in \mathcal{M}$, an $F$-stable modified generalized Gelfand-Graev character $\varGamma^{(m)}_{c,\xi,\theta}$, and its extension to $G^{F^m}\langle\sigma\rangle$, where $\sigma = F|_{G^{F^m}}$. The case where $\theta = 1$ is discussed in [S2, 1.8]. For $c \in A_\lambda^F$, we choose $\dot{c} \in L^F$. We construct the linear character $\varLambda_c^{(m)}$ of $U^{F^m}$ as in 2.1, i.e., we choose $\beta_c \in L$ such that $\beta_c^{-1} F^m(\beta_c) = \dot{c}$, and define $\lambda_c$ by $\lambda_c = \lambda \circ \operatorname{Ad} \beta_c^{-1}$, and put $\varLambda_c^{(m)} = \psi_m \circ \lambda_c \circ \log$, where $\psi_m = \psi \circ \operatorname{Tr}_{\mathbf{F}_{q^m}/\mathbf{F}_q}$. Put $\hat{c} = \beta_c F(\beta_c^{-1}) \in L^{F^m}$. Then $\varLambda_c^{(m)}$ turns out to be $\hat{c}F$-stable. (Note that it is possible to choose $\dot{c} \in T^F$. Then we can choose $\beta_c \in T$.)

On the other hand, it can be checked that $\hat{c}F$ acts on $Z_L(\lambda_c)$ commuting with $F^m$, and that under the isomorphism

$$\operatorname{ad} \beta_c^{-1} : Z_L(\lambda_c)^{F^m}/Z_L^0(\lambda_c)^{F^m} \simeq Z_L(\lambda)^{\dot{c}F^m}/Z_L^0(\lambda)^{\dot{c}F^m} \simeq A_\lambda,$$

the action of $\hat{c}F$ on $Z_L(\lambda_c)^{F^m}$ is transferred to the action of $F$ on $A_\lambda$. Hence if we take $\xi \in (A_\lambda^\wedge)^F$, it produces a $\hat{c}F$-stable linear character $\xi^{\natural}$ on $Z_L(\lambda_c)^{F^m}$.

Furthermore, we take a linear character $\theta$ of $Z_L(\lambda)^{F^m}$ of the following type.

(2.2.1) $\theta$ is the restriction to $Z_L(\lambda)^{F^m}$ of an $F$-stable linear character of $Z_{\widetilde{L}}(\lambda)^{F^m}$ as in (2.1.3) by replacing $F$ by $F^m$.

Then $\theta$ is regarded as an $F$-stable linear character of $Z_L(\lambda)^{\dot{c}F^m}$. It follows, under the isomorphsim $\operatorname{ad}\beta_c^{-1} : Z_L(\lambda_c)^{F^m} \simeq Z_L(\lambda)^{\dot{c}F^m}$, that $\theta$ determines a $\hat{c}F$-stable linear character of $Z_L(\lambda_c)^{F^m}$, which we denote also by $\theta$. Thus $\theta\xi^\natural \otimes \varLambda_c^{(m)}$ is $\hat{c}F$-stable for $(c,\xi) \in \mathcal{M}$, and we conclude that $\varGamma_{c,\xi,\theta}^{(m)}$ is $F$-stable.

Put $\hat{c}_0 = (\hat{c}\sigma)^m \in L^{F^m}$. Then we have

$$\hat{c}_0 = \beta_c F^m(\beta_c^{-1}) = \dot{c}^{-1}$$

since $\beta_c$ and $F^m(\beta_c)$ commutes. We note that $\hat{c}_0 \in Z_L(\lambda_c)^{F^m} = Z_L(\varLambda_c^{(m)})^{F^m}$. In fact, since $\varLambda_c^{(m)}$ is $\hat{c}F$-stable, it is stable by $(\hat{c}\sigma)^m = \hat{c}_0$. Put $M_c = Z_L(\lambda_c)^{F^m}$ and $M_c^0 = Z_L^0(\lambda_c)^{F^m}$. We consider a subgroup $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ of $G^{F^m}\langle\sigma\rangle$ generated by $M_cU^{F^m}$ and $\hat{c}\sigma$. Since $\theta\xi^\natural \in M_c^\wedge$ is $\hat{c}\sigma$-stable, and $(\hat{c}\sigma)^m = \hat{c}_0 \in M_c$, $\theta\xi^\natural$ may be extended to a linear character $\widetilde{\theta\xi}^\natural$ of $M_c\langle\hat{c}\sigma\rangle$ in $m$ distinct way. The extension $\widetilde{\theta\xi}^\natural$ is determined by the value $\widetilde{\theta\xi}^\natural(\hat{c}\sigma) = \mu_{c,\theta\xi}$, where $\mu_{c,\theta\xi}$ is any $m$-th root of $\theta\xi^\natural(\dot{c}^{-1}) = \theta(\dot{c}^{-1})\xi(c^{-1})$.

We fix an extension $\widetilde{\theta\xi}^\natural$ of $\theta\xi^\natural$ to $M_c\langle\hat{c}\sigma\rangle$. Since $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ is the semidirect product of $M_c\langle\hat{c}\sigma\rangle$ with $U^{F^m}$, $\widetilde{\theta\xi}^\natural$ may be regarded as a character of $M_cU^{F^m}\langle\hat{c}\sigma\rangle$. On the other hand, since $\varLambda_c^{(m)}$ is $\hat{c}\sigma$-stable, it can be extended to a linear character $\widetilde{\varLambda}_c^{(m)}$ on $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ by $\widetilde{\varLambda}_c^{(m)}(\hat{c}\sigma) = 1$. Thus we have a character $\widetilde{\theta\xi}^\natural \otimes \widetilde{\varLambda}_c^{(m)}$ of $M_cU^{F^m}\langle\hat{c}\sigma\rangle$ which is an extension of $\theta\xi^\natural \otimes \varLambda_c^{(m)}$ on $M_cU^{F^m}$. We put

$$\widetilde{\varGamma}_{c,\xi,\theta}^{(m)} = \operatorname{Ind}_{M_cU^{F^m}\langle\hat{c}\sigma\rangle}^{G^{F^m}\langle\sigma\rangle}(\widetilde{\theta\xi}^\natural \otimes \widetilde{\varLambda}_c^{(m)}).$$

Then $\widetilde{\varGamma}_{c,\xi,\theta}^{(m)}$ gives rise to an extension of $\varGamma_{c,\xi,\theta}^{(m)}$ to $G^{F^m}\langle\sigma\rangle$. Note that $\mu_{c,\theta\xi}^{-1}\widetilde{\varGamma}_{c,\xi,\theta}^{(m)}|_{G^{F^m}\sigma}$ depends only on the choice of $(c,\xi)$ and $\theta$.

**2.3.** In [L1], Lusztig defined, for a connected reductive group $H$ with connected center, a map from the set of irreducible characters of $H^F$ to the set of $F$-stable unipotent classes in $H$. It is shown in [L7] that this map coincides with the map defined by Kawanaka [K1, K2, K3] in terms of generalized Gelfand-Graev characters. We denote by $C_\rho$ (resp. $\mathcal{O}_\rho$ ) the unipotent class in $H$ (resp. the nilpotent orbit in $\operatorname{Lie} H$) corresponding to $\rho \in \operatorname{Irr} H$ under this map. We call $C_\rho$ the unipotent class associated to $\rho$ (the wave front set associated to $\rho$ in the sense of Kawanaka).

In the case of $\widetilde{G}$, this map is given as follows. Let $\widetilde{\rho} = \widetilde{\rho}_{\dot{s},E} \in \mathcal{E}(\widetilde{G}^F, \{\dot{s}\})$. Put $E' = E \otimes \varepsilon$ for the sign character $\varepsilon$ of $W_{\dot{s}}$. Then $\operatorname{Ind}_{W_{\dot{s}}}^W E'$ contains a unique irreducible character $\widehat{E}$ of $W$ such that $b_{E'} = b_{\widehat{E}}$. $C_{\widetilde{\rho}}$ is defined as the unipotent class in $\widetilde{G}$ corresponding to $\widehat{E}$ under the Springer correspondence. More precisely, we have the following. Assume that $\widetilde{G} = GL_n$. Then $W_{\dot{s}}$ is a product of various symmetric groups. Accordingly, $E \in W_{\dot{s}}^\wedge$ is parametrized by a multipartition $\beta = (\beta_1, \dots, \beta_k)$ of $n$. By mixing and rearranging the parts in $\beta_1, \dots, \beta_k$, we regard $\beta$ as a partition of $n$ which we denote by $\bar{\beta}$. Let $\bar{\beta}^*$ be the dual partition of $\bar{\beta}$. Then $C_{\widetilde{\rho}}$ is the unipotent class in $\widetilde{G}$ corresponding to $\bar{\beta}^*$ through Jordan's normal form. For general $\widetilde{G}$, the description of $C_{\widetilde{\rho}}$ is reduced to the case of $GL_n$ through the decomposition $\widetilde{G} = GL_{n_1} \times \cdots \times GL_{n_r}$.

**2.4.** Let $\widetilde{\mathfrak{g}} = \operatorname{Lie}\widetilde{G}$. For a nilpotent element $N \in \widetilde{\mathfrak{g}}^F$, we denote by $\mathcal{O}_N$ the nilpotent orbit in $\widetilde{\mathfrak{g}}$ containing $N$. Let $P = LU_1$ be as in 2.1, and let $\widetilde{L} \subset \widetilde{P}$ be as before. For each irreducible character $\theta'$ of $Z_{\widetilde{L}}(\lambda)^F$, the modified generalized Gelfand-Grave character $\widetilde{\varGamma}_{N,\theta'}$ is defined as $\widetilde{\varGamma}_{N,\theta'} = \operatorname{Ind}_{Z_{\widetilde{L}}(\lambda)^F U^F}^{\widetilde{G}^F}(\theta' \otimes \varLambda_N)$.

Let $\widetilde{\rho} = \widetilde{\rho}_{\dot{s},E}$ be an irreducible character of $\widetilde{G}^F$, such that $\mathcal{O}_{\widetilde{\rho}} = \mathcal{O}_N$. Then it is known by [S2, Prop. 2.14] that there exists a unique linear character $\varphi$ of $Z_{\widetilde{L}}(\lambda)^F$ such that

$$\langle \widetilde{\varGamma}_{N,\theta'}, \widetilde{\rho}\rangle_{\widetilde{G}^F} = \begin{cases} 1 & \text{if } \theta' = \varphi, \\ 0 & \text{if } \theta' \neq \varphi. \end{cases}$$

We denote by $\varDelta(\widetilde{\rho})$ the linear character $\varphi$ determined as above. We note that

(2.4.1) $\varDelta(\widetilde{\rho})$ is a linear character of $Z_{\widetilde{L}}(\lambda)^F$ which is trivial on $(Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^F$.

In fact, by [S2, 2.13], $\varphi = \varDelta(\widetilde{\rho})$ is determined in the following way. There exists an $F$-stable Levi subgroup $\widetilde{M}$ of a parabolic subgroup of $\widetilde{G}$ such that $Z_{\widetilde{L}}(\lambda) \subset \widetilde{M}$ and that $\dot{s} \in Z_{\widetilde{M}^*}$, where $\widetilde{M}^* \subset \widetilde{G}^*$ is the dual group of $\widetilde{M}$. We choose an integer $m > 0$ such that $\dot{s} \in Z_{\widetilde{M}^*}^{F^m}$, and let $\widehat{\varphi}$ be a linear character of $Z_{\widetilde{L}}(\lambda)^{F^m}$ obtained by restricting the linear character of $\widetilde{M}^{F^m}$ corresponding to $\dot{s}$. Then $\widehat{\varphi}$ is $F$-stable, and the Shintani descent $Sh_{F^m/F}(\widehat{\varphi})$ coincides with $\varDelta(\widetilde{\rho})$. Since the linear character of $\widetilde{M}^{F^m}$ corresponding to $\dot{s}$ has a trivial restriction on $\widetilde{M}_{\mathrm{der}}^{F^m}$, we see that $\widehat{\varphi}$ has a trivial restriction on $(Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^{F^m}$, and so $\varphi$ is trivial on $(Z_{\widetilde{L}}(\lambda)_{\mathrm{der}})^F$. This shows (2.4.1).

In view of (2.4.1), the restriction $\theta$ of $\varDelta(\widetilde{\rho})$ to $Z_L(\lambda)^F$ satisfies the property in (2.1.3). Hnece we can consider $\varGamma_{c,\xi,\theta}$ as in 2.1, which tursn out to be a direct summand of $\widetilde{\varGamma}_{N,\theta'}|_{G^F}$.

**2.5.** Let $(s, E)$ be as in 1.6. Then $\dot{s} \in \widetilde{G}^* = GL_{n_1} \times \cdots \times GL_{n_r}$ is written as $\dot{s} = (\dot{s}_1, \dots, \dot{s}_r)$ with $\dot{s}_i \in GL_{n_i}$, and we have $W_{\dot{s}} = W_{1,\dot{s}_1} \times \cdots \times W_{r,\dot{s}_r}$. We now consider the following special setting for the pair $(s, E)$.

(2.5.1) Let $t$ be a common divisor of $n_1, \dots, n_r$ which is prime to $p$. We have $\Omega_s \simeq \langle w_0\rangle$, where $w_0 \in W_s$ is an element of order $t$ permuting the factors of $W_{i,\dot{s}_i}$ transitively, and $W_{i,\dot{s}_i}$ is isomorphic to $\mathfrak{S}_{b_i} \times \cdots \times \mathfrak{S}_{b_i}$ ($t$ times) with $b_i = n_i/t$. Moreover, $E \in (W_{\dot{s}}^\wedge)^{F'}$ is of the form

$$E = E_1 \boxtimes \cdots \boxtimes E_r \quad \text{where} \quad E_i \simeq E_i' \boxtimes \cdots \boxtimes E_i' \in W_{i,\dot{s}_i}^\wedge \text{ with } E_i' \in \mathfrak{S}_{b_i}^\wedge.$$

Assume that the pair $(s, E)$ satisfies the condition (2.5.1). Then $E$ is $\Omega_s$-stable, and in particular, $\gamma_x$-stable for $x \in \Omega_s$. Since $E$ is $\Omega_s$-stable, we have $E \in (W_{\dot{s}}^\wedge/\Omega_s)^{F'}$ and $(\widetilde{\Omega}_{s,E})_{F'} = (\Omega_s)_{F'}$ with $a_E = 1$. Hence the bijection in (1.6.1) leaves the pair $(x, E)$ invariant for $x \in (\Omega_s)_{F'}$. It follows that the set $\mathcal{T}_{s,E}$ coincides with the disjoint union of $\mathcal{T}_{\dot{s}_x,E}$ for $x \in (\Omega_s)_{F'}$.

Let $\widetilde{\rho} = \widetilde{\rho}_{\dot{s},E}$. It is known that $\widetilde{\rho}|_{G^F}$ is multiplicity free. Let $\mathcal{T}_{\widetilde{\rho}} = \mathcal{T}_{\dot{s},E}$ be as in 1.5. Then $\mathcal{T}_{\widetilde{\rho}}$ consists of $t'$ elements, where $t'$ is the order of $\Omega_{s,E}^{F'} = \Omega_s^{F'}$. It follows from [S2] that $(A_\lambda)_F$ acts transitively on the set $\mathcal{T}_{\widetilde{\rho}}$. Thus there exists a quotient $(A_\lambda)'_F$ of $(A_\lambda)_F$ such that $(A_\lambda)'_F$ is in bijection with $\mathcal{T}_{\widetilde{\rho}}$. $(A_\lambda)'_F$ can be written also as $(\bar{A}_\lambda)_F$ with some quotient $\bar{A}_\lambda$ of $A_\lambda$, where $\bar{A}_\lambda$ is a cyclic group of order $t$ (see [S2, 2.19]). It can be checked from the proof in [S2, 2.21] that the map $A_\lambda^F \to \bar{A}_\lambda^F$ is surjective. Let us define a set $\overline{\mathcal{M}}_{s,N}$ by

$$\overline{\mathcal{M}}_{s,N} = (\bar{A}_\lambda)_F \times (\bar{A}_\lambda^F)^\wedge. \tag{2.5.2}$$

The set $(\bar{A}_\lambda^F)^\wedge$ is regarded as a subset of $(A_\lambda^F)^\wedge$ through the map $A_\lambda^F \to \bar{A}_\lambda^F$. Also we have a surjective map $(A_\lambda)_F \to (\bar{A}_\lambda)_F$. We define a subset $\overline{\mathcal{M}}_0$ of $\overline{\mathcal{M}}$ by $\overline{\mathcal{M}}_0 = (A_\lambda)_F \times (\bar{A}_\lambda^F)^\wedge$. Thus we have a natural surjective map $f : \overline{\mathcal{M}}_0 \to \overline{\mathcal{M}}_{s,N}$. The following result, which gives a parametrization of $\mathcal{T}_{s,E}$ in terms of generalized Gelfand-Graev characters, is a generalization of the results 2.16 and 2.21 in [S2]. The proof is done in a similar way as in [S2].

**Theorem 2.6.** *Assume that the pair $(s, E)$ satisfies (2.5.1). Let $\widetilde{\rho} = \widetilde{\rho}_{\dot{s},E} \in \operatorname{Irr} \widetilde{G}^F$. Let $\mathcal{O}_N$ be the nilpotent orbit in $\widetilde{\mathfrak{g}}$ containing $N$. Let $\theta$ be a linear character of $Z_L(\lambda)^F$ as in (2.1.3), and $\theta_0$ the restriction of $\theta$ to $Z_L^0(\lambda)^F$. Then for each pair $(c, \xi) \in \overline{\mathcal{M}}$, the following holds.*

(i) *$\langle \varGamma_{c,\xi,\theta}, \widetilde{\rho}|_{G^F} \rangle_{G^F} = 0$ unless $\mathcal{O}_N \subseteq \overline{\mathcal{O}}_{\widetilde{\rho}}$.*

(ii) *Assume that $\mathcal{O}_{\widetilde{\rho}} = \mathcal{O}_N$, and let $\Delta(\widetilde{\rho})$ be as in 2.4.*

(a) *If $\Delta(\widetilde{\rho})|_{Z_L^0(\lambda)^F} \neq \theta_0$, then $\langle \varGamma_{c,\xi,\theta}, \widetilde{\rho}|_{G^F} \rangle_{G^F} = 0$.*

(b) *If $\Delta(\widetilde{\rho})|_{Z_L^0(\lambda)^F} = \theta_0$, then there exists a bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,N}$ satisfying the following; Let $\rho_{c,\xi} \in \mathcal{T}_{s,E}$ be the character corresponding to $(c, \xi) \in \overline{\mathcal{M}}_{s,E}$. For each pair $(c', \xi') \in \overline{\mathcal{M}}_0$ we have*

$$\langle \varGamma_{c',\xi',\theta}, \rho_{c,\xi} \rangle_{G^F} = \begin{cases} 1 & \text{if } f((c', \xi')) = (c, \xi), \\ 0 & \text{if } f((c', \xi')) \neq (c, \xi). \end{cases}$$

*We have $\langle \varGamma_{c',\xi',\theta}, \rho_1 \rangle_{G^F} = 0$ for any $\rho_1 \in \mathcal{T}_{s,E}$, if the pair $(c', \xi') \in \overline{\mathcal{M}}$ is not contained in $\overline{\mathcal{M}}_0$.*

*Furthermore, $\Delta(\widetilde{\rho})|_{Z_L(\lambda)^F}$ is expressed as $\theta\xi_1^\natural$ for a character $\xi_1$ of $A_\lambda^F$. Then $\xi_1$ is contained in $(\bar{A}_\lambda^F)^\wedge$, and we have*

$$\widetilde{\rho}|_{G^F} = \sum_{c \in (\bar{A}_\lambda)_F} \rho_{c,\xi_1}.$$

**2.7.** The above parametrization of $\mathcal{T}_{s,E}$ is also interpreted in terms of (not modified) generalized Gelfand-Graev characters of $G^F$ as follows.

(2.7.1) For each $x \in (\Omega_s)_{F'}$ and $c \in (A_\lambda)_F$, we have $\langle \varGamma_c, \widetilde{\rho}_{\dot{s}_x,E}|_{G^F} \rangle_{G^F} = 1$, i.e., there exists a unique irreducible character of $G^F$ which occurs both in the decomposition of $\widetilde{\rho}_{\dot{s}_x,E}|_{G^F}$ and of $\varGamma_c$. Under the parametrization in Theorem 2.6, this character is given by $\rho_{c,\xi_x}$ for some $\xi_x \in (\bar{A}_\lambda^F)^\wedge$. In particular, we have

$$\widetilde{\rho}_{\dot{s}_x,E}|_{G^F} = \sum_{c \in (\bar{A}_\lambda)_F} \rho_{c,\xi_x}.$$

By using (2.7.1) we can identify $\overline{\mathcal{M}}_{s,N}$ with $\overline{\mathcal{M}}_{s,E}$ in (1.6.3). Note that in this case, $\overline{\mathcal{M}}_{s,E}$ is nothing but the set $(\Omega_s^{F'})^\wedge \times (\Omega_s)_{F'}$. Also note that the map $x \mapsto \xi_x$ gives a bijection $h : (\Omega_s)_{F'} \to (\bar{A}_\lambda^F)^\wedge$, where $\xi_x$ is given by $\varDelta(\widetilde{\rho}_{\dot{s}_x,E})|_{Z_L(\lambda)^F} = \theta\xi_x$. By the discussion in 2.4, we can choose $\theta$ such that $\varDelta(\widetilde{\rho}_{\dot{s},E})|_{Z_L(\lambda)^F} = \theta$. Then $\xi_1$ (the case where $x = 1$) is the trivial character of $\bar{A}_\lambda^F$. Let us write $\dot{s}_x = \dot{s}z_x$ with $z_x \in Z_{\widetilde{G}^*}$. Since $\dot{s}_x$ is $xF'$-stable, we have

$$\dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} = z_x F(z_x)^{-1}, \tag{2.7.2}$$

where $\dot{x} \in N_{G^*}(T^*)$ is a representative of $x \in (\Omega_s)_{F'}$. We may assume that $z_x \in Z_{\widetilde{G}^*}^{F^m}$ for a large $m$. Let $\widehat{\psi}'_x$ be the linear character of $\widetilde{G}^{*F^m}$ corresponding to $z_x$. Since $\dot{s}_x, \dot{s} \in \widetilde{T}^*$, and $\dot{s}_x$ is $xF'$-stable, $\dot{s}$ is $F'$-stable, we see that $\widehat{\psi}'_x$ is also $F'$-stable. As explained in [S2, 2.13], there exists an $F$-stable Levi subgroup $\widetilde{M}$ of $\widetilde{G}$ containing $\widetilde{T}$ such that $\widetilde{M}$ contains $Z_{\widetilde{L}}(\lambda)$ and that $\dot{s}$ is contained in the center of the dual group of $\widetilde{M}$. This implies that the restriction $\widehat{\psi}_x$ of $\widehat{\psi}'_x$ on $Z_{\widetilde{L}}(\lambda)^{F^m}$ is $F$-stable (cf. [S2, Prop. 2.14]). We define a linear character $\psi_x$ of $Z_{\widetilde{L}}(\lambda)^F$ by $\psi_x = Sh_{F^m/F}(\widehat{\psi}_x)$. Since $\xi_1 = 1$, we see that $\xi_x$ is obtained from the restriction of $\psi_x$ to $Z_L(\lambda)^F$.

Next, we shall describe the bijection between $(\bar{A}_\lambda)_F$ and $(\Omega_s^{F'})^\wedge$. There exists a surjective homomorphism $f_1 : \widetilde{G}^F/G^F \to (\bar{A}_\lambda)_F$ defined as follows (cf. [S2, 2.19]). For $g \in \widetilde{G}^F$, we can write $g = g_1 z$, with $g_1 \in G, z \in Z_{\widetilde{G}}$. Then $g_1^{-1}F(g_1) \in Z_G$, and it determines an element in $A_\lambda = Z_L(\lambda)/Z_L^0(\lambda)$, and so an element in $\bar{A}_\lambda$, which is unique up to $F$-conjugacy. On the other hand, we construct $f_2 : \widetilde{G}^F/G^F \to (\Omega_s^{F'})^\wedge$ as follows. From (2.7.2), we have $\dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} \in Z_{\widetilde{G}^*}^F$ (we may choose $\dot{x} \in N_{G^*}(T^*)^F$), and this defines a well-defined injective homomorphism $f_2^* : \Omega_s^{F'} \to Z_{\widetilde{G}^*}^F, x \mapsto \dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1}$. Since $Z_{\widetilde{G}^*}^F \simeq (\widetilde{G}^F/G^F)^\wedge$, we have a surjective map $f_2$ as the transpose of $f_2^*$. Then $\operatorname{Ker} f_1 = \operatorname{Ker} f_2$, and these maps induce the bijection $f : (\Omega_s^{F'})^\wedge \to (\bar{A}_\lambda)_F$.

Now the parametrization is given as follows. There exists a unique $\rho_0 \in \operatorname{Irr} G^F$ such that $\rho_0$ occurs in $\widetilde{\rho}_{\dot{s}_x,E}|_{G^F}$ and in $\varGamma_N$. In our parametrization, then $\rho_0 = \rho_{1,\xi_x} = \rho_{1,x}$ ($(1,\xi_x) \in \overline{\mathcal{M}}_{s,N}$, $(1,x) \in \overline{\mathcal{M}}_{s,E}$). Then any $\rho$ contained in $\widetilde{\rho}_{\dot{s}_x,E}|_{G^F}$ is obtained as ${}^g\rho_0$ with $g \in \widetilde{G}^F/G^F$. We then have $\rho = \rho_{c,\xi_x} = \rho_{\eta,x}$ with $c = f_1(g)$ and $\eta = f_2(g)$.

By summing up the above argument, we obtain a bijection

$$(\Omega_s^{F'})^\wedge \times (\Omega_s)_{F'} \to (\bar{A}_\lambda)_F \times (\bar{A}_\lambda^F)^\wedge \quad (\eta, x) \mapsto (f(\eta), \xi_x).$$

This gives the required bijection $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}$.

**2.8.** Slightly modifying the arguments in [S2, 4.5], (see the remark below), we establish a parametrization of $\mathcal{E}(G^F, \{s\})$ as in (1.6.4). We give a bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$ for each pair $(s, E)$ such that $E \in (W_{\dot{s}}^\wedge/\Omega_s)^{F'}$.

(a) First we consider the case where the pair $(s, E)$ satisfies the property (2.5.1). If we put $\theta = \varDelta(\widetilde{\rho}_{\dot{s},E})|_{Z_L(\lambda)^F}$, then $\theta$ satisfies the property (2.1.3). Hence we have a natural bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,N} \leftrightarrow \overline{\mathcal{M}}_{s,E}$ by Theorem 2.6 (ii), (b) together with the argument in 2.7.

(b) Next we consider the case where $W_s$ satisfies the same assumption as in (2.5.1), but $E$ is not of the form there. So we assume that $\Omega_s(E) \neq \Omega_s$, and put $t' = |\Omega_s(E)|$. Replacing $(s, E)$ by a certain $N_W(W_{\dot{s}})$-conjugate, we may assume that $E$ can be written as $E \simeq E_1 \boxtimes \cdots \boxtimes E_r$, $(E_i \in W_{i,\dot{s}}^{\wedge})$ with

$$E_i = (E_{i1} \boxtimes \cdots \boxtimes E_{i1}) \boxtimes \cdots \boxtimes (E_{ik} \boxtimes \cdots \boxtimes E_{ik}),$$

where $E_{i1}, \dots, E_{ik}$ are distinct irreducible characters of $\mathfrak{S}_{b_i}$ with $k = t/t'$, and $E_{ij}$ appears $t'$ times in the components of $E_i$. Moreover, $\Omega_s(E)$ acts transitively on the factors $E_{ij}$. Since $E \in (W_{\dot{s}}^{\wedge}/\Omega_s)^{F'}$, there exists $a_E \in \Omega_s$ such that $E \in (W_{\dot{s}}^{\wedge})^{F''}$ with $F'' = a_E F'$. Let $\widetilde{L} = \widetilde{L}_1 \times \cdots \times \widetilde{L}_r$ be an $F$-stable Levi subgroup of $\widetilde{G}$ according to the decomposition of $E$, where $\widetilde{L}_i = \widetilde{L}_{i1} \times \cdots \times \widetilde{L}_{ik}$ with $\widetilde{L}_{ij} \simeq GL_{b_i t'}$. Then $W_{\dot{s}}$ coincides with $W_{\widetilde{L}^*,\dot{s}}$, the stabilizer of $s$ in $W_{\widetilde{L}^*}$, and $F''$ can be written as $F'' = Fw_2$ with $w_2 \in W_{\widetilde{L}^*}$. Moreover, we have $\Omega_s(E) = \Omega_{s,L}$, a similar group as $\Omega_s$ for $L = \widetilde{L} \cap G$, and the pair $(s, E)$ satisfies the condition in (2.5.1) with respect to $L$. Hence by (a), the set $\mathcal{T}_{s,E}^L$ is parametrized by $\overline{\mathcal{M}}_{s,E}^L$ (the super script $L$ denotes the corresponding object in $L$). Let $\widetilde{P}$ be the standard parabolic subgroup of $\widetilde{G}$ containing $\widetilde{L}$ and put $P = \widetilde{P} \cap G$. Then by Lemma 4.2 in [S2], the map $\rho_0 \mapsto \operatorname{Ind}_{P^F}^{G^F} \rho_0$ gives a bijection between $\mathcal{T}_{s,E}^L$ and $\mathcal{T}_{s,E}$. Since

$$\overline{\mathcal{M}}_{s,E}^L = (\Omega_{s,L}^{F''})^{\wedge} \times (\Omega_{s,L})_{F''} = (\Omega_{s,E}^{F'})^{\wedge} \times (\Omega_{s,E})_{F'} = \overline{\mathcal{M}}_{s,E},$$

this gives a bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$.

(c) We consider the general case. Let $W_{\dot{s}} = W_{1,\dot{s}_1} \times \cdots \times W_{r,\dot{s}_r}$, and $W_s = W_{\dot{s}}\Omega_s$. Here we assume that there exists $i$ such that $\Omega_s$ acts non-transitively on $W_{i,\dot{s}_i}$. In this case, there exists a proper Levi subgroup $L^*$ of $G^*$ such that $W_s$ is contained in $W_{L^*}$ and that $L^*$ is both $F$-stable and $F'$-stable. Then $Z_{G^*}(s)$ is contained in $L^*$. Under this condition, it is known that the twisted induction $R_L^G(\dot{w}_1)$ (see, e.g., [S2, 3.1]) induces a bijection between $\mathcal{E}(L^{F'}, \{s\})$ and $\mathcal{E}(G^F, \{s\})$. By induction hypothesis, we may assume that there exists a bijection $\mathcal{T}_{s,E}^L \leftrightarrow \overline{\mathcal{M}}_{s,E}^L$. Since $\Omega_s(E) = \Omega_{s,L}$, $\overline{\mathcal{M}}_{s,E}^L$ is identified with $\overline{\mathcal{M}}_{s,E}$. Hence we have a bijection $\mathcal{T}_{s,E} \leftrightarrow \overline{\mathcal{M}}_{s,E}$ as asserted.

**Remark 2.9.** In [S2], 4.5, the parametrization is done through three steps as above. However, in the step (a), only the pair $(s, E)$ such that $\varDelta(\widetilde{\rho}_{\dot{s},E}) = 1$ is treated, and it is stated that other cases are reduced to this the case by considering the linear character $\theta$ of $G^F$ corresponding to the central element $\dot{z} \in Z_{G^*}^F$. But this is not true in general. In fact, if the $F$-stable class $\{\dot{s}'\}$ in $G^*$ satisfies the property in (2.5.1), then $\dot{s}'$ can be written as $\dot{s}' = \dot{z}\dot{s}$ for an $F$-stable class $\{\dot{s}\}$ such that $\varDelta(\widetilde{\rho}_{\dot{s},E}) = 1$ with $\dot{z} \in Z_{G^*}$. However, it occurs that $\dot{z} \notin Z_{G^*}^F$ even if the classes $\{\dot{s}\}$ and $\{\dot{z}\dot{s}\}$ are $F$-stable. In that case one cannot find a linear character $\theta$ of $G^F$ corresponding to $\dot{z}$. Hence the step (a) in [S2] does not cover all the cases, and one needs to consider the cases where $\varDelta(\widetilde{\rho}_{\dot{s},E}) \neq 1$ discussed as in 2.8.

## 3. Character formula for generalized Gelfand-Graev characters

**3.1.** For later use, we shall prove a character formula for $\varGamma_{c,\xi}$, which is a variant of the formula given in [K3]. Note that $\varGamma_{c,\xi}$ is constructed by using a specific Lagrangian subspace $\mathfrak{s}$ of $\mathfrak{u}_1$. Following [S2, 2.3], we recall the construction of $\mathfrak{s}$. Assume, for simplicity, that $G = SL_n$. The weighted Dynkin diagram of $N$ is given as follows. Let $\Pi \subset \varSigma^+$ be the set of simple roots and the set of positive roots of $G$, which is written in the form $\varSigma^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$ for certain basis vectors $\varepsilon_1, \dots, \varepsilon_n$ of $\mathbf{R}^n$, and $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$ with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Assume that $N$ corresponds to a partition $\mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r > 0)$ of $n$ via Jordan's normal form. For each $\mu_i$, put

$$Y_i = \{\mu_i - 1, \mu_i - 3, \dots, -\mu_i + 1\}$$

consisting of $\mu_i$ integers. Then $Y = \coprod_i Y_i$ is a set of $n$ integers (with multiplicities), and we arrange its elements in a decreasing order,

$$Y = \{\nu_1 \ge \nu_2 \ge \cdots \ge \nu_n\}. \tag{3.1.1}$$

The weighted Dynkin diagram $h : \Pi \to \mathbf{Z}$ is given by $h(\alpha_i) = \nu_i - \nu_{i+1}$ for $1 \le i \le n-1$. Let $\Pi_1$ (resp. $\varSigma_1$) be the set of $\alpha \in \Pi$ (resp. $\alpha \in \varSigma^+$) such that $h(\alpha) = 1$. Clearly we have $\mathfrak{g}_1 = \bigoplus_{\alpha \in \varSigma_1} \mathfrak{g}_\alpha$. The set $\varSigma_1$ is described as follows. For a given $\alpha_i \in \Pi_1$, let $j$ be the smallest integer such that $j > i$ and that $h(\alpha_j) > 0$, and let $k$ be the largest integer such that $k < i$ and that $h(\alpha_k) > 0$. We define a subset $\Psi_i$ of $\varSigma^+$ by

$$\Psi_i = \{\varepsilon_p - \varepsilon_q \mid k + 1 \le p \le i, i + 1 \le q \le j\}.$$

(If $j$ or $k$ does not exist, we put $k = 0$ or $j = n$.) Then it is easy to see that $\Psi_i$ are mutually disjoint and that

$$\varSigma_1 = \coprod_{\alpha_i \in \Pi_1} \Psi_i.$$

For $\alpha_i, \alpha_j \in \Pi_1$ such that $i < j$, we say that $\Psi_i$ and $\Psi_j$ are adjacent if $\alpha_k \notin \Pi_1$ for $i < k < j$. There exists a subset $\Psi$ of $\varSigma_1$ satisfying the following properties; $\Psi$ is a union of the $\Psi_i$ which are not adjacent each other, and $\varSigma_1 = \Psi \coprod \sigma(\Psi)$, where $\sigma$ is the permutation of $\varSigma^+$ induced from the graph automorphism of $\Pi$. Note that $\Psi$ is uniquely determined up to the action of $\sigma$. Put $\mathfrak{s} = \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$. Then it was shown in [S2, 2.3] that $\mathfrak{s}$ is a Lagrangian subspace in $\mathfrak{g}_1$, stable by the action of $L$.

In the discussion below, we follow the notation in 2.1. Let $V$ be an $n$-dimensional vector space over $k$ on which $G$ acts naturally. We can find a basis $\{N^j v_i \mid 1 \le i \le r, 0 \le j < \mu_i\}$ of $V$ such that $N^{\mu_i} v_i = 0$ and that $H$ acts on $N^j v_i$ by a scalar multiplication $-\mu_i + 1 + 2j$.

