GREEN FUNCTIONS ASSOCIATED TO COMPLEX REFLECTION GROUPS $G(e, 1, n)$

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0. Introduction

Green polynomials $Q^{\mu}_{\lambda}(q)$ of $GL_n(\mathbf{F}_q)$, where λ, μ are partitions of n, were first introduced by J.A.Green [G] in 1955 in a combinatorial framework of symmetric functions. They are obtained as the transition matrix between Schur functions and Hall-Littlewood functions. Those Green functions play a crucial role in [G] in describing irreducible characters of $GL_n(\mathbf{F}_q)$. In 1976, P. Deligne and G. Lusztig constructed in [DL] Green functions Q^G_T for finite reductive groups $G(\mathbf{F}_q)$ in general, as the restriction to unipotent classes of $G(\mathbf{F}_q)$ of Deligne-Lusztig's virtual character $R^G_T(\theta)$, associated to a maximal torus T of G defined over $\mathbf{F}_q$. The Green function Q_{χ}, for an irreducible character χ of the Weyl group W of G, is defined as a certain linear combination of Q^G_T for various T. In the case of $GL_n(\mathbf{F}_q)$, $Q_{\chi^{\lambda}}(u_{\mu})$ coincides with $Q^{\mu}_{\lambda}(q)$, where χ^{λ} is the irreducible character of $W = \mathfrak{S}_n$ corresponding to λ, and u_{μ} is the unipotent class corresponding to μ. Lusztig proved in [L3], generalizing the result in [S1], that there exists a simple algorithm of computing Green functions. In this context, Green functions arise as a unique solution of a certain type of matrix equation

$$(*) \qquad P\varLambda\,{}^tP = \Omega$$

with unknown $P, \varLambda$. (The entries of the matrix P give Green functions $Q_{\chi}(u)$). Note that the matrix Ω is completely determined by the property of irreducible characters of W, while the shape of P and $\varLambda$ are determined by the property of the Springer correspondence between unipotent classes of G and irreducible characters of W. It is known, by the geometric interpretation of Green functions in [L5], that the entries of the matrices P and $\varLambda$ are in $\mathbb{Z}[q]$.

In [GM], Geck and Malle considered an analogy of the equation in (*) by making use of a-functions of irreducible characters of Weyl groups and families of unipotent characters, and showed that they have a unique solution with coefficients in $\mathbb{Q}(t)$. They verified in the case of exceptional groups that the matrices $P, \varLambda$ have coefficients in $\mathbb{Z}[t]$, and also the matrix $\varLambda$ provides the polynomials conjectured in [L6] expressing the cardinality of special pieces in $G(\mathbf{F}_q)$. They conjecture that similar facts will hold for reductive groups in general.

In the case of classical groups, the Springer correspondence is described in terms of certain combinatorial objects called "u-symbols" introduced by Lusztig [L2]. The remarkable fact is that the matrix equation (*) is completely described by the combinatorial property of such u-symbols. The notion of u-symbols was generalized by

Malle [Ma1] so that it fits to the case of complex reflection groups $G(e, 1, n)$ as a generalization of Weyl group of type C_n. Now the ingredients used to construct the equation (*) make sense even for this case, and one can define a matrix equation by using the combinatorics of symbols. By using a similar argument as in [GM], it is shown that this equation has a unique solution. So it would be natural to call the solution of this equation as the Green function associated to $G(e, 1, n)$.

After Green [G], the character table of $GL_n(\mathbf{F}_q)$, especially the part corresponding to the values at unipotent elements of unipotent characters, is completely dominated by Green functions. Thanks to Lusztig [L3], similar facts hold in general; the character values at unipotent elements of finite reductive groups, say, $Sp_{2n}(\mathbf{F}_q)$ or $SO_{2n+1}(\mathbf{F}_q)$, of unipotent characters are dominated by Green functions (or rather by generalized Green functions). Lusztig showed that his method of classifying unipotent characters of finite reductive groups makes sense even for other Coxeter groups W such as H_3, H_4, and determined their "degrees", even there does not exist a finite group $G(\mathbf{F}_q)$ whose Weyl group is W. Inspired by this, Malle [Ma1] classified, in the case of $G(e, 1, n)$, the "unipotent characters" and described their degrees. One might expect that Green functions associated to $G(e, 1, n)$ will provide us the values of unipotent characters at other "unipotent elements".