Put $M = Z_L(\lambda)$. Take $t \in M$ such that $t$ stabilizes each basis vector $N^j v_i$ up to scalar. It follows that $t \in T_1$, where $T_1$ is a maximal torus in $L$ related to the weighted Dynkin diagram of $N$. Since $G$ is simply connected, $Z_G(t)$ is connected. Put $\mathfrak{z}_t = \operatorname{Lie} Z_G(t)$. Since $M = Z_G(N) \cap Z_G(N^*)$, $\mathfrak{z}_t$ contains $N, N^*$, and so it contains $H$. If we put $(\mathfrak{z}_t)_j = \mathfrak{g}_j \cap \mathfrak{z}_t$, $\mathfrak{z}_t = \bigoplus_j (\mathfrak{z}_t)_j$ gives the grading of $\mathfrak{z}_t$ associated to $N \in \mathfrak{z}_t$. Put $(\mathfrak{u}_t)_j = \bigoplus_{j' \ge j} (\mathfrak{z}_t)_{j'}$. Then we have $(\mathfrak{u}_t)_j = \mathfrak{u}_j \cap \mathfrak{z}_t$. In particular, $P_t = P \cap Z_G(t)$ (resp. $L_t = L \cap Z_G(t)$) is the parabolic subgroup of $Z_G(t)$ (resp. its Levi subgroup) associated to $N$. Moreover, the restriction of $\lambda : \mathfrak{u}_1 \to k$ to $(\mathfrak{u}_t)_1$ coincides with the corresponding linear map $\lambda_t$ with respect to $N$. We have the following lemma.

**Lemma 3.2.** *$\mathfrak{s}_t = \mathfrak{s} \cap \mathfrak{z}_t$ is a Lagrangian subspace of $(\mathfrak{z}_t)_1$, which is stable by $L_t$.*

*Proof.* From the above discussion, the symplectic form on $(\mathfrak{z}_t)_1$ is obtained as the restriction to $(\mathfrak{z}_t)_1$ of the symplectic form on $\mathfrak{g}_1$. Hence $\mathfrak{s} \cap \mathfrak{z}_t$ is an anisotropic subspace of $(\mathfrak{z}_t)_1$. Also it is clear that $\mathfrak{s} \cap \mathfrak{z}_t$ is stable by $L_t$. Note that $T_1$ is a maximal torus on which the root system $\varSigma$ is defined. Then the subroot system $\varSigma_t$ of $\varSigma$ associated to the group $Z_G(t)$ consists of roots $\varepsilon_p - \varepsilon_q \in \varSigma$ such that the corresponding basis vectors $N^k v_l$ and $N^{k'} v_{l'}$ in $V$ have the eigenvalue 0 for $t$. (We identify the basis $\{\varepsilon_i\}$ and $\{N^j v_i\}$ via the total order $\nu_1, \dots, \nu_n$ in (3.1.1)). Since $t \in M$, $N^j v_i$ have the same eigenvalue for all $1 \le j \le \mu_i - 1$ ($i$ is fixed). It follows that the subroot system $\varSigma_t$ is invariant under the action of $\sigma$. Hence $\varSigma_1 \cap \varSigma_t$ is also $\sigma$-invariant. If we put $\Psi_t = \Psi \cap \varSigma_t$, then we have $\varSigma_1 \cap \varSigma_t = \Psi_t \coprod \sigma(\Psi_t)$. In particular, $|\Psi_t| = |\varSigma_1 \cap \varSigma_t|/2$. Since $\mathfrak{s} \cap \mathfrak{z}_t = \bigoplus_{\alpha \in \Psi_t} \mathfrak{g}_\alpha$, we see that $\mathfrak{s} \cap \mathfrak{z}_t$ is a Lagrangian subspace of $(\mathfrak{z}_t)_1$. The lemma is proved. $\square$

**3.3.** For $c \in A_\lambda$, we choose $\dot{c} \in Z_L \subset T_1$, and take $\alpha_c \in T_1$ such that $\alpha_c^{-1} F(\alpha_c) = \dot{c}$. We consider the $c$-twisted version of the previous results. In the following, we denote by $X_c$ the object obtained from $X$ related to $G$ or $\mathfrak{g}$ by the conjugation or adjoint action by $\alpha_c$. Then $M_c = \alpha_c M \alpha_c^{-1}$ coincides with $Z_L(\lambda_c)$, and we have $M_c = Z_G(N_c) \cap Z_G(N_c^*)$, where $N_c, N_c^*, H_c$ are $F$-stable TDS-triple. Since $\alpha_c \in L$, we have $\mathfrak{s}_c = \mathfrak{s}$ and $\mathfrak{u}_{2,c} = \mathfrak{u}_2$. It follows that $\mathfrak{u}_c = \mathfrak{u}$ for $\mathfrak{u} = \mathfrak{s} + \mathfrak{u}_2$, and so $U_c = U$. Let $t$ be a semisimple element in $M_c^F$. Put $\mathfrak{z}_t = \operatorname{Lie} Z_G(t)$, etc., as before. If we put $t' = \alpha_c^{-1} t \alpha_c \in M$, $t'$ is conjugate to an element in $T_1$ under $M^0 \subset L$ (since $M_c = M_c^0 Z_G$). It follows that $\mathfrak{s} \cap \mathfrak{z}_{t'}$ is a Lagrangian subspace of $(\mathfrak{z}_{t'})_1$, and so $\mathfrak{s} \cap \mathfrak{z}_t = {}^{\alpha_c}(\mathfrak{s} \cap (\mathfrak{z}_{t'})_1)$ is a Lagrangian subspace of $(\mathfrak{z}_t)_1$, which is stable by $F$. We put $\mathfrak{s}_t = \mathfrak{s} \cap \mathfrak{z}_t$. Let $\mathfrak{u}_t = \mathfrak{s}_t + (\mathfrak{u}_t)_2$. Then we have an $F$-stable subgroup $U_t$ of $Z_G(t)$ such that $\operatorname{Lie} U_t = \mathfrak{u}_t$, which is stable by $L_t$. Moreover, $\mathfrak{u}_t = \mathfrak{u} \cap \mathfrak{z}_t$. It follows from this that

$$U \cap Z_G(t) = U_t. \tag{3.3.1}$$

Now $\lambda_c : \mathfrak{u}_1 \to k$ is the linear map defined as $\lambda$, by using $N_c$ instead of $N$. Then $\lambda_{c,t} = \lambda_c|_{(\mathfrak{u}_t)_1}$ is the linear map on $(\mathfrak{u}_t)_1$ defined by $N_c \in \mathfrak{z}_t$. It follows that the restriction of $\varLambda_c : U^F \to \bar{\mathbf{Q}}_l^*$ on $U_t^F$ coincides with the linear character of $U_t^F$ defined in terms of $N_c$, which we denote by $\varLambda_{c,t}$.

**3.4.** Take a semisimple element $s \in G^F$, and assume that there exists $g \in G^F$ such that $g^{-1} s g \in M_c^F$. (Do not confuse $s$ with an element in the dual group $G^*$). By fixing $s$, we put $P_g = gPg^{-1} \cap Z_G(s)$ and $L_g = gLg^{-1} \cap Z_G(s)$. We apply the previous argument for $t = g^{-1} s g$. Then $N_g = {}^g N_c$ is an $F$-stable nilpotent element in $\operatorname{Lie} Z_G(s)$, and $U_g = g U_t g^{-1}$ is the unipotent subgroup of $Z_G(s)$ associated to $N_g$. We have $P_g = g P_t g^{-1}$ and $L_g = g L_t g^{-1}$. Moreover $\lambda_g = \operatorname{Ad} g \circ \lambda_{c,t}$ is the linear map of ${}^g(\mathfrak{u}_t)_1 = (\mathfrak{u}_s)_1$, and $\varLambda_g = \operatorname{ad} g \circ \varLambda_{c,t}$ coincides with the linear character of $U_g^F$ associated to $N_g$. We define a modified generalized Gelfand-Grave character $\varGamma_{N_g,1}^{Z_G(s)}$ of $Z_G(s)$ by

$$\varGamma_{N_g,1}^{Z_G(s)} = \operatorname{Ind}_{Z_{L_g}(\lambda_g)^F U_g^F}^{Z_G(s)^F} (1 \otimes \varLambda_g) \tag{3.4.1}$$

associated to $N_g \in \mathrm{Lie}\, Z_G(s)$.

The following result gives a description of modified generalized Gelfand-Graev characters in terms of various modified generalized Gelfand-Graev characters of smaller groups, which is an extended version of the formula stated in [K3, Lemma 2.3.5].

**Proposition 3.5.** *Assume that $s, v \in G^F$ such that $sv = vs$, where $s$ is semisimple and $v$ is unipotent. Then we have*

$$\varGamma_{c,\xi,\theta}(sv) = \frac{1}{|Z_G(s)^F|} \sum_{\substack{g \in G^F \\ g^{-1}sg \in Z_L(\lambda_c)^F}} \frac{|Z_{L_g}(\lambda_g)^F|}{|Z_L(\lambda_c)^F|} \theta\xi^{\natural}(g^{-1}sg) \varGamma^{Z_G(s)}_{N_g,1}(v).$$

*Proof.* By definition,

$$\begin{aligned} \varGamma_{c,\xi,\theta}(sv) &= \big(\mathrm{Ind}_{M_c^F U^F}^{G^F} \theta\xi^{\natural} \otimes \varLambda_c\big)(sv) \\ &= |M_c^F U^F|^{-1} \sum_{\substack{g \in G^F \\ g^{-1}svg \in M_c^F U^F}} (\theta\xi^{\natural} \otimes \varLambda_c)(g^{-1}svg). \end{aligned} \tag{3.5.1}$$

Here the condition $g^{-1}svg \in M_c^F U^F$ in the sum is equivalent to the condition that $g^{-1}vg \in M_c^F U^F$ and $g^{-1}sg \in M_c^F U^F$. We note that

(3.5.2) Any semisimple element in $M_c^F U^F$ is contained in $\bigcup_{x \in U^F} x M_c^F x^{-1}$.

In fact, let $T_1$ be an $F$-stable maximal torus in $L$ as in 3.1. Then $T_2 = T_1 \cap M$ is a maximal torus in $M$. Since we can choose $\dot{c} \in T_1$, $T_2$ is also contained in $M_c$. Thus $T_2$ is a maximal torus in $M_cU$. We have $N_{M_cU}(T_2) \simeq N_{M_c}(T_2)Z_U(T_2)$. Since $U$ is a product of one parameter subgroups $U_\alpha$ associated to roots $\alpha$ with respect to $T_1$, $Z_U(T_2)$ is a product of $U_\alpha$ such that $\alpha|_{T_2} = \mathrm{Id}$. It follows that $Z_U(T_2)$ is connected, and $N_{M_cU}(T_2)^0 = T_2 Z_U(T_2)$. We see that $N_{M_cU}(T_2)/N_{M_cU}(T_2)^0 \simeq N_{M_c}(T_2)/T_2$. This implies that any $F$-stable maximal torus in $M_cU$ is taken from $M_c$ up to $U^F$-conjugate. Since any semisimple element in $M_c^F U^F$ is contained in an $F$-stable maximal torus, we obtain (3.5.2).

It follows from (3.5.2) that $g^{-1}sg \in xM_c^F x^{-1}$ for some $x \in U^F$, i.e., $(gx)^{-1}s(gx) \in M_c^F$. It is easy to see that the set $\{x_1 \in U^F \mid (gx_1)^{-1}s(gx_1) \in M_c^F\}$ is given by $xZ_U((gx)^{-1}s(gx))^F$ for some $x \in U^F$ such that $(gx)^{-1}s(gx) \in M_c^F$. Hence the last formula in (3.5.1) implies that

$$\varGamma_{c,\xi,\theta}(sv) = |M_c^F U^F|^{-1} \sum_{\substack{g \in G^F, x \in U^F \\ g^{-1}vg \in M_c^F U^F \\ g^{-1}sg \in xM_c^F x^{-1}}} |Z_U((gx)^{-1}s(gx))^F|^{-1} (\theta\xi^{\natural} \otimes \varLambda_c)(g^{-1}svg).$$

By replacing $gx$ by $g$, we have

$$\begin{aligned}
\varGamma_{c,\xi,\theta}(sv) &= |M_c^F|^{-1} \sum_{\substack{g\in G^F \\ g^{-1}vg\in M_c^FU^F \\ g^{-1}sg\in M_c^F}} |Z_U^F(g^{-1}sg)|^{-1}\theta\xi^\natural(g^{-1}sg)(\theta\xi^\natural\otimes\varLambda_c)(g^{-1}vg) \\
&= |M_c^F|^{-1}\sum_t \sum_{\substack{y\in Z_G(t)^F \\ y^{-1}v_1y\in M_c^FU^F}} |Z_U^F(t)|^{-1}\theta\xi^\natural(t)(\theta\xi^\natural\otimes\varLambda_c)(y^{-1}v_1y),
\end{aligned}$$

where in the first sum in the last formula, $t$ runs over all the semisimple element in $M_c^F$ such that $t = g^{-1}sg$ for some $g \in G^F$. We fix such $g$ for each $t$, and put $v_1 = g^{-1}vg$.

Hence we have $v_1 \in Z_G(t)^F$. Since $t$ normalizes $M_c^F$ and $U^F$, we have

$$\begin{aligned}
Z_G(t)^F \cap M_c^FU^F &= (Z_G(t)^F \cap Z_L(\lambda_c)^F)(Z_G(t)^F \cap U^F) \\
&= Z_{M_c}(t)^FU_t^F
\end{aligned}$$

by (3.3.1). Also we have $Z_U(t) = U_t$ by (3.3.1). It follows that we have

$$\tag{3.5.3}
\begin{aligned}
\varGamma_{c,\xi,\theta}(sv) = &\sum_{\substack{t\in M_c^F \\ g^{-1}sg=t}} \frac{|Z_{M_c}(t)^F|}{|M_c^F|}\theta\xi^\natural(t)\times \\
&\times\Biggl\{|Z_{M_c}(t)^F|^{-1}|U_t^F|^{-1}\sum_{\substack{y\in Z_G(t)^F \\ y^{-1}v_1y\in Z_{M_c}(t)^FU_t^F}}(\theta\xi^\natural\otimes\varLambda_c)(y^{-1}v_1y)\Biggr\}.
\end{aligned}$$

Here we note that $y^{-1}v_1y$ is unipotent. Hence the component of $y^{-1}v_1y$ in $Z_{M_c}(t)^F$ is unipotent. Since $\xi^\natural$ is a character of $M_c^F$ which is trivial on $M_c^{0F}$, it is trivial on the set of unipotent elements in $M_c^F$. Also by (2.1.3) $\theta$ is trivial on the set of unipotent elements in $M_c^F$. It follows that

$$(\theta\xi^\natural\otimes\varLambda_c)(y^{-1}v_1y) = (1\otimes\varLambda_c)(y^{-1}v_1y).$$

Then the expression in the parenthesis in (3.5.3) coincides with

$$\mathrm{Ind}_{Z_{M_c}(t)^FU_t^F}^{Z_G(t)^F}(1\otimes\varLambda_c)(v_1) = \varGamma_{N_g,1}^{Z_G(s)}(v)$$

under the conjugation by $g \in G^F$. Substituting this into (3.5.3) we obtain the proposition. $\square$

## 4. Shintani descent and almost characters

**4.1.** We consider the group $G^{F^m}$ for a positive integer $m$. We denote by $G^{F^m}/{\sim_F}$ the set of $F$-twisted conjugacy classes in $G^{F^m}$. (In the case where $m = 1$, the set of $F$-twisted classes coincides with the set of conjugacy classes, which we denote simply by $G^F/\!\sim$.) A norm map

$$N_{F^m/F} : G^{F^m}/\!\sim_F \to G^F/\!\sim$$

is defined by attaching $x = F^m(\alpha)\alpha^{-1}$ to $\hat{x} = \alpha^{-1}F(\alpha)$ where $x \in G^F, \hat{x} \in G^{F^m}$ and $\alpha \in G$. Let $C(G^{F^m}/\!\sim_F)$ (resp. $C(G^F/\!\sim)$) be the space of $F$-twisted class functions on $G^{F^m}$ (resp. class functions on $G^F$). A Shintani descent map

$$Sh_{F^m/F} : C(G^{F^m}/\!\sim_F) \to C(G^F/\!\sim)$$

is given by $Sh_{F^m/F} = (N^*_{F^m/F})^{-1}$, which is a linear isomorphism of vector spaces.

Let $\sigma = F|_{G^{F^m}}$. We consider the semidirect product $G^{F^m}\langle\sigma\rangle$ of $G^{F^m}$ with the cyclic group $\langle\sigma\rangle$ of order $m$ generated by $\sigma$. Then the coset $G^{F^m}\sigma$ is invariant under the conjugation action of $G^{F^m}$, and the set $G^{F^m}\sigma/\!\sim$ is identified with the set $G^{F^m}/\!\sim_F$ via the map $x\sigma \leftrightarrow x$. Now each $F$-stable irreducible character $\rho$ of $G^{F^m}$ can be extended to an irreducible character $\widetilde{\rho}$ of $G^{F^m}\langle\sigma\rangle$ (in $m$-distinct way), and the restriction $\widetilde{\rho}|_{G^{F^m}\sigma}$ to the coset $G^{F^m}\sigma$ determines an element in $C(G^{F^m}/\!\sim_F)$ under the above bijection. The function $\widetilde{\rho}|_{G^{F^m}\sigma}$ does not depend on the choice of the extension up to a scalar multiple, and the collection of those $\widetilde{\rho}|_{G^{F^m}\sigma}$ for $\rho \in (\operatorname{Irr} G^{F^m})^F$ gives a basis of $C(G^{F^m}/\!\sim_F)$. In what follows, we often regard a character $f$ of $G^{F^m}\langle\sigma\rangle$ as an element in $C(G^{F^m}/\!\sim_F)$ by considering its restriction to $G^{F^m}\sigma$, if there is no fear of confusion.

**4.2.** We shall describe the Shintani descent of the modified generalized Gelfand-Graev characters. We follow the setting in 2.2. Recall the set $\overline{\mathcal{M}}$ in (2.1.2) and $\mathcal{M}$ in 2.2. Let $\theta$ be as in (2.2.1). Hence it is the restriction to $Z_L(\lambda)^{F^m}$ of an $F$-stable linear character $\theta'$ of $Z_{\widetilde{L}}(\lambda)^{F^m}$. We denote by $\theta_0$ the linear character of $Z_L(\lambda)^F$ obtained by restricting the linear character $Sh_{F^m/F}(\theta')$ of $Z_{\widetilde{L}}(\lambda)^F$. Hence $\theta_0$ satisfies the condition in (2.1.3). We consider the modified generalized Gelfand-Graev characters $\varGamma^{(m)}_{c,\xi,\theta}$ and $\varGamma_{c_1,\xi_1,\theta_0}$ for $(c,\xi) \in \mathcal{M}$ and $(c_1,\xi_1) \in \overline{\mathcal{M}}$.

Let us consider an extension $\widetilde{\varGamma}^{(m)}_{c,\xi,\theta}$ as in 2.2, which is determined by the choice of an extension $\widetilde{\theta\xi^\natural}$ of $\theta\xi^\natural$ to $M_c\langle\hat{c}\sigma\rangle$. Since $c \in A_\lambda^F$, we may choose $\dot{c} \in Z_L(\lambda)^F$. Note that, under the isomorphism $\operatorname{ad}\beta_c^{-1} : Z_L(\lambda_c)^{F^m} \simeq Z_L(\lambda)^{\dot{c}F^m}$, the linear character $\theta\xi^\natural$ corresponds to a linear character $\theta\xi$ of $Z_L(\lambda)^{\dot{c}F^m}$, and $\widetilde{\theta\xi^\natural}$ corresponds to its extension $\widetilde{\theta\xi}$ to $Z_L(\lambda)^{\dot{c}F^m}\langle\sigma\rangle$. Take $c_1 \in (A_\lambda)_F$. As $A_\lambda = A_\lambda^{F^m}$, we may choose an element $\dot{c}_1 \in Z_L(\lambda)^{F^m}$ whose image on $A_\lambda$ gives a representative of $c_1 \in (A_\lambda)_F$. Now the following proposition describes the Shintani descent of $\widetilde{\varGamma}^{(m)}_{c,\xi,\theta}$ in terms of $\varGamma_{c_1,\xi_1,\theta_0}$. The proof is done in a similar way as in [S2]. In fact, Theorem 1.10 in [S2] can be extended to our setting, and the proposition is the direct consequence of the theorem (cf. [S2, 4.11]).

**Proposition 4.3.** *Let the notations be as above. Assume that $m$ is sufficiently divisible. Then we have*

$$Sh_{F^m/F}(\mu^{-1}_{c,\theta\xi}\widetilde{\varGamma}^{(m)}_{c,\xi,\theta}) = |A_\lambda^F|^{-1}\widetilde{\xi}(\sigma)\sum_{(c_1,\xi_1)\in\overline{\mathcal{M}}}\xi(c_1)\xi_1(c)\varGamma_{c_1,\xi_1,\theta_0}.$$

**4.4.** We shall describe the set of $F$-stable irreducible characters of $G^{F^m}$ in the case where $m$ is sufficiently divisible. Let $\{s\}$ be an $F$-stable class in $G^*$, and we assume that $s \in T^*$. As in the case of $G^F$, one can find $\dot{s} \in \widetilde{T}^*$ such that $\pi(\dot{s}) = s$ and that the class $\{s\}$ is $F$-stable. Hence $F'(\dot{s}) = \dot{s}$ for $F' = Fw_1$. We choose $m$ large enough so that $\dot{s} \in \widetilde{T}^{*F^m}$ and that $F^m$ acts trivially on $\Omega_s$. For each $E \in W_{\dot{s}}^{\wedge}$, we denote by $\overline{\mathcal{M}}^{(m)}_{s,E}$ and $\mathcal{T}^{(m)}_{s,E}$ the set $\overline{\mathcal{M}}_{s,E}$ and $\mathcal{T}_{s,E}$ as given in 1.4, but replacing $F'$ by $F^m$. Then $\mathcal{E}(G^{F^m}, \{s\})$ is a disjoint union of various $\mathcal{T}^{(m)}_{s,E}$, and the latter set is in bijection with $\overline{\mathcal{M}}^{(m)}_{s,E}$. By our assumption on $m$, we have $\overline{\mathcal{M}}^{(m)}_{s,E} = \Omega^{\wedge}_{s,E} \times \Omega_{s,E}$. Let us define a subset $\mathcal{M}_{s,E}$ of $\overline{\mathcal{M}}^{(m)}_{s,E}$ by $\mathcal{M}_{s,E} = (\Omega^{\wedge}_{s,E})^{F'} \times \Omega^{F'}_{s,E}$, where $(\Omega^{\wedge}_{s,E})^{F'}$ means the set of $F'$-stable irreducible characters of $\Omega_{s,E}$. Then by [S2, (4.6.1)], the set $(\mathcal{T}^{(m)}_{s,E})^F$ of $F$-stable irreducible characters in $\mathcal{T}^{(m)}_{s,E}$ is parametrized by $\mathcal{M}_{s,E}$, and so $\mathcal{E}(G^{F^m}, \{s\})^F$ can be described as

$$\mathcal{E}(G^{F^m}, \{s\})^F = \coprod_{E \in (W_{\dot{s}}^{\wedge}/\Omega_s)^{F'}} \mathcal{M}_{s,E}. \tag{4.4.1}$$

In the case where $(s, E)$ is of the form 2.8 (a), the set $\mathcal{T}_{s,E}$ is also parametrized in terms of $\overline{\mathcal{M}}_{s,N}$. Since $m$ is large enough, $\mathcal{T}^{(m)}_{s,E}$ is parametrized by $\overline{\mathcal{M}}^{(m)}_{s,N} = \bar{A}_\lambda \times \bar{A}^{\wedge}_\lambda$. Then under this parametrization, $(\mathcal{T}^{(m)}_{s,E})^F$ is parametrized by $\bar{A}^F_\lambda \times (\bar{A}^{\wedge}_\lambda)^F$.

**4.5.** We define a pairing $\{\ ,\ \} : \mathcal{M}_{s,E} \times \overline{\mathcal{M}}_{s,E} \to \bar{\mathbf{Q}}_l^*$ as follows. For $x = (\eta, z) \in \mathcal{M}_{s,E}$ and $y = (\eta', z') \in \overline{\mathcal{M}}_{s,E}$,

$$\{x, y\} = |\Omega^{F'}_{s,E}|^{-1} \eta(z')\eta'(z). \tag{4.5.1}$$

(Note that $\eta \in (\Omega^{\wedge}_{s,E})^{F'}$ can be viewed as a character of the group $(\Omega_{s,E})_{F'}$.)

We define a function $R_x \in C(G^F/\!\sim)$ for each $x \in \mathcal{M}_{s,E}$ by

$$R_x = \sum_{y \in \overline{\mathcal{M}}_{s,E}} \{x, y\}\rho_y. \tag{4.5.2}$$

In the case where $(s, E)$ satisfies the property in 2.8 (a), the set $\mathcal{T}_{s,E}$ is also parametrized by $\overline{\mathcal{M}}_{s,N} = (\bar{A}_\lambda)_F \times (\bar{A}^F_\lambda)^{\wedge}$, and we have a bijection between $\overline{\mathcal{M}}_{s,E}$ and $\overline{\mathcal{M}}_{s,N}$ by 2.7. Put $\mathcal{M}_{s,N} = \bar{A}^F_\lambda \times (\bar{A}^{\wedge}_\lambda)^F$. Then the set $(\mathcal{T}^{(m)}_{s,E})^F$ is parametrized by $\mathcal{M}_{s,N}$. By modifying the argument in 2.7 appropriately to the situation in $G^{F^m}$, we have a bijection between $\mathcal{M}_{s,E}$ and $\mathcal{M}_{s,N}$. Let $\theta_0$ be the linear character of $Z_L(\lambda)^F$ obtained by restricting $\varDelta(\widetilde{\rho}_{\dot{s},E})$ to $Z_L(\lambda)^F$. The linear character $\theta$ of $Z_L(\lambda)^{F^m}$ is also defined by using the Shintani descent of $Z_{\widetilde{L}}(\lambda)$ (cf. 4.2). We say that $\theta$ (resp. $\theta_0$) is the linear character associated to $\mathcal{M}_{s,N}$ (resp. $\overline{\mathcal{M}}_{s,N}$).

We define a pairing $\{\ ,\ \} : \mathcal{M}_{s,N} \times \overline{\mathcal{M}}_{s,N} \to \bar{\mathbf{Q}}_l^*$ for $x = (c, \xi) \in \mathcal{M}_{s,N}$ and $y = (c', \xi') \in \overline{\mathcal{M}}_{s,N}$,

$$\{x, y\} = |\bar{A}^F_\lambda|^{-1} \xi(c')\xi'(c). \tag{4.5.3}$$

Then the bijections $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}$, etc., are compatible with those pairings. This property was used in [S2, 4.11] to connect almost characters defined in terms of $\overline{\mathcal{M}}_{s,N}$ to that of $\overline{\mathcal{M}}_{s,E}$ (in the case where $\theta = 1$, but the proof was omitted there). We give a proof of this property.

**Lemma 4.6.** *Assume that $(s, E)$ is as in 2.8 (a). Then under the bijections $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}, (\eta, z) \leftrightarrow (c, \xi)$ and $\mathcal{M}_{s,E} \simeq \mathcal{M}_{s,N}$, $(\eta', z') \leftrightarrow (c', \xi')$, we have*

$$|\Omega_{s,E}^{F'}|^{-1}\eta(z')\eta'(z) = |\bar{A}_\lambda^F|^{-1}\xi(c')\xi'(c).$$

*Proof.* By our assumption, we have $\Omega_{s,E}^{F'} = \Omega_s^{F'}$. It follows from the parametrization of $\operatorname{Irr} G^F$ in 1.6 and 2.7, we see that $|\Omega_s^{F'}| = |\bar{A}_\lambda^F|$, which coincides with the number of irreducible components in $\widetilde{\rho}_{\dot{s},E}|_{G^F}$. Thus, in order to prove the lemma, it is enough to show that

$$\eta(z') = \xi'(c), \quad \eta'(z) = \xi(c'). \tag{4.6.1}$$

We recall the bijection $\overline{\mathcal{M}}_{s,E} \simeq \overline{\mathcal{M}}_{s,N}$ given by $h : (\Omega_s)_{F'} \to (\bar{A}_\lambda^F)^\wedge$ and $f : (\Omega_s^{F'})^\wedge \to (\bar{A}_\lambda)_F$ in 2.7. A similar construction gives bijections

$$\begin{aligned} h' &: \Omega_s \to \bar{A}_\lambda^\wedge, &\quad h'' &: (\Omega_s)^{F'} \to (\bar{A}_\lambda^\wedge)^F, \\ f' &: \Omega_s^\wedge \to \bar{A}_\lambda, &\quad f'' &: (\Omega_s^\wedge)^{F'} \to \bar{A}_\lambda^F, \end{aligned}$$

and $h'' \times f'' : (\Omega_s^\wedge)^{F'} \times \Omega_s^{F'} \to \bar{A}_\lambda^F \times (\bar{A}_\lambda^\wedge)^F$ gives the bijection $\mathcal{M}_{s,E} \to \mathcal{M}_{s,N}$. We have inclusions $(\Omega_s)^{F'} \hookrightarrow \Omega_s, (\Omega_s^\wedge)^{F'} \hookrightarrow \Omega_s^\wedge$ and natural surjections $\Omega_s \to (\Omega_s)_{F'}, \Omega_s^\wedge \to (\Omega_s^{F'})^\wedge$. Also we have inclusions $(\bar{A}_\lambda^\wedge)^F \hookrightarrow \bar{A}_\lambda^\wedge, \bar{A}_\lambda^F \hookrightarrow \bar{A}_\lambda$, and natural surjections $(\bar{A}_\lambda)^\wedge \to (\bar{A}_\lambda^F)^\wedge, \bar{A}_\lambda \to (\bar{A}_\lambda)_F$.

We want to show that the maps $h, h', h''$ and $f, f', f''$ are compatible with various inclusions and surjections given above. First we note that the map $h : (\Omega_s)_{F'} \to (\bar{A}_\lambda^F)^\wedge$ is compatible with the extension of the filed, i.e., the following diagram commutes.

$$\begin{array}{ccc} (\Omega_s)_{F'} & \xrightarrow{h} & (\bar{A}_\lambda^F)^\wedge \\ \uparrow & & \uparrow \\ (\Omega_s)_{F^k} & \xrightarrow{h^0} & (\bar{A}_\lambda^{F^k})^\wedge, \end{array} \tag{4.6.2}$$

where $F^k$ is the map such that $F^k(\dot{s}) = \dot{s}$ and that $F^k$ acts trivially on $\Omega_s$, and on $\bar{A}_\lambda$. $h^0$ is a similar map as $h$ defined by replacing $F'$ by $F^k$.

We show (4.6.2). We choose $m$ large enough so that $m$ is divisible by $k$. Let $\psi_x, \widehat{\psi}_x, \widehat{\psi}'_x$, be the maps given in 2.7 with respect to $F'$. Let $y \in (\Omega_s)_{F^k}$ such that its canonical image in $(\Omega_s)_{F'}$ coincides with $x$. We may assume that $\dot{x} = \dot{y} \in N_{G^*}(T)$. We denote by $\psi_y, \widehat{\psi}_y, \widehat{\psi}'_y$ similar maps constructed by using $F^k$ instead of $F'$. In particular, $\widehat{\psi}'_y$ is the linear character of $\widetilde{G}^{F^m}$ corresponding to $z_y \in Z_{\widetilde{G}^*}^{F^m}$ such that $\dot{s}^{-1}\dot{x}\dot{s}\dot{x}^{-1} = z_y F^k(z_y{}^{-1})$. Since $z_y F^k(z_y{}^{-1}) = z_x F(z_x^{-1})$, one can choose $z_x$ and $z_y$ so that they satisfy the relation

$$z_x = z_y F(z_y) \cdots F^{k-1}(z_y).$$

It follows that

$$\widehat{\psi}_x = \widehat{\psi}_y F^{-1}(\widehat{\psi}_y) \cdots F^{-k+1}(\widehat{\psi}_y).$$

Put $\psi'_x = Sh_{F^m/F^k}(\widehat{\psi}_x)$. Since $\psi_y = Sh_{F^m/F^k}(\widehat{\psi}_y)$, we have

$$\psi'_x = \psi_y F^{-1}(\psi_y) \cdots F^{-k+1}(\psi_y),$$

and $\psi_x = Sh_{F^k/F}(\psi'_x)$. We shall compute the value $\psi_x(t)$ for $t \in T^F$. Take $\alpha \in T$ such that $t = F^k(\alpha)\alpha^{-1}$, and put $\hat{t} = \alpha^{-1}F(\alpha)$. Then $\hat{t} \in T^{F^k}$, and we have

$$\psi_x(t) = \psi'_x(\hat{t}) = \psi_y(\hat{t}F(\hat{t}) \cdots F^{k-1}(\hat{t})) = \psi_y(\alpha^{-1}F^k(\alpha)) = \psi_y(t)$$

since $\alpha^{-1}F^k(\alpha) = F^k(\alpha)\alpha^{-1} = t$. It follows that

$$\psi_x|_{T^F} = \psi_y|_{T^F}. \tag{4.6.3}$$

Now for $c \in \bar{A}_\lambda^F$, one can choose a representative $\dot{c} \in Z_L(\lambda)^F$ of $c$ so that $\dot{c} \in T^F$. Then by (4.6.3), we have

$$h(x)(c) = \psi_x(\dot{c}) = \psi_y(\dot{c}) = h^0(y)(c).$$

This proves the commutativity of (4.6.2).

Next we show that the map $f : (\Omega_s)_{F'} \to (\bar{A}_\lambda)_F$ is compatible with the extension of the field, i.e., the following diagram commutes.

$$\begin{array}{ccc} (\Omega_s^{F'})^\wedge & \xrightarrow{f} & (\bar{A}_\lambda)_F \\ \uparrow & & \uparrow \\ (\Omega_s^{F^k})^\wedge & \xrightarrow{f^0} & (\bar{A}_\lambda)_{F^k}, \end{array} \tag{4.6.4}$$

where $f^0$ is a similar map as $f$ defined by replacing $F'$ by $F^k$.

In fact, we consider the following diagram

$$\begin{array}{ccccc} \widetilde{G}^F/G^F & \xrightarrow{\pi_1} & Z_F & \xrightarrow{\pi_2} & (\bar{A}_\lambda)_F \\ {\scriptstyle N_{F^k/F}}\uparrow & & \uparrow & & \uparrow \\ \widetilde{G}^{F^k}/G^{F^k} & \xrightarrow{\pi_1^0} & Z_{F^k} & \xrightarrow{\pi_2^0} & (\bar{A}_\lambda)_{F^k}, \end{array} \tag{4.6.5}$$

where $\pi_2 \circ \pi_1 = f_2$, $\pi_2^0 \circ \pi_1^0 = f_2^0$, and the second and the third vertical maps are natural surjections. This diagram turns out to be commutative. In order to show this, it is enough to see the commutativity of the left square. Take $g \in \widetilde{G}^F$ and write it as $g = g_1 z$ with $g_1 \in G, z \in Z_{\widetilde{G}}$. Then one can find $\beta \in G, \gamma \in Z_{\widetilde{G}}$ such that $g_1 = F^k(\beta)\beta^{-1}$, $z = F^k(\gamma)\gamma^{-1}$. Put $\hat{g}_1 = \beta^{-1}F(\beta), \hat{z} = \gamma^{-1}F(\gamma)$. Since $\gamma \in Z_{\widetilde{G}}$, we see that $\hat{g} = \hat{g}_1\hat{z}$ satisfies the condition that $N_{F^k/F}(\hat{g}) = g$. Hence we have $\pi_1(g) =$

$g_1^{-1}F(g_1) = zF(z^{-1})$ and $\pi_1^0(\hat{g}) = \hat{z}F^k(\hat{z}^{-1})$. But by using $\hat{z} = \gamma^{-1}F(\gamma)$, we see easily that $zF(z^{-1}) = \hat{z}F^k(\hat{z}^{-1})$. This shows that the diagram (4.6.5) is commutative.

On the other hand, since the map $f_2^* : \Omega_s^{F'} \to Z_{\widetilde{G}^*}^F$ is compatible with the inclusions $\Omega_s^{F'} \hookrightarrow \Omega_s^{F^k}, Z_{\widetilde{G}^*}^F \hookrightarrow Z_{\widetilde{G}}^{F^k}$, we have the commutative diagram

$$
\begin{array}{ccc}
\widetilde{G}^F/G^F & \longrightarrow & (\Omega_s^{F'})^\wedge \\
{\scriptstyle N_{F^k/F}}\uparrow & & \uparrow \\
\widetilde{G}^{F^k}/G^{F^k} & \longrightarrow & (\Omega_s^{F^k})^\wedge.
\end{array}
\tag{4.6.6}
$$

The commutativity of the diagram (4.6.4) follows from (4.6.5) and (4.6.6).

Finally, it is easy to check that the maps $h', h''$ are compatible with $(\Omega_s)^{F'} \hookrightarrow \Omega_s$ and $(\bar{A}_\lambda^\wedge)^F \hookrightarrow \bar{A}_\lambda^\wedge$, and the corresponding results hold also for $f', f''$.

Now by using the commutativity of $h, h', h''$ and $f, f', f''$, one can check that (4.6.1) is reduced to showing that

(4.6.7) Assume that $F(s) = s$, and that $F$ acts trivially on $\Omega_s$ and on $A_\lambda$. Then, for $(\eta, z) \in \Omega_s^\wedge \times \Omega_s$ and $(c, \xi) \in \bar{A}_\lambda \times \bar{A}_\lambda^\wedge$, we have $\eta(z) = \xi(c)$.

But (4.6.7) is proved in a similar way as the proof of Lemma 3.16 in [ShS], where a similar problem for $(\mathcal{T}_{s,E}^{(m)})^{F^2}$ is discussed. Thus Lemma 4.6 is proved. $\square$

The following result gives a description of the Shintani descent of $G^{F^m}$ in the case where $m$ is sufficiently divisible. In the following, we denote by $\rho_x^{(m)}$ the $F$-stable irreducible character of $G^{F^m}$ belonging to the set $(\mathcal{T}_{s,E}^{(m)})^F$ corresponding to $x \in \mathcal{M}_{s,E}$.

**Theorem 4.7** ([S2, Theorem 4.7]). *Assume that $m$ is sufficiently divisible. For each $\rho_x^{(m)} \in (\operatorname{Irr} G^{F^m})^F$ corresponding to $x = (\eta, z) \in \mathcal{M}_{s,E}$, we fix an extension $\tilde{\rho}_x^{(m)}$ of $\rho_x^{(m)}$ to $G^{F^m}\langle\sigma\rangle$. Then we have*

$$
Sh_{F^m/F}(\tilde{\rho}_x^{(m)}|_{G^{F^m}\sigma}) = \mu_x R_x,
$$

*where $\mu_x$ is a certain rot of unity. In the case where $(s, E)$ is in (a) of 2.8, $\mu_x$ is given by an $m$-th root of unity of $\theta(\dot{c}^{-1})\xi(c^{-1})$ under the correspondence $(\eta, z) \leftrightarrow (c, \xi)$ (see 2.2 and 4.5 for the notation).*

**Remark 4.8.** $\mu_x$ is not given explicitly in [S2]. But the determination of $\mu_x$ is reduced to the case where $(s, E)$ is in (a) of 2.8. In this case the extension $\tilde{\rho}_x^{(m)}$ of $\rho_x^{(m)}$ is determined by the extension $\widetilde{\varGamma}_{c,\xi,\theta}^{(m)}$ of $\varGamma_{c,\xi,\theta}^{(m)}$ for $x = (c, \xi) \in \mathcal{M}_{s,N}$, which is determined by the choice of $\mu_{c,\theta\xi}$ as in 2.2. Then the argument in 4.11 in [S2] gives the description of $\mu_x = \mu_{c,\theta\xi}$.

**4.9.** Let $L$ be a Levi subgroup of a standard parabolic subgroup $P$ of $G$ containing $T$. Let $\delta = \delta^{(m)}$ be an irreducible cuspidal character of $L^{F^m}$. Let $\mathcal{W} = N_G(L)/L$, and put

$$
\begin{aligned}
\mathcal{W}_\delta &= \{w \in \mathcal{W} \mid {}^w\delta = \delta\}, \\
\mathcal{Z}_\delta &= \{w \in \mathcal{W} \mid {}^{Fw}\delta = \delta\}.
\end{aligned}
$$

$\mathcal{W}_\delta$ is naturally regarded as a subgroup of $W$, and according to Howlett and Lehrer [HL], $\mathcal{W}_\delta$ can be decomposed as $\mathcal{W}_\delta = \mathcal{W}_\delta^0\Omega_\delta$, where $\mathcal{W}_\delta^0$ is a normal subgroup of $\mathcal{W}_\delta$ which is a reflection group with a set of simple reflections associated to some root system $\varGamma \subset \varSigma$, and $\Omega_\delta$ is given by

$$\Omega_\delta = \{w \in \mathcal{W}_\delta \mid w(\varGamma^+) \subset \varGamma^+\},$$

where $\varGamma^+ = \varGamma \cap \varSigma^+$ is the set of positive roots of $\varGamma^+$. Assume that $\mathcal{Z}_\delta \neq \emptyset$. Then $\mathcal{Z}_\delta$ can be written as $\mathcal{Z}_\delta = w_\delta\mathcal{W}_\delta$ for some $w_\delta \in \mathcal{W}$. We choose $w_\delta$ so that $Fw_\delta(\varGamma^+) \subset \varGamma^+$, and let $\dot{w}_\delta \in N_G(L)$ be a representative of $w_\delta$. Note that this condition determines $w_\delta$ only up to the coset of $\Omega_\delta$. Let $\gamma_\delta : \mathcal{W}_\delta \to \mathcal{W}_\delta$ be the automorphism induced by the map $Fw_\delta$. Then $\gamma_\delta$ stabilizes $\mathcal{W}_\delta^0$. Let $\widetilde{\mathcal{W}_\delta} = \mathcal{W}_\delta\langle\gamma_\delta\rangle$ be the semidirect product of $\mathcal{W}_\delta$ with the cyclic group generated by $\gamma_\delta$. We denote by $(\mathcal{W}_\delta^\wedge)^{\gamma_\delta}$ the set of $\gamma_\delta$-stable irreducible characters of $\mathcal{W}_\delta$.

Let $\mathcal{P}_\delta = \operatorname{Ind}_{P^{F^m}}^{G^{F^m}} \delta$ be the Harish-Chandra induction of $\delta$. We review the results from [S2, 3.5, 3.6]. The irreducible characters of $G^{F^m}$ appearing in the decomposition of $\mathcal{P}_\delta$ are parametrized by $\mathcal{W}_\delta^\wedge$. We denote by $\rho_E = \rho_E^{(m)}$ the irreducible character of $G^{F^m}$ corresponding to $E \in \mathcal{W}_\delta^\wedge$. Let $M$ be the subgroup of $N_G(L)$ generated by $L$ and $w \in \mathcal{W}_\delta$. Then it is known ([G], [Le]) that $\delta$ can be extended to a representation $\widetilde{\delta}$ of $M^{F^m}$. $F\dot{w}_\delta$ stabilizes $M^{F^m}$, and the restriction of $F\dot{w}_\delta$ on $M^{F^m}$ is written as $\sigma\dot{w}_\delta$. $F\dot{w}_\delta$ stabilizes $\widetilde{\delta}$, and one can extend $\widetilde{\delta}$ to a representation of $M^{F^m}\langle\sigma\dot{w}_\delta\rangle$. We fix such an extension of $\widetilde{\delta}$, and denote it also by $\widetilde{\delta}$.

Now we have an action of $F$ on $\mathcal{P}_\delta$. $\rho_E$ is $F$-stable if and only if $E \in (\mathcal{W}_\delta^\wedge)^{\gamma_\delta}$. The choice of an extension $\widetilde{E}$ of $E$ to $\widetilde{\mathcal{W}_\delta}$-module (and of $\widetilde{\delta}$) determines an extension of $\rho_E$ to $G^{F^m}\langle\sigma\rangle$, which we denote by $\widetilde{\rho}_{\widetilde{E}}$. We consider the Shintani descent of $\widetilde{\rho}_{\widetilde{E}}$. Then by Theorem 3.4, one can write $Sh_{F^m/F}(\widetilde{\rho}_{\widetilde{E}}|_{G^{F^m}\sigma}) = \mu_{\widetilde{E}}R_E$, where $R_E$ is a certain almost character of $G^F$, and $\mu_{\widetilde{E}}$ is a root of unity depending on the choice of $\widetilde{E}$. Similarly, for each $y \in W_\delta$, $\widetilde{\delta}$ is a character of $L^{F^m}\langle\sigma\dot{w}_\delta\dot{y}\rangle$. Hence the Shintani descent of $\widetilde{\delta}$ can be written as

$$Sh_{F^m/F\dot{w}_\delta\dot{y}}(\widetilde{\delta}|_{L^{F^m}\sigma\dot{w}_\delta\dot{y}}) = \mu_{\widetilde{\delta},y}R_{\delta,y}$$

where $R_{\delta,y}$ is the almost character of $L^{F\dot{w}_\delta\dot{y}}$, and $\mu_{\widetilde{\delta},y}$ is a root of unity depending on the choice of $\widetilde{\delta}$ and on $y$.

Now the twisted induction $R^G_{L(\dot{w})} : C(L^{F\dot{w}}/\sim) \to C(G^F/\sim)$ is defined as in [S2, 3.1]. By using the specialization argument of the Shintani descent identity (see [S2, Remark 4.13]), we obtain the following.

**Proposition 4.10.** *For each $w = w_\delta y \in \mathcal{Z}_\delta$, we have*

$$R^G_{L(\dot{w})}(\mu_{\widetilde{\delta},y}R_{\delta,y}) = \sum_{E \in (\mathcal{W}_\delta^\wedge)^{\gamma_\delta}} \operatorname{Tr}(\gamma_\delta y, \widetilde{E})\mu_{\widetilde{E}}R_E.$$

**Remark 4.11.** The formula in [S2, Remark 4.13] contains a linear character $\varepsilon : \mathcal{W}_\delta \to \{\pm 1\}, y \mapsto \varepsilon_y$ which is trivial on $\Omega_\delta$. However, we have $\varepsilon = 1$ in our case. In fact, since $\varepsilon$ is a character of $\mathcal{W}_\delta^0$, $\varepsilon$ is determined by the corresponding formula for $R^{\widetilde{G}}_{\widetilde{L}(w)}$ with $y \in \mathcal{W}_\delta^0$. In that case, the formula is nothing but the decomposition of the

Deligne-Lusztig character $R^{\widetilde G}_{\widetilde T_x}(\theta)$ into irreducible characters for some $\theta \in (\widetilde T_x^F)^\wedge$, and the assertion is verified by using the explicit description in 1.4.

## 5. Unipotently supported functions

**5.1.** Let $G_{\mathrm{uni}}$ be the unipotent variety of $G$. Let $\mathcal{I}_G$ be the set of all pairs $(C, \mathcal{E})$ where $C$ is a unipotent class in $G$ and $\mathcal{E}$ is an irreducible $G$-equivariant local system on $C$. If we fix $u \in C$, the set of $G$-equivariant local systems on $C$ is in bijection with $A_G(u)^\wedge$. Thus the pair $(C, \mathcal{E})$ is represented by the pair $(u, \tau)$ for $\tau \in A_G(u)^\wedge$. Let $\mathcal{M}_G$ be the set of triples $(L, C_0, \mathcal{E}_0)$, up to $G$-conjugacy, where $L$ is a Levi subgroup of some parabolic subgroup $P$ of $G$, and $\mathcal{E}_0$ is a cuspidal local system on a unipotent class $C_0$ in $L$. It is known by Lusztig [L2, 6.5] that there exists a natural bijection

$$\mathcal{I}_G \simeq \coprod_{(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G} (N_G(L)/L)^\wedge, \tag{5.1.1}$$

which is called the generalized Springer correspondence between unipotent classes in $G$ and irreducible characters of various Coxeter groups. (Note that $N_G(L)/L$ is a Coxeter group for any $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G$.) The set $\mathcal{M}_G$ gives a partition of $\mathcal{I}_G$. A subset of $\mathcal{I}_G$ corresponding to some triple $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G$ is called a block. The correspondence in (5.1.1) is given more precisely as follows. For each triple $(L, C_0, \mathcal{E}_0)$, one can associate a semisimple perverse sheaf $K$ on $G$ such that $\operatorname{End} K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}]$ with $\mathcal{W} = N_G(L)/L$. Let $K_E$ be the simple component of $K$ corresponding to $E \in \mathcal{W}$. Then

$$K_E|_{G_{\mathrm{uni}}} = \mathrm{IC}(\overline{C}, \mathcal{E})[\dim C + \dim Z_L^0] \tag{5.1.2}$$

for some pair $(C, \mathcal{E}) \in \mathcal{I}_G$. The correspondence $(C, \mathcal{E}) \leftrightarrow E$ gives the required bijection.

Now $F$ acts naturally on $\mathcal{I}_G$ and $\mathcal{M}_G$ by $(C, \mathcal{E}) \mapsto (F^{-1}(C), F^*\mathcal{E}), (L, C_0, \mathcal{E}_0) \mapsto (F^{-1}(L), F^{-1}(C_0), F^*\mathcal{E}_0)$. Let $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G^F$ and $\mathcal{I}_0$ the block corresponding to it. Then one can choose $L$ an $F$-stable Levi subgroup of an $F$-stable parabolic subgroup $P$ of $G$. In that case, $C_0$ is an $F$-stable unipotent class and $\mathcal{E}_0$ is an $F$-stable local system. Then $F$ acts on $\mathcal{W}$, and we consider the semidirect product $\widetilde{\mathcal{W}} = \mathcal{W}\langle c \rangle$ , where $c$ is the automorphism on $\mathcal{W}$ induced by $F$. For each $\iota = (C, \mathcal{E}) \in \mathcal{I}_0$, we put $K_\iota = K_E$ if $\iota = (C, \mathcal{E})$ corresponds to $E \in \mathcal{W}^\wedge$ under (5.1.1). Then $K_\iota$ is $F$-stable if and only if $E$ is $F$-stable. We choose an isomorphism $\phi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$ so that it induces a map of finite order at the stalk of each point in $C_0^F$. Then it induces an isomorphism $F^*K \xrightarrow{\sim} K$, and by choosing a preferred extension of $E$ to $\widetilde{\mathcal{W}}$, induces an isomorphism $\phi_E : F^*K_E \xrightarrow{\sim} K_E$. Since $\mathcal{H}^{a_0}(K_E)|_C = \mathcal{E}$ with $a_0 = -\dim C - \dim Z_L^0$, $\phi_E$ induces an isomorphism $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. We define $\psi_\iota : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ by the condition that $q^{(a_0+r)/2}\psi_\iota$ coincides with the map $\phi_E : F^*\mathcal{H}^{a_0}(K_E) \xrightarrow{\sim} \mathcal{H}^{a_0}(K_E)$, where $r = \dim \operatorname{supp} K_E$. Note that we have

$$a_0 + r = (\dim G - \dim C) - (\dim L - \dim C_0).$$

Then by [L3, 24.2], $\psi_\iota$ induces a map of finite order at the stalk of each point in $C^F$.

**5.2.** Let $\mathcal{V}_G = C(G^F/\sim)$ be the space of $G^F$-invariant functions on $G^F$, and $\mathcal{V}_{\mathrm{uni}}$ the subspace of $\mathcal{V}_G$ consisting of functions whose supports lie in $G^F_{\mathrm{uni}}$. For each pair

$\iota = (C, \mathcal{E}) \in \mathcal{I}_G$, we define $\mathcal{Y}_\iota \in \mathcal{V}_{\text{uni}}^F$ by

$$\mathcal{Y}_\iota(v) = \begin{cases} \operatorname{Tr}(\psi_\iota, \mathcal{E}_v) & \text{if } v \in C^F \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{E}_v$ is the stalk of $\mathcal{E}$ at $v$. Then $\{\mathcal{Y}_\iota \mid \iota \in \mathcal{I}_G\}$ gives rise to a basis of $\mathcal{V}_{\text{uni}}$. We have a natural decomposition

$$\mathcal{V}_{\text{uni}} = \bigoplus_{\mathcal{I}_0} \mathcal{V}_{\mathcal{I}_0}, \tag{5.2.1}$$

where $\mathcal{I}_0$ runs over all the $F$-sable blocks, and $\mathcal{V}_{\mathcal{I}_0}$ is the subspace of $\mathcal{V}_{\text{uni}}$ spanned by $\mathcal{Y}_\iota$ for $\iota \in \mathcal{I}_0$.

Let $\mathcal{I}_0$ be an $F$-stable block associated to the triple $(L, C_0, \mathcal{E}_0)$. We assume that $L$ is an $F$-stable Levi subgroup of an $F$-stable parabolic subgroup of $G$. We denote by $L_w$ an $F$-stable Levi subgroup twisted by $w \in \mathcal{W} = N_G(L)/L$. For a pair $\iota = (C, \mathcal{E}) \in \mathcal{I}_G$, we put $\operatorname{supp}(\iota) = C$. For each $\iota, \iota' \in \mathcal{I}_0^F$, put

$$\begin{aligned} \omega_{\iota,\iota'} = &|\mathcal{W}|^{-1} q^{-(\operatorname{codim} C + \operatorname{codim} C')/2 + \dim Z_L^0} \\ &\times |G^F| \sum_{w \in \mathcal{W}} |Z_{L_w}^{0F}|^{-1} \operatorname{Tr}(w, E_\iota) \operatorname{Tr}(w, E_{\iota'}), \end{aligned} \tag{5.2.2}$$

where $C = \operatorname{supp}(\iota), C' = \operatorname{supp}(\iota')$, and $E_\iota, E_{\iota'} \in \mathcal{W}^\wedge$ are the ones corresponding to $\iota, \iota'$ via the generalized Springer correspondence. If $\iota, \iota' \in \mathcal{I}_G^F$ are not in the same block, we put $\omega_{\iota,\iota'} = 0$.

For $K_\iota = K_E$, put $\phi_\iota = \phi_E$. We define $\mathcal{X}_\iota \in \mathcal{V}_{\text{uni}}$ by

$$\mathcal{X}_\iota(g) = \sum_a (-1)^{a+a_0} \operatorname{Tr}(\phi_\iota, \mathcal{H}_g^a(K_\iota)) q^{-(a_0+r)/2} \qquad (g \in G_{\text{uni}}^F).$$

We define an equivalence relation $\sim$ in $\mathcal{I}_G$ by $\iota \sim \iota'$ if $\operatorname{supp} \iota = \operatorname{supp} \iota'$. Also we define a partial order on $\mathcal{I}_G$ by $\iota \le \iota'$ if $\overline{\operatorname{supp} \iota} \subseteq \overline{\operatorname{supp} \iota'}$. Assume that $\iota \in \mathcal{I}_0^F$. Then it is known that $\mathcal{X}_\iota$ can be written as

$$\mathcal{X}_\iota = \sum_{\iota' \in \mathcal{I}_0} P_{\iota',\iota} \mathcal{Y}_{\iota'}, \tag{5.2.3}$$

where $P_{\iota',\iota} \in \mathbf{Z}$. Actually there exists a polynomial $\mathbf{P}_{\iota',\iota}(t) \in \mathbf{Z}[t]$ such that $P_{\iota',\iota} = \mathbf{P}_{\iota',\iota}(q)$. Moreover, $P_{\iota',\iota} = 0$ if $\iota' \not\le \iota$ or if $\iota' \sim \iota, \iota' \ne \iota$. $P_{\iota,\iota} = 1$. In particular, $\{\mathcal{X}_\iota \mid \iota \in \mathcal{I}_0^F\}$ gives rise to a basis of $\mathcal{V}_{\mathcal{I}_0}$. Moreover, we have

$$\langle \mathcal{X}_\iota, \mathcal{X}_{\iota'} \rangle_{G^F} = |G^F|^{-1} \omega_{\iota,\iota'}. \tag{5.2.4}$$

**5.3.** Take $(L, C_0, \mathcal{E}_0) \in \mathcal{M}^F$. In our case (i.e., $G$ is given as in 1.1), $C_0$ is the regular unipotent class in $L_0$. We choose $u_0 \in C_0^F$ as Jordan's normal form, and define $\phi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$ by the condition that it induces the identity map on the stalk at $u_0$.

Let $\mathcal{I}_0$ be the block corresponding to $(L, C_0, \mathcal{E}_0)$. For $\iota = (C, \mathcal{E}) \in \mathcal{I}_0^F$, we fix $u_1 \in C^F$ in Jordan's normal form. $A_G(u_1)$ is abelian, on which $F$ acts naturally. The set of $G^F$-conjugacy classes in $C^F$ is in bijective correspondence with the group $A_G(u_1)_F$. We denote by $u_a$ a representative of the $G^F$-class in $C^F$ corresponding to $a \in A_G(u_1)_F$. Assume that $\mathcal{E}$ corresponds to an $F$-stable irreducible character $\tau \in A_G(u_1)^\wedge$. We define a function $\chi_{u_1,\tau} \in \mathcal{V}_{\mathrm{uni}}$ by

$$\chi_{u_1,\tau}(g) = \begin{cases} \tau(u_a) & \text{if } g \sim_{G^F} u_a, \\ 0 & \text{if } g \notin C^F. \end{cases}$$

The following result determines the function $\mathcal{Y}_\iota$ explicitly.

**Proposition 5.4** ([S3])**.** *Assume that $\iota = (C, \mathcal{E})$ is represented by $(u_1, \tau)$ as above. Then we have $\mathcal{Y}_\iota = \chi_{u_1,\tau}$.*

**5.5.** By making use of the map $\log : G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}$ (see 2.1), we identify $\mathcal{V}_{\mathrm{uni}}$ the space of $G^F$-invariant functions $\mathfrak{g}_{\mathrm{nil}}^F$. Then the function $\mathcal{Y}_\iota$ can be regarded as a function on $\mathcal{O}^F$, where $\mathcal{O}$ is the nilpotent orbit corresponding to $C$ such that $C = \operatorname{supp}(\iota)$. For each $F$-stable nilpotent orbit $\mathcal{O}$, we choose a representative $N \in \mathcal{O}^F$ via Jordan's normal form corresponding to $u_1 \in C^F$. Let $\{N, N^*, H\}$ be the TDS-triple. Then the associated parabolic subgroup $P_N$ and its Levi subgroup $L_N$ are defined, and we have the group $A_\lambda = Z_{L_N}(\lambda)/Z^0_{L_N}(\lambda)$ as in 2.1. For $c \in (A_\lambda)_F$, we consider the twisted element $N_c$. Then the generalized Gelfand-Graev character $\mathit{\Gamma}_c$ associated to $N_c$ is defined as in 2.1, which gives an element of $\mathcal{V}_{\mathrm{uni}}$. Now Lusztig gave a formula expressing $\mathit{\Gamma}_c$ in terms of the linear combination of $\mathcal{X}_\iota$ as follows.

**Theorem 5.6** ([L7, Theorem 7.3])**.** *Let $\mathcal{I}_0$ be an $F$-stable block corresponding to $(L, C_0, \mathcal{E}_0)$. Let $(\mathit{\Gamma}_c)_{\mathcal{I}_0}$ be the projection of $\mathit{\Gamma}_c$ onto the subspace $\mathcal{V}_{\mathcal{I}_0}$ in (5.2.1). Then*

$$\begin{aligned}(\mathit{\Gamma}_c)_{\mathcal{I}_0} = \sum_{\iota,\iota',\iota_1 \in \mathcal{I}_0} & q^{f(\iota,\iota_1)} \zeta_{\mathcal{I}_0}^{-1} |\mathcal{W}|^{-1} \sum_{w \in \mathcal{W}} \operatorname{Tr}(w, E_\iota) \operatorname{Tr}(w, E_{\iota_1} \otimes \varepsilon) \\ & \times |Z^{0F}_{L_w}| \mathbf{P}_{\iota',\iota}(q^{-1}) \overline{\mathcal{Y}_{\iota'}(-N_c^*)} \mathcal{X}_{\iota_1},\end{aligned} \tag{5.6.1}$$

*where*

$$\begin{aligned} f(\iota, \iota_1) = & -\dim \operatorname{supp}(\iota_1)/2 + \dim \operatorname{supp}(\iota)/2 \\ & - \dim \mathcal{O}_N/2 + \dim (G/Z_L^0)/2, \end{aligned}$$

*and $\zeta_{\mathcal{I}_0}$ is a fourth root of unity attached to the block $\mathcal{I}_0$. $\varepsilon$ is the sign representation of $\mathcal{W}$ (cf. [L7, 5.5]).*

**Remark 5.7.** The restriction of the Fourier transform of $\mathcal{X}_\iota$ ($\iota \in \mathcal{I}_0$) on $\mathfrak{g}_{\mathrm{nil}}$ coincides with $\mathcal{X}_\iota$ up to scalar. The fourth root of unity $\zeta_{\mathcal{I}_0}$ occurs in the description of this scalar ([L7, Proposition 7.2]). $\zeta_{\mathcal{I}_0}$ depends only on the pair $(C_0, \mathcal{E}_0)$ and does not depend on $G$. In our case, $\mathcal{I}_0$ is always a regular block, i.e., $C_0$ is the regular unipotent class in $L$. In such a case, Digne, Lehrer and Michel [DLM1, Proposition 2.8] determined the value $\zeta_{\mathcal{I}_0}$ explicitly.

**5.8.** In order to apply the formula (5.6.1), we need to describe $-N_c^*$ for a nilpotent element $N \in \mathfrak{g}^F$. Since $-N^* \in \mathfrak{g}^F$ is $G$-conjugate to $N$, one can write $-N^* = N_{c_0}$ for some $c_0 \in (A_\lambda)_F$, i.e., $N_{c_0} = \operatorname{Ad}(\alpha_{c_0})N$ with $\alpha_{c_0}^{-1}F(\alpha_{c_0}) = \dot{c}_0$. We consider $N_c$ for $c \in (A_\lambda)_F$. Then $N_c = \operatorname{Ad}(\alpha_c)N$ with $\alpha_c^{-1}F(\alpha_c) = \dot{c}$. Since $Z_G \to A_\lambda$ is surjective, we may choose $\dot{c} \in Z_G$. Now $-N_c^*$ is obtained as $-N_c^* = \operatorname{Ad}(\alpha_c)(-N^*)$. Hence $-N_c^*$ is $G^F$-conjugate to $\operatorname{Ad}(\alpha_c\alpha_{c_0})N$. But since $\dot{c} \in Z_G$, we have

$$(\alpha_c\alpha_{c_0})^{-1}F(\alpha_c\alpha_{c_0}) = \alpha_{c_0}^{-1}\dot{c}F(\alpha_{c_0}) = \dot{c}\dot{c}_0,$$

It follows that $-N_c^*$ is $G^F$-conjugate to $N_{cc_0}$.

**5.9.** Following [L2, LS], we describe the generalized Springer correspondence for $G$ explicitly. Let $n'$ be the largest common divisor of $n_1, \dots, n_r$ which is prime to $p$. Then $Z_G$ is the cyclic group of order $n'$. Let $u = u_\mu$ be a unipotent element in $G$ corresponding to $\mu = (\mu_1, \dots, \mu_r)$, where $\mu_i = (\mu_{i1} \ge \mu_{i2} \ge \cdots)$ is a partition of $n_i$. Put $n'_\mu$ be the greatest common divisor of $n'$ and $\{\mu_{ij}\}$. Then $A_G(u)$ is a cyclic group of order $n'_\mu$. For each $\tau \in A_G(u)^\wedge$, $Z_G/Z_G^0$ acts on the representation space $V_\tau$ of $\tau$ via the homomorphism $Z_G/Z_G^0 \to A_G(u)$. For each $\eta \in (Z_G/Z_G^0)^\wedge$, we denote by $A_G(u)^\wedge_\eta$ the set of irreducible characters $\tau \in A_G(u)^\wedge$ such that $Z_G/Z_G^0$ acts on $V_\tau$ via the character $\eta$. We have

$$|A_G(u)^\wedge_\eta| = \begin{cases} 1 & \text{if } d|n'_\mu, \\ 0 & \text{otherwise,} \end{cases} \tag{5.9.1}$$

where $d$ is the order of $\eta$. Now the generalized Springer correspondence in (5.1.1) is described as follows: We have a partition

$$\mathcal{I}_G = \coprod_{\eta \in (Z_G/Z_G^0)^\wedge} (\mathcal{I}_G)_\eta,$$

where $(\mathcal{I}_G)_\eta$ is the set of pairs $(u, \tau)$ with $\tau \in A_G(u)^\wedge_\eta$. Note that $\tau$ is uniquely determined by $u$ if $(u, \tau) \in (\mathcal{I}_G)_\eta$ by (5.9.1), which we denote by $\tau(u)$. For each $\eta \in (Z_G/Z_G^0)^\wedge$ of order $d$, there exists a unique Levi subgroup up to conjugacy such that the type of $L$ is $A_{d-1} + \cdots + A_{d-1}$, and a unique cuspidal pair $(L, C_0, \mathcal{E}_0)$. Here $C_0$ is regular unipotent in $L$ and for $u_0 \in C_0$, $A_L(u_0) \simeq Z_L/Z_L^0$. $\mathcal{E}_0$ is the unique local system on $C_0$ corresponding to $\eta_0 \in (Z_L/Z_L^0)^\wedge$ such that $\eta_0 \circ f = \eta$ for a natural homomorphism $f : Z_G/Z_G^0 \to Z_L/Z_L^0$. Then $N_G(L)/L \simeq \mathfrak{S}_{n_1/d} \times \cdots \times \mathfrak{S}_{n_r/d}$, and the map $E_\mu \mapsto (u_{d\mu}, \tau(u_{d\mu}))$ ($d\mu = (d\mu_{ij})$ for $\mu = (\mu_{ij})$) gives the generalized Springer correspondence

$$(N_G(L)/L)^\wedge \simeq (\mathcal{I}_G)_\eta. \tag{5.9.2}$$

**5.10.** Assume that $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$ with $n_1 = \cdots = n_r = t$. In this case, $n'$ is the largest divisor of $t$ which is prime to $p$. From the description of the generalized Springer correspondence, the partition of $\mathcal{I}_G$ into blocks is nothing but the partition of $\mathcal{I}_G$ into $(\mathcal{I}_G)_\eta$. Assume that $\mathcal{I}_0 = (\mathcal{I}_G)_\eta$ with $\eta \in (Z_G/Z_G^0)^\wedge$ of order $d$. We shall make the formula (5.6.1) more explicit in the case where $N$ is regular nilpotent, i.e., where $\varGamma_c$ is the modified Gelfand-Graev characters.

**Lemma 5.11.** *Let $G$ be as before. Assume that $N$ is regular nilpotent. Then for any $\iota \in \mathcal{I}_0 = (\mathcal{I}_G)_\eta$,*

$$\langle (\varGamma_c)_{\mathcal{I}_0}, \mathcal{X}_\iota \rangle_{G^F} = \begin{cases} q^{(\dim Z_L^0 - \operatorname{codim} \operatorname{supp}(\iota))/2} \zeta_{\mathcal{I}_0}^{-1} \eta(cc_0)^{-1} & \text{if } E_\iota = \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We apply the formula (5.6.1). Since $-N_c^*$ is regular nilpotent, the non-zero contribution of $\mathcal{Y}_{\iota'}(-N_c^*)$ occurs only when $\operatorname{supp}(\iota') = \mathcal{O}_N$, the regular nilpotent orbit. By 5.8, $-N_c^*$ is $G^F$-conjugate to $N_{cc_0}$. Since $A_G(N) \simeq Z_G/Z_G^0$, we see that $\iota' = (N, \eta)$ under the identification $\eta \in A_G(N)^\wedge$. This implies that $\mathcal{Y}_{\iota'}(-N_c^*) = \eta(cc_0)$ by Proposition 5.4. On the other hand, since $\operatorname{supp}(\iota') = \mathcal{O}_N$, $P_{\iota',\iota} = 0$ unless $\iota = \iota'$, and we have $P_{\iota',\iota'} = 1$ by the property in 5.2. Moreover, under the generalized Springer correspondence, $E_{\iota'}$ is the identity character of $\mathcal{W}$. Thus (5.6.1) can be written as

$$(\varGamma_c)_{\mathcal{I}_0} = \eta(cc_0)^{-1} \sum_{\iota_1 \in \mathcal{I}_0} q^{f(\iota',\iota_1)} \zeta_{\mathcal{I}_0}^{-1} |\mathcal{W}|^{-1} \sum_{w \in \mathcal{W}} |Z_{L_w}^{0F}| \operatorname{Tr}(w, E_{\iota_1} \otimes \varepsilon) \mathcal{X}_{\iota_1}.$$

Now by (5.2.2) and (5.2.4), we have

$$\begin{aligned} \langle (\varGamma_c)_{\mathcal{I}_0}, \mathcal{X}_\iota \rangle_{G^F} &= \zeta_{\mathcal{I}_0}^{-1} \eta(cc_0)^{-1} |\mathcal{W}|^{-2} \sum_{\iota_1 \in \mathcal{I}_0} q^{(\dim Z_L^0 - \operatorname{codim} \operatorname{supp}(\iota))/2} \\ &\quad \times \sum_{w, w' \in \mathcal{W}} |Z_{L_w}^{0F}| |Z_{L_{w'}}^{0F}|^{-1} \varepsilon(w) \operatorname{Tr}(w, E_{\iota_1}) \operatorname{Tr}(w', E_{\iota_1}) \operatorname{Tr}(w', E_\iota) \\ &= q^{(\dim Z_L^0 - \operatorname{codim} \operatorname{supp}(\iota))/2} \zeta_{\mathcal{I}_0}^{-1} \eta(cc_0)^{-1} |\mathcal{W}|^{-1} \sum_{w \in \mathcal{W}} \varepsilon(w) \operatorname{Tr}(w, E_\iota). \end{aligned}$$

The lemma follows from this. □

For later applications, we also consider the general case where $N$ is arbitrary.

**Lemma 5.12.** *Assume that $N$ is an arbitrary nilpotent element.*

(i) *For any $\iota \in \mathcal{I}_0^F$, we have*

$$\langle \varGamma_c, \mathcal{X}_\iota \rangle_{G^F} = q^{g(\iota',\iota)} \zeta_{\mathcal{I}_0}^{-1} \mathbf{P}_{\iota'',\iota'}(q^{-1}) \overline{\mathcal{Y}_{\iota''}(-N_c^*)},$$

*where*

$$g(\iota', \iota) = (-\operatorname{codim}_G \operatorname{supp}(\iota') + \dim Z_L)/2 + (\dim \operatorname{supp}(\iota) - \dim \mathcal{O}_N)/2$$

*and $\iota' \in \mathcal{I}_0$ is such that $E_\iota = E_{\iota'} \otimes \varepsilon$, $\iota''$ is the unique element in $\mathcal{I}_0$ such that* $\operatorname{supp}(\iota'') = \mathcal{O}_N$.

(ii) *$t^{(\dim \operatorname{supp}(\iota') - \dim \operatorname{supp}(\iota''))/2} \mathbf{P}_{\iota'',\iota'}(t^{-1})$ is a polynomial in $t$. It is divisible by $t$ if $\iota'' \neq \iota'$.*

(iii) *There exists the ring of integers $\mathcal{A}$ of some fixed cyclotomic field independent of the field $\mathbf{F}_q$ such that*

$$\langle \varGamma_c, \mathcal{X}_\iota\rangle_{G^F} \in q^{(\dim Z_L - \operatorname{codim}_G \operatorname{supp}(\iota))/2}\mathcal{A}.$$

*Proof.* First we show (i). By applying Theorem 5.6, we have

$$\begin{aligned}\langle \varGamma_c, \mathcal{X}_\iota\rangle_{G^F} =& \langle (\varGamma_c)_{\mathcal{I}_0}, \mathcal{X}_\iota\rangle_{G^F} = \sum_{\iota',\iota'',\iota_1} q^{f(\iota',\iota_1)}\zeta_{\mathcal{I}_0}^{-1}|\mathcal{W}|^{-1}\times \\ &\times \sum_{w\in\mathcal{W}} \operatorname{Tr}(w, E_{\iota'})\operatorname{Tr}(w, E_{\iota_1})\varepsilon(w)|Z_{L^w}^{0F}|\mathbf{P}_{\iota'',\iota'}(q^{-1})\overline{\mathcal{Y}_{\iota''}(-N_c^*)}\langle \mathcal{X}_{\iota_1}, \mathcal{X}_\iota\rangle_{G^F}.\end{aligned}$$

Substituting (5.2.4) into $\langle \mathcal{X}_{\iota_1}, \mathcal{X}_\iota\rangle_{G^F}$, we have

$$\begin{aligned}\langle \varGamma_c, \mathcal{X}_\iota\rangle_{G^F} &= \sum_{\iota',\iota''} q^{g(\iota',\iota)}\zeta_{\mathcal{I}_0}^{-1}|\mathcal{W}|^{-2} \\ &\qquad \times \sum_{w,w'\in\mathcal{W}} |Z_{L^w}^{0F}||Z_{L^{w'}}^{0F}|^{-1}\biggl(\sum_{\iota_1}\operatorname{Tr}(w, E_{\iota_1})\operatorname{Tr}(w', E_{\iota_1})\biggr) \\ &\qquad \times \varepsilon(w)\operatorname{Tr}(w, E_{\iota'})\operatorname{Tr}(w', E_\iota)\mathbf{P}_{\iota'',\iota'}(q^{-1})\overline{\mathcal{Y}_{\iota''}(-N_c^*)} \\ &= \sum_{\iota',\iota''} q^{g(\iota',\iota)}\zeta_{\mathcal{I}_0}^{-1}\biggl(|\mathcal{W}|^{-1}\sum_{w\in\mathcal{W}}\operatorname{Tr}(w, E_{\iota'}\otimes\varepsilon)\operatorname{Tr}(w, E_\iota)\biggr) \\ &\qquad \times \mathbf{P}_{\iota'',\iota'}(q^{-1})\overline{\mathcal{Y}_{\iota''}(-N_c^*)}\end{aligned}$$

with $g(\iota',\iota)$ as in (i). Hence in the sum, the only $\iota'$ such that $E_\iota = E_{\iota'}\otimes\varepsilon$ gives the contribution. On the other hand, the condition $\mathcal{Y}_{\iota''}(-N_c^*) \neq 0$ implies that $\operatorname{supp}(\iota'') = \mathcal{O}_N$. It follows that

$$\langle \varGamma_c, \mathcal{X}_\iota\rangle_{G^F} = q^{g(\iota',\iota)}\zeta_{\mathcal{I}_0}^{-1}\mathbf{P}_{\iota'',\iota'}(q^{-1})\overline{\mathcal{Y}_{\iota''}(-N_c^*)},$$

where $\iota''$ is the unique elements in $(\mathcal{I}_0)^F$ such that $\operatorname{supp}(\iota'') = \mathcal{O}_N$. This proves the first statement.