In this note, we try to find a combinatorial theory of symmetric functions which fits to the situation for $G(e, 1, n)$. In our theory, the role of partitions is replaced by symbols. In particular, we construct Hall-Littlewood functions associated to $G(e, 1, n)$, which are parametrized by symbols. We show that a large part of arguments used to construct Green polynomials of $GL_n(\mathbf{F}_q)$ can be generalized to our setting, and that the Green functions are obtained as the transition matrix between Schur functions and Hall-Littlewood functions. Thus obtained Green functions are rational functions in $\mathbb{Q}(t)$. As some examples show, it is very likely that these Green functions are actually polynomials in t with positive integral coefficients.

The details of this note will be found in [S3].

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1. A background from finite reductive groups

1.1. Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$, with Frobenius map $F : G \to G$. We assume that G is split over $\mathbf{F}_q$. We are interested in the representation theory of the finite group G^F over $\bar{\mathbf{Q}}_l$, an algebraic closure of the l-adic number field $\mathbf{Q}_l$ with $l \neq \operatorname{ch} F_q$. For an F-stable maximal torus T and a character $\theta : T^F \to \bar{\mathbf{Q}}_l^*$, Deligne-Lusztig's virtual character $R_T^G(\theta) : G^F \to \bar{\mathbf{Q}}_l$ is defined. The Green function Q_T^G is defined as the restriction of $R_T^G(\theta)$ to the set of unipotent elements in G^F, which is independent of the choice of θ. The functions $R_T^G(\theta)$ play a crucial role in the representation theory of G^F, and it is known that the computation of $R_T^G(\theta)$ is reduced to the computation of various Green functions of smaller reductive groups.

Let W be the Weyl group of G. Then the G^F-conjugacy classes of F-stable maximal tori are parametrized by the conjugacy classes of W. We denote by T_w the F-stable maximal torus corresponding to $w \in W$ (up to conjugacy). Let $\mathcal{V}_{\text{uni}}$ be the space of G^F-invariant functions on the set of unipotent elements in G^F. Thus $Q_T^G \in \mathcal{V}_{\text{uni}}$. Let $W^\wedge$ be the set of irreducible characters of W. For $\chi \in W^\wedge$, let us define a function

Q_χ on $\mathcal{V}_{\text{uni}}$ by

$$Q_\chi = |W|^{-1} \sum_{w \in W} \chi(w) Q^G_{T_w}.$$

We refer Q_χ also as Green functions. Clearly, the determination of $Q^G_{T_w}$ for any $w \in W$ is equivalent to that of Q_χ for any $\chi \in W^\wedge$.

1.2. Let I be the set of pairs (C, ρ), where C is a unipotent class in G, and ρ is an irreducible character of $A_G(u)$. (Here C is F-stable, and u is an element in C^F. $A_G(u)$ is the component group $Z_G(u)/Z^0_G(u)$). It is known, by a suitable choice of $u \in C^F$, that the G^F-conjugacy classes in C^F is in bijection with the set of conjugacy classes in $A_G(u)$. We denote by u_a the class in C^F corresponding to $a \in A_G(u)$. For each $i = (C, \rho)$, we define a function $Y_i \in \mathcal{V}_{\text{uni}}$ by

$$Y_i(v) = \begin{cases} \rho(a) & \text{if } v \sim u_a \ (G^F\text{-conjugate}), \\ 0 & \text{if } v \notin C^F. \end{cases}$$

Then $\{Y_i \mid i \in I\}$ forms a basis of the space $\mathcal{V}_{\text{uni}}$.