Next we show (ii). We may assume that $G = SL_n$. By the generalized Springer correspondence $(\mathcal{I}_G)_\eta \leftrightarrow \mathfrak{S}_{n/d}$, we have $\operatorname{supp}(\iota') = \mathcal{O}_\lambda$ and $\operatorname{supp}(\iota'') = \mathcal{O}_\mu$, where $\lambda, \mu$ are partitions of $n$ such that each part is divisible by $d$. Put $\iota' = \iota_\lambda, \iota'' = \iota_\mu$. We denote by $\lambda/d, \mu/d$ the partitions of $n/d$ obtained from $\lambda, \mu$ by dividing each part by $d$. We consider the group $GL_{n/d}$ and nilpotent orbits $\mathcal{O}_{\lambda/d}, \mathcal{O}_{\mu/d}$ of $\mathfrak{gl}_{n/d}$. We have a polynomial $\mathbf{P}_{\iota_{\mu/d},\iota_{\lambda/d}}$ associated to $GL_{n/d}$, defined in a similar way as $\mathbf{P}_{\iota_\mu,\iota_\lambda}$, where $\iota_{\mu/d}, \iota_{\lambda/d}$ are elements in $\mathcal{I}_{GL_{n/d}}$ corresponding to $\mathcal{O}_{\lambda/d}, \mathcal{O}_{\mu/d}$ under the Springer correspondence. Then it is known by [DLM2, Theorem 8.1] that

$$t^{(\dim\mathcal{O}_\mu - \dim\mathcal{O}_\lambda)/2}\mathbf{P}_{\iota_\mu,\iota_\lambda}(t) = t^{(\dim\mathcal{O}_{\mu/d} - \dim\mathcal{O}_{\lambda/d})/2}\mathbf{P}_{\iota_{\mu/d},\iota_{\lambda/d}}(t).$$

It follows that

$$t^{(\dim \mathcal{O}_\lambda - \dim \mathcal{O}_\mu)/2} \mathbf{P}_{\iota_\mu, \iota_\lambda}(t^{-1}) = t^{(\dim \mathcal{O}_{\lambda/d} - \dim \mathcal{O}_{\mu/d})/2} \mathbf{P}_{\iota_{\mu/d}, \iota_{\lambda/d}}(t^{-1}).$$

But the right hand side of this formula coincides with the Kostka polynomial $K_{\lambda/d,\mu/d}(t)$ associated to partitions $\lambda/d, \mu/d$ of $n/d$ (cf. [M, III]), hence it is a polynomial in $t$. Then the second assertion of the lemma follows from the well-known properties of Kostka polynomials.

Finally we show (iii). We may assume that $\zeta_{\mathcal{I}_0} \in \mathcal{A}$ and $\overline{\mathcal{Y}_{\iota''}(-N_c^*)} \in \mathcal{A}$. Thus (iii) follows from (i) and (ii). The lemma is proved. □

## 6. Character sheaves

**6.1.** Following [L3, IV], we review the classification of character sheaves in the case of type $A$. So let $G$ be as before, and we denote by $\widehat{G}$ the set of character sheaves on $G$. Let $\mathcal{S}(T)$ be the set of local systems of rank 1 on $T$ such that $\mathcal{L}^{\otimes m} \simeq \bar{\mathbf{Q}}_l$ for some $m$ prime to $p$. Then for each $\mathcal{L} \in \mathcal{S}(T)$, the subset $\widehat{G}_{\mathcal{L}}$ of $\widehat{G}$ is defined, and we have

$$\widehat{G} = \coprod_{\mathcal{L} \in \mathcal{S}(T)} \widehat{G}_{\mathcal{L}}.$$

For each $\mathcal{L} \in \mathcal{S}(T)$, we put $W_{\mathcal{L}} = \{w \in W \mid w^*\mathcal{L} \simeq \mathcal{L}\}$. Then there exists a subroot system $\varSigma_{\mathcal{L}}$ of $\varSigma$, and $W_{\mathcal{L}}$ can be written as $W_{\mathcal{L}} = \Omega_{\mathcal{L}} \ltimes W^0_{\mathcal{L}}$, where $W^0_{\mathcal{L}}$ is the reflection group associated to the root system $\varSigma_{\mathcal{L}}$, and $\Omega_{\mathcal{L}} = \{w \in W_{\mathcal{L}} \mid w(\varSigma^+_{\mathcal{L}}) = \varSigma^+_{\mathcal{L}}\}$. If $\mathcal{L} \in \mathcal{S}(T)$ is $F^m$-stable for some integer $m > 0$, we fix an isomorphism $\phi : (F^m)^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ so that it induces an identity map on $\mathcal{L}_1$ (the stalk at $1 \in T^{F^m}$). Then the characteristic function $\chi_{\mathcal{L},\phi}$ gives rise to a character $\theta \in (T^{F^m})^\wedge$, which induces an isomorphism between $\mathcal{S}(T)^{F^m}$ and $(T^{F^m})^\wedge$. Thus we have $\mathcal{S}(T)^{F^m} \simeq (T^*)^{F^m}$. Since this isomorphism is compatible with the extension of the filed $\mathbf{F}_{q^m}$, we can identify $\mathcal{S}(T)$ with $T^*$ in this way. Now assume that $\mathcal{L} \in \mathcal{S}(T)$ corresponds to $s \in T^*$. Then we have $W_{\mathcal{L}} = W_s, W^0_{\mathcal{L}} = W^0_s$ and $\Omega_{\mathcal{L}} = \Omega_s$ in the notation of Section 1.

In [L3, 17], families in $(W^0_{\mathcal{L}})^\wedge$ and in $W^\wedge_{\mathcal{L}}$ were introduced. In our case, $W^0_{\mathcal{L}}$ is a product of symmetric groups and a family $\mathcal{F}$ in $(W^0_{\mathcal{L}})^\wedge$ is of the form $\mathcal{F} = \{E\}$ with $E \in (W^0_{\mathcal{L}})^\wedge$. Let $\Omega_{\mathcal{L},E}$ be the stabilizer of $E$ in $\Omega_{\mathcal{L}}$. Then $E$ can be extended to a character $\widetilde{E}$ on $\Omega_{\mathcal{L},E}W^0_{\mathcal{L}}$. (We choose a canonical extension so that for each $\sigma \in \Omega_{\mathcal{L},E}$, $\widetilde{E}$ gives the preferred extension of $E$ to $\langle\sigma\rangle\, W^0_{\mathcal{L}}$). For each $\theta \in \Omega^\wedge_{\mathcal{L},E}$, put $\widetilde{E}_\theta = \operatorname{Ind}^{\Omega_{\mathcal{L}}W^0_{\mathcal{L}}}_{\Omega_{\mathcal{L},E}W^0_{\mathcal{L}}}(\theta \otimes \widetilde{E})$. Then $\widetilde{E}_\theta$ is irreducible, and the family $\mathcal{F}'$ in $W_{\mathcal{L}}$ associated to $\mathcal{F}$ consists of $\{\widetilde{E}_\theta \mid \theta \in \Omega^\wedge_{\mathcal{L},E}\}$. Put

$$\mathcal{M}_{\mathcal{L},E} = \Omega_{\mathcal{L},E} \times \Omega^\wedge_{\mathcal{L},E}.$$

We have an embedding $\mathcal{F}' \hookrightarrow \mathcal{M}_{\mathcal{L},E}$ by $\widetilde{E}_\theta \mapsto (1,\theta)$.

**6.2.** For each $\mathcal{L} \in \mathcal{S}(T)$ and $w \in W_{\mathcal{L}}$, we choose a representative $\dot{w} \in N_G(T)$. By [L3, 2.4], a complex $K^{\mathcal{L}}_{\dot{w}} \in \mathcal{D}G$ is defined. The set $\widehat{G}_{\mathcal{L}}$ is defined as the set of character sheaves $A$ such that $A$ is a consistuent of ${}^pH^i(K^{\mathcal{L}}_{\dot w})$ for some $w \in W_{\mathcal{L}}$ and some $i \in \mathbf{Z}$. Let $E \in (W^0_{\mathcal{L}})^\wedge$ and $\widetilde{E}_\theta \in W^\wedge_{\mathcal{L}}$ be as in 6.1. We define

$$R^{\mathcal{L}}_{\widetilde{E}_\theta} = |W_{\mathcal{L}}|^{-1} \sum_{y \in W_{\mathcal{L}}} \mathrm{Tr}\,(y^{-1}, \widetilde{E}_\theta) \sum_i (-1)^{i+\dim G}\, {}^pH^i(K^{\mathcal{L}}_y),$$

which is an element of the subgroup of the Grothendieck group of the perverse sheaves on $G$ spanned by the character sheaves of $G$.

By [L3, Corollary 16.7], a natural map from $\widehat{G}_{\mathcal{L}}$ to the set of two sided cells of $W_{\mathcal{L}}$ was constructed. We denote by $\widehat{G}_{\mathcal{L},\mathcal{F}'}$ the set of character sheaves $A \in \widehat{G}_{\mathcal{L}}$ such that the corresponding two sided cell coincides with $\mathcal{F}'$. In our case, the family $\mathcal{F}'$ is determined by the choice of $E \in (W^0_{\mathcal{L}})^\wedge$. Thus we write $\widehat{G}_{\mathcal{L},\mathcal{F}'}$ as $\widehat{G}_{\mathcal{L},E}$. We have the following partition of $\widehat{G}_{\mathcal{L}}$.

$$\widehat{G}_{\mathcal{L}} = \coprod_{E \in (W^0_{\mathcal{L}})^\wedge/\Omega_{\mathcal{L}}} \widehat{G}_{\mathcal{L},E}.$$

The following result gives a parametrization of $\widehat{G}$.

**Proposition 6.3** ([L3, Proposition 18.5]). *There exists a bijection $\widehat{G}_{\mathcal{L},E} \leftrightarrow \mathcal{M}_{\mathcal{L},E}, A \leftrightarrow (x_A, \theta_A)$ satisfying the following property: for any $\theta \in \Omega^\wedge_{\mathcal{L},E}$,*

$$(A : R^{\mathcal{L}}_{\widetilde{E}_\theta}) = |\Omega_{\mathcal{L},E}|^{-1}\varepsilon_A \theta(x_A)^{-1},$$

*where $\varepsilon_A = (-1)^{l(x_A)}$. ($l$ is the restriction of the length function of $W$ to $\Omega_{\mathcal{L},E}$).*

**6.4.** Let $\widehat{G}_0$ be the set of cuspidal character sheaves on $G$. We shall describe the set $\widehat{G}_0$ explicitly. By Lemma 18.4 and by the proof of Proposition 18.5 in [L3], $\widehat{G}_{\mathcal{L}}$ contains a cuspidal character sheaf if and only if $W^0_{\mathcal{L}} = \{1\}$ and $\Omega_{\mathcal{L}} = W_{\mathcal{L}}$ is a cyclic group generated by a Coxeter element in $W$, which is isomorphic to $Z_G/Z^0_G$. They are indexed by the pair $(x, z)$ where $x$ is a generator of $\Omega_{\mathcal{L}}$ and $z$ is a representative of $Z_G/Z^0_G$. In particular, $\widetilde{G}$ is of the form $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$, where $\widetilde{G}_i \simeq GL_t$ with $t = n/r$. Also we note that the character sheaf $A_{x,z}$ corresponding to the pair $(x, z)$ has its support in $zZ^0_G \times G_{\mathrm{uni}}$. Under the parametrization in Proposition 6.3, this is also given in the following form.

(6.4.1) Let $\widetilde{G}$ be as above. Assume that $W^0_{\mathcal{L}} = \{1\}$ and that $\Omega_{\mathcal{L}}$ is a cyclic group generated by a Coxeter element, which is isomorphic to $Z_G/Z^0_G$. Hence $\mathcal{M}_{\mathcal{L},E} = \Omega_{\mathcal{L}} \times \Omega^\wedge_{\mathcal{L}}$. Let $(\widehat{G}_{\mathcal{L}})_0$ be the set of cuspidal character sheaves in $\widehat{G}_{\mathcal{L}}$. Then we have

$$(\widehat{G}_{\mathcal{L}})_0 = \{A_{x,\theta} \mid x \in \Omega_{\mathcal{L},0}, \theta \in \Omega^\wedge_{\mathcal{L}}\},$$

where $\Omega_{\mathcal{L},0}$ is the set of $x \in \Omega_{\mathcal{L}}$ such that $x$ is a generator of $\Omega_{\mathcal{L}}$.

**6.5.** The set $(\widehat{G}_{\mathcal{L}})_0$ is also given in terms of intersection cohomology complexes as follows. Let $C$ be the regular unipotent class in $G$, and we choose a representative $u_1 \in G^F$. For each $z \in Z_G/Z^0_G$, we choose a representative $\dot z \in Z_G$. Then $\dot zC$ is a conjugacy class of $G$ containing $\dot zu_1$ and the component group $A_G(\dot zu_1)$ coincides with $A_G(u_1) \simeq Z_G/Z^0_G$. We denote by $\mathcal{E}_\eta$ the irreducible local system on $\dot zC$ corresponding to $\eta \in A_G(\dot z u_1)^\wedge$. Put $\varSigma = zZ_G^0C = Z_G^0 \times \dot zC$, and consider a local system $\bar{\mathbf{Q}}_l \boxtimes \mathcal{E}_\eta$ on $\varSigma$ associated to $(z, \eta) \in Z_G/Z_G^0 \times A_G(\dot z u_1)^\wedge$. We define a perverse sheaf $A_{z,\eta}$ on $\varSigma$ by

$$A_{z,\eta} = \mathrm{IC}(\overline{\varSigma}, \bar{\mathbf{Q}}_l \boxtimes \mathcal{E}_\eta)[\dim \varSigma] = \bar{\mathbf{Q}}_l \boxtimes \mathrm{IC}(\overline{\dot zC}, \mathcal{E}_\eta)[\dim Z_G^0 + \dim C]. \tag{6.5.1}$$

Then $A_{z,\eta} \in \widehat{G}$. Let $\mathcal{E}$ be a local system of $G$ of rank 1 obtained as the inverse image under the map $G \to G/G_{\mathrm{der}}$ of a local system $\mathcal{E}' \in \mathcal{S}(G/G_{\mathrm{der}})$. We have $\mathcal{E} \otimes A_{z,\eta} \in \widehat{G}$. Let $A_G(\dot z u_1)_0^\wedge$ be the subset of $A_G(\dot z u_1)^\wedge$ consisting of faithful characters of $A_G(\dot z u_1)$. Then we have

$$\widehat{G}_0 = \{\mathcal{E} \otimes A_{z,\eta} \mid z \in Z_G/Z_G^0, \eta \in A_G(\dot z u_1)_0^\wedge, \mathcal{E}' \in \mathcal{S}(G/G_{\mathrm{der}})\}. \tag{6.5.2}$$

We now consider the $\mathbf{F}_q$-structure of $G$, and let $\widehat{G}^F$ (resp. $\widehat{G}_0^F$) be the set of $F$-stable character sheaves (resp. $F$-stable cuspidal character sheaves ) of $G$. The regular unipotent class $C$ is $F$-stable, and we choose $u_1 \in C^F$ such that $u_1$ is given by Jordan's normal form. Also we can choose a representative $\dot z \in Z_G^F$. Then $\widehat{G}_0^F$ is given as

$$\widehat{G}_0^F = \{\mathcal{E} \otimes A_{z,\eta} \mid z \in (Z_G/Z_G^0)^F, \eta \in (A_G(\dot z u_1)_0^\wedge)^F, \mathcal{E}|_{\dot z Z_G^0} : F\text{-stable}\}. \tag{6.5.3}$$

Put $y = (z, \eta)$, and assume that $A_y = A_{z,\eta}$ is $F$-stable. We have $F^*\mathcal{E}_\eta \simeq \mathcal{E}_\eta$, and choose an isomorphism $\varphi_0 : F^*\mathcal{E}_\eta \xrightarrow{\sim} \mathcal{E}_\eta$ by the requirement that $\varphi_0$ induces an identity map on the stalk at $\dot z u_1 \in (\dot zC)^F$. $\varphi_0$ induces an isomorphism $\widetilde{\varphi}_0 : F^*A_y \simeq A_y$. Note that $\mathcal{H}^{-d}(A_y)|_{zZ_G^0C} = \bar{\mathbf{Q}}_l \boxtimes \mathcal{E}_\eta$, where $d = \dim Z_G^0 + \dim C$. Then we define $\phi_y : F^*A_y \xrightarrow{\sim} A_y$ by the condition that $\phi_y = q^{(\dim G - d)/2}\widetilde{\varphi}_0 = q^{(\operatorname{codim} C - \dim Z_G^0)/2}\widetilde{\varphi}_0$ on $\mathcal{H}_g^{-d}(A_y)$ $(g \in (zZ_G^0\overline{C})^F)$. We denote by $\chi_y$ the characteristic function $\chi_{A_y,\phi_y}$ associated to $A_y$ and $\phi_y$. Now, for each $\dot z \in Z_G^F$, the left multiplication $\dot z : C \to \dot zC$ induces, under the identification $A_G(\dot z u_1) \simeq A_G(u_1)$, the isomorphism $\dot z^* \mathrm{IC}(\overline{\dot zC}, \mathcal{E}_\eta) \xrightarrow{\sim} \mathrm{IC}(\overline{C}, \mathcal{E}_\eta)$ compatible with the $\mathbf{F}_q$-structure. Recall that for the cuspidal pair $\iota_0 = (C, \mathcal{E}_\eta) \in \mathcal{I}_G$, one can attach the function $\mathcal{X}_{\iota_0} \in \mathcal{V}_{\mathrm{uni}}$ as in 5.2. Then we see easily that

$$\chi_y(g) = \begin{cases} q^{(\operatorname{codim} C - \dim Z_G^0)/2}\mathcal{X}_{\iota_0}(v) & \text{if } g = \dot z z_1 v \text{ with } z_1 \in Z_G^{0F}, v \in \overline{C}^F, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5.4}$$

More generally, we consider $\mathcal{E} \otimes A_y$ with $\mathcal{E}' \in \mathcal{S}(G/G_{\mathrm{der}})$ such that $\mathcal{E}|_{\dot z Z_G^0}$ is $F$-stable. Put $\mathcal{E}_1 = \mathcal{E}|_{\dot z Z_G^0}$. The isomorphism $\varphi_1 : F^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$ is described as follows. Put $\mathcal{E}_2 = \mathcal{E}|_T$. There exists an integer $m > 0$ such that $\mathcal{E}_2$ is $F^m$-stable. We choose $\varphi_2 : (F^m)^*\mathcal{E}_2 \xrightarrow{\sim} \mathcal{E}_2$ so that it induces the identity map on the stalk at $1 \in T^{F^m}$. Then the characteristic function $\chi_{\mathcal{E}_2,\varphi_2}$ coincides with a linear character $\psi \in (T^{F^m})^\wedge$. Let $V$ be a one dimensional $T^{F^m}$-module affording $\psi$. We consider the quotient $T \times^{T^{F^m}} V$ of $T \times V$ by $T^{F^m}$ under the action $t_1 : (t, v) \mapsto (tt_1^{-1}, t_1v)$. Then $\mathcal{E}_2$ is realized as the local system associated to the locally trivial fibration $f : T \times^{T^{F^m}} V \to T$, $(t, v) \mapsto t^{q^m-1}$. For $\dot z \in T^F$, one can choose $\alpha \in T$ such that $\alpha^{q^m-1} = \dot z$. Now $f^{-1}(\dot zZ_G^0)$ can be identified with $\alpha Z_G^0 \times^{Z_G^{0F^m}} V$, and $\mathcal{E}_1$ is the local system associated to $f_1 : \alpha Z_G^0 \times^{Z_G^{0F^m}} V \to \dot z Z_G^0$

with $f_1 = f|_{f^{-1}(\dot z Z_G^0)}$. Since $\mathcal{E}|_{\dot z Z_G^0}$ is $F$-stable, the restriction of $\psi$ on $\dot z Z_G^{0F^m}$ is $F$-stable. Since $\dot z \in Z_G^F$, $\psi|_{Z_G^{0F^m}}$ is also $F$-stable. Hence we may assume that $V$ satisfies the property that $F(t)v = tv$ for $t \in Z^{0F^m}, v \in V$. Moreover, since $\dot z \in T^F$, we have $\hat z = \alpha^{-1}F(\alpha) \in T^{F^m}$. It follows that we have a well-defined automorphism $F$ on $\alpha Z_G^0 \times^{Z_G^{0F^m}} V$ given by

$$F(\alpha t, v) = (F(\alpha t), v) = (\alpha F(t), \hat z v) \tag{6.5.5}$$

for $(\alpha t, v) \in \alpha Z_G^0 \times^{Z_G^{0F^m}} V$. The map $f_1$ is compatible with the actions of $F$ on $\alpha Z_G^0 \times^{Z_G^{0F^m}} V$ and on $\dot z Z_G^0$ (the natural action). Hence $F$ defines an isomorphism $\varphi_1 : F^*\mathcal{E}_1 \,\tilde\to\, \mathcal{E}_1$.

We define a character $\theta_0 \in (Z_G^{0F})^\wedge$ by $\theta_0 = Sh_{F^m/F}(\psi|_{Z_G^{0F^m}})$. Then the previous argument implies, in view of (6.5.5), that

$$\chi_{\mathcal{E}_1,\phi_1}(\dot z z_1) = \psi(\hat z)\theta_0(z_1) \quad (z_1 \in Z_G^{0F}). \tag{6.5.6}$$

We now define an isomorphism $\phi_0 : F^*(\mathcal{E}_1 \otimes A_y) \,\tilde\to\, \mathcal{E}_1 \otimes A_y$ by $\phi_0 = \varphi_1 \otimes \phi_y$, which is regarded as an isomorphism $F^*(\mathcal{E} \otimes A_y) \,\tilde\to\, \mathcal{E} \otimes A_y$. We denote by $\chi_{\mathcal{E},y}$ the characteristic function $\chi_{\mathcal{E}\otimes A_y,\phi_0}$. It follows from (6.5.6) that we have

$$\chi_{\mathcal{E},y}(z_1 g) = \psi(\hat z)\theta_0(z_1)\chi_y(g) \qquad (z_1 \in Z_G^{0F}, g \in \dot z \overline{C}^F). \tag{6.5.7}$$

**6.6.** We consider the map $f : Z_G^0 \hookrightarrow G \to G/G_{\mathrm{der}}$. Then $f(Z_G^0)$ can be identified with $Z_G^0/Z_G^0 \cap G_{\mathrm{der}}$, and $Z_G^0 \to Z_G^0/Z_G^0 \cap G_{\mathrm{der}}$ is a finite étale covering. It follows that we have an isomorphism $\mathcal{S}(Z_G^0) \simeq \mathcal{S}(Z_G^0/Z_G^0 \cap G_{\mathrm{der}})$ and a surjective map $\mathcal{S}(G/G_{\mathrm{der}}) \to \mathcal{S}(Z_G^0/Z_G^0 \cap G_{\mathrm{der}})$. This implies that all the tame local systems (resp. $F$-stable tame local systems) on $Z_G^0$ are obtained as the pull back from tame local systems on $G/G_{\mathrm{der}}$ (resp. tame local systems on $G/G_{\mathrm{der}}$ such that its restriction on $Z_G^0$ is $F$-stable) by the map $f$.

This fact can be explained in the following way from a view point of Shintani descent. We consider a map $f : Z_G^{0F^m} \hookrightarrow G^{F^m} \to G^{F^m}/G_{\mathrm{der}}^{F^m}$ for sufficiently divisible integer $m$. Let $\theta'$ be a linear character of $G^{F^m}/G_{\mathrm{der}}^{F^m}$. Then $\theta_1 = \theta' \circ f$ is a linear character of $Z_G^{0F^m}$ whose restriction to $Z_G^{0F^m} \cap G_{\mathrm{der}}^{F^m}$ is trivial, and all such characters of $Z_G^{0F^m}$ are obtained by the pull back by $f$ from a linear character of $G^{F^m}/G_{\mathrm{der}}^{F^m}$. We now consider the norm map $N_{F^m/F} : Z_G^{0F^m} \to Z_G^{0F}$, which is a surjective homomorphism with kernel $K = \{z \in Z^{0F^m} \mid zF(z)\cdots F^{m-1}(z) = 1\}$. Now the Shintani descent gives a bijection between the set of irreducible characters of $Z_G^{0F}$ and the set of irreducible characters of $Z_G^{0F^m}$ whose restriction on $K$ is trivial. Since $Z_G^0 \cap G_{\mathrm{der}}$ is an $F$-stable finite set, we see that $Z_G^0 \cap G_{\mathrm{der}} \subset K$ if $m$ is chosen large enough. In particular, any irreducible character $\theta_0$ of $Z_G^{0F}$ can be obtained as $Sh_{F^m/F}(\theta_1)$ with $\theta_1 = \theta' \circ f$, where $\theta'$ is a linear character of $G^{F^m}/G_{\mathrm{der}}^{F^m}$ such that $\theta_1$ is $F$-stable.

**6.7.** Let $\widetilde L$ be a Levi subgroup of the standard parabolic subgroup $\widetilde P$ of $\widetilde G$ containing $T$, and put $L = G \cap \widetilde L$. Assume that $L$ contains a cuspidal character sheaf. Then by 6.4, $\widetilde L$ is of the form $\widetilde L = \widetilde L_1 \times \cdots \times \widetilde L_r$, where $\widetilde L_i \simeq GL_d \times \cdots \times GL_d$ ($n_i/d$-times) for a fixed $d$. Under the notation of 6.5 applied to $L$, let $A_0 = \mathcal{E} \otimes A_{z,\eta}$ be the cuspidal character sheaf on $L$, where $z \in Z_L/Z_L^0$ and $\eta \in A_L(\dot{z}u_1)^\wedge$ with $u_1 \in C$ ($C$ is the regular unipotent class in $L$), and $\mathcal{E}' \in \mathcal{S}(L/L_{\mathrm{der}})$. Let $K = \mathrm{ind}_P^G A_0$ be the induced complex of $G$. Then $K$ is a semisimple perverse sheaf whose simple components are character sheaves on $G$. All the characters sheaves on $G$ are obtained in this way by decomposing a suitable $K$. Let $\mathcal{E}_1$ be the tame local system on $zZ_L^0$ obtained by restricting $\mathcal{E}$ on $zZ_L^0$. Let $W_L$ be the Weyl subgroup of $W$ corresponding to $L$, and put

$$\mathcal{W}_{\mathcal{E}_1} = \{w \in N_W(W_L) \mid w^*\mathcal{E}_1 \simeq \mathcal{E}_1\}/W_L.$$

Let $\mathrm{End}\, K$ be the endomorphism algebra of $K$ in $\mathcal{M}G$, and $\bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_1}]$ be the group algebra of $\mathcal{W}_{\mathcal{E}_1}$ over $\bar{\mathbf{Q}}_l$. We note that

$$\mathrm{End}\, K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_1}]. \tag{6.7.1}$$

In fact it is known by [L3, 10.2] that $\mathrm{End}\, K$ is isomorphic to the twisted group algebra of $\mathcal{W}_{\mathcal{E}_1}$. By a general principle, we have only to show that $K$ contains a character sheaf with multiplicity one. Let $A_1 = \mathcal{E} \otimes A_{1,\eta}$ be the cuspidal character sheaf in the case where $z = 1$, and put $K_1 = \mathrm{ind}_P^G A_1$. We may choose a representative $\dot{z}$ of $Z_L/Z_L^0$ so that $\dot{z} \in Z_G$. Then we have $\dot{z}^*K \simeq K_1$, and it is enough to consider the case where $z = 1$, i.e., $K = K_1$. But in this case, it is known by [L4, 2.4] that $\mathrm{End}\, K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_1}]$. This shows (6.7.1).

In view of the discussion in 6.6, (6.7.1) is interpreted also in the following form. Take $m$ large enough so that $\mathcal{E}'$ is $F^m$-stable. Let $\theta'$ be the linear character of $(L/L_{\mathrm{der}})^{F^m}$ corresponding to $\mathcal{E}'$, and $\theta$ the linear character of $L^{F^m}$ obtained by the pull back of $\theta'$. Let $\theta_1$ be the restriction of $\theta$ on $Z_L^{0F^m}$ and put

$$\mathcal{W}_{\theta_1} = \{w \in N_W(W_L) \mid {}^w\theta_1 = \theta_1\}/W_L.$$

Then we have

$$\mathcal{W}_{\mathcal{E}_1} \simeq \mathcal{W}_{\theta_1}. \tag{6.7.2}$$

**6.8.** We keep the notation in 6.7. We denote by $A_E$ the character sheaf occurring as the simple component in $K$ corresponding to $E \in \mathrm{End}\, K$. For each $A_E \in \widehat{G}^F$, we shall determine $\phi_{A_E} : F^*A_E \xrightarrow{\sim} A_E$. Assume that $A_{z,\eta}$ is $F$-stable, and put

$$\mathcal{Z}_{\mathcal{E}_1} = \{w \in N_W(W_L) \mid (Fw)^*\mathcal{E}_1 \simeq \mathcal{E}_1\}/W_L.$$

By [L3, II, 10.2], we see that

$$\mathcal{Z}_{\mathcal{E}_1} \simeq \{w \in N_W(W_L) \mid (Fw)^*A_0 \simeq A_0\}/W_L.$$

Now any $w \in \mathcal{Z}_{\mathcal{E}_1}$ can be written as $w = w_{\mathcal{E}_1}y$ with $y \in \mathcal{W}_{\mathcal{E}_1}$. $Fw_{\mathcal{E}_1}$ induces an automorphism $\gamma_{\mathcal{E}_1} : \mathcal{W}_{\mathcal{E}_1} \to \mathcal{W}_{\mathcal{E}_1}$. For each $w \in \mathcal{Z}_{\mathcal{E}_1}$, let $L^w \subset P^w$ be the $F$-stable Levi, and parabolic subgroup of $G$ obtained from $L \subset P$ by twisting $w$, so that $(L^w)^F \simeq L^{F\dot{w}}$, etc. (Note that this definition of $L^w$ etc. is not the same as $L^w$ given in [L3, II, 10.6]. $L^w$ defined there coincides with our $L^{w'}$ with $w' = F^{-1}(w^{-1})$. The formulas below are derived from [L3, II] under a suitable modification.) We denote by

$A_0^w, K^w$ the corresponding cuspidal character sheaf on $L^w$ and the induced complex on $G$. Then we have $K^w = \operatorname{ind}_{P^w}^G A_0^w$. By applying the arguments in 6.5 to the $Fw$-stable subgroups $L^w \subset P^w$, one can construct $\varphi_0^w = \varphi_1^w \otimes \widetilde{\varphi}_0^w : F^*A_0^w \xrightarrow{\sim} A_0^w$, where $\varphi_1^w, \widetilde{\varphi}_0^w$ are constructed from $\varphi_1 : (Fw)^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1$, $\widetilde{\varphi}_0 : (Fw)F^*A_y \xrightarrow{\sim} A_y$ as in 6.5 (replacing $F$ by $Fw$). Put $\phi_0^w = q^{b_0}\varphi_0^w$, where

$$b_0 = (\operatorname{codim}_L C_0 - \dim Z_L^0)/2$$

with the regular unpotent class $C_0$ in $L$. Now $\varphi_0^w : F^*A_0^w \xrightarrow{\sim} A_0^w$ induces an isomorphism $\varphi^w : F^*K^w \xrightarrow{\sim} K^w$.

We look at the special case where $w = w_{\mathcal{E}_1}$. Then $A_E$ occurring in $K^w$ is $F$-stable if and only if $E \in \mathcal{W}_{\mathcal{E}_1}$ is $\gamma_{\mathcal{E}_1}$-stable. We fix a preferred extension $\widetilde{E}$ of $E$ for each $E \in (\mathcal{W}_{\mathcal{E}_1}^\wedge)^{\omega_{\mathcal{E}_1}}$. Then $\varphi^w : F^*K^w \xrightarrow{\sim} K^w$ induces an isomorphism $\varphi_{A_E} : F^*A_E \xrightarrow{\sim} A_E$ satisfying the following formula (cf. [L3, 10.4, 10.6]). For each $w = w_{\mathcal{E}_1}y \in \mathcal{Z}_{\mathcal{E}_1}$, we have

$$\chi_{K^w,\varphi^w} = \sum_{E \in (\mathcal{W}_{\mathcal{E}_1}^\wedge)^{\gamma_{\mathcal{E}_1}}} \operatorname{Tr}(\gamma_{\mathcal{E}_1}y, \widetilde{E})\chi_{A_E,\varphi_{A_E}}, \tag{6.8.1}$$

We define a normalized isomorphism $\phi_{A_E} : F^*A_E \xrightarrow{\sim} A_E$ by $\phi_{A_E} = q^{b_0}\varphi_{A_E}$. In general, for an $F$-stable character sheaf $A = A_E$, we put $\phi_A = \phi_{A_E}$, and we denote by $\chi_A$ the characteristic function $\chi_{A,\phi_A}$ for $A \in \widehat{G}^F$.

## 7. Lusztig's conjecture

**7.1.** We give a description of cuspidal irreducible characters of $G^F$. Assume that $\overline{\mathcal{M}}_{s,E}$ contains a cuspidal character. Then there exists $\dot{s}_x \in \widetilde{T}$ such that $\pi(\dot{s}_x) = s$, $\dot{s}_x$ is a regular semisimple element such that $Z_{\dot{s}_x}$ consists of a Coxeter element in $W$. $W_{\dot{s}_x} = \{1\}$, and $\widetilde{\rho}_{\dot{s}_x,E}$ gives a cuspidal character of $\widetilde{G}^F$, where $E$ is the trivial character of $W_{\dot{s}_x} = W_s^0$. We assume further that the pair $(s,E) = (s,1)$ is of the form as in (2.5.1). Then $W_s = \Omega_s$, and $\Omega_s$ is a cyclic group generated by $w_0$. $\widetilde{G}$ is of the form $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$, where $\widetilde{G}_i = GL_t$ with $t = n/r$ for an integer $t$ prime to $p$. Moreover, $w_0 = (w_c, \dots, w_c)$ with a cyclic permutation $w_c = (1,2,\dots,t) \in \mathfrak{S}_t$. Then $\{s\}$ is an $F$-stable regular semisimple class in $G^*$ such that $\Omega_s \simeq \mathbf{Z}/t\mathbf{Z}$, which is unique modulo $Z_{G^*}$. We make the following specific choice of $s$.

(7.1.1) Let $\dot{s} = (\dot{s}_1, \dots, \dot{s}_r) \in \widetilde{T}^* = \widetilde{T}_1^* \times \cdots \times \widetilde{T}_r^*$ such that $\dot{s}_i = \operatorname{Diag}(1, \zeta, \dots, \zeta^{t-1})$, where $\zeta$ is a primitive $t$-th root of unity in $\overline{\mathbf{F}}_q$. Put $s = \pi(\dot{s})$.

Let $s'$ be a regular semisimple element such that $\{s'\}$ is $F$-stable and that $\Omega_{s'} \simeq \mathbf{Z}/t\mathbf{Z}$. Since they are unique modulo $Z_{G^*}$, $s'$ can be written as $s' = zs$ with $z \in Z_{G^*}$. If $z \in Z_{G^*}^F$, then $z$ determines a linear character $\theta$ of $G^F$ under the natural isomorphism $Z_{G^*}^F \simeq (G^F/G_{\text{der}}^F)^\wedge$. The parameter set $\overline{\mathcal{M}}_{s,E}$ and $\overline{\mathcal{M}}_{sz,E}$ are then naturally identified, and we have a bijection $\mathcal{T}_{s,E} \simeq \mathcal{T}_{sz,E}$ via $\rho_y \leftrightarrow \theta \otimes \rho_y$ with $y \in \overline{\mathcal{M}}_{s,E}$.