There exists a natural injection f, the so-called "Springer correspondence" from $W^\wedge$ to I. We denote by I_0 the image of $W^\wedge$ under the map f. If $f(\chi) = i$, we write Q_χ also as Q_i. Then it is known that Q_χ can be written as

$$Q_\chi = Q_i = \sum_{j \in I_0} p_{ij} Y_j$$

with coefficients $p_{ij} \in \bar{\mathbf{Q}}_l$. In fact, it is clear that Q_i is written as a linear combination of Y_j with $j \in I$, and it is a deep result due to Lusztig that $p_{ij} = 0$ if $j \notin I_0$. We fix a total order on I_0 which is compatible with the closure relations of unipotent classes, and consider the square matrix $P = (p_{ij})$. The determination of Q_i for various $i \in I_0$ is equivalent to the determination of the matrix P. Lusztig showed that P is determined as a unique solution of the matrix equation of the form $P\Lambda\,{}^tP = \Omega$ as in (*), where Ω is a known matrix obtained from the data on $W^\wedge$. In fact, we define an equivalence relation $\sim$ on I_0, by the condition that $i \sim i'$ if $i = (C, \rho), i' = (C', \rho')$ with $C = C'$, and consider the block matrix with respect to this equivalence relation. Then it is shown that P is a block lower triangular matrix with diagonal blocks of the form $q^{d_u} I$ if the block corresponds to $C \ni u$. Here d_u is the dimension of the variety $\mathcal{B}_u = \{B_1 \in G/B \mid u \in B_1\}$. Furthermore, the matrix Λ is a non-singular block diagonal matrix. By this condition, P and Λ are uniquely determined from the equation (*).

1.3. In the case where $G = Sp_{2n}$ or SO_{2n+1}, all of the ingredients involved in the above arguments are described in a purely combinatorial way; the set I is in bijection with certain combinatorial objects, "u-symbols". Then the subset I_0 of I corresponds to the u-symbols of defect 1. Furthermore, the equivalence class in I with respect to $\sim$ corresponds to the similarity class of u-symbols. The assignment $u \to d_u$ is defined as a function $a : I_0 \to \mathbb{Z}_{\geq 0}$ combinatorially.

As explained in the introduction, these formalism can be extended to the case of complex reflection groups $W = G(e, 1, n)$. In the next section, we formulate the equation corresponding to (*) for such W.

2. Green functions associated to $G(e, 1, n)$

2.1. Let $W = \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$ be the imprimitive complex reflection group $G(e, 1, n)$, and $V = \mathbb{C}^n$ the natural reflection representation of W. Let $S(V)$ be the symmetric algebra of V, and I_+ the ideal of $S(V)$ generated by the homogeneous W-invariant vectors of strictly positive degrees. We denote by $R = \oplus R_i$ the coinvariant algebra of W, which is a graded algebra defined as the the quotient of $S(V)$ by I_+. Let t be an indeterminate. The Poincaré polynomial $P_W(t)$ is defined as $P_W(t) = \sum_{i \ge 0}(\dim_{\mathbb{C}} R_i)t^i$, which is explicitly given as

$$P_W(t) = \prod_{i=1}^{n} \frac{t^{ei} - 1}{t - 1}.$$

For any class function f on W, we define $R(f)$ by

$$R(f) = (t-1)^n P_W(t) \frac{1}{|W|} \sum_{w \in W} \frac{\det_V(w) f(w)}{\det_V(t \cdot \mathrm{id}_V - w)}, \tag{2.1.1}$$

where $\det_V$ denotes the determinant on V. If χ is an irreducible character of W, then $R(\chi)$ coincides with $\sum_i \langle \chi, R_i \rangle_W t^i$, where $\langle\ ,\ \rangle_W$ denotes the inner product of class functions. Hence $R(\chi)$ coincides with the fake degree of χ, and $R(f) \in \mathbb{Z}[t]$ for a generalized character f of W.

Let N^* be the number of complex reflections in W. Then N^* is the maximal degree in R, and the W-module R_{N^*} coincides with $\overline{\det_V}$.

2.2. An e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ is an e-tuple of partitions $\alpha^{(k)}$. The size $|\boldsymbol{\alpha}|$ of $\boldsymbol{\alpha}$ is defined as $|\boldsymbol{\alpha}| = \sum_{i=0}^{e-1} |\alpha^{(k)}|$. We denote by $\mathcal{P}_{n,e}$ the set of e-partitions of size n. Then the set $W^\wedge$ of irreducible characters of W is in bijection with the set $\mathcal{P}_{n,e}$. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. In particular, the unit character corresponds to $\boldsymbol{\alpha} = ((n), -, \dots, -)$, and $\overline{\det_V}$ corresponds to $\boldsymbol{\alpha} = (-. \dots, -, (1^n))$.

Let us fix $\mathbf{m} = (m_0, \dots, m_{e-1})$, where m_k are positive integers such that $m_k \ge n$ for any k. We denote by $Z_n^{0,0} = Z_n^{0,0}(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha}$ with size n, such that $\alpha^{(k)}$ is regarded as an element in $\mathbb{Z}^{m_k}$ written in the form, $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{m_k}^{(k)} \ge 0$. We fix integers $r \ge s \ge 0$, and consider an e-tuple of partitions $\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\mathbf{m}) = (\Lambda_0, \dots, \Lambda_{e-1})$ defined by

$$\begin{aligned} &\Lambda_0 : (m_0 - 1)r \ge \cdots \ge 2r \ge r \ge 0, \\ &\Lambda_i : s + (m_i - 1)r \ge \cdots \ge s + 2r \ge s + r \ge s \qquad \text{for } 1 \le i \le e - 1. \end{aligned} \tag{2.2.1}$$

We denote by $Z_n^{r,s} = Z_n^{r,s}(\mathbf{m})$ the set of e-partitions of the form $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha} \in Z_n^{0,0}$ and the sum is taken entry-wisely. We denote by $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda}$ is as above, and call it the e-symbol of type (r, s) corresponding to $\boldsymbol{\alpha}$. Let us define a shift operation $Z_n^{r,s}(\mathbf{m}) \to Z_n^{r,s}(\mathbf{m}')$ by associating $\boldsymbol{\Lambda}' = (\Lambda'_0, \dots, \Lambda'_{e-1}) \in Z_n^{r,s}(\mathbf{m}')$ to $\boldsymbol{\Lambda} = (\Lambda_0, \dots, \Lambda_{e-1}) \in Z_n^{r,s}(\mathbf{m})$, where $\mathbf{m}' = (m_0 + 1, \dots, m_{e-1} + 1)$ with $\Lambda'_0 = (\Lambda_0 + r) \cup \{0\}$ and $\Lambda'_k = (\Lambda_k + r) \cup \{s\}$ for $k = 1, \dots, e - 1$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $Z_n^{r,s}$ is said to be similar if all the entries of them coincide each other with multiplicities. The set of symbols which are similar to a fixed symbol is called a similarity class in $Z_n^{r,s}$.

We shall define a function $a : Z_n^{r,s} \to \mathbb{Z}_{\geq 0}$. Take $\boldsymbol{\Lambda} \in Z_n^{r,s}$ and let $\boldsymbol{\Lambda}^0$ be as before. We put

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda,\lambda' \in \boldsymbol{\Lambda}} \min(\lambda, \lambda') - \sum_{\mu,\mu' \in \boldsymbol{\Lambda}^0} \min(\mu, \mu'), \tag{2.2.2}$$

where in the first sum, we assume that $\lambda \neq \lambda'$ if λ and λ' are contained in the same Λ_i, and similarly for the second sum (this occurs only when $r = 0$). The function a on $Z_n^{r,s}$ is invariant under the shift operation, and takes a constant value on each similarity class in $Z_n^{r,s}$.