Assume that $\dot{s}$ is as in (7.1.1). We assume further that $F$ acts trivially on $Z_G/Z_G^0$. Since $Z_G/Z_G^0 \simeq \mathbf{Z}/t\mathbf{Z}$, $F$ stabilizes $\zeta$. Hence $F(\dot{s}) = \dot{s}$ and so $F' = F$. Moreover, $F'$

acts trivially on $\Omega_s$, and we have

$$\overline{\mathcal{M}}_{s,E} = \Omega_s^\wedge \times \Omega_s.$$

By 2.8 (a), $\overline{\mathcal{M}}_{s,E}$ is in bijection with $\overline{\mathcal{M}}_{s,N}$. Since $\dot{s}$ is regular semisimple, $\mathcal{O}_{\widetilde{\rho}}$ for $\widetilde{\rho} = \widetilde{\rho}_{\dot{s},E}$ is the regular nilpotent orbit. So we assume that $N$ is regular nilpotent. In this case, $\widetilde{P}_N = \widetilde{B}$ and $\widetilde{L}_N = \widetilde{T}$, and $A_\lambda \simeq Z_G/Z_G^0$. Since $A_\lambda^F \simeq \mathbf{Z}/t\mathbf{Z}$, we have $\overline{A}_\lambda = A_\lambda$, and so $\overline{\mathcal{M}}_{s,N} = A_\lambda \times A_\lambda^\wedge$. Let $(A_\lambda)_0^\wedge$ be the set of faithful characters of $A_\lambda$. We show

**Lemma 7.2.** *Let $\dot{s}$ and $s = \pi(\dot{s})$ be as in (7.1.1). Assume that $F$ acts trivially on $Z_G/Z_G^0$.*

(i) *For each $x \in \Omega_s$, we have $\Delta(\widetilde{\rho}_{\dot{s}_x,E})|_{Z_G^{0F}} = 1$.*

(ii) *Assume further that $q \equiv 1 \pmod 4$ if $t = 2$. Then for each $(c,\xi) \in \overline{\mathcal{M}}_{s,N}$, $\rho_{c,\xi} \in \mathcal{T}_{s,E}$ is cuspidal if and only if $\xi \in (A_\lambda)_0^\wedge$.*

*Proof.* For each $x \in (\Omega_s)_{F'} = \Omega_s$, we consider $\widetilde{\rho}_{\dot{s}_x,E} \in \operatorname{Irr} \widetilde{G}^F$. $\Delta(\widetilde{\rho}_{\dot{s}_x,E})$ is determined from $\dot{s}_x$ as in 2.4 by using an $F$-stable Levi subgroup $\widetilde{M}$ of an $F$-stable parabolic subgroup of $\widetilde{G}$. Assume that $\dot{s}_x \in \widetilde{T}^{*xF}$ for $x \in \Omega_s$. In the case where $N$ is regular nilpotent, $\widetilde{M}$ coincides with $\widetilde{T}$. We now choose $m \ge 1$ such that $\dot{s}_x \in \widetilde{T}^{*F^m}$ and that $(xF)^m = F^m$. Then by the duality of the torus, there exists an $xF$-stable linear character $\varTheta$ of $\widetilde{T}^{F^m}$ corresponding to $\dot{s}_x$. The Shintani descent $Sh_{F^m/xF}(\varTheta) = \varTheta_0$ gives rise to a linear character of $\widetilde{T}^{xF}$. The restriction of $\varTheta_0$ on $Z_G^{xF} = Z_G^F$ gives the character $\Delta(\widetilde{\rho}_{\dot{s}_x,E})$.

Following the arguments in [S2, Corollary 2.21], we give a more precise description of $\varTheta$ and $\varTheta_0$. Since $\pi(\dot{s}_x) = \pi(\dot{s}) = s$, there exists $z_x \in (\ker \pi)^{F^m} \subset Z_{\widetilde{G}^*}^{F^m}$ such that $\dot{s}_x = \dot{s}z_x$. Hence there exists an $F$-stable linear character $\widehat{\varTheta}$ of $\widetilde{T}^{F^m}$ and a linear character $\omega_x$ of $\widetilde{G}^{F^m}$ such that $\varTheta = \widehat{\varTheta}\omega_x$. Now there exists a decomposition $\widetilde{T} = \widetilde{T}_1^+ \times \cdots \times \widetilde{T}_t^+$ such that $\widetilde{T}_i^+$ are all $F$-stable, and $x$ permutes the factors $\widetilde{T}_i^+$. According to this decomposition of $\widetilde{T}$, $\omega_x$ can be written as

$$\omega_x|_{\widetilde{T}^{F^m}} = \omega_x^1 \boxtimes \cdots \boxtimes \omega_x^1,$$

where $\omega_x^1$ is a linear character of $\widetilde{T}_i^{+F^m} \simeq \widetilde{T}_1^{+F^m}$. Here $\omega_x^1$ is expressed as $\omega_x^1(y_1) = \overline{\omega}_x(\det y_1)$ for $y_1 \in \widetilde{T}_1^{+F^m}$, where $\overline{\omega}_x$ is a homomorphism $\mathbf{F}_{q^m}^* \to \bar{\mathbf{Q}}_l^*$. Under the decomposition of $\widetilde{T}$ into $\widetilde{T}_i^+$, $Z_G^0$ can be identified with the set of $y = (y_1, y_1, \dots, y_1)$ ($t$-times) such that $\det y_1 = 1$. It follows that $\omega_x|_{Z_G^{0F^m}} = 1$. On the other hand, our choice of $\dot{s}$ implies that $\widehat{\varTheta}$ is written on $\widetilde{T}^{F^m}$ as

$$\widehat{\varTheta} = 1 \boxtimes a \boxtimes \cdots \boxtimes a^{t-1}$$

with $a^t = 1$. We may assume that the generator $w_0 \in \Omega_s$ permutes the factors $\widetilde{T}_i^+$ by $w_0(\widetilde{T}_i^+) = \widetilde{T}_{i+1}^+$ for $i \in \mathbf{Z}/t\mathbf{Z}$. Put $x = w_0$. Since $\widehat{\varTheta}$ is $F$-stable and $\widehat{\varTheta}\omega_x$ is $xF$-stable, we see that $a = (\omega_x^1)^{-1}F(\omega_x^1) = (\omega_x^1)^{q-1}$. This implies that $\widehat{\varTheta}|_{Z_G^{0F^m}} = 1$. Hence $\varTheta|_{Z_G^{0F^m}} = 1$, and we see that $\varTheta_0|_{Z_G^{0F}} = 1$. This proves the first assertion.

By the previous argument, the restriction of $\Theta_0$ on $Z_G^F$ gives rise to a character $\xi_x \in A_\lambda^\wedge$. We know that $\widetilde{\rho}_{\dot{s}_x,E}$ is cuspidal if and only if $x$ is a generator of $\Omega_s$, and that all the cuspidal irreducible characters in $\mathcal{T}_{s,E}$ are obtained as the irreducible components $\rho_{c,\xi_x}$ in $\widetilde{\rho}_{\dot{s}_x,E}|_{G^F}$. Hence in order to prove the second assertion, we have only to show that $\xi_x \in (A_\lambda)_0^\wedge$ if $x$ is a generator of $\Omega_s$. First we show the following.

(7.2.1) Assume that $x = w_0$. Then $\xi_x$ is a faithful character.

It is easy to see that $\omega_x|_{Z_{\widetilde{G}}^{F^m}}$ is $F$-stable. Put $\omega_0 = Sh_{F^m/F}(\omega_x|_{Z_{\widetilde{G}}^{F^m}})$. Let us choose $y \in Z_G^F$ such that the image of $y$ in $Z_G/Z_G^0 = A_\lambda$ is a generator of $A_\lambda$. Then $y = (y_1, \dots, y_1) \in Z_G^F$ with $\det y_1$ a primitive $t$-th root of unity. By a similar argument as in [S2, p.208 - p.209], we see that $\omega_0(y)$ is a primitive $t$-th root of unity. On the other hand, put $\Theta_1 = Sh_{F^m/F}(\widehat{\Theta}|_{Z_{\widetilde{G}}^{F^m}})$. Any element $z \in Z_{\widetilde{G}}^{F^m}$ can be written as $z = (z_1, \dots, z_1)$ with $z_1 \in \widetilde{T}_1^{+F^m}$ and

$$\widehat{\Theta}(z) = a^{t(t-1)/2}(z_1) = (\omega_x^1)^{t(t-1)(q-1)/2}(z_1).$$

Since $a^t = 1$, we have $(\omega_x^1)^{t(q-1)} = 1$. It follows that $\widehat{\Theta}^2|_{Z_{\widetilde{G}}^{F^m}} = 1$ and so $\Theta_1^2 = 1$. Since $\Theta_0|_{Z_{\widetilde{G}}^F} = \Theta_1\omega_0$, we see that $\xi_x$ is a faithful character if $t > 2$. So assume that $t = 2$. Let $y = (y_1, y_1) \in Z_{\widetilde{G}}^F$ be such that $\det y_1 = -1$. We have only to show that $\Theta_1(y) = 1$. Take $z = (z_1, z_1) \in Z_{\widetilde{G}}^{F^m}$ such that $N_{F^m/F}(z) = y$. Then it is easy to see that $N_{F^m/F}(\det z_1) = \det y_1 = -1$. Then we have

$$\Theta_1(y) = \widehat{\Theta}(z) = (\omega_x^1)^{q-1}(z_1) = \overline{\omega}_x^{q-1}(\det z_1).$$

Since $(\omega_x^1)^{2(q-1)} = 1$, we may assume that $(\omega_x^1)^{q-1}$ is a character of order 2. Hence $(\overline{\omega}_x)^{q-1}$ is the unique character of order 2 of $\mathbf{F}_{q^m}^*$, and one can write as $(\overline{\omega}_x)^{q-1} = \theta \circ N_{F^m/F}$, where $\theta$ is the unique character of order 2 of $\mathbf{F}_q^*$, i.e., $\theta(x) = x^{(q-1)/2}$ for $x \in \mathbf{F}_q^*$. It follows that

$$\overline{\omega}_x^{q-1}(\det z_1) = \theta(-1) = (-1)^{(q-1)/2} = 1$$

since $q \equiv 1 \pmod 4$ by our assumption. Hence we have $\Theta_1(y) = 1$, and $\xi_x$ is faithful in this case also. Thus (7.2.1) is proved.

If we replace $x$ by $x^j$ for some $j$, then we have $Sh_{F^m/F}(\omega_{x^j}|_{Z_{\widetilde{G}}^{F^m}}) = \omega_0^j$ and the previous argument shows that $Sh_{F^m/x^jF}(\Theta|_{Z_{\widetilde{G}}^{F^m}}) = \Theta_1\omega_0^j$. Since $\omega_0^j(y^i)$ are all distinct for $i = 1, \dots, t$ if $j$ is prime to $t$, we see that $\xi_{x^j}$ is faithful if $j$ is a generator of $\Omega_s$. This proves the second assertion, and the lemma follows. □

**7.3.** We preserve the setting in 7.1. Removing the assumption on $F$, we consider the sets $\overline{\mathcal{M}}_{s,N} = (A_\lambda)_F \times (A_\lambda^F)^\wedge$ and $\mathcal{M}_{s,N} = A_\lambda^F \times (A_\lambda^\wedge)^F$, which are in bijection with $\overline{\mathcal{M}}_{s,E}$ and $\mathcal{M}_{s,E}$. Since $A_\lambda = \bar{A}_\lambda$, we have $\overline{\mathcal{M}}_0 = \overline{\mathcal{M}}_{s,N}$. Applying Lemma 7.2 to the situation in $G^{F^m}$, we have the following corollary.

**Corollary 7.4.** *Assume that $m$ is sufficiently divisible so that $F^m$ satisfies the assumption in Lemma 7.2 with respect to $F$. Let $s'$ be a regular semisimple element in $G^*$ such that $\mathcal{M}_{s',E}$ contains an $F$-stable cuspidal irreducible character of $G^{F^m}$. Then*

(i) *The pair $(s', E)$ is of the from as in 2.8 (a), and $\widetilde{G} \simeq \widetilde{G}_1 \times \cdots \times \widetilde{G}_r$, where $\widetilde{G}_i \simeq GL_t$ with $t = n/r$, and $\{s'\}$ is the unique regular semisimple class modulo $Z_{G^*}$ such that $W_{s'} = \Omega_{s'} \simeq \mathbf{Z}/t\mathbf{Z}$. In particular, $(\mathcal{T}^{(m)}_{s',E})^F$ is parametrized by $\mathcal{M}_{s',N}$, where $N$ is a regular nilpotent element in $\mathfrak{g}^F$.*

(ii) *Let $(A_\lambda^\wedge)^F_0$ be the set of $F$-stable, faithful characters of $A_\lambda = Z_G/Z_G^0$. Then under the parametrization $(\mathcal{T}^{(m)}_{s',E})^F \leftrightarrow \mathcal{M}_{s',N}$, $\rho^{(m)}_{c,\xi}$ is cuspidal if and only if $\xi \in (A_\lambda^\wedge)^F_0$.*

*Proof.* Since $s'$ is regular semisimple, the pair $(s', E)$ is of type (a) or (c) in 2.8. But it is easy to see that in case (c), $\mathcal{M}_{s,E}$ does not contain a cuspidal irreducible character. (Note that by the Shintani descent theory, Lusztig induction $R^G_L(\dot{w}_1)$ corresponds to the Harish-Chandra induction from $L^{F^m}$ to $G^{F^m}$). Hence $(s', E)$ is of type (a), and (i) follows from 7.1. Now by 7.1, $s'$ can be written as $s' = zs$, where $s$ is as in (7.1.1) and $z \in Z_{G^*}$. We may assume that $z \in Z_{G^*}^{F^m}$ by choosing $m$ large enough. Thus we have a natural bijection $\mathcal{T}^{(m)}_{s',E} \simeq \mathcal{T}^{(m)}_{s,E}$ by 7.1 under the identification $\mathcal{M}^{(m)}_{s',N} = \mathcal{M}^{(m)}_{s,N}$. Since $F^m$ acts trivially on $A_\lambda$, (ii) follows from Lemma 7.2. □

We note the following lemma.

**Lemma 7.5.** *Let $s'$ be as in 7.1 and $\theta = \Delta(\widetilde{\rho}_{\dot{s}',E})|_{Z_G^F}$ the linear character of $Z_G^F$. Let $R_{z,\eta}$ be the almost character of $G^F$ for $(z, \eta) \in \mathcal{M}_{s',N}$ under the bijection $\mathcal{M}_{s',N} \simeq \mathcal{M}_{s',E}$. Let $\varGamma_{c,\xi,\tau'}$ be the modified generalized Gelfand-Graev character associated to $(c, \xi) \in \overline{\mathcal{M}}_0 = \overline{\mathcal{M}}_{s',N}$ and to a linear character $\theta'$ of $Z_G^F$. Then we have*

$$\langle \varGamma_{c,\xi,\theta'}, R_{z,\eta}\rangle_{G^F} = \begin{cases} \eta(c)\xi(z)|(Z_G/Z_G^0)^F|^{-1} & \text{if } \theta'|_{Z_G^{0F}} = \theta|_{Z_G^{0F}}, \\ 0 & \text{if } \theta'|_{Z_G^{0F}} \neq \theta|_{Z_G^{0F}}. \end{cases}$$

*Proof.* In our case, $\overline{A}_\lambda = A_\lambda = Z_G/Z_G^0$. Thus by applying (4.5.2) and (4.5.3), one can write as

$$R_{z,\eta} = |(Z_G/Z_G^0)^F|^{-1} \sum_{(c_1,\xi_1)} \eta(c_1)\xi_1(z)\rho_{c_1,\xi_1}.$$

Then by Theorem 2.6, (ii)-(b), we have

$$\langle \varGamma_{c,\xi,\theta}, R_{z,\eta}\rangle_{G^F} = |(Z_G/Z_G^0)^F|^{-1}\eta(c)\xi(z)$$

if $\theta'|_{Z_G^{0F}} = \theta|_{Z_G^{0F}}$. This proves the first formula. The second formula also follows from Theorem 2.6, (ii)-(a) together with (4.5.2). □

**7.6.** We preserve the notations $\overline{\mathcal{M}}_{s,N}$, etc. as in 7.1 for regular nilpotent element $N$. Recall that $\mathcal{E} \otimes A_{z,\eta}$ is an $F$-stable cuspidal character sheaf on $G$ as given in (6.5.3) for $z \in (Z_G/Z_G^0)^F$, $\eta \in (A_G(\dot{z}u_1)^\wedge_0)^F$ and $\mathcal{E}|_{\dot{z}Z_G^0}$: $F$-stable. Under the identification $A_G(\dot{z}u_1) = A_G(u_1) \simeq A_\lambda$, we regard $\eta$ as an element in $(A_\lambda^\wedge)^F$. Hence $(z, \eta)$ is regarded as an element in $\mathcal{M}_{s,N}$. Let $\chi_{\mathcal{E},y} = \chi_{\mathcal{E},z,\eta}$ be the characteristic function of $\mathcal{E} \otimes A_{z,\eta}$

defined in 6.5. Also recall the linear character $\theta_0$ of $Z_G^{0F}$ associated to $\mathcal{E}'$ in 6.5. Let $\varGamma_{c,\xi,\theta'}$ be as in Lemma 7.5. We shall compute the inner product of $\varGamma_{c,\xi,\theta'}$ with $\chi_{\mathcal{E},z,\eta}$.

**Lemma 7.7.** *Under the setting in 7.6, we have*

$$
\begin{aligned}
&\langle \varGamma_{c,\xi,\theta'}, \chi_{\mathcal{E},z,\eta}\rangle_{G^F} \\
&\qquad = \begin{cases} \zeta_{\mathcal{I}_0}^{-1}\xi(z)\eta(cc_0)^{-1}\psi(\hat{z}^{-1})\theta'(\dot{z})|(Z_G/Z_G^0)^F|^{-1} & \text{if } \theta'|_{Z_G^{0F}} = \theta_0, \\ 0 & \text{if } \theta'|_{Z_G^{0F}} \neq \theta_0. \end{cases}
\end{aligned}
$$

*where $\zeta_{\mathcal{I}_0}$ is the fourth root of unity associated to the block $\mathcal{I}_0 = \{\iota_0\}$ for the cuspidal pair $\iota_0 = (C, \mathcal{E}_\eta)$. $\dot{z} \in Z_G^F$ is a representative of $z$, and $\hat{z}$ is as in 6.5.*

*Proof.* We follow the notation in 2.1. Since $N$ is regular nilpotent, $U$ is the maximal unipotent subgroup of $G$ and $Z_L(\lambda_c) = Z_G$. $\xi^\natural$ is a linear character of $Z_G^F$ which is trivial on $Z_G^{0F}$, hence it is naturally identified with $\xi \in (A_\lambda^F)^\wedge$. $\varGamma_{c,\xi,\theta'}$ is given as $\varGamma_{c,\xi,\theta'} = \operatorname{Ind}_{Z_G^F U^F}^{G^F}(\theta'\xi^\natural \otimes \varLambda_c)$. It is easy to see that $\varGamma_{c,\xi,\theta'}$ has the support in $Z_G^F G_{\text{uni}}^F$, and that

$$
\varGamma_{c,\xi,\theta'}(z'v) = \theta'\xi(z')|Z_G^F|^{-1}\varGamma_c(v) \qquad (z' \in Z_G^F, v \in G_{\text{uni}}^F). \tag{7.7.1}
$$

Then by (6.5.4) and (6.5.7), we have

$$
\langle \varGamma_{c,\xi,\theta'}, \chi_{\mathcal{E},z,\eta}\rangle_{G^F} = \begin{cases} A\langle \varGamma_c, \mathcal{X}_{\iota_0}\rangle_{G^F} & \text{if } \theta'|_{Z_G^{0F}} = \theta_0, \\ 0 & \text{if } \theta'|_{Z_G^{0F}} \neq \theta_0 \end{cases} \tag{7.7.2}
$$

with

$$
A = q^{(\operatorname{codim} C - \dim Z_G^0)/2}\psi(\hat{z}^{-1})\theta'(\dot{z})\xi(z)|(Z_G/Z_G^0)^F|^{-1}.
$$

Hence we have only to compute $\langle \varGamma_c, \mathcal{X}_{\iota_0}\rangle_{G^F}$. We compute it by making use of Lusztig's formula (Theorem 5.6). By (5.2.4), $\mathcal{X}_{\iota_0}$ is orthogonal to any function in $\mathcal{V}_{\mathcal{I}_1}$ such that $\mathcal{I}_1 \neq \mathcal{I}_0$. It follows that $\langle \varGamma_c, \mathcal{X}_{\iota_0}\rangle_{G^F} = \langle (\varGamma_c)_{\mathcal{I}_0}, \mathcal{X}_{\iota_0}\rangle_{G^F}$. But since $\mathcal{I}_0 = \{\iota_0\}$, we have $L = G, \mathcal{W} = \{1\}$ and $\operatorname{supp}(\iota_0) = C$. Hence by Lemma 5.11, we have

$$
\langle (\varGamma_c)_{\mathcal{I}_0}, \mathcal{X}_{\iota_0}\rangle_{G^F} = q^{(-\operatorname{codim} C + \dim Z_G^0)/2}\zeta_{\mathcal{I}_0}^{-1}\eta(cc_0)^{-1}.
$$

Substituting this into (7.7.2), we obtain the lemma. □

**7.8.** Returning to the original setting in 1.1, we consider an $Fw_\delta$-stable cuspidal character $\delta$ of $L^{F^m}$, and $\mathcal{P}_\delta = \operatorname{Ind}_{P^{F^m}}^{G^{F^m}}\delta$ as in 4.9. We shall describe $H(\delta) = \operatorname{End}\mathcal{P}_\delta$ more precisely. By Corollary 7.4, $(L, \delta)$ is given as follows; $L = \widetilde{L} \cap G$, where $\widetilde{L}$ is a Levi subgroup of $\widetilde{G}$ such that $\widetilde{L} = \widetilde{L}_1 \times \cdots \times \widetilde{L}_r$ with $\widetilde{L}_i \simeq GL_t \times \cdots \times GL_t$ ($n_i/t$-times) for a fixed integer $t$. There exists a (unique) cuspidal character $\widetilde{\delta}$ of $\widetilde{L}^{F^m}$ belonging to $\mathcal{E}(\widetilde{L}^{F^m}, \{\dot{s}\})$, where $\dot{s}$ is as in (7.1.1) (by replacing $G$ by $L$). Then the restriction of $\widetilde{\delta}$ is a sum of cuspidal characters of $L^{F^m}$, and $\delta$ is obtained in the form $\delta = \theta' \otimes \delta_0$, where $\delta_0$ is an irreducible consistent of $\widetilde{\delta}|_{L^{F^m}}$ and $\theta'$ is a linear character of $L^{F^m}$ corresponding to $z \in Z_{L^*}^{F^m}$. By Lehrer [Le, Theorem 10], $\mathcal{W}_\delta \simeq \mathcal{W}_\delta^0 \rtimes \Omega_\delta$, where $\Omega_\delta \simeq \mathbf{Z}/t_1\mathbf{Z}$ for some integer $t_1 > 0$, and $\mathcal{W}_\delta$ acts on $L^{F^m}$ as a permutation of factors in the direct product.

Moreover, $\mathcal{W}_\delta^0$ is isomorphic to the ramification group $\mathcal{W}_{\widetilde{\delta}}$ of $\widetilde{\delta}$ in $\widetilde{L}^{F^m}$. In our case, it is easy to see that $\mathcal{W}_{\widetilde{\delta}}$ is isomorphic to $\mathcal{W}$.

Now assume that $\delta$ is a constituent of $\widetilde{\delta}|_{L^{F^m}}$. Then $\delta \in \mathcal{E}(L^{F^m}, \{s\})$. Moreover we have $\mathcal{W}_\delta = \mathcal{W}$ since $\mathcal{W}_\delta$ is a subgroup of $\mathcal{W}$. Let $\theta$ be a linear character of $L^{F^m}$ as above. Then $\theta \otimes \delta$ is a cuspidal character belonging to $\mathcal{E}(L^{F^m}, \{zs\})$.

We can now prove Lusztig's conjecture for $G^F$.

**Theorem 7.9.** *Lusztig's conjecture holds for $G^F$. More precisely, there exists a bijection between the set of $F$-stable character sheaves and the set of almost characters of $G^F$ satisfying the following:*

(i) *For each almost character $R_x$ of $G^F$, we denote by $A_x$ the corresponding character sheaf of $G$, and by $\phi_x$ the isomorphism $F^*A_x \xrightarrow{\sim} A_x$ as given in 6.8. Then*

$$\chi_{A_x,\phi_x} = \nu_x R_x$$

*for a certain constant $\nu_x \in \bar{\mathbf{Q}}_l^*$. Here $\nu_x$ is a root of unity contained in a fixed cyclotomic field independent of $q$.*

(ii) *Let $\chi_{\mathcal{E},z,\eta}$ be the characteristic function of the $F$-stable cuspidal character sheaf $\mathcal{E} \otimes A_{z,\eta}$ as given in (6.5.3). Let $z_1 \in Z_{G^*}$ be the element corresponding to $\mathcal{E}' \in \mathcal{S}(G/G_{\mathrm{der}})$. Let $s$ be as in (7.1.1), and put $s' = z_1 s$. Let $R_{z,\eta^{-1}}$ be the almost character of $G^F$ corresponding to $(z, \eta^{-1}) \in \mathcal{M}_{s',N}$. Then we have*

$$\chi_{\mathcal{E},z,\eta} = \zeta_{\mathcal{I}_0}^{-1}\eta(c_0)^{-1}R_{z,\eta^{-1}}.$$

*Proof.* Let $L$ be the $F$-stable Levi subgroup containing $T$ of a proper standard parabolic subgroup $P$ of $G$. By induction on $\dim G$, we may assume that Lusztig's conjecture holds for any $L^{F'}$, where $F' = F\dot{w}_1$ for some $\dot{w}_1 \in N_G(L)$. Assume that $A_0$ is an $F'$-stable cuspidal character sheaf on $L$. Then $A_0$ can be written as $A_0 = \mathcal{E} \otimes A_{z,\eta}$, where $z \in (Z_L/Z_L^0)^{F'}$, $u_1 \in C_0^{F'}$ ($C_0$ is the regular unipotent class in $L$), $\eta \in A_L(\dot{z}u_1)^\wedge_{F'}$ and $\mathcal{E}$ is a local system on $L$ such that $\mathcal{E}|_{\dot{z}Z_L^0}$ is $F'$-stable. By (ii), there exists an $F'$-stable cuspidal character $\delta$ of $L^{F^m}$ associated to $(z, \eta, \mathcal{E})$ such that the corresponding almost character of $L^{F'}$ is given by $R_{z,\eta^{-1}}$. This holds for any $F' = Fw$ for $w \in N_W(W_L)$ such that $(Fw)^*A_0 \simeq A_0$.

On the other hand, let $\mathcal{E}_1$ be the tame local system on $Z_L^0$ obtained by restricting $\mathcal{E}$ to $Z_L^0$, and let $\mathcal{Z}_{\mathcal{E}_1}$ be as in 6.8. The above discussion then shows that $\mathcal{Z}_{\mathcal{E}_1} = \mathcal{Z}_\delta$. Since $\mathcal{Z}_\delta$ is a coset of $\mathcal{W}_\delta$ and $\mathcal{Z}_{\mathcal{E}_1}$ is a coset of $\mathcal{W}_{\mathcal{E}_1}$, we see that

$$\mathcal{W}_\delta = \mathcal{W}_{\mathcal{E}_1}. \tag{7.9.1}$$

Now $w_{\mathcal{E}_1}$ given in 6.8 coincides with $w_\delta$ given in 4.9, and so $\gamma_{\mathcal{E}_1} = \gamma_\delta$. Let $K^w, \varphi^w$ etc. be as in 6.8. Then by (6.8.1), we have

$$\chi_{K^w,\varphi^w} = q^{-b_0} \sum_{E \in (\mathcal{W}_\delta^\wedge)^{\gamma_\delta}} \mathrm{Tr}\,(\gamma_\delta y, \widetilde{E})\chi_{A_E,\phi_{A_E}}, \tag{7.9.2}$$

where $A_E$ is the simple component of $K^w$ corresponding to $E \in \mathcal{W}_{\mathcal{E}_1}^\wedge$. On the other hand, under the isomorphism $(L^w)^F \simeq L^{F\dot{w}}$, $\chi_{A_0^w,\phi_0^w} \in C((L^w)^F/\sim)$ is regarded as an element in $C(L^{F\dot{w}}/\sim)$. Then it is known by [L6, Prop. 9.2] that

$$\chi_{K^w,\varphi^w} = (-1)^{\dim C_0 + \dim Z_L^0} R^G_{L(w)}(q^{-b_0}\chi_{A_0^w,\phi_0^w}). \tag{7.9.3}$$

(Note that $\chi_{\mathcal{E}^\sharp}$ in [loc. cit.] coincides with $\chi_{A_0^w,\varphi_0^w} = q^{-b_0}\chi_{A_0^w},\phi_0^w$. Also we note that (7.9.3) holds only under some restriction that $q > q_0$ for some constant $q_0$ (see [loc. cit.]). However, this restriction on $q$ can be removed by using a similar method as in [S1] based on the Shintani descent identity of character sheaves. Since this argument will appear in the proof of Proposition 9.12 in a more extended form, we omit the details here.)

Since $\chi_{A_0^w,\phi_0^w}$ coincides with the almost character $R_{z,\eta^{-1}}$ up to scalar, we see by Proposition 4.10 together with (7.9.2) and (7.9.3), that

(7.9.4) $\chi_{A_E,\phi_{A_E}}$ coincides with $R_E$ up to scalar.

The above argument implies that $\chi_{A,\phi_A}$ is identified with some almost character of $G^F$ up to scalar unless $A$ is cuspidal. So we assume that $\widehat{G}_0^F \neq \emptyset$. Let $\mathcal{V}_0$ be the subspace of $\mathcal{V}_G$ spanned by $\chi_{A,\phi_A} \in \widehat{G}_0^F$. Then in view of the previous discussion, $\mathcal{V}_0$ coincides with the subspace of $\mathcal{V}_G$ spanned by almost characters of $G^F$ obtained from $F$-stable cuspidal irreducible characters of $G^{F^m}$ by Shintani descent. Now by Lemma 7.5, $R_{z,\eta}$ is characterized as the unique function in $\mathcal{V}_0$ satisfying the property of inner product with various $\varGamma_{c,\xi,\theta}$ for $(c,\xi) \in \overline{\mathcal{M}}_{s',N}$ with $\theta = \varDelta(\widetilde{\rho}_{\dot{s}',E})|_{Z_G^F}$. By Lemma 7.7, $\chi_{\mathcal{E},z,\eta}$ is also characterized by the inner product with $\varGamma_{c,\xi,\theta}$. Hence by comparing the formulas in Lemma 7.5 and Lemma 7.7, we see that

$$\chi_{\mathcal{E},z,\eta} = \zeta_{\mathcal{I}_0}^{-1}\eta(c_0)^{-1}\psi(\hat{z}^{-1})\theta(\dot{z})R_{z,\eta^{-1}}.$$

Here we claim that $\psi(\hat{z}) = \theta(\dot{z})$. In fact, let $z_0$ be an element in $Z_{G^*}$ corresponding to $\mathcal{E}' \in \mathcal{S}(G/G_{\mathrm{der}})$, and $\dot{z}_0 \in Z_{\widetilde{G}^*}$ a representative of $z_0$. Let $\widetilde{\psi}'$ be the linear character of $\widetilde{G}^{F^m}$ corresponding to $\dot{z}_0$ for a large $m$. The restriction of $\widetilde{\psi}'$ on $T^{F^m}$ gives the character $\psi$. But it follows from the discussion on $\varDelta(\widetilde{\rho}_{\dot{s}',E})$ (see 2.4) that the restriction $\widetilde{\psi}$ of $\widetilde{\psi}'$ on $Z_{\widetilde{G}}^{F^m}$ is also $F$-stable, and the restriction of $Sh_{F^m/F}(\widetilde{\psi})$ on $Z_G^F$ gives the character $\theta$. This shows that $\psi(\hat{z}) = \widetilde{\psi}(\hat{z}) = \theta(\dot{z})$, and the claim follows. Thus we have proved

$$\chi_{\mathcal{E},z,\eta} = \zeta_{\mathcal{I}_0}^{-1}\eta(c_0)^{-1}R_{z,\eta^{-1}}, \tag{7.9.5}$$

and the assertion (ii) follows.

In order to complete the proof, we have only to show the assertion on the scalars $\nu_x$. The assertion is certainly true for the case of cuspidal character sheaves by (ii). In the general case, this scalar $\nu_x$ is given by Proposition 4.10 as $\nu_x = \nu_0\mu_{\widetilde{\delta},y}^{-1}\mu_{\widetilde{E}}$ under the notation of Proposition 4.10, where $\chi_{A_0^{w\delta y}} = \nu_0 R_{\delta,y}$, and $\chi_{A_x} = \nu_x R_x$ (here $R_x = R_E$). Note that $\mu_{\widetilde{\delta},y}$ and $\mu_{\widetilde{E}}$ are determined as in Theorem 4.7, by the choice a representative $\dot{c} \in T^F$ of $c \in A_\lambda^F$ (for a fixed $F$) and of an extension field $\mathbf{F}_{q^m}$; $m$ is chosen sufficiently divisible so that the Shintani descent gives the almost characters. But the argument in [S2] shows that the requirement for $m$ only comes from the Shintani descent of modified generalized Gelfand-Graev characters, and so we can choose $m$ independent of the base field $\mathbf{F}_q$. Hence $\mu_{\widetilde{\delta},y}$ and $\mu_{\widetilde{E}}$ are root of unities contained in a fixed cyclotomic field (cf. Theorem 4.7). This proves the assertion (i), and the theorem follows. $\square$

## 8. Parametrization of almost characters

**8.1.** Theorem 7.9 is based on the parametrization in terms of the induction from cuspidal character sheaves, and its counter part for almost characters. However, in order to decompose almost characters into irreducible characters, one needs the parametrization of almost characters given in 4.5. In this section, we discuss the relationship between these two parametrizations. In the remainder of this paper, we assume that $\widetilde{G} = GL_n$ and $G = SL_n$, for simplicity. The general case is dealt with similarly.

We consider the following semisimple element $\dot{s}$ in $\widetilde{G}^*$, which is a more general type than (7.1.1).

(8.1.1) Let $t$ be a divisor of $n$ prime to $p$. Take $\dot{s} \in \widetilde{T}^*$ such that

$$\dot{s} = \mathrm{Diag}(\underbrace{1, \dots, 1}_{n/t\text{-times}}, \underbrace{\zeta, \dots, \zeta}_{n/t\text{-times}}, \dots, \underbrace{\zeta^{t-1}, \dots, \zeta^{t-1}}_{n/t\text{-times}}),$$

where $\zeta$ is a primitive $t$-th root of unity in $k$. Put $s = \pi(\dot{s})$.