Remark 2.3. The notion of symbols already appeared in several articles in various forms. If $e = 1$, $Z_n^{r,s}$ is in bijection with the set of partitions of n. Next consider the case where $e = 2$. Then the set $Z_n^{1,0}$ (up to shift operation, all the same below) was used in [L1] to parameterize unipotent characters of $Sp_{2n}(\mathbf{F}_q)$ or $SO_{2n+1}(\mathbf{F}_q)$. In that case, the a-function defined in (2.2.2) coincides with the original a-function, i.e., the exact power of q dividing the degree of the corresponding unipotent character. The set $Z_n^{2,1}$ (resp. $Z_n^{2,0}$) was used in [L2] to describe the generalized Springer correspondence for $G = Sp_{2n}(k)$ (resp. $G = SO_{2n+1}(k)$) subject to the condition that $\operatorname{ch} k \neq 2$. In this case, the similarity classes are in 1 to 1 correspondence with the unipotent classes of G, and the value $a(\boldsymbol{\Lambda})$ for such a symbol $\boldsymbol{\Lambda}$ coincides with $\dim \mathcal{B}_u$, (see, for example [L4, 4.4]). While, the set $Z_n^{4,2}$ was used in [LS] to describe the generalized Springer correspondence for $Sp_{2n}(k)$ with ch $k = 2$. On the other hand, for arbitrary e, the set $Z_n^{1,0}$ was used in [Ma1] to parameterize unipotent degrees associated to the complex reflection group $G(e, 1, n)$.

2.4. Under the identification of $Z_n^{0,0}$ with $\mathcal{P}_{n,e}$, we consider the a function as the function on the set $\mathcal{P}_{n,e}$. Also the similarity classes in $\mathcal{P}_{n,e}$ are defined by inheriting the classes in $Z_n^{r,s}$. We denote by $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$ are in the same similarity class. We choose a total order $\succ$ on $\mathcal{P}_{n,e}$ such that $a(\boldsymbol{\alpha}) \leq a(\boldsymbol{\beta})$ whenever $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$, and that each similarity class in $\mathcal{P}_{n,e}$ forms an interval. In the following, we consider the matrices of degree $|\mathcal{P}_{n,e}|$ with respect to this order. For each $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$, we define a polynomial $\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Z}[t]$ by

$$\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*} R(\chi^{\boldsymbol{\alpha}} \otimes \chi^{\boldsymbol{\beta}} \otimes \overline{\det}_V), \tag{2.4.1}$$

and put $\Omega = (\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, the matrix of degree $|\mathcal{P}_{n,e}|$. Let $P = (p_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $\Lambda = (\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be matrices of degree $|\mathcal{P}_{n,e}|$. We consider the following system of equations with respect to the unknown variables $\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}, p_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

$$\begin{aligned}
&\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{unless } \boldsymbol{\alpha} \sim \boldsymbol{\beta}, \\
&p_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{unless either } \boldsymbol{\alpha} \succ \boldsymbol{\beta} \text{ and } \boldsymbol{\alpha} \not\sim \boldsymbol{\beta}, \text{ or } \boldsymbol{\alpha} = \boldsymbol{\beta}, \\
&p_{\boldsymbol{\alpha},\boldsymbol{\alpha}} = t^{a(\boldsymbol{\alpha})}, \\
&P \Lambda\, {}^t\!P = \Omega.
\end{aligned} \tag{2.4.2}$$

2.5. There exists a system of equations which is closely related to (2.4.2). Let $\Omega' = (\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be a matrix defined by

$$\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*} R(\chi^{\boldsymbol{\alpha}} \otimes \overline{\chi}^{\boldsymbol{\beta}} \otimes \overline{\det}_V), \tag{2.5.1}$$

where $\overline{\chi}^{\boldsymbol{\beta}}$ is the complex conjugate of $\chi^{\boldsymbol{\beta}}$. We consider a matrix equation of the form

$$P' \varLambda' \, {}^t P'' = \Omega', \tag{2.5.2}$$

where either of $P' = (p'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $P'' = (p''_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ satisfies similar conditions as in the second and the third one in (2.4.2), and $\varLambda' = (\lambda'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ as in the first one. Let $\sigma = (\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the permutation matrix realizing the complex conjugation of irreducible characters of W, i.e., $\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$ if $\chi^{\boldsymbol{\beta}} = \overline{\chi}^{\boldsymbol{\alpha}}$ and $\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ otherwise. Then we have $\Omega' = \Omega\sigma$, and the equation $P\varLambda^t P = \Omega$ implies that

$$P \cdot \varLambda\sigma \cdot \sigma^{-1t} P\sigma = \Omega'.$$

It is easily checked that the equation (2.4.2) has a solution if and only if the equation (2.5.2) has a solution, and in that case, we have $P' = P, \varLambda' = \varLambda\sigma, P'' = \sigma^{-1t}P\sigma$. In the remainder of this note, we report that the solution of (2.5.2) can be described in terms of certain symmetric functions as in the case of GL_n.