Then $W_{\dot{s}} \simeq \mathfrak{S}_{n/t} \times \cdots \times \mathfrak{S}_{n/t}$ ($t$-times), and $\Omega_s$ is a cyclic group of order $t$ generated by $w_0 \in W$ which permutes the factors of $W_{\dot{s}}$ cyclicly. Hence $W_s$ is of the form as given in (2.5.1). We note that the class $\{s\}$ is the unique class in $G^*$ satisfying (2.5.1) for a fixed $t$. We now assume that $F^m$ acts trivially on $Z_G$. Since $F^m(\zeta) = \zeta$, we have $F^m(\dot{s}) = \dot{s}$. Then for $x = w_0^i \in \Omega_s$, $\dot{s}_x \in \widetilde{T}^{*xF^m}$ is defined as $\dot{s}_x = \dot{s}z_x$ with $z_x \in Z_{\widetilde{G}^*}$ such that $z_x^{q^m-1} = \dot{s}^{-1}x^{-1}\dot{s}x = \mathrm{Diag}(\zeta^{-i}, \dots, \zeta^{-i}) \in Z_{\widetilde{G}^*}$. Let $d$ be a divisor of $t$, and consider the Levi subgroup $\widetilde{L}$ in $\widetilde{G}$ such that $\widetilde{L} \simeq GL_d \times \cdots \times GL_d$ ($n/d$-times). Let $\dot{s}_L$ be a regular semisimple element in $\widetilde{L}^*$ defined as follows; under the isomorphism $\widetilde{L}^* \simeq GL_d \times \cdots \times GL_d$, $\dot{s}_L$ is a product of $\mathrm{Diag}(1, \zeta_0, \dots, \zeta_0^{d-1})$, where $\zeta_0 = \zeta^{t/d}$. Put $s_L = \pi(\dot{s}_L)$ under the natural map $\pi : \widetilde{L}^* \to L^*$. Then one sees easily that there exists $\dot{z}_L \in Z_{\widetilde{L}^*}$ such that $\dot{s}_L\dot{z}_L$ is $W$-conjugate to $\dot{s}$. Here $\Omega_{s_L}$ is a cyclic group of order $d$ generated by $w_{0,L} \in W_L$, and there exists an injective homomorphism $\Omega_{s_L} \to \Omega_s$ such that the image of $w_{0,L}$ coincides with $w_0^{t/d}$. For any $y \in \Omega_{s_L}$, one can define $(\dot{s}_L)_y = \dot{s}_L z'_y$ for some $z'_y \in Z_{\widetilde{G}^*}$ as in the case of $G$. It follows that $(\dot{s}_L)_y z_L$ is $W$-conjugate to $\dot{s}z'_y$. If $y = (w_{0,L})^i$, then $(z'_y)^{q^m-1} = \mathrm{Diag}(\zeta_0^i, \dots, \zeta_0^i)$, and we may choose $z'_y = z_{y^{t/d}}$. Summing up the above argument, we have

(8.1.2) For each generator $y$ of $\Omega_{s_L}$, the class $\{(\dot{s}_L)_y\dot{z}_L\}$ in $\widetilde{G}^*$ gives rise to a class $\{\dot{s}_x\}$ for some $x \in \Omega_s$ such that the order of $x$ is $d$. The correspondence $\{(\dot{s}_L)_y\} \to \{\dot{s}_x\}$ gives a bijection between the classes in $\widetilde{L}^*$ associated to $y \in \Omega_{s_L}$ of order $d$, and the classes in $\widetilde{G}^*$ associated to $x \in \Omega_s$ of order $d$.

**8.2.** We write $\dot{s}_L\dot{z}_L$ in a more explicit way. $\dot{z}_L \in Z_{\widetilde{L}^*}$ can be written as

$$\dot{z}_L = (a, \dots, a) \quad \text{with } a = (1, \zeta, \dots, \zeta^{t/d-1}) \in (k^*)^{t/d}$$

under the isomorphism $Z_{\widetilde{L}^*} \simeq (k^*)^{t/d} \times \cdots \times (k^*)^{t/d}$ ($n/t$-times). Then we have

$$\dot{s}_L\dot{z}_L = (a_1, \dots, a_{t/d-1}, a_1, \dots, a_{t/d-1}, \dots, a_1, \dots, a_{t/d-1})$$

with

$$a_j = \mathrm{Diag}(\zeta^{j-1}, \zeta^{t/d+j-1}, \zeta^{2t/d+j-1}, \dots, \zeta^{(d-1)t/d+j-1}) \in GL_d.$$

From this, we see easily that there exists $w_L \in N_W(W_L)$ such that $Fw_L(\dot{s}_L\dot{z}_L) = \dot{s}_L\dot{z}_L$. Then $\dot{z}_L$ is also $Fw_L$-stable modulo $(Z_{\widetilde{L}^*})_d$, where $(Z_{\widetilde{L}^*})_d = \{z \in Z_{\widetilde{L}^*} \mid z^d = 1\}$. We choose $w_L$ in a standard way. Hence $w_L$ gives the permutation of factors in each block $a_1, \dots, a_{t/d-1}$. We put $F'' = Fw_L$. Take $m$ large enough so that $\dot{z}_L \in Z_{\widetilde{L}^*}^{F^m}$. Let $\widetilde{\theta}$ be the linear character of $\widetilde{L}^{F^m}$ corresponding to $\dot{z}_L \in Z_{\widetilde{L}^*}^{F^m}$, and $\theta$ the restriction of $\widetilde{\theta}$ to $L^{F^m}$. Then by Lemma 1.2, $\widetilde{\theta}|_{Z_{\widetilde{L}}^{F^m}}$ is $F''$-stable, and so $\theta_1 = \theta|_{Z_L^{0F^m}}$ is also $F''$-stable. We put $\widetilde{\theta}_1 = \widetilde{\theta}|_{Z_{\widetilde{L}}^{F^m}}$.

Let us denote by $z_L \in L^*$ the image of $\dot{z}_L$ under the natural map $\widetilde{L}^* \to L^*$. We consider the parameter set $\mathcal{M}^L_{s_Lz_L,N_0}$ (a similar set as $\mathcal{M}_{s,N}$ in 4.5, defined by replacing $G, F$ by $L, F''$) with respect to the $F''$-stable regular semisimple class $\{s_Lz_L\}$ in $L^*$, where $N_0$ is a regular nilpotent element in $\mathrm{Lie}\, L$. Let $\delta_{z,\eta} (= \rho_{z,\eta})$ be a cuspidal irreducible character of $L^{F^m}$, stable by $F''$ corresponding to $(z, \eta) \in \mathcal{M}^L_{s_Lz_L,N_0}$. Then there exists a cuspidal irreducible character $\widetilde{\delta} = \widetilde{\rho}_{(\dot{s}_L)_y\dot{z}_L,1}$ of $\widetilde{L}^{F^m}$ such that $\delta_{z,\eta}$ is an irreducible constituent of the restriction of $\widetilde{\delta}$ to $L^{F^m}$. Note that $\widetilde{\delta}$ can be written as $\widetilde{\delta} = \widetilde{\theta} \otimes \widetilde{\delta}'$ for $\widetilde{\delta}' = \widetilde{\rho}_{(\dot{s}_L)_y,1}$. The class $\{s_L\}$ is $F''$-stable, and there exists a cuspidal irreducible character $\delta'_{z,\eta}$ parametrized by $(z, \eta) \in \mathcal{M}^L_{s_L,N_0}$, which is a constituent of $\widetilde{\delta}'|_{L^{F^m}}$, such that $\delta_{z,\eta} = \theta \otimes \delta'_{z,\eta}$. We consider the Harish-Chandra induction $\widetilde{I} = \mathrm{Ind}_{\widetilde{P}^{F^m}}^{\widetilde{G}^{F^m}}(\widetilde{\delta})$ and its restriction $I$ on $G^{F^m}$. By (8.1.1), irreducible components of $I$ belong to $\mathcal{E}(G^{F^m}, \{s\})$.

By (6.7.2) and (7.9.1) we have $\mathcal{W}_\delta \simeq \mathcal{W}_{\theta_1}$. Since $\mathcal{W}_{\widetilde{\delta}} \simeq \mathcal{W}^0_\delta$, we see that $\mathcal{W}_{\widetilde{\delta}} \simeq \mathcal{W}^0_{\theta_1}$. We have $\mathcal{W}_{\theta_1} \simeq \mathcal{W}^0_{\theta_1} \rtimes \Omega_{\theta_1}$, where $\mathcal{W}^0_{\theta_1}$ is given as in

$$\mathcal{W}^0_{\theta_1} \simeq \mathfrak{S}_{n/t} \times \cdots \times \mathfrak{S}_{n/t} \quad (t/d\text{-times}). \tag{8.2.1}$$

and $\Omega_{\theta_1}$ is the cyclic group of order $t/d$ generated by an element $y_0 \in \mathcal{W}$ permuting the factors $\mathfrak{S}_{n/t}$ cyclicly. Now $\widetilde{I}$ is decomposed as

$$\widetilde{I} = \sum_{E_1 \in (\mathcal{W}^0_{\theta_1})^\wedge} (\dim E_1)\widetilde{\rho}_{E_1}, \tag{8.2.2}$$

where $\widetilde{\rho}_{E_1}$ is the irreducible characters of $\widetilde{G}^{F^m}$. But since $\widetilde{\rho}_{E_1}$ is contained in $\mathcal{E}(\widetilde{G}^{F^m}, \{\dot{s}_x\})$, they are expressed as $\widetilde{\rho}_{\dot{s}_x,E'_1}$ with $E'_1 \in W_{\dot{s}}$. The relationship of these two parametrizations are given as follows. We have $W_{\dot{s}_x} \simeq (\mathfrak{S}_{n/t})^t$, and there exists a natural embedding $\mathcal{W}^0_{\theta_1} \to W_{\dot{s}_x}$ via the diagonal embedding

$$(\mathfrak{S}_{n/t})^{t/d} \hookrightarrow (\mathfrak{S}_{n/t})^t \simeq (\mathfrak{S}_{n/t})^{t/d} \times \cdots \times (\mathfrak{S}_{n/t})^{t/d}.$$

Then one can define a map $f : (\mathcal{W}_{\theta_1}^0)^\wedge \to W_{\dot{s}}^\wedge$ through $E_1 \mapsto E_1 \boxtimes \cdots \boxtimes E_1$. We note that

(8.2.3) $\widetilde{\rho}_{E_1} = \widetilde{\rho}_{\dot{s}_x, E_1'}$ with $E_1' = f(E_1) \in W_{\dot{s}_x}^\wedge$.

In fact, let $w_1$ be the Coxeter element in $W_L$, and $\psi_0$ a regular character of $\widetilde{T}_{w_1}^F \simeq \widetilde{T}^{Fw_1}$ obtained by the Shintani descent $Sh_{F^m/Fw_1}(\theta|_{\widetilde{T}^{F^m}})$. The cuspidal character $\widetilde{\delta}$ can be expressend as $\pm R_{\widetilde{T}_{w_1}}^{\widetilde{L}}(\psi_0)$. Thus $\widetilde{I}$ coincides with $\pm R_{\widetilde{L}}^{\widetilde{G}}(R_{\widetilde{T}_{w_1}}^{\widetilde{L}}(\psi_0))$, and

$$R_{\widetilde{L}}^{\widetilde{G}}(R_{\widetilde{T}_{w_1}}^{\widetilde{L}}(\psi_0)) = R_{\widetilde{T}_{w_1}}^{\widetilde{G}}(\psi_0) = \sum_{E_1' \in W_{\dot{s}}^\wedge} \mathrm{Tr}\,(w_1, E_1')\widetilde{\rho}_{E_1'}. \tag{8.2.4}$$

Then we have $\mathrm{Tr}\,(w_1, E_1') = \dim E_1$ if $E_1' = E_1 \boxtimes \cdots \boxtimes E_1$ for some $E_1 \in (\mathcal{W}_{\theta_1}^0)^\wedge$, and $\mathrm{Tr}\,(w_1, E_1') = 0$ otherwise. Comparing (8.2.2) with (8.2.4), we see that $\widetilde{\rho}_{E_1} = \widetilde{\rho}_{E_1'}$ for $E_1 \in (\mathcal{W}_{\theta_1}^0)^\wedge$. This proves (8.2.3).

Put $\delta = \delta_{z,\eta}$. Then $\mathcal{W}_\delta \simeq \mathcal{W}_{\theta_1}$. Take $E \in (\mathcal{W}_\delta^\wedge)^{F''}$, and put $E_1 = E|_{\mathcal{W}_\delta^0}$. We assume that $E_1$ is of the form that

$$E_1 = E_2 \boxtimes \cdots \boxtimes E_2 \in (\mathcal{W}_{\theta_1}^0)^\wedge \tag{8.2.5}$$

with $E_2 \in (\mathfrak{S}_{n/t})^\wedge$. Then $E_1' = f(E_1) \in W_{\dot{s}}^\wedge$ satisfies the condition in (2.5.1). Let $\rho_E$ be the irreducible component of $\mathrm{Ind}_{P^{F^m}}^{G^{F^m}} \delta$ corresponding to $E \in (\mathcal{W}_\delta^\wedge)^F$. We denote by $R_E$ the almost character of $G^F$ corresponding to $\rho_E$ through the Shintani descent. Let $\mathcal{O}_{\widetilde{\rho}}$ be the nilpotent orbit in $\mathfrak{g}$ associated to $\widetilde{\rho} = \widetilde{\rho}_{\dot{s}_x, E_1'}$. We choose a nilpotent element $N$ such that $\mathcal{O}_{\widetilde{\rho}} = \mathcal{O}_N$. Let $\widetilde{P}_N$ and $\widetilde{L}_N$ be the associated parabolic subgroup and its Levi subgroup in $\widetilde{G}$. We put $\widetilde{M} = \widetilde{L}_N$ and $M = \widetilde{M} \cap G$. We consider the modified generalized Gelfand-Graev characters $\varGamma_{c,\xi}$ of $G^F$ associated to $(c, \xi) \in \overline{\mathcal{M}}_{s,N}$. A similar formula as in Lemma 7.5 holds also for the general $\overline{\mathcal{M}}_{s,N}$, which implies the following lemma. (Note that we don't need to consider $\varGamma_{c,\xi,\theta}$ since $\varDelta(\widetilde{\rho})|_{Z_M(\lambda)^F}$ is trivial for any $\widetilde{\rho} = \widetilde{\rho}_{\dot{s}_x, E_1'}$ in the case of $G = SL_n$ by [S2, Lemma 2.18].)

**Lemma 8.3.** *The almost character $R_E$ is written as $R_{z',\eta'}$ with $(z', \eta') \in \mathcal{M}_{s,N}$. The correspondence $E \leftrightarrow (z', \eta')$ is determined by computing the inner product $\langle \varGamma_{c,\xi}, R_E \rangle$ for various $\varGamma_{c,\xi}$.*

**8.4.** Let $C_0$ be the regular unipotent class in $L$, and $\mathcal{E}$ the local system of $L$ corresponding to the linear character $\theta$ of $L^{F^m}$ in 8.2. Let $A_0 = A_{\mathcal{E},z,\eta}$ be the $F''$-stable cuspidal character sheaf on $L$ associated to $(z, \eta) \in \mathcal{M}_{s_L z_L, N_0}^L$ and $\mathcal{E}$, defined as in 6.5, by replacing $G$ by $L$. Put $K = \mathrm{ind}_P^G A_0$. Then $\mathrm{End}\, K \simeq \bar{\mathbf{Q}}_l[\mathcal{W}_{\mathcal{E}_1}]$ with $\mathcal{W}_{\mathcal{E}_1} \simeq \mathcal{W}_{\theta_1}$. We denote by $A_E$ the character sheaf of $G$ which is a direct summand of $K$ corresponding to $E \in \mathcal{W}_{\theta_1}^\wedge$. Put $\mathcal{Z}_{\theta_1} = \{w \in \mathcal{W} \mid {}^{Fw}\theta_1 = \theta_1\}$. Then $\mathcal{Z}_{\theta_1} = w_L \mathcal{W}_{\theta_1}$, and $F'' = Fw_L$ acts on $\mathcal{W}_{\theta_1}$. For each $w \in \mathcal{Z}_{\theta_1}$, we denote by $\theta_0^w$ the linear character of $Z_L^{0Fw}$ given by $\theta_0^w = Sh_{F^m/Fw}(\theta_1)$.

Let $E \in \mathcal{W}_{\theta_1}^\wedge$ and $\mathcal{O}_N$ be as in 8.2. By Theorem 7.9 (see (7.9.4)), $\chi_{A_E}$ coincides with $R_E$ up to scalar. Then we see that $N = N_\mu \in \mathfrak{g}$ is of the form such that $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ with $\mu_j$ divisible by $t$. We choose $N_\mu$ in Jordan's normal form corresponding to the partition $\mu$. Then $A_\lambda = Z_M(\lambda)/Z_M^0(\lambda)$ is a cyclic group of order $t'$, where $t'$ is the largest common divisor of $\mu_1, \dots, \mu_k$, coprime to $p$. Note that $Z_M(\lambda)$ is $F$-stable. By our choice of $\dot{s}z_L$ and of $w_L$, $w_L$ acts on $Z_M(\lambda)$, and so we have an action of $F'' = Fw_L$ on $Z_M(\lambda)$.

There exists an $F''$-stable subgroup $Z_M^1(\lambda)$ of $Z_M(\lambda)$ such that $\bar{A}_\lambda = Z_M(\lambda)/Z_M^1(\lambda)$ is a cyclic group of order $t$ (see [S2, 2.19]), which is given as follows; if we write the partition $\mu$ as $\mu = (1^{m_1}, 2^{m_2}, \dots)$. then we have

$$Z_M^1(\lambda) \simeq \{(\dots, x_i, \dots) \in \prod_{m_i > 0} GL_{m_i} \mid \prod_i (\det x_i)^{i/t} = 1\}.$$

(Note that $m_i = 0$ if $i$ is not divisible by $t$.) We may choose $T \subset M \cap L$. Then we have

$$\bar{A}_\lambda \simeq Z_M(\lambda)/Z_M^1(\lambda) \simeq (Z_M(\lambda) \cap T)/(Z_M^1(\lambda) \cap T).$$

By our choice of $N$, we have $Z_M(\lambda) \cap T \subset Z_L$. On the other hand, the above description of $Z_M^1(\lambda)$ implies that $Z_M^1(\lambda) \cap T \subset Z_L^0$. It follows that we have a natural homomorphism

$$\bar{A}_\lambda \to Z_L/Z_L^0 \tag{8.4.1}$$

compatible with the action of $F$. Note that $\eta$ is an $F$-stable linear character of $Z_L/Z_L^0$. Thus one can define an $F$-stable linear character $\eta_1 \in (\bar{A}_\lambda^\wedge)^F$ as the pull back of $\eta$ under the above homomorphism..

Since $T \subset M$, we have $Z_L \subset M$. One can write

$$Z_L^0 = \{(a_1, \dots, a_{n/d}) \in (k^*)^{n/d} \mid \prod_i a_i = 1\} \tag{8.4.2}$$

under the identification

$$Z_L \simeq \{(a_1, \dots, a_{n/d}) \in (k^*)^{n/d} \mid \prod_i a_i^d = 1\}.$$

Let $zZ_L^0$ be a coset in $Z_L/Z_L^0$. We may take a representative $z$ in $Z_G$. Then we have $Z_M(\lambda) \cap zZ_L^0 = z(Z_M(\lambda) \cap Z_L^0)$, and

$$\begin{aligned} Z_M(\lambda) \cap Z_L^0 &= \{(\underbrace{b_1, \dots, b_1}_{\mu_1/d\text{-times}}, \underbrace{b_2, \dots, b_2}_{\mu_2/d\text{-times}}, \dots, \underbrace{b_r, \dots, b_r}_{\mu_r/d\text{-times}}) \in Z_L^0 \mid b_i \in k^*\} \\ &\simeq \{(b_1, \dots, b_r) \in (k^*)^r \mid \prod_i b_i^{\mu_i/d} = 1\}, \end{aligned} \tag{8.4.3}$$

and

$$Z_M^1(\lambda) \cap Z_L^0 \simeq \{(b_1, \dots, b_r) \in (k^*)^r \mid \prod_i b_i^{\mu_i/t} = 1\}.$$

Now $Z^1_M(\lambda)\cap Z^0_L$ acts on $z(Z_M(\lambda)\cap Z^0_L)$ by a left multiplication. Put $X_M=(Z_M(\lambda)\cap zZ^0_L)/(Z^1_M(\lambda)\cap Z^0_L)$. It follows from the above argument that there exists a bijection

$$X_M \simeq \mathbf{Z}/(t/d)\mathbf{Z}. \tag{8.4.4}$$

If $zZ^0_L$ is $F''$-stable, then $F''$ acts naturally on $X_M$, which is compatible with the natural action of $F$ on $\mathbf{Z}/(t/d)\mathbf{Z}\simeq\langle\zeta^d\rangle$.

Assume that the coset $zZ^0_L$ is $Fw$-stable for $w\in\mathcal{W}$. Then we choose a representative $\dot z_w\in Z^{Fw}_L$ of $zZ^0_L$. Since $\dot z_w\in T^{Fw}$, there exists $\alpha_w\in T$ such that $F^m(\alpha_w)\alpha_w^{-1}=\dot z_w$. We put $\hat z_w=\alpha_w^{-1Fw}\alpha$. Then we have $\hat z_w\in T^{F^m}$. Let $\psi$ be a linear character of $T^{F^m}$ obtained by restricting $\theta$ to $T^{F^m}$. The value $\psi(\hat z_w)$ does not depend on the choice of $\alpha_w\in T$.

We show the following lemma.

**Lemma 8.5.** *Let the notations be as before.*

(i) *Let $t_1\in Z_G\cap zZ^0_L$. There exists a linear character $\Psi_{t_1}$ of $\mathcal{W}_{\theta_1}$ satisfying the following properties; $\Psi_{t_1}$ is trivial on $\mathcal{W}^0_{\theta_1}$, and is regarded as a character of $\Omega_{\theta_1}$. If $t_1\in(zZ^0_L)^{F''}$, then $\Psi_{t_1}$ is $F''$-stable, and in that case, we have $t_1\in(zZ^0_L)^{Fw}$ for $w=w_Ly$ with $y\in\mathcal{W}_{\theta_1}$, and*

$$\psi(\hat z_w)\theta^w_0(t_1\dot z_w^{-1})=\psi(\hat z_{w_L})\theta^{w_L}_0(t_1\dot z_{w_L}^{-1})\Psi_{t_1}(y).$$

(ii) *$\Psi_{t_1}$ depends only on the coset in $X_M$ to which $t_1$ belongs, and we have*

$$\{\Psi_{t_1}\mid t_1\in X^{F''}_M\}=(\Omega^\wedge_{\theta_1})^{F''}.$$

*Proof.* Put $w=w_Ly$ for $y\in\mathcal{W}_{\theta_1}$. For $t_1\in Z_G\cap zZ^0_L$, choose $\alpha\in Z_{\widetilde L}$ such that $t_1=F^m(\alpha)\alpha^{-1}$. We choose $m'$, a multiple of $m$, such that $\alpha$ is $F^{m'}$-stable. One can find a linear character $\widetilde\theta'_1$ on $Z^{F^{m'}}_{\widetilde L}$, which is an extension of $\widetilde\theta_1$ on $Z^{F^m}_{\widetilde L}$ (see 8.2), stable by $F''$, such that $\mathcal{W}^0_{\theta_1}\subseteq\mathcal{W}_{\widetilde\theta'_1}$. We put $\Psi_{t_1}(y)=\widetilde\theta'_1(\alpha^{-1y}\alpha)$ (for a fixed $\alpha$ and $\widetilde\theta'_1$). We have $\widetilde\theta'_1(\alpha^{-1y}\alpha)=\widetilde\theta'_1(\alpha^{-1})\widetilde\theta'_1({}^y\alpha)=1$ for $y\in\mathcal{W}^0_{\theta_1}$, and so $\Psi_{t_1}$ is trivial on $\mathcal{W}^0_{\theta_1}$. It follows that $\Psi_{t_1}(y)=\Psi_{t_1}(y_2)$ if $y=y_1y_2$ with $y_1\in\mathcal{W}^0_{\theta_1},y_2\in\Omega_{\theta_1}$. Hence in order to show that $\Psi_{t_1}$ is a homomorphism $\mathcal{W}_{\theta_1}\to\bar{\mathbf{Q}}_l^*$, it is enough to see that (*) $\Psi_{t_1}$ is a homomorphism on $\Omega_{\theta_1}$. This will be shown soon later.

Now assume that $t_1\in(zZ^0_L)^{F''}$. Since $z\in Z_G$, we have $t_1\in(zZ^0_L)^{Fw}=\dot z_wZ^{0Fw}_L$. Choose $\beta_w\in Z^0_L$ such that $F^m(\beta_w)\beta_w^{-1}=t_1\dot z_w^{-1}\in Z^{0Fw}_L$. Then $\theta^w_0(t_1\dot z_w^{-1})=\theta_1(\beta_w^{-1Fw}\beta_w)$. Since $t_1=F^m(\alpha)\alpha^{-1}$, we may assume that $\alpha_w=\alpha\beta_w^{-1}$. In particular, we have $\alpha^{-1Fw}\alpha\in T^{F^m}$ and $\psi(\hat z_w)\theta^w_0(t_1\dot z_w^{-1})=\psi(\alpha^{-1Fw}\alpha)$. A similar formula also holds for $w_L$. Since $\alpha^{-1Fw}\alpha=\alpha^{-1Fw_L}\alpha\cdot{}^{Fw_L}(\alpha^{-1y}\alpha)$, we see that $\alpha^{-1y}\alpha\in T^{F^m}$. Since $\widetilde\theta_1$ and $\psi$ coincides with each other on $T^{F^m}\cap Z^{F^m}_{\widetilde L}$, we have

$$\begin{aligned}\psi(\hat z_w)\theta^w_0(t_1\dot z_w^{-1})&=\psi(\alpha^{-1Fw}\alpha)\\&=\psi(\alpha^{-1Fw_L}\alpha)\psi({}^{Fw_L}(\alpha^{-1y}\alpha))\\&=\psi(\hat z_{w_L})\theta^{w_L}_0(t_1\dot z_{w_L}^{-1})\Psi_{t_1}(y).\end{aligned}$$

Hence the formula in (i) holds. In particular, $\Psi_{t_1}(y)$ does not depend on the choice of $\alpha \in Z_{\widetilde{L}}$. Put $\alpha' = F''(\alpha)$. We have $t_1 = F^m(\alpha)\alpha^{-1} = F^m(\alpha')(\alpha')^{-1}$ since $t_1$ is $F''$-stable, and one can replace $\alpha$ by $\alpha'$ in defining $\Psi_{t_1}$. Since $\widetilde{\theta}'_1$ is $F''$-stable, we have

$$\Psi_{t_1}(y) = \widetilde{\theta}'_1(\alpha^{-1y}\alpha) = \widetilde{\theta}'_1(\alpha'^{-1F''(y)}\alpha') = \Psi_{t_1}(F''(y)).$$

This shows that $\Psi_{t_1}$ is $F''$-stable. Thus the assertion in (i) was proved modulo (*).

Next we show (ii). Take $t_1 \in Z_G \cap (Z_M(\lambda) \cap zZ_L^0)$. Then as in (8.4.2) $t_1$ can be written as $t_1 = zb$ with $b \in Z_M(\lambda) \cap Z_L^0$. Hence $b = (b_1, \dots, b_r)$ such that $\prod_i b_i^{\mu_i/d} = 1$ with $b_i \in k^*$. One can choose $\alpha \in T$ such that $F^m(\alpha)\alpha^{-1} = t_1$ as follows; $\alpha = \beta\gamma$ with $\beta \in Z_L^0$ and $\gamma \in Z_{\widetilde{G}}$ such that $F^m(\gamma)\gamma^{-1} = z$, $F^m(\beta)\beta^{-1} = b$. More precisely, we can choose $\beta$ as

$$\beta = (\nu(t_1)^{-1}\beta_1, \underbrace{\beta_1, \dots, \beta_1}_{\mu_1/d-1\text{-times}}, \underbrace{\beta_2, \dots, \beta_2}_{\mu_2/d\text{-times}}, \dots, \underbrace{\beta_r, \dots, \beta_r}_{\mu_r/d\text{-times}}), \tag{8.5.1}$$

where $\beta_i^{q^m-1} = b_i$ for $i = 1, \dots, r$ and $\nu(t_1) = \prod_{i=1}^r \beta_i^{\mu_i/d} \in \mathbf{F}_{q^m}^*$. Let $y_0$ be the generator of $\Omega_\theta$ as in 8.2. Since $y_0$ is a cyclic permutation of order $t/d$ of consecutive factors, it makes no change except the part $\nu(t_1)^{-1}\beta_1, \beta_1, \dots \beta_1$. Hence we see that

$$\beta^{-1y_0^j}\beta = (\nu(t_1), 1, \dots, 1, \nu(t_1)^{-1}, 1, \dots, 1) \in Z_L^{0F^m} \tag{8.5.2}$$

for $1 \le j \le t/d - 1$, where $\nu(t_1)^{-1}$ occurs in the $(j+1)$th factors. Moreover, since $\gamma \in Z_{\widetilde{G}}$, we have ${}^{y_0}\gamma = \gamma$. It follows that

$$\alpha^{-1y_0^j}\alpha = \beta^{-1y_0^j}\beta.$$

On the other hand, $\theta$ is the restriction of the linear character $\widetilde{\theta}$ of $(\widetilde{L}/\widetilde{L}_{\text{der}})^{F^m}$ such that $\widetilde{\theta} = (1 \boxtimes \widetilde{\Theta} \boxtimes \cdots \boxtimes \widetilde{\Theta}^{t/d-1})^{\boxtimes n/t}$, where $\widetilde{\Theta}$ is a linear character of $GL_d^{F^m}$ of order $t$. Since $\widetilde{L}/\widetilde{L}_{\text{der}} \simeq Z_{\widetilde{L}}/(Z_{\widetilde{L}} \cap \widetilde{L}_{\text{der}})$, and $\widetilde{L}_{\text{der}} = L_{\text{der}} = SL_d \times \cdots \times SL_d$, the linear character $\widetilde{\theta}_1$ on $Z_{\widetilde{L}}^{F^m}$ can be written as

$$\widetilde{\theta}_1 = (1 \boxtimes \Theta \boxtimes \cdots \boxtimes \Theta^{t/d-1})^{\boxtimes n/t}, \tag{8.5.3}$$

where $\Theta$ is a linear character of $Z_{GL_d}^{F^m}$ which is trivial on $Z_{SL_d}^{F^m}$. Hence $\Theta$ is identified with a linear character of $\mathbf{F}_{q^m}^*$ of order $t/d$. If we replace $m$ by $m'$, $\Theta$ can be extended to a linear character $\Theta'$ of $\mathbf{F}_{q^{m'}}^*$, and $(1 \boxtimes \Theta' \boxtimes \cdots \boxtimes \Theta'^{t/d-1})^{\boxtimes n/t}$ gives rise to a linear character of $Z_{\widetilde{L}}^{F^{m'}}$, which gives $\widetilde{\theta}'_1$. It follows that

$$\Psi_{t_1}(y_0^j) = \widetilde{\theta}'_1(\alpha^{-1y_0^j}\alpha) = \Theta(\nu(t_1))^{-j}. \tag{8.5.4}$$

This proves (*) since if $t_1 \in Z_G \cap zZ_L^0$, then $t_1 \in Z_G \cap (Z_M(\lambda) \cap zZ_L^0)$.

Now assume that $t_1 \in Z_M^1(\lambda) \cap Z_L^0$. Since $\prod_i b_i^{\mu_i/t} = 1$, we have $\nu(t_1)^{(q^m-1)d/t} = 1$. It follows that there exists $\nu_1 \in \mathbf{F}_{q^m}$ such that $\nu(t_1) = \nu_1^{t/d}$, and we have $\Psi_{t_1}(y_0^j) = 1$.

This implies that $\Psi_{t_1}$ depends only on $t_1 \in X_M$. Now $X_M$ is in bijection with a cyclic group of order $t/d$, and we can choose a representative $x_0$ of a generator of $X_M$ as $x_0 = zc_0$ with $c_0 = (c, \dots, c) \in Z_G \cap Z_L^0$ such that $c^{n/t}$ is a primitive $t/d$-th root of unity in $k$. Put $\nu_0 = \nu(x_0)$. Then $\nu_0$ is a generator of the cyclic group $\mathbf{F}_{q^m}^*$, and we have $\nu(x_i) = \nu_0^i$ for $x_i = zc_0^i$. It follows from (8.5.1) that we have $\Psi_{x_i}(y_0) = \varTheta(\nu_0)^{-i}$ for $i = 0, \dots, t/d-1$. Since $\nu_0$ is a generator of $\mathbf{F}_{q^m}^*$, $\varTheta(\nu_0)$ is of order $t/d$ in $\bar{\mathbf{Q}}_l^*$. Hence $\Psi_x(y_0)$ are all distinct for $x \in X_M$. Since $\Omega_{\theta_1}$ is a cyclic group of order $t/d$, we see that there exists a bijection between $\{\Psi_x \mid x \in X_M\}$ and $\Omega_{\theta_1}^\wedge$. We note that $\Psi_x$ is $F''$-stable for $x \in X_M^{F''}$. In fact, take a representative $t_1 \in Z_M(\lambda) \cap zZ_L^0$ of $x \in X_M^{F''}$. Then we have $F''(t_1) = t_1t_2$ with $t_2 \in Z_M^1(\lambda) \cap zZ_L^0$. It follows that $\Psi_{F''(t_1)} = \Psi_{t_1t_2} = \Psi_{t_1}$. Now take $\alpha \in Z_{\widetilde{L}}$ such that $F^m(\alpha)\alpha^{-1} = t_1$, and put $\alpha' = F''(\alpha)$. Since $\widetilde{\theta}_1'$ is $F''$-stable, we have

$$\Psi_{t_1}(y) = \widetilde{\theta}_1'(\alpha^{-1y}\alpha) = \widetilde{\theta}_1'(\alpha'^{-1F''(y)}\alpha').$$

But since $F''(t_1) = F^m(\alpha')\alpha'^{-1}$, we have

$$\widetilde{\theta}_1'(\alpha'^{-1F''(y)}\alpha') = \Psi_{F''(t_1)}(F''(y)) = \Psi_{t_1t_2}(F''(y)) = \Psi_{t_1}(F''(y)).$$

This implies that $\Psi_{t_1}(y) = \Psi_{t_1}(F''(y))$, and so $\Psi_x$ is $F''$-invariant. Actually this argument shows that $\Psi_x$ is $F''$-invariant if and only if $x \in X_M^{F''}$. Thus we have $\{\Psi_x \mid x \in X_M^{F''}\} = (\Omega_{\theta_1}^\wedge)^{F''}$, and (ii) is proved. $\square$

**Theorem 8.6.** *Let $G, s$ and $L, s_L, z_L$ be as in 8.1, 8.2. In particular, $L$ is a Levi subgroup of $G$ such that $L/Z_L$ is a product of $PGL_d$ with $d|t$. Let $\theta$ be the linear character of $L^{F^m}$ corresponding to $z_L \in Z_{L^*}^{F^m}$, and let $\theta_1 = \theta|_{Z_L^{0F^m}}$ be the $F''$-stable linear character. Let $\mathcal{E}$ be the local system on $L$ corresponding to $\theta$.*

(i) *Let $A_{\mathcal{E},z,\eta} = \mathcal{E} \otimes A_{z,\eta}$ be an $F''$-stable cuspidal character sheaf of $L$ as in 7.6 for $(z, \eta) \in \mathcal{M}^L_{z_Ls_L,N_0}$, and $A_E$ be the character sheaf corresponding to $E \in (\mathcal{W}_{\theta_1}^\wedge)^{F''}$, where $E_1 = E|_{\mathcal{W}_{\theta_1}^0}$ satisfies the condition in (8.2.5). Then there exists $z_E \in \bar{A}_\lambda^F$ satisfying the following.*

(8.6.1)
$$\chi_{A_E} = \nu_E R_{z_E,\eta_1^{-1}},$$

*where $(z_E, \eta_1^{-1}) \in \mathcal{M}_{s,N}$ is given as follows; $\eta_1$ is the $F$-stable character of $\bar{A}_\lambda$ defined as the pull back of $\eta$ in 8.4. $z_E$ is determined uniquely by the following condition; let $\iota$ be the unique element of $\mathcal{I}_0$ such that* $\operatorname{supp}(\iota) = \mathcal{O}_N$, *and let $E_\iota \in \mathcal{W}^\wedge$ the corresponding character under the generalized Springer correspondence. Then there exists a unique class $x_E \in X_M^{F''}$ such that*

$$\langle E \otimes \Psi_{t^{-1}}, E_\iota \rangle_{\mathcal{W}_{\theta_1}} = \begin{cases} 1 & \text{if } t \in x_E, \\ 0 & \text{otherwise.} \end{cases}$$

*$z_E$ is the image of $x_E$ under the natural map $X_M^{F''} \to \bar{A}_\lambda^F$. $\nu_E$ is a root of unity, which is determined explicitly (see (9.10.6)).*

*All the almost characters $R_{z',\eta'}$ associated to $(z'\eta') \in \mathcal{M}_{s,N}$ such that $\eta'$ is of order $d$ is obtained in this way from some $A_E$.*

(ii) *Let $\delta = \delta_{z,\eta} = \theta \otimes \delta'_{z,\eta} \in \mathcal{T}^{(m)}_{s_L z_L, N_0}$ be the cuspidal irreducible character of $L^{F^m}$ as in 8.2. Let $\rho_E$ be the $F$-stable irreducible character of $G^{F^m}$ contained in $\mathrm{Ind}_{P^{F^m}}^{G^{F^m}} \delta$ corresponding to $E \in \mathcal{W}_\delta^\wedge$. Then we have $\rho_E = \rho_{z_E, \eta_1^{-1}}$ with $(z_E, \eta_1^{-1}) \in \mathcal{M}_{s,N}$. In particular, we have $R_E = R_{z_E, \eta_1^{-1}}$.*

**Remark 8.7.** Theorem 8.6 gives an identification of two parametrizations of almost characters of $G^F$, one given by the parameter set $\mathcal{M}_{s,N}$ and the other by the Harish-Chandra induction, in the case where $(s, E)$ satisfies the property (2.5.1). The parametrizations of irreducible characters, and of almost characters, are described in 2.8, (a) - (c). The above result covers the case (a). The case (b) is obtained by applying the Harish-Chandra induction for the case (a). Thus the above parametrization can be extended also for this case. The case (c) is obtained by applying the twisted induction for the cases (a) or (b). Since the twisted induction of almost characters corresponds to the induction of character sheaves, our result gives an enough information for decomposing the characteristic functions of character sheaves in terms of irreducible characters of $G^F$ based on our parametrization $\overline{\mathcal{M}}_{s,E}$.

**8.8.** The proof of the theorem will be done in the next section. Here we prove some preliminary results. As discussed in 5.5, we identify the set $\mathcal{I}_G$ with the set of pairs $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and $\mathcal{E}$ is a simple $G$-equivariant local system on $\mathcal{O}$. For $\iota \in \mathcal{I}_G$ belonging to $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G$, we define two integers $b(\iota)$ and $b_0$ as follows (cf. 5.1 and 6.8).

$$
\begin{aligned}
(8.8.1) \qquad b(\iota) &= (a_0 + r)/2 = (\mathrm{codim}_G \,\mathrm{supp}\,(\iota) - \mathrm{codim}_L C_0)/2, \\
b_0 &= (\dim G - \dim \mathrm{supp}\ K_\iota)/2 = (\mathrm{codim}_L C_0 - \dim Z_L)/2.
\end{aligned}
$$

Suppose that $\iota$ belongs to the triple $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_G^F$. Let $z \in (Z_L/Z_L^0)^F$, and we choose a representative $\dot z \in Z_L^F$. By the translation $C_0 \xrightarrow{\sim} \dot z C_0$, we may regard $\mathcal{E}_0$ the $F$-stable local system on $\dot z C_0$. Let $\varSigma = zZ_L^0 \times C_0 = Z_L^0 \times \dot z C_0$, and we follow the setting in 6.7 and 6.8. In particular, $A_0 = \mathcal{E} \otimes A_{z,\eta}$ is the cuspidal character on $L$ associated to $\mathcal{E}_0 = \mathcal{E}_\eta$, and to the local system $\mathcal{E}$ on $L$ which is the pull back of $\mathcal{E}' \in \mathcal{S}(L/L_{\mathrm{der}})$. We put $\mathcal{E}_1 = \mathcal{E}|_{zZ_L^0}$ as before, and consider the corresponding character $\theta_1$ of $Z_L^{0F^m}$, etc. We identify $\mathcal{W}_{\mathcal{E}_1}$ with $\mathcal{W}_{\theta_1}$ and with $\mathcal{W}_\delta$, and similarly for $\mathcal{Z}_{\mathcal{E}_1}$ with $\mathcal{Z}_{\theta_1}$, $\mathcal{Z}_\delta$. We write the automorphism $\gamma_{\mathcal{E}_1}$ on $\mathcal{W}_{\mathcal{E}_1}$ as $\gamma_\delta$. Let $K = \mathrm{Ind}_P^G A_0$ be the induced complex on $G$, and we consider $\varphi^w : (Fw)^*K^w \xrightarrow{\sim} K^w$ for $w \in \mathcal{Z}_{\mathcal{E}_1}$. Then by (7.9.2), we have

$$
(8.8.2) \qquad \chi_{A_E} = q^{b_0} |\mathcal{W}_{\theta_1}|^{-1} \sum_{y \in \mathcal{W}_\delta} \mathrm{Tr}\,((\gamma_\delta y)^{-1}, \widetilde E) \chi_{K^w, \varphi^w}
$$

with $w = w_\delta y$.

Let $\mathcal{E}'_0 = \bar{\mathbf{Q}}_l \boxtimes \mathcal{E}_0$ be the local system on $\varSigma = Z_L^0 \times C_0$. Let $\mathcal{F} = \mathcal{E} \otimes \mathcal{E}'_0$ be the local system on $\varSigma$. Then $A_0$ coincides with $\mathrm{IC}(\overline{\varSigma}, \mathcal{F})[\dim \varSigma]$. For each $w \in \mathcal{Z}_{\mathcal{E}_1}$, take $\alpha \in G$ such that $\alpha^{-1}F(\alpha) = \dot w$, where $\dot w$ is a representative of $w$ in $N_G(L)$. Then $A_0^w$ is constricted from the twisted data $(L^w, \varSigma^w, \mathcal{F}^w)$, where $L^w = \alpha L \alpha^{-1}$, $\varSigma^w = \alpha \varSigma \alpha^{-1}$ and $\mathcal{F}^w = \mathrm{ad}(\alpha^{-1})^*\mathcal{F}$. By applying the argument in 6.5 (see also 6.8), we can construct an isomorphism $F^*\mathcal{F}^w \xrightarrow{\sim} \mathcal{F}^w$ which induces $\varphi_0^w : F^*A_0^w \xrightarrow{\sim} A_0^w$. We denote this map also by $\varphi_0^w$. We put $C_0^w = \alpha C_0 \alpha^{-1}$. The set $zZ_L^0$ is $Fw$-stable for $w \in \mathcal{Z}_{\mathcal{E}_1}$ and one can choose a representative $\dot{z}_w \in Z_L^{Fw}$ of the coset $zZ_L^0$. We have an $F$-stable set $\varSigma^w = \dot{z}^w Z_{L^w}^0 \times C_0^w$ with $\dot{z}^w = \alpha \dot{z}_w \alpha^{-1} \in Z_{L^w}^F$. For each $w \in \mathcal{Z}_{\theta_1}$, let $\theta_0^w = Sh_{F^m/Fw}(\theta_1)$ be the linear character of $Z_L^{0Fw}$. Under the isomorphism $Z_L^{0Fw} \simeq Z_{L^w}^{0F}$, we regard $\theta_0^w$ as the character of $Z_{L^w}^{0F}$, which we denote by $\bar{\theta}_0^w$.

Now take $t \in G^F$ and fix it. Let $x \in G^F$ be an element such that $x^{-1}tx \in \dot{z}^w Z_{L^w}^0$. Then $L_x^w = xL^w x^{-1}$ is a Levi subgroup of some parabolic subgroup of $Z_G(t)$. Let $C_x^w = xC_0^w x^{-1}$ be the unipotent class in $L_x^w$. We denote by $\mathcal{F}_x^w$ the local system on $C_x^w$ obtained by the pull back of $\mathcal{E}_0^w = \mathrm{ad}(\alpha^{-1})^*\mathcal{E}_0$ by $\mathrm{ad}\, x^{-1} : C_x^w \to C_0^w$. By the map $\beta : C_x^w \to \varSigma^w$, $v \mapsto x^{-1}tvx$, $\varphi_0^w : F^*\mathcal{F}^w \xrightarrow{\sim} \mathcal{F}^w$ induces an isomorphism $\varphi_x' = \beta^*\varphi_0^w : F^*\mathcal{F}_x^w \xrightarrow{\sim} \mathcal{F}_x^w$. Let $\varphi_x : F^*\mathcal{F}_x^w \xrightarrow{\sim} \mathcal{F}_x^w$ be the isomorphism defined by $\varphi_x = (\mathrm{ad}\, x^{-1})^*\varphi_0^w$. Then we have

$$\varphi_x' = \bar{\theta}_0^w(x^{-1}tx(\dot{z}^w)^{-1})\psi(\hat{z}_w)\varphi_x$$

by (6.5.6), where $\psi$ is a character of $T^{F^m}$ as given in (6.5.6), and $\hat{z}_w$ is as in 8.4. Assume that $t, v \in G^F$ such that $tv = vt$, where $t$ is semisimple and $v$ is unipotent. Then by the character formula [L3, II, Theorem 8.5] for the function $\chi_{K^w,\varphi^w}$, we have

$$\chi_{K^w,\varphi^w}(tv) = |Z_G(t)^F|^{-1} \sum_{\substack{x \in G^F \\ x^{-1}tx \in \dot{z}^w Z_{L^w}^0}} Q^{Z_G(t)}_{L_x^w, C_x^w, \mathcal{F}_x^w, \varphi_x}(v)\bar{\theta}_0^w(x^{-1}tx(\dot{z}^w)^{-1})\psi(\hat{z}_w), \tag{8.8.3}$$

where $Q^{Z_G(t)}_{L_x^w, C_x^w, \mathcal{F}_x^w, \varphi_x}$ is the generalized Green function of $Z_G(t)^F$ (note that $Z_G(t)$ is connected).

**8.9.** Recall that $\mathcal{X}_\iota$ is the function on $G^F_{\mathrm{uni}}$ associated to $\iota$ given in 5.2. It is known by [L3, V, (24.2.8)] that $\mathcal{X}_\iota$ is expressed as

$$\mathcal{X}_\iota = |\mathcal{W}|^{-1} q^{-b(\iota)} \sum_{w \in \mathcal{W}} \mathrm{Tr}\,(w^{-1}, E_\iota) Q^G_{L^w, C_0^w, \mathcal{E}_0^w, \varphi_0^w}, \tag{8.9.1}$$

where $\varphi_0^w : F^*\mathcal{E}_0^w \xrightarrow{\sim} \mathcal{E}_0^w$ is given by $\varphi_0^w = \mathrm{ad}(\alpha^{-1})^*\varphi_0$ from $\varphi_0 : F^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$. For each linear character $\theta$ of $Z_{L^w}^{0F}$, we denote by $K_\theta^w$ the complex $K^w$ given in 8.8 (subject to the condition that $z = 1$, i.e., $\varSigma = Z_L^0 \times C_0$) such that $\theta_0^w = \theta$, and denote by $\chi_{K_\theta^w}$ the characteristic function $\chi_{K^w,\varphi^w}$ Then by the character formula (8.8.3), we see that

$$Q^G_{L^w, C_0^w, \mathcal{E}_0^w, \varphi_0^w} = |Z_{L^w}^{0F}|^{-1} \sum_{\theta \in (Z_{L^w}^{0F})^\wedge} \chi_{K_\theta^w}, \tag{8.9.2}$$

where we regard the left hand side as the class function on $G^F$ by extending by 0 outside of $G^F_{\mathrm{uni}}$. Under the isomorphism $Z_{L^w}^{0F} \simeq Z_L^{0Fw}$, $\theta$ determines an $Fw$-stable character $\theta_1 \in (Z_L^{0F^m})^\wedge$ such that $Sh_{F^m/Fw}(\theta_1) = \theta$. Put $\mathcal{Z}_{\theta_1} = \{w' \in \mathcal{W} \mid Fw'(\theta_1) = \theta_1\}$. Then there exists a $w_1 \in \mathcal{W}$ such that $\mathcal{Z}_{\theta_1} = w_1\mathcal{W}_{\theta_1}$, and we define $\gamma_1 : \mathcal{W}_{\theta_1} \to \mathcal{W}_{\theta_1}$ by

$\gamma_1 = \operatorname{ad} w_1$. Hence, as in the case of $K^w$, one can decompose $K^w_\theta$ by

$$\chi_{K^w_\theta} = q^{-b_0} \sum_{E' \in (\mathcal{W}^\wedge_{\theta_1})^{\gamma_1}} \operatorname{Tr}(\gamma_1 y, \widetilde{E}')\chi_{A_{E',\theta}}, \tag{8.9.3}$$

for $y \in \mathcal{W}_{\theta_1}$ such that $w = w_1 y$, where $\chi_{A_{E',\theta}}$ is the (normalized) characteristic function of the character sheaf $A_{E',\theta}$.

**8.10.** For each irreducible character $\eta$ of $Z_M(\lambda_c)^F$, put $\varGamma_{c,\eta} = \operatorname{Ind}_{Z_M(\lambda_c)^F}^{G^F}(\eta \otimes \varLambda_c)$. Then we have

$$\varGamma_c = \sum_\eta (\deg \eta)\varGamma_{c,\eta},$$

where $\eta$ runs over all the irreducible characters in $Z_M(\lambda_c)^F$. We denote by $\varGamma'_c$ the sum of $\varGamma_{c,\eta}$, where $\eta$ runs over all the linear characters of $Z_M(\lambda_c)^F$, and by $\varGamma''_c$ the complement of $\varGamma'_c$ in $\varGamma_c$ so that $\varGamma_c = \varGamma'_c + \varGamma''_c$. Then we have the following lemma.

**Lemma 8.11.** *The inner product* $\langle \varGamma''_c, Q^G_{L^w, C^w_0, \mathcal{E}^w_0, \varphi^w_0}\rangle_{G^F}$ *can be expressed as*

$$\langle \varGamma''_c, Q^G_{L^w, C^w_0, \mathcal{E}^w_0, \varphi^w_0}\rangle_{G^F} = q^{-b_0+1}|Z^{0F}_{L^w}|^{-1} n_0^{-1}\beta,$$

*where $\beta$ is an algebraic integer contained in a fixed cyclotomic field $\mathcal{A}$ independent of $q$, and $n_0$ is an integer independent of $q$.*

*Proof.* By (8.9.2) and (8.9.3), we have

$$Q^G_{L^w, C^w_0, \mathcal{E}^w_0, \phi^w_0} = |Z^{0F}_{L^w}|^{-1} q^{-b_0} \sum_{\theta \in (Z^{0F}_{L^w})^\wedge} \sum_{E' \in (\mathcal{W}^\wedge_{\theta_1})^{\gamma_1}} \operatorname{Tr}(\gamma_1 y, \widetilde{E}')\chi_{A_{E',\theta}}, \tag{8.11.1}$$

where $A_{E',\theta}$ is the character sheaf which is a direct summand of $K^w_\theta$. Here we may assume that $\operatorname{Tr}(\gamma_1 y, \widetilde{E}') \in \mathcal{A}$. Moreover, by Theorem 7.9, $\chi_{A_{E',\theta}}$ coincides with the almost character $R_x$ up to a scalar $\nu_x$, and we may assume that $\nu_x$ is a unit in $\mathcal{A}$. On the other hand, it is known that $\deg \eta$ is a polynomial in $q$, and that $\deg \eta$ is divisible by $q$ if $\eta$ is not a linear character. It follows that $\langle \varGamma''_c, \rho\rangle_{G^F} \in n_1^{-1}q\mathbf{Z}$ for any $\rho \in \operatorname{Irr} G^F$. In particular we have $\langle \varGamma''_c, R_x\rangle_{G^F} \in n_2^{-1}q\mathcal{A}$ for any almost character of $G^F$, where $n_1, n_2$ are integers independent of $q$. Then the lemma follows from (8.11.1). □

## 9. Proof of Theorem 8.6

**9.1.** In this section, we prove Theorem 8.6. First we note that (ii) follows from (i). In fact, since $\rho_E$ and $A_E$ have the same parametrizaition via the decomposition of $\operatorname{Ind}_{P^{F^m}}^{G^{F^m}} \delta$ and of $\operatorname{ind}_P^G A_{\mathcal{E},z,\eta}$, we see that $R_E$ coincides with $\chi_{A_E}$ up to scalar. Hence $R_E$ coincides with $R_{zz_E,\eta_1^{-1}}$ by Theorem 8.6 (i). This also shows that $\rho_E = \rho_{zz_E,\eta_1^{-1}}$, and (ii) follows.

In order to prove (i), first we show the following.

**Proposition 9.2.** *Suppose that $q$ is large enough (but we don't assume that $q$ is sufficiently divisible). Then the statement of (i) in Theorem 8.6 holds.*

**9.3.** The proof of the proposition will be done through 9.3 to 9.10. We shall prove it by computing the inner product $\langle \varGamma_{c,\xi}, \chi_{A_E}\rangle$ under the assumption that $q$ is large enough. By Lemma 8.3, we have only to compare the inner products of $R_{z',\eta'}$ and of $\chi_{A_E}$ with various $\varGamma_{c,\xi}$ associated to the nilpotent element $N$ such that $N \in \mathcal{O}_{\widetilde{\rho}_{\dot{s}_x,E'}}$.

First we shall compute the inner product $\langle \varGamma_{c,\xi}, \chi_{K^w,\varphi^w}\rangle_{G^F}$. By (8.8.3) and Proposition 3.5, we have

$$\begin{aligned}\langle \varGamma_{c,\xi},\chi_{K^w,\varphi^w}\rangle_{G^F} &= \frac{1}{|G^F|}\sum_{t',v'\in G^F}\frac{1}{|Z_G(t')^F|^2}\sum_{\substack{g\in G^F\\ g^{-1}t'g\in Z_M(\lambda_c)^F}}\frac{|Z_M(\lambda_c)^F\cap Z_G(g^{-1}t'g)^F|}{|Z_M(\lambda_c)^F|}\\ &\times\sum_{\substack{x\in G^F\\ x^{-1}t'x\in \dot{z}^wZ^{0F}_{L^w}}}\xi^\natural(g^{-1}t'g)\overline{\bar\theta^w_0(x^{-1}t'x(\dot{z}^w)^{-1})\psi(\hat{z}_w)}\varGamma^{Z_G(t')}_{N_g,1}(v')Q^{Z_G(t')}_{L^w_x,C^w_x,\mathcal{E}^w_x,\varphi_x}(v'),\end{aligned}$$

where in the first sum, $t',v'\in G^F$ runs over semisimple elements $t'$, unipotent elements $v'$ such that $t'v'=v't'$. Then the right hand side of the above expression can be written as

$$\begin{aligned}&\frac{1}{|G^F|}\sum_{t\in\dot{z}^wZ^{0F}_{L^w}}\frac{1}{|Z_G(t)^F|^2}\sum_{\substack{g,x\in G^F\\ g^{-1}xtx^{-1}g\in Z_M(\lambda_c)^F}}\frac{|Z_M(\lambda_c)^F\cap Z_G({}^{g^{-1}x}t)^F|}{|Z_M(\lambda_c)^F|}\xi^\natural({}^{g^{-1}x}t)\bar\theta^w_0(t^{-1}\dot{z}^w)\psi(\hat{z}_w^{-1})\\ &\times\sum_{v\in Z_G(t)^F_{\mathrm{uni}}}\varGamma^{Z_G(xtx^{-1})}_{N_g,1}(xvx^{-1})Q^{Z_G(xtx^{-1})}_{L^w_x,C^w_x,\mathcal{E}^w_x,\varphi_x}(xvx^{-1}).\end{aligned}$$

Note that $\varGamma^{Z_G(xtx^{-1})}_{N_g,1}(xvx^{-1}) = \varGamma^{Z_G(t)}_{N_{x^{-1}g},1}(v)$ and that

$$Q^{Z_G(xtx^{-1})}_{L^w_x,C^w_x,\mathcal{E}^w_x,\varphi_x}(xvx^{-1}) = Q^{Z_G(t)}_{L^w,C^w_0,\mathcal{E}^w_0,\varphi^w_0}(v).$$

Hence by replacing $x^{-1}g$ by $g$ in the previous expression, we have

$$\tag{9.3.1}\begin{aligned}\langle \varGamma_{c,\xi},\chi_{K^w,\varphi^w}\rangle_{G^F} &= \sum_{t\in\dot{z}^wZ^{0F}_{L^w}}\sum_{\substack{g\in G^F\\ g^{-1}tg\in Z_M(\lambda_c)^F}}\frac{|Z_M(\lambda_c)^F\cap Z_G({}^{g^{-1}}t)^F|}{|Z_G(t)^F||Z_M(\lambda_c)^F|}\times\\ &\times\xi^\natural_g(t)\bar\theta^w_0(t^{-1}\dot{z}^w)\psi(\hat{z}_w^{-1})\langle\varGamma^{Z_G(t)}_{N_g,1},Q^{Z_G(t)}_{L^w,C^w_0,\mathcal{E}^w_0,\varphi^w_0}\rangle_{Z_G(t)^F},\end{aligned}$$

where $\xi^\natural_g(t) = \xi^\natural(g^{-1}tg)$.

**9.4.** Returning to the original setting, we consider $A_E$ as in the theorem, i.e., $A_E$ is a direct summand of $K^w_\theta$, where $\theta\in(Z^{0F}_{L^w})^\wedge$ corresponds to $\theta^w_0$ under the isomorphism $Z^{0F}_{L^w}\simeq Z^{0Fw}_L$. Hence $\mathcal{W}_{\theta_1},\mathcal{Z}_{\theta_1}$, etc. are nothing but the objects discussed in 8.2. So we write $w_\delta = w_L$ and $\gamma_\delta=\gamma$. We continue the computation of $\langle\varGamma_{c,\xi},\chi_{A_E}\rangle$ under this setting. By (8.8.2) and (9.3.1), we have

$$\langle \varGamma_{c,\xi}, \chi_{A_E}\rangle = |\mathcal{W}_{\theta_1}|^{-1}q^{b_0}\sum_{y\in\mathcal{W}_{\theta_1}}\operatorname{Tr}((\gamma y)^{-1},\widetilde{E})\sum_{t\in\dot z^w Z^{0F}_{L^w}}\sum_{\substack{g\in G^F\\ g^{-1}tg\in Z_M(\lambda_c)^F}}\frac{|Z_M(\lambda_c)^F\cap Z_G({}^{g^{-1}}t)^F|}{|Z_G(t)^F||Z_M(\lambda_c)^F|}$$
$$\times\, \xi^{\natural}_g(t)\bar\theta^w_0(t^{-1}\dot z^w)\psi(\hat z_w^{-1})\langle \varGamma^{Z_G(t)}_{N_g,1}, Q^{Z_G(t)}_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{Z_G(t)^F}\,.$$

Let $H$ be an $F$-stable reductive subgroup of $G$ containing $L$. Put $\mathcal{W}_H = N_H(L)/L\subset \mathcal{W}$. Recall that $\theta_1\in(Z_L^{0F^m})^\wedge$, and $\mathcal{Z}_{\theta_1}=\{w\in\mathcal{W}\mid {}^{Fw}\theta_1=\theta_1\}=w_L\mathcal{W}_{\theta_1}$. Then $\mathcal{W}_H\cap\mathcal{Z}_{\theta_1}=w_L'\mathcal{W}_{H,\theta_1}$, where $w'_L=w_Lw_H\in\mathcal{W}_H$ with $w_H\in\mathcal{W}_{\theta_1}$ and $\mathcal{W}_{H,\theta_1}$ is the stabilizer of $\theta_1$ in $\mathcal{W}_H$. Then $H$ contains $L^{w'_Ly}$ for any $y\in\mathcal{W}_{H,\theta_1}$. Put

$$\Delta^{(g)}_H = |\mathcal{W}_{\theta_1}|^{-1}q^{b_0}\sum_{y\in\mathcal{W}_{H,\theta_1}}\operatorname{Tr}((\gamma w_Hy)^{-1},\widetilde{E})\,\times \tag{9.4.1}$$
$$\times\sum_t \xi^{\natural}_g(t)\bar\theta^w_0(t^{-1}\dot z^w)\psi(\hat z_w^{-1})\langle\varGamma^H_{N_g,1}, Q^H_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{H^F},$$

where in the second sum, $t$ runs over all the elements in $\dot z^wZ^{0F}_{L^w}$ with $w=w'_Ly=w_Lw_Hy$ such that $Z_G(t)=H$ and that $g^{-1}tg\in Z_M(\lambda_c)$ for $g\in G^F$. Then we have

$$\langle\varGamma_{c,\xi},\chi_{A_E}\rangle=\sum_H\sum_g\Delta^{(g)}_H, \tag{9.4.2}$$

where in the first sum, $H$ runs over all the $F$-stable reductive subgroups of $G$ containing $L$ such that $H=Z_G(t)$ for a fixed $t\in Z_L$, and in the second sum $g$ runs over all the elements in the double cosets $H^F\backslash G^F/Z_M(\lambda_c)^F$ such that $g^{-1}tg\in Z_M(\lambda_c)^F$. Here we note the following lemma, which is a stronger version of Lemma 8.11.

**Lemma 9.5.** *Assume that $q$ is large enough. Let $\varGamma''_c$ be as in 8.10. Then we have*

$$\langle\varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F}=q^{-b_0+1}n_0^{-1}\beta,$$

*where $\beta\in\mathcal{A}$, and $n_0$ is an integer independent of $q$.*

*Proof.* First we note that

$$\langle\varGamma_{c,1}, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F}=q^{-b_0}n_1^{-1}\beta', \tag{9.5.1}$$

where $\beta'$ is an algebraic integer contained in a fixed cyclotomic field $\mathcal{A}$ independent of $q$, and $n_1$ is an integer independent of $q$. We show (9.5.1). We apply (9.3.1) to the situation that $\xi=1$ and $\theta_1=1$, i.e, $\mathcal{W}_{\theta_1}=\mathcal{W}$. Then by a similar argument as in the proof of (9.4.2), we have

$$\langle\varGamma_{c,1},\chi_{K^w,\varphi^w}\rangle_{G^F}=|Z_G^F|\langle\varGamma_{c,1}, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F}+\sum_{H\ne G}\sum_g\Xi^{(g)}_H, \tag{9.5.2}$$

where $\Xi_H^{(g)} = \sum_t \langle \varGamma^H_{N_g,1}, Q^H_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{H^F}$. By induction on the rank of $G$, we may assume that $\Xi_H^{(g)}$ can be expressed as $\Xi_H^{(g)} = q^{-b_0}n_H^{-1}\beta'_H$, where $n_H, \beta'_H$ are similar elements as $n_1, \beta'$ in (9.5.1). (Note that $b_0$ has common value for all $H$ containing $L$.) On the other hand, by Theorem 7.9, $\chi_{A_E}$ coincides with an almost character $R_x$ of $G^F$ up to a scalar $\nu_x$ which is a root of unity in $\mathcal{A}$. It follows, by (7.9.2), that

$$\langle \varGamma_{c,\xi}, \chi_{K^w,\varphi^w}\rangle_{G^F} \in q^{-b_0}n_2^{-1}\mathcal{A}$$

with some $n_2 \in \mathbf{Z}$ independent of $q$. (9.5.1) now follows from (9.5.2).

Next we note that

$$\langle \varGamma_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} \in q^{-b_0}\mathcal{A}. \tag{9.5.3}$$

In fact, by Lemma 5.12 (iii), we have

$$\langle \varGamma_c, \mathcal{X}_\iota\rangle_{G^F} \in q^{-b_0-b(\iota)}\mathcal{A}.$$

Since

$$Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w} = \sum_{\iota\in\mathcal{I}_0^F}\operatorname{Tr}(w, E_\iota)q^{b(\iota)}\mathcal{X}_\iota$$

by (8.9.1), we obtain (9.5.3).

Now we have

$$\langle \varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} = \langle \varGamma_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} - f_{N_c}\langle \varGamma_{c,1}, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F},$$

where $f_{N_c}$ is the number of linear characters of $Z_M(\lambda_c)^F$. Hence by (9.5.1) and (9.5.3), we have

$$\langle \varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} = q^{-b_0}n_1^{-1}\beta' \tag{9.5.4}$$

with $n_1 \in \mathbf{Z}$ independent of $q$, and $\beta' \in \mathcal{A}$.

On the other hand, by Lemma 8.11, we have

$$\langle \varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} = q^{-b_0+1}|Z^{0F}_{L^w}|^{-1}n_2^{-1}\beta'' \tag{9.5.5}$$

with $\beta'' \in \mathcal{A}$. We may assume that $\langle \varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} \neq 0$. Then by (9.5.4) and (9.5.5), we see that

$$n_2\beta' = q|Z^{0F}_{L^w}|^{-1}n_1\beta'' \in \mathcal{A}.$$

Since $q$ and $|Z^{0F}_{L^w}|$ are prime to each other, $n_1\beta''$ is divisible by $|Z^{0F}_{L^w}|$. Hence $\beta'$ can be written as $\beta' = qn_2^{-1}\beta$ with $\beta \in \mathcal{A}$, and we have

$$\langle \varGamma''_c, Q^G_{L^w,C_0^w,\mathcal{E}_0^w,\varphi_0^w}\rangle_{G^F} = q^{-b_0+1}(n_1n_2)^{-1}\beta.$$

This proves the lemma. □

**9.6.** We continue the computation in 9.4. By (8.9.1) applied to the reductive group $H$, we have

$$Q^H_{L^w,C_0^w,\mathcal{E}_0^w,\phi_0^w} = \sum_{\iota\in(\mathcal{I}_0')^F}\mathrm{Tr}\,(w,E_\iota)q^{b_H(\iota)}\mathcal{X}^H_\iota$$

for any $w\in\mathcal{W}_H$, where $b_H(\iota)$ is given as in (8.8.1) by replacing $G$ by $H$, and $(\mathcal{I}_H)_0$ is a block in $\mathcal{I}_H$ corresponding to $(L,C_0,\mathcal{E}_0)$. Substituting this into the previous formula, we see that

$$\Delta^{(g)}_H = q^{b_0}\sum_{\iota\in(\mathcal{I}_H)_0^F}\sum_{\substack{t\in z_1Z^{0F}_{L_1}\\ Z_G(t)=H}} a_{\iota,H}(t)\xi^\natural_g(t)q^{b_H(\iota)}\langle\varGamma^H_{N_g,1},\mathcal{X}^H_\iota\rangle, \tag{9.6.1}$$

where $L_1=L^{w_L'}$ and $z_1=\dot z^{w_L'}$. Here $a_{\iota,H}$ is given as follows; take $\alpha\in H$ such that $\alpha^{-1}F(\alpha)=\dot w_L'\in N_H(L)$. Then we have $L_1=\alpha L\alpha^{-1}$ and $z_1=\alpha\dot z_{w_L'}\alpha^{-1}$. By putting $t_1=\alpha^{-1}t\alpha\in(\dot zZ^0_L)^{Fw_L'}$ we have

$$a_{\iota,H}(t)=|\mathcal{W}_{\theta_1}|^{-1}\sum_{y\in\mathcal{W}_{H,\theta_1}}\theta^w_0(t_1^{-1}\dot z_w)\psi(\hat z_w^{-1})\,\mathrm{Tr}\,((\gamma w_Hy)^{-1},\widetilde{E})\,\mathrm{Tr}\,(w,E_\iota) \tag{9.6.2}$$

with $w=w_L'y=w_Lw_Hy$.

In the formula (9.6.1), we shall replace the inner product $\langle\varGamma^H_{N_g,1},\mathcal{X}^H_\iota\rangle$ by $\langle\varGamma^H_{N_g},\mathcal{X}^H_\iota\rangle$, which is easier to handle with. Now it is easy to see, by making use of Lemma 9.5, that

$$\langle\varGamma''_c,\mathcal{X}_\iota\rangle_{G^F}=q^{-b_0-b(\iota)+1}n_0^{-1}\beta_\iota$$

for an appropriate integer $n_0$ and $\beta_\iota\in\mathcal{A}$. We also note the relation

$$\langle\varGamma_c,\mathcal{X}_\iota\rangle_{G^F}=f_{N_c}\langle\varGamma_{c,1},\mathcal{X}_\iota\rangle_{G^F}+\langle\varGamma''_c,\mathcal{X}_\iota\rangle_{G^F}\,.$$

Applying these formulas to the case of $H$, we have

$$\Delta^{(g)}_H=\sum_{\iota\in(\mathcal{I}_H)_0^F}\sum_{\substack{t\in z_1Z^{0F}_{L_1}\\ Z_G(t)=H}} a_{\iota,H}(t)\xi^\natural_g(t)f^{-1}_{N_g}\bigl\{q^{b_H(\iota)+b_0}\langle\varGamma^H_{N_g},\mathcal{X}^H_\iota\rangle+qn_H^{-1}\beta_{H,\iota}\bigr\}, \tag{9.6.3}$$

where $N_g=\mathrm{ad}(g)N_c$ is a nilpotent element in $\mathrm{Lie}\,H$, $n_H$ is an integer independent of $q$, and $\beta_{H,\iota}\in\mathcal{A}$. Moreover, $f_{N_g}$ is the number of linear characters of $Z_{{}^gM\cap H}({}^g\lambda)^F$.

Next we shall compute $\langle\varGamma^H_{N_g},\mathcal{X}^H_\iota\rangle$. Recall that $Z_G(g^{-1}tg)\supset L$, and $N_g=\mathrm{Ad}(g)N_c\in\mathrm{Lie}\,H$. It follows that $\mathrm{Ad}(g)N\in\mathrm{Lie}\,H$. Since $g\in G^F$, we may replace $N$ by $N_1=\mathrm{Ad}(g)N$ in parameterizing the nilpotent orbits $\mathcal{O}^F_N$ in $\mathfrak{g}$. Then we can identify $A_{H,\lambda}=Z_H(N_1)/Z^0_H(N_1)$ as a subgroup of $A_G(N)$. As in 5.8, there exists $c_0\in A_\lambda$ such that $-N_c^*$ is $G^F$-conjugate to $N_{cc_0}$. Then $N_g=\mathrm{Ad}(g)N_c$ is $H^F$-conjugate to $(N_1)_c$, and so $-N_g^*=\mathrm{Ad}(g)(-N_c^*)$ is $H^F$-conjugate to $(N_1)_{cc_0}$, with $c,c_0\in A_{H,\lambda}$. In particular, for $\iota''\in\mathcal{I}_H$ such that $\mathrm{supp}\,\iota''=\mathcal{O}_{N_g}$, we see that $\mathcal{Y}_{\iota''}(-N_g^*)=\eta(cc_0)$, which is independent from $H$. Let $\mathcal{O}_1$ be the nilpotent orbit in $\mathrm{Lie}\,H$ such that $\mathrm{supp}\,\iota'=\mathcal{O}_1$

with $E_{\iota'} = E_\iota \otimes \varepsilon$. Then we have

$$q^{b_H(\iota)+b_0}\langle \varGamma^H_{N_g}, \mathcal{X}^H_\iota\rangle = \begin{cases} \zeta_{\mathcal{I}_0}^{-1}\eta(cc_0)^{-1} & \text{if } \mathcal{O}_{N_g} = \mathcal{O}_1, \\ q\beta'_{H,\iota} & \text{if } \mathcal{O}_{N_g} \neq \mathcal{O}_1, \end{cases} \tag{9.6.4}$$

for some $\beta'_{H,\iota} \in \mathcal{A}$.

In fact, by Lemma 5.12 (i) applying to $H$, we have

$$q^{b_H(\iota)+b_0}\langle \varGamma^H_{N_g}, \mathcal{X}^H_\iota\rangle = \zeta_{\mathcal{I}_0}^{-1}q^{(\dim\operatorname{supp}(\iota')-\dim\operatorname{supp}(\iota''))/2}\mathbf{P}_{\iota'',\iota'}(q^{-1})\overline{\mathcal{Y}_{\iota''}(-N_g^*)}.$$

(Note that $\zeta_{(\mathcal{I}_H)_0}$ for $\mathcal{I}_H$ coincides with $\zeta_{\mathcal{I}_0}$ for $\mathcal{I}_G$ since both of them correspond to $(L, C_0, \mathcal{E}_0)$.) If $\operatorname{supp}(\iota') = \operatorname{supp}(\iota'')$, then we have $\iota' = \iota''$ and $\mathbf{P}_{\iota'',\iota'} = 1$. This implies the first equality. Now assume that $\iota' \neq \iota''$. By Lemma 5.12 (ii), we have

$$q^{(\dim\operatorname{supp}(\iota')-\dim\operatorname{supp}(\iota''))/2}\mathbf{P}_{\iota'',\iota'}(q^{-1}) \in q\mathbf{Z}.$$

Since we may assume that $\zeta_{\mathcal{I}_0}^{-1}, \overline{\mathcal{Y}_{\iota''}(-N_g^*)} \in \mathcal{A}$, we obtain the second equality.

**9.7.** We consider the sum of $a_{\iota,H}(t)\xi^\natural_g(t)f_{N_g}^{-1}$ in (9.6.3) for a fixed $\iota$. Recall that $L_1 = \alpha L\alpha^{-1}$ and that $t$ is an element of $z_1 Z_{L_1}^{0F}$ such that $Z_G(t) = H$ and that $g^{-1}tg \in Z_M(\lambda_c)$. Put $M_H = gMg^{-1} \cap H$. Then $M_H$ is the Levi subgroup in $H$ associated to the nilpotent element $N_g$, and $t \in (Z_{M_H}(N_g) \cap z_1 Z^0_{L_1})^F$. Put $M'_H = \alpha^{-1}M_H\alpha = {}^{\alpha^{-1}g}M \cap H$, which is stable by $F''_H = Fw'_L$, and put

$$X_{M_H} = (Z_{M'_H}(N_{\alpha^{-1}g}) \cap \dot{z}Z^0_L)/(Z^1_{M'_H}(N_{\alpha^{-1}g}) \cap Z^0_L).$$

Then

$$(Z_{M_H}(N_g) \cap z_1 Z^0_{L_1})/(Z^1_{M_H}(N_g) \cap Z^0_{L_1}) \simeq X_{M_H} \tag{9.7.1}$$

and the action of $F$ on the left hand side corresponds to the action of $Fw'_L$ on $X_{M_H}$.