3. A combinatorial setting for $G(e, 1, n)$

3.1. For a given $\mathbf{m} = (m_0, \dots, m_{e-1})$ as in 2.2, we introduce indeterminates $x_j^{(k)}$ $(0 \le k \le e-1, 1 \le j \le m_k)$. We denote by x the whole variables $(x_j^{(k)})$, and also denote by $x^{(k)}$ the variables $x_1^{(k)}, \dots, x_{m_k}^{(k)}$. Power sum symmetric functions and Schur functions are defined as in [M, Appendix B]. Let ζ be a primitive e-th root of unity in $\mathbb{C}$. For each integer $r \ge 1$ and i such that $0 \le i \le e-1$, put

$$p_r^{(i)}(x) = \sum_{j=0}^{e-1} \zeta^{-ij} p_r(x^{(j)}),$$

where $p_r(x^{(j)})$ denotes the r-th power sum symmetric function with respect to the variables $x^{(j)}$. We put $p_r^{(i)}(x) = 1$ for $r = 0$. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ with $\alpha^{(k)} : \alpha_1^{(k)} \ge \dots \ge \alpha_{m_k}^{(k)}$, we define a function $p_{\boldsymbol{\alpha}}(x)$ by

$$p_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} p^{(k)}_{\alpha_j^{(k)}}(x). \tag{3.1.1}$$

Next, we define the Schur function $s_{\boldsymbol{\alpha}}(x)$ and the monomial symmetric function $m_{\boldsymbol{\alpha}}(x)$ by

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}), \qquad m_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} m_{\alpha^{(k)}}(x^{(k)}), \tag{3.1.2}$$

where $s_{\alpha^{(k)}}(x^{(k)})$ (resp. $m_{\alpha^{(k)}}(x^{(k)})$) denotes the usual Schur function (resp. monomial symmetric function) associated to the partition $\alpha^{(k)}$ with respect to the variables $x^{(k)}$.

3.2. Let t be an indeterminate. In what follows we regard the variables $x_i^{(k)}$ defined for $k \in \mathbb{Z}/e\mathbb{Z} \simeq \{0, 1, \dots, e-1\}$. For each $0 \le k \le e-1$ and an integer $r \ge 0$, we define a function $q_{r,\pm}^{(k)}(x;t)$ by

$$q_{r,\pm}^{(k)}(x;t) = \sum_{i \ge 1} (x_i^{(k)})^{r+\delta} \frac{\prod_j x_i^{(k)} - t x_j^{(k\pm 1)}}{\prod_{j \ne i} x_i^{(k)} - x_j^{(k)}} \qquad (r \ge 1), \tag{3.2.1}$$

where $\delta = m_k - 1 - m_{k\pm 1}$, and by $q_{r,\pm}^{(k)}(x;t) = 1$ for $r = 0$. In the product of the denominator, $x_j^{(k)}$ runs over all the variables in $x^{(k)}$ except $x_i^{(k)}$, while in the numerator, $x_j^{(k\pm 1)}$ runs over all the variables in $x^{(k\pm 1)}$. Note that if $x_i^{(k)} = x_i^{(k\pm 1)} = x_i$ and $m_k = m_{k\pm 1}$, $q_{r,\pm}^{(k)}(x;t)$ coincide with the function $q_r(x;t)$ given in [M, III, 2.9]. It can be shown that $q_{r,\pm}^{(k)}(x;t)$ is a polynomial in x, t with integral coefficients, homogeneous of degree r with respect to the variable x.

For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)}) \in \mathcal{P}_{n,e}$, we define a function $q_{\boldsymbol{\alpha},\pm}(x)$ by

$$q_{\boldsymbol