We apply Lemma 8.5 (i) to $H$. Then one can write, for $w = w'_L y$,

$$\theta_0^w(t_1^{-1}\dot{z}_w)\psi(\hat{z}_w^{-1}) = \theta_0^{w'_L}(t_1^{-1}\dot{z}_{w'_L})\psi(\hat{z}_{w'_L}^{-1})\Psi_{t_1^{-1}}(y),$$

where $\Psi_{t_1^{-1}}$ is a linear character of $\mathcal{W}_{H,\theta_1}$. Then (9.6.2) can be rewritten as

$$a_{\iota,H}(t) = \frac{|\mathcal{W}_{H,\theta_1}|}{|\mathcal{W}_{\theta_1}|}\theta_0^{w'_L}(t_1^{-1}\dot{z}_{w'_L})\psi(\hat{z}_{w'_L}^{-1})\langle \widetilde{E} \otimes \Psi_{t_1^{-1}}, E_\iota\rangle_{w'_L\mathcal{W}_{H,\theta_1}}, \tag{9.7.2}$$

where $\langle\ ,\ \rangle_{w'_L\mathcal{W}_{H,\theta_1}}$ is the inner product on the $\mathcal{W}_{H,\theta_1}$-invariant functions on $w'_L\mathcal{W}_{H,\theta_1}$. Put $Y = Z_{M_H}(N_g) \cap z_1 Z^0_{L_1}$ and $Y_1 = Z^1_{M_H}(N_g) \cap Z^0_{L_1}$. Then by Lemma 8.5 (ii), together with the isomorphism in (9.7.1), $\Psi_{t_1}$ depends only on $t \mod Y_1^F$.

On the other hand, we have an isomorphism

$$Y/Y_1 \simeq (Z_{M\cap {}^{g^{-1}}H}(\lambda_c) \cap \dot{z}Z^0_{{}^{g^{-1}}L_1})/(Z^1_{M\cap {}^{g^{-1}}H}(\lambda_c) \cap Z^0_{{}^{g^{-1}}L_1})$$

and a natural map

$$(Z_{M\cap {}^{g^{-1}}H}(\lambda_c)\cap \dot z Z^0_{{}^{g^{-1}}L_1})/(Z^1_{M\cap {}^{g^{-1}}H}(\lambda_c)\cap Z^0_{{}^{g^{-1}}L_1}) \to Z_M(\lambda_c)/Z^1_M(\lambda_c)\simeq \bar A_\lambda.$$

Hence we have a map

$$Y^F/Y_1^F \to (Y/Y_1)^F \to \bar A_\lambda^F \tag{9.7.3}$$

satisfying the property that $\xi^\natural(g^{-1}tg)=\xi(a)$, where $a\in \bar A_\lambda^F$ is the image of $t\in Y^F$ under this map. In particular, we see that $\xi_g^\natural(t)$ also depends only on $t \mod Y_1^F$.

Put $\bar t = t \mod Y_1^F \in Y^F/Y_1^F$. It follows from the above argument, together with (9.7.2), that

$$\begin{aligned}\sum_{t\in\bar t} & a_{\iota,H}(t)\xi_g^\natural(t)\\ &= \frac{|\mathcal W_{H,\theta_1}|}{|\mathcal W_{\theta_1}|}\xi_g^\natural(\bar t)\langle \widetilde E\otimes\Psi_{\bar t^{-1}},E_\iota\rangle_{w'_L\mathcal W_{H,\theta_1}}\psi(\hat z_{w'_L})\sum_{t\in\bar t}\theta_0^{w'_L}(t_1^{-1}z_{w'_L}),\end{aligned}\tag{9.7.4}$$

where $\Psi_{\bar t^{-1}}$ (resp. $\xi_g^\natural(\bar t)$) denotes $\Psi_{t_1^{-1}}$ (resp. $\xi_g^\natural(t)$) for $t\in\bar t$. Since $\theta_0^{w'_L}$ is a linear character on $Z_L^{0Fw'_L}$, we have

$$\sum_{t\in\bar t}\theta_0^{w'_L}(t_1^{-1}z_{w'_L}) = \begin{cases} |Y_1^F|\bar\theta_0^{w'_L}(\bar t^{-1}z_1) & \text{if } \bar\theta_0^{w'_L}|_{Y_1^F} \text{ is trivial},\\ 0 & \text{otherwise},\end{cases}\tag{9.7.5}$$

where $\bar\theta_0^{w'_L}(\bar t^{-1}z_1)$ denotes a common value of $\theta_0^{w'_L}(t_1^{-1}z_{w'_L})$ for $t\in\bar t$.

By definition, $f_{N_g}$ is the number of linear characters of $Z_{M_H}(N_g)^F$, which coincides with the order of $Z_{M_H}(N_g)^F/(Z_{M_H}(N_g)^F)_{\mathrm{der}}$. Hence one can write $f_{N_g}$ as

$$f_{N_g} = f_H|Z_1^{0F}|,$$

where $Z_1 = Z(Z_{M_H}(N_g))$ is the center of $Z_{M_H}(N_g)$, and $f_H$ is contained in a finite subset of $\mathbf{Q}$ independent of $q$. On the other hand, the description of $Y_1$ in 8.4 (applied to the case $M_H$) implies that $Z_1^0\subseteq Y_1$. Thus $Y_1^F$ is divisible by $Z_1^{0F}$ and $|Y_1^F|/|Z_1^{0F}|$ is a polynomial in $q$ (in fact, it is the order of the group of $F$-fixed points of the group $Y_1/Z_1^0$ whose connected component is a torus). Let $g_H$ be the constant term of the polynomial $|Y_1^F|/|Z_1^{0F}|$.

Note that we may assume that $\theta_0^w(t)\in\mathcal A$ for any $w\in\mathcal W$ since $\theta_0^w$ comes from a linear character $\theta_1$ of $(L/L_{\mathrm{der}})^{F^m}$, and $L/L_{\mathrm{der}}$ is a finite group. Thus summing up the above argument, we have the following formula.

$$\begin{aligned}\sum_{t\in Y^F} & a_{\iota,H}(t)\xi_g^\natural(t)f_{N_g}^{-1} = n_1^{-1}q\beta\\ &+\frac{|\mathcal W_{H,\theta_1}|}{|\mathcal W_{\theta_1}|}f_H^{-1}g_H\psi(\hat z_{w'_L}^{-1})\sum_{\bar t\in Y^F/Y_1^F}\bar\theta_0^{w'_L}(\bar t^{-1}z_1)\xi_g^\natural(\bar t)\langle\widetilde E\otimes\Psi_{\bar t^{-1}},E_\iota\rangle_{w'_L\mathcal W_{H,\theta_1}},\end{aligned}\tag{9.7.6}$$

where $n_1 \in \mathbf{Z}$ is independent of $q$, and $\beta \in \mathcal{A}$. (We understand that $\bar\theta_0^{w'_L}(\bar t^{-1}z_1) = 0$ if $\bar\theta_0^{w'_L}|_{Y_1^F}$ is non-trivial).

**9.8.** We return to the setup in 9.6. In view of (9.6.4) and (9.7.6), one can rewrite the equation (9.6.3) in the form

$$
\begin{aligned}
(9.8.1)\qquad \varDelta_H^{(g)} = n_1^{-1}q\beta + \frac{|\mathcal{W}_{H,\theta_1}|}{|\mathcal{W}_{\theta_1}|} f_H^{-1}g_H\zeta_{\mathcal{I}_0}^{-1}\eta(cc_0)^{-1}\psi(\hat z_{w'_L}^{-1})\times \\
\times \sum_{\bar t\in Y^F/Y_1^F}\bar\theta_0^{w'_L}(\bar t^{-1}z_1)\xi_g^\natural(\bar t)\langle \widetilde E\otimes\Psi_{\bar t^{-1}}, E_\iota\rangle_{w'_L\mathcal{W}_{H,\theta_1}},
\end{aligned}
$$

where $n_1 \in \mathbf{Z}$ is independent of $q$ and $\beta \in \mathcal{A}$, and $\iota$ is the unique element in $(\mathcal{I}_H)_0$ such that $\operatorname{supp}(\iota') = \mathcal{O}_{N_g}$ with $E_{\iota'} = E_\iota\otimes\varepsilon$.

We now compute the inner product $\langle \widetilde E\otimes\Psi_{\bar t^{-1}}, E_\iota\rangle_{w'_L\mathcal{W}_{H,\theta_1}}$ in the right hand side of (9.8.1). Note that $E \in \mathcal{W}_{\theta_1}^\wedge$ is an extension of the $\Omega_{\theta_1}$-stable character $E_1$ of $\mathcal{W}_{\theta_1}^0$ to $\mathcal{W}_{\theta_1}\rtimes\Omega_{\theta_1}$, where $E_1 \in (\mathcal{W}_{\theta_1}^0)^\wedge$ is of the form $E_1 = E^\mu\boxtimes\cdots\boxtimes E^\mu$ ($t/d$-times) with $E^\mu \in \mathfrak{S}_{n/t}^\wedge$ corresponding to a partition $\mu$ of $n/t$. Let $\mu^*$ be the partition dual to $\mu$. Then $N = N_\lambda$ with $\lambda = t\mu^*$ by 8.2. (In general, for a partition $\rho = (\rho_1,\dots,\rho_k)$ and $a \in \mathbf{Z}_{>0}$, we put $a\rho = (a\rho_1,\dots,a\rho_k)$). By our assumption, $\operatorname{supp}(\iota') = \mathcal{O}_{N_g}$. Then $\mathcal{W}_H$ is of the form

$$
(9.8.2)\qquad \mathcal{W}_H \simeq \mathfrak{S}_{\nu_1}\times\cdots\times\mathfrak{S}_{\nu_k},
$$

where $\nu = (\nu_1,\dots,\nu_k)$ is a partition of $n/d$ such that $(t/d)\mu^*$ is a refinement of $\nu$, i.e., $\nu_i$ is a sum of parts of $(t/d)\mu^*$. Moreover $\mathcal{W}_{H,\theta_1}$ is given as

$$
(9.8.3)\qquad \mathcal{W}_{H,\theta_1} \simeq \mathcal{W}_{H,\theta_1}^0\rtimes\Omega_{\theta_1}
$$

with

$$
\mathcal{W}_{H,\theta_1}^0 \simeq \mathfrak{S}_{\nu'}\times\cdots\times\mathfrak{S}_{\nu'} \quad (t/d\text{-times}),
$$

where $\nu'$ is a partition of $n/t$ such that $(t/d)\nu' = \nu$. Since $\nu' = (\nu'_1,\dots,\nu'_k)$ is a partition of $n/t$ whose parts are sums of parts of $\mu^*$, $\mu^*$ determines a partition $(\mu^{(i)})^*$ of $\nu'_i$ for $i = 1,\dots,k$. We denote by $\mu^{(i)}$ its dual partition. Hence $\mu = \mu^{(1)}+\cdots+\mu^{(k)}$. We define an irreducible character $E^0$ of $\mathcal{W}_{H,\theta_1}^0$ by $E^0 = E_1^0\boxtimes\cdots\boxtimes E_1^0$ with $E_1^0 \in \mathfrak{S}_{\nu'}^\wedge$ such that

$$
E_1^0 = E^{\mu^{(1)}}\boxtimes\cdots\boxtimes E^{\mu^{(k)}}.
$$

We remark that

(9.8.4) $\langle E, E_\iota\rangle_{\mathcal{W}_{H,\theta_1}^0} = 1$, and $E^0$ is the unique irreducible character of $\mathcal{W}_{H,\theta_1}^0$ which appears in the restrictions to $\mathcal{W}_{H,\theta_1}^0$ of both of $E$ and $E_\iota$.

In fact, one can write $E|_{\mathcal{W}_{H,\theta_1}^0} = E^0 + \sum E'$ with $a(E') > a(E^0)$. On the other hand, $E_\iota$ is given as $E_\iota = E^{\rho^{(1)}}\boxtimes\cdots\boxtimes E^{\rho^{(k)}}$ with $E^{\rho^{(i)}} \in \mathfrak{S}_{\nu_i}^\wedge$, where $\rho^{(i)}$ is a partition of $\nu_i$ such that $(\rho^{(i)})^* = (t/d)(\mu^{(i)})^*$. It follows that $\rho^{(i)} = \mu^{(i)}\cup\cdots\cup\mu^{(i)}$ ($t/d$-times). (For a partition $\lambda$ and $\mu$, we denote by $\lambda\cup\mu$ the partition obtained by rearranging the parts of $\lambda$ and $\mu$ in decreasing order.) Then one can write as $E_\iota|_{\mathcal{W}^0_{H,\theta_1}} = E^0 + \sum E''$ with $a(E'') < a(E^0)$. Hence (9.8.4) holds.

Since $E^0$ is $\Omega_{\theta_1}$-stable, (9.8.4) implies that there exists a unique extension $\widetilde{E}^0$ of $E^0$ to $\mathcal{W}_{H,\theta_1}$ which appears in the decomposition of $E_\iota|_{\mathcal{W}_{H,\theta_1}}$ with multiplicity one. On the other hand, again by (9.8.4), the restriction of $E$ to $\mathcal{W}_{H,\theta_1}$ also contains a certain extesnion $(\widetilde{E}^0)'$ of $E^0$ with multiplicity one. Hence we can write $(\widetilde{E}^0)' = \widetilde{E}^0 \otimes \omega$ with some $\omega \in \Omega^\wedge_{\theta_1}$. Since $(\mathcal{I}_H)^F_0 = (\mathcal{I}_H)^{F''_H}_0$, $E_\iota$ is stable by $F''_H$. Since $E$ is also $F''_H$-stable, we see that $\widetilde{E}^0$ and $(\widetilde{E}^0)'$ are $F''_H$-stable. It follows that $\omega$ is $F''_H$-stable.

Now by applying Lemma 8.5 (ii) to the group $H$, we see that

(9.8.5) There exists a unique class $x_H \in X^{F''_H}_{M_H}$ satisfying the following.

$$\langle E \otimes \Psi_{t^{-1}}, E_\iota\rangle_{\mathcal{W}_{H,\theta_1}} = \begin{cases} 1 & \text{if } t \in x_H, \\ 0 & \text{otherwise.} \end{cases}$$

We pass to the extension $\widetilde{E}$. Then the above arguments show that the restrictions of $\widetilde{E} \otimes \Psi_t^{-1}$ and $E_\iota$ to $\langle w'_L\rangle\, \mathcal{W}_{H,\theta_1}$ contain a unique irreducible character which is an extension of $\widetilde{E}^0$ to $\langle w'_L\rangle\, \mathcal{W}_{H,\theta_1}$ for $t \in x_H$.

Under the notation in 9.7, $Y^F/Y^F_1$ is regarded as a subset of $M_H^{F''_H}$. If $x_H$ is contained in $Y^F/Y_1^F$, $x_H$ determines an element in $\bar{A}^F_\lambda$ by (9.7.3), which we denote by $z_H \in \bar{A}^F_\lambda$. Then summing up the above arguments, we have

$$\tag{9.8.6} \begin{aligned} &\sum_{\bar{t} \in Y^F/Y^F_1} \bar{\theta}_0^{w'_L}(\bar{t}^{-1}z_1)\xi^\natural_g(\bar{t})\langle \widetilde{E} \otimes \Psi_{\bar{t}^{-1}}, E_\iota\rangle_{w'_L W_{H,\theta_1}} \\ &\qquad = \begin{cases} \bar{\theta}_0^{w'_L}(x_H^{-1}z_1)\xi(z_H)\alpha_H & \text{if } x_H \in Y^F/Y^F_1, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\alpha_H$ is a root of unity determined by the extension $\widetilde{E}$, which is independent of $q$. Substituting this into (9.8.1), we have

$$\tag{9.8.7} \varDelta^{(g)}_H \in \delta_H \frac{|\mathcal{W}_{H,\theta_1}|}{|\mathcal{W}_{\theta_1}|} f_H^{-1} g_H \zeta^{-1}_{\mathcal{I}_0}\eta(cc_0)^{-1}\psi(\hat{z}^{-1}_{w'_L})\bar{\theta}_0^{w'_L}(x_H^{-1}z_1)\xi(z_H)\alpha_H + n_1^{-1}q\mathcal{A},$$

where $\delta_H = 1$ if $x_H \in Y^F/Y^F_1$ and $\bar{\theta}_0^{w'_L}|_{Y^F_1}$ is trivial, and $\delta_H = 0$ otherwise.

**9.9.** We shall compare $x_H$ for various $H$ appearing in 9.8. Since

$$Z_{M'_H}(N_{\alpha^{-1}g}) \cap \dot{z}Z^0_L = Z_{\alpha^{-1}gM}(N_{\alpha^{-1}g}) \cap \dot{z}Z^0_L$$

and similarly for $Z^1_{M'_H}(N_{z^{-1}g}) \cap Z^0_L$, a similar argument as in 8.4 implies that

$$X_{M_H} \simeq \mathbf{Z}/(t/d)\mathbf{Z} \simeq X_M.$$

We have a natural isomorphism $f : X_{M_H} \to X_M$ which is compatible with the action of $F''_H$ and of $F''$ (the both coincide with the action of $F$ on $\mathbf{Z}/(t/d)\mathbf{Z}$). Assume that $H$ satisfies the property (9.8.2). Then it is easy to check that $w_L \in \mathcal{W}_H$, and so we have

$$w_L = w'_L. \tag{9.9.1}$$

We note that

$$f(x_H) = x_G. \tag{9.9.2}$$

In fact, let $E_\iota \in \mathcal{W}_H^\wedge, E^0 \in (\mathcal{W}^0_{H,\theta_1})^\wedge$ be as in 9.8 for $H$, and $E_{\iota_G} \in \mathcal{W}^\wedge$, $E^0_G \in (\mathcal{W}^0_{\theta_1})^\wedge$ the corresponding objects for $G$. Here $\iota$ (resp. $\iota_G$) is the unique element in $(\mathcal{I}_H)_0$ (resp. $\mathcal{I}_0$ ) such that supp $\iota' = \mathcal{O}_{N_g}$ (resp. supp $\iota'_G = \mathcal{O}_N$). Then $E_\iota$ is contained in $E_{\iota_G}|_{\mathcal{W}_H}$ with multiplicity one. On the other hand, we have $E|_{\mathcal{W}^0_{\theta_1}} = E^0_G$, and the restriction of $E^0_G$ on $\mathcal{W}^0_{H,\theta_1}$ contains $E^0$ with multiplicity one by the property (9.8.4). Let $\widetilde{E}^0$ be the extension of $E^0$ to $\mathcal{W}_{H,\theta_1}$ appearing in $E_\iota|_{\mathcal{W}_{H,\theta_1}}$, and let $\widetilde{E}^0_G$ a similar object as $\widetilde{E}^0$ for $E^0_G$. Then the above fact shows that $\widetilde{E}^0$ occurs in the restriction of $\widetilde{E}^0_G$ with multiplicity one. This implies that $E$ coincides with $\widetilde{E}^0_G \otimes \omega$, where $\omega \in \Omega^\wedge_{\theta_1}$ is given as in 9.8 for $H$. Thus (9.9.2) is proved.

It follows from (9.9.2) that $x_H$ in (9.8.5) depends only on $E$, and does not depend on the choice of $H$, which we denote by $x_E$. We also denote by $z_E$ the element $z_H \in \bar{A}^F_\lambda$ determined from $x_H$ (see (9.8.6)). A similar argument as above shows that the root of unity $\alpha_H$ in (9.8.6) also depends only on $E$, and not on $H$, which we denote by $\alpha_E$.

Summing up the above argument, (9.8.7) can be written as

$$\varDelta^{(g)}_H \in \delta_H \frac{|\mathcal{W}_{H,\theta_1}|}{|\mathcal{W}_{\theta_1}|} f_H^{-1} g_H \zeta_{\mathcal{I}_0}^{-1}\eta(cc_0)^{-1}\psi(\hat{z}^{-1}_{w'_L})\bar{\theta}_0^{w_L}(x_E^{-1}z_1)\xi(z_E)\alpha_E + qn_H^{-1}\mathcal{A}, \tag{9.9.3}$$

where $\delta_H = 1$ if $x_E \in Y^F/Y_1^F$ and $\bar{\theta}_0^{w_L}$ is trivial on $Y_1^F$, and $\delta_H = 0$ otherwise, and $n_H$ is an integer independent of $q$.

**9.10** We are now ready to prove Proposition 9.2, by completing the computation of $\langle \varGamma_{c,\xi}, \chi_{A_E}\rangle$. We now look at the formula (9.4.2). Take $H$ such that $H = Z_G(t)$ for $t \in Z^{0F}_{L_1}$. Then the set of $g$ in the second sum in (9.4.2) corresponds to the set of semisimple conjugacy classes in $Z_M(\lambda_c)^F$ which are conjugate to a fixed $t$ in $G^F$. Let $e_H$ be the number of $g$ occurring in the second sum in (9.4.2). Then the above observation implies that $e_H$ is bounded by a positive integer independent of $q$.

Now by substituting (9.9.3) into (9.4.2), together with the above remark, we have

$$\langle \varGamma_{c,\xi}, \chi_{A_E}\rangle_{G^F} = Q\xi(z_E)\eta(cc_0)^{-1} + qm^{-1}\beta \tag{9.10.1}$$

with some integer $m$ independent of $q$ and $\beta \in \mathcal{A}$. Here

$$Q = \alpha_E\zeta^{-1}_{\mathcal{I}_0}\psi(\hat{z}^{-1}_{w'_L})\bar{\theta}_0^{w_L}(x_E^{-1}z_1)|\mathcal{W}_{\theta_1}|^{-1}\sum_H |\mathcal{W}_{H,\theta_1}|e_H f_H^{-1} g_H, \tag{9.10.2}$$

where $H$ runs over the subgroups such that $\delta_H = 1$. $Q$ is independent of $c, \xi$, and also is contained in a finite subset of $\mathcal{A}$ independent of $q$.

By Theorem 7.9, $\chi_{A_E}$ coincides with $\nu_E R_x$ for a certain $x = (c', \xi') \in \bar{A}_\lambda^F \times (\bar{A}_\lambda^\wedge)^F$, where $\nu_E \in \bar{\mathbf{Q}}_l^*$ is a certain root of unity. By (4.5.1) and (4.5.2), together with Theorem 2.6, we see that

$$\langle \varGamma_{c,\xi}, \nu_x R_x \rangle_{G^F} = \nu_E |\bar{A}_\lambda^F|^{-1} \xi(c')\xi'(c). \tag{9.10.3}$$

On the other hand, suppose that $\beta \neq 0$ in (9.10.1). Then the absolute value of $qm^{-1}\beta$ turns out to be very large if we choose $q$ large enough since $\beta$ is contained in the ring $\mathcal{A}$ of algebraic integers of the fixed cyclotomic field. This implies that the absolute value of $\langle \varGamma_{c,\xi}, \chi_{A_E} \rangle_{G^F}$ becomes very large since $Q\xi(z_E)\eta(cc_0)^{-1}$ is contained in a finite subset of $\mathcal{A}$ independent of $q$. This contradicts the formula (9.10.3), and we conclude that $\beta = 0$, and we have

$$\langle \varGamma_{c,\xi}, \chi_{A_E} \rangle_{G^F} = Q\xi(z_E)\eta(cc_0)^{-1}. \tag{9.10.4}$$

Comparing (9.10.3) and (9.10.4), we see that

$$c' = z_E, \quad \xi' = \eta^{-1}, \quad \nu_E = Q\eta(c_0)^{-1}|\bar{A}_\lambda^F|.$$

Note that the last equality implies that

$$|\mathcal{W}_{\theta_1}|^{-1} \sum_H |\mathcal{W}_{H,\theta_1}| e_H f_H^{-1} g_H = \pm |\bar{A}_\lambda^F|^{-1}, \tag{9.10.5}$$

where $H$ runs over the $F$-stable reductive subgroups of $G$ containing $L$ such that $\delta_H = 1$, which is of the form $H = Z_G(t)$ for $t \in Z_L^0$ satisfying the property that there exists $g \in G^F$ such that $g^{-1}tg \in Z_M(\lambda_c)^F$. Then we have

$$\nu_E = \pm \zeta_{\mathcal{I}_0}^{-1} \eta(c_0)^{-1} \alpha_E \psi(\hat{z}_{w_L'}^{-1}) \bar{\theta}_0^{w_L}(x_E^{-1}), \tag{9.10.6}$$

and the signature $\pm 1$ is determined by (9.10.5). Thus Proposition 9.2 is proved.

**9.11.** Returning to the setting in Theorem 8.6, we shall show that the statement (i) in Theorem 8.6 holds without the restriction on $q$. The argument is divided into two steps. First we show that Lusztig's conjecture holds without restriction on $q$, and that the scalar constants are determined explicitly, by applying the specialization argument based on the Shintani descent identities of character sheaves in [S1, Corollary 2.12]. In the second step, we show that the parametrization of almost characters as given in Theorem 8.6 holds without the restriction on $q$.

**9.12.** For a positive integer $c$, we put $\mathcal{P}_c = \{r \in \mathbf{Z}_{\geq 1} \mid r \equiv 1 \pmod{c}\}$. Let $s \in G^*$ be as in (8.1.1). We choose a positive integer $c$ such that $F^c$ acts trivially on $A_\lambda$, and that $F^c$ stabilizes $\dot{s}$ and $z_L$. Then for any $r \in \mathcal{P}_c$, $\overline{\mathcal{M}}_{s,N}^{(r)}, \mathcal{M}_{s,N}^{(r)}$ (objects for $G^{F^r}$) are naturally identified with $\overline{\mathcal{M}}_{s,N}, \mathcal{M}_{s,N}$ (objects for $G^F$). We denote by $\rho_x^{(r)}$ (resp. $R_y^{(r)}$ the irreducible character (resp. the almost character) of $G^{F^r}$ corresponding to

$x \in \overline{\mathcal{M}}^{(r)}_{s,N}$ (resp. $y \in \mathcal{M}^{(r)}_{s,N}$). In particular, the set $\mathcal{E}(G^{F^{mr}}, \{s\})^{F^r}$ is naturally identified with the set $\mathcal{E}(G^{F^m}, \{s\})^F$ for a sufficiently divisible $m$.

Similarly, the set $\mathcal{M}^{L,(r)}_{s_L,N_0}$ (for $L^{F^r}$) is identified with the set $\mathcal{M}^L_{s_L,N_0}$ (for $L^F$). Let $\delta^{(mr)} = \delta^{(mr)}_{z,\eta}$ be a cuspidal irreducible character of $L^{F^{mr}}$ corresponding to $(z,\eta) \in \mathcal{M}^L_{s_L,N_0}$. Then $\mathcal{W}_{\delta^{(mr)}}$ and $\mathcal{Z}_{\delta^{(mr)}}$ are independent of $r$, which we denote by $\mathcal{W}_\delta, \mathcal{Z}_\delta$. By (6.7.2) and (7.9.1), we have $\mathcal{W}_\delta = \mathcal{W}_{\theta_1}$. For each $E \in (\mathcal{W}^\wedge_{\theta_1})^{F''}$, we consider the character sheaf $A_E$, and its characteristic function $\chi^{(r)}_{A_E}$ with respect to $F^r$. We also consider the almost character $R^{(r)}_E$ of $G^{F^r}$. It follows from the proof of Theorem 7.9 (see (7.9.4)) that $\chi^{(r)}_{A_E}$ coincides with $R^{(r)}_E$ up to scalar. Then by Proposition 9.2 that there exists a positive integer $r_0$ such that this scalar $\nu^{(r)}_E$ is described by (9.10.6) and (9.10.5) if $r > r_0$. We put $\mathcal{P}'_c = \{r \in \mathcal{P}_c \mid r > r_0\}$. One can check that $\nu^{(r)}_E$ is independent of the choice of $r \in \mathcal{P}'_c$, by replacing $c$ by its appropriate multiple, if necessary. We denote by $\nu_E$ this common value $\nu^{(r)}_E$. We have the following proposition.

**Proposition 9.13.** *Let the notations be as in 9.12. Assume that $q$ is arbitrary. Then for each $E \in (\mathcal{W}^\wedge_{\theta_1})^{F''}$, we have*

$$\chi_{A_E} = \nu_E R_E.$$

*Proof.* By Theorem 4.7, we have

$$Sh_{F^{mr}/F^r w}(\tilde\delta^{(mr)}|_{L^{F^{mr}}\sigma^r w}) = \mu_0 (R^{L_w}_{z,\eta})^{(r)} \tag{9.13.1}$$

for $w \in \mathcal{Z}_\delta$, where $(R^{L_w}_{z,\eta})^{(r)}$ is the almost character of $L_w^{F^r}$ corresponding to $(z,\eta)$. Note that by Theorem 4.7, $\mu_0$ is independent of the choice of $r \in \mathcal{P}_c$ under the appropriate choice of the extension $\widetilde\delta^{(mr)}$ of $\delta^{(mr)}$.

Let $A_0 = \mathcal{E} \otimes A_{z,\eta}$ be the cuspidal character sheaf on $L$ as in Theorem 7.9. We have $\mathcal{W}_{\mathcal{E}_1} = \mathcal{W}_\delta$ and $\mathcal{Z}_{\mathcal{E}_1} = \mathcal{Z}_\delta$. Then $A_0$ gives rise to an $F^r$-stable character sheaf $A^w_0$ of $L_w$ for $w \in \mathcal{Z}_\delta$. We denote by $\chi^{(r)}_{A^w_0}$ the characteristic function on $L_w^{F^r}$ induced from the isomorphism $\phi^w_0 : (F^r)^*(A^w_0) \xrightarrow{\sim} A^w_0$. Then by Theorem 7.9, we have

$$\chi^{(r)}_{A^w_0} = \nu_0 (R^{L_w}_{z,\eta^{-1}})^{(r)}, \tag{9.13.2}$$

where $\nu_0 = \zeta_{\mathcal{I}_0}^{-1}\eta(c_0)^{-1}$ is independent of the choice of $r \in \mathcal{P}_c$.

Now the map $a_{F^r w} : C(L^{F^r}/\sim_{F^r w}) \to C(G^{F^{mr}}/\sim_{F^r})$ is defined as in [S2, 3.3]. By [S2, (3.6.3)], for $w = w_\delta y \in \mathcal{Z}_\delta$, we have

$$\begin{aligned} a_{F^r w}&(\widetilde\delta^{(mr)}|_{L^{F^{mr}}\sigma^r w}) \\ &= \sum_{E \in (\mathcal{W}^\wedge_\delta)^{F^r}} q_y^{-mr/2} q^{-\tilde l(w)mr/2}\,\mathrm{Tr}\,(T_{\gamma_\delta y}, \widetilde E(q^{mr}))(\widetilde\rho^{(mr)}_E|_{G^{F^{mr}}\sigma^r}),\end{aligned}$$

where $q_y$ is a power of $q$. (For the notation, see [S2, 3.5]). By applying the Shintani descent operator to the above equality, and by using Theorem 4.7, we have

$$
\begin{aligned}
&Sh_{F^{mr}/F^r} \circ a_{F^r w}(\widetilde{\delta}^{(mr)}|_{L^{F^{mr}}\sigma^r w}) \\
&\qquad = \sum_{E \in (\mathcal{W}_\delta^\wedge)^{F^r}} q_y^{-mr/2} q^{-\tilde{l}(w)mr/2} \operatorname{Tr}(T_{\gamma_\delta y}, \widetilde{E}(q^{mr}))\mu_E R_E^{(r)},
\end{aligned}
\tag{9.13.3}
$$

where $\mu_E$ is also independent of the choice of $r \in \mathcal{P}_c$.

On the other hand, by the Shintani descent identity for character sheaves ([S1, Corollary 2.12]), the following formula holds for each $r \in \mathcal{P}_c$;

$$
(-1)^{d_{\mathbf{w}}} Sh_{F^{mr}/F^r} \circ a_{F^r w} \circ N^*_{F^{mr}/F^r w}(\chi^{(r)}_{A_0^w}) = \sum_A \Bigl(\sum_{i \in I_A} c_i \xi_i^r\Bigr) \chi_A^{(r)},
$$

where $A$ runs over all the elements in $\widehat{G}^F$, and $\{\xi_i \mid i \in I_A\}$ and $\{c_i \mid i \in I_A\}$ are certain finite subsets of $\bar{\mathbf{Q}}_l$ associated to $A$. Then combining (9.13.1), (9.13.2) and (9.13.3), we see that

$$
\begin{aligned}
(-1)^{d_{\mathbf{w}}} \nu_0 \mu_0^{-1} &\sum_{E \in (\mathcal{W}_\delta^\wedge)^{F^r}} q_y^{-mr/2} q^{-\tilde{l}(w)mr/2} \operatorname{Tr}(T_{\gamma_\delta y}, \widetilde{E}(q^{mr}))\mu_E R_E^{(r)} \\
= \sum_A \Bigl(\sum_{i \in I_A} c_i \xi_i^r\Bigr) \chi_A^{(r)}.&
\end{aligned}
$$

By using the orthogonality relations for Hecke algebras (see [S2, (3.6.4)]), we can deduce a formula

$$
P_{\mathcal{W}_\delta}(q^{mr})\mu_E R_E^{(r)} = \sum_A \Bigl(\sum_{i \in I_A} d_i \zeta_i^r\Bigr) \chi_A^{(r)}
\tag{9.13.4}
$$

for certain subsets $\{d_i \mid i \in I_A\}$ $\{\zeta_i \mid i \in I_A\}$ of $\bar{\mathbf{Q}}_l$ (depending on $E$), where $P_{\mathcal{W}_\delta}(t)$ is a polynomial in $t$ (a generalization of Poincaré polynomial).

As discussed in 9.12 we have

$$
\chi_{A_E}^{(r)} = \nu_E R_E^{(r)}
\tag{9.13.5}
$$

for $r \in \mathcal{P}'_c$. Substituting (9.13.5) into (9.13.4), we see that

$$
\sum_{i \in I_A} d_i \zeta_i^r = \begin{cases} \mu_E \nu_E^{-1} P_{\mathcal{W}_\delta}(q^{mr}) & \text{if } A = A_E, \\ 0 & \text{otherwise} \end{cases}
\tag{9.13.6}
$$

for any $r \in \mathcal{P}'_c$. By applying a variant of Dedekind's theorem ([S1, (3.7.6)]), we see that (9.13.6) holds for any $r \in \mathcal{P}_c$. In particular, substituting (9.13.6) into (9.13.4), we see that (9.13.5) holds for any $r \in \mathcal{P}_c$. By putting $r = 1$, we obtain the proposition. □

Next we show that

**Lemma 9.14.** *For each $E \in (\mathcal{W}_{\theta_1}^{\wedge})^{F''}$, we have*

$$R_E = R_{z_E, \mu_1^{-1}}.$$

*Proof.* Let $\mathcal{P}'_c$ be as in the proof of Proposition 9.13. We choose $r \in \mathcal{P}'_c$ and take $m$ large enough so that $F^{mr}$ is a sufficiently divisible extension of $F^r$. Then $F^{mr}$ is also sufficiently divisible for $F$. Let $\delta = \delta^{(mr)}$ be a cuspidal irreducible character of $L^{F^{mr}}$. Then by the definition of $\mathcal{P}'_c$, $\mathcal{Z}_\delta^{(r)}$ is identified with $\mathcal{Z}_\delta^{(1)}$. It follows that for any $E \in (\mathcal{W}_\delta^{\wedge})^{F''}$, we have

$$\begin{aligned} Sh_{F^{mr}/F^r}(\widetilde{\rho}_E^{(mr)}|_{G^{F^{mr}}\sigma^r}) &= \mu_E^{(r)} R_E^{(r)}, \\ Sh_{F^{mr}/F}(\widetilde{\rho}_E^{(mr)}|_{G^{F^{mr}}\sigma}) &= \mu_E^{(1)} R_E^{(1)} \end{aligned}$$

with some constants $\mu_E^{(r)}, \mu_E^{(1)}$. But by Proposition 9.2, we know that for $r \in \mathcal{P}'_c$, $R_E^{(r)}$ coincides with $R^{(r)}_{z_E, \eta_1^{-1}}$ with $(z_E, \eta_1^{-1}) \in \mathcal{M}_{s,N}$. It follows that $\rho_E^{(mr)}$ coincides with $\rho^{(mr)}_{z_E, \eta_1^{-1}}$, and so we have $R_E^{(1)} = R_{z_E, \eta_1^{-1}}$. This proves the lemma. $\square$

**9.15.** Combining Proposition 9.13 and Lemma 9.14, we see that (8.6.1) holds for any $q$. This completes the proof of Theorem 8.6.

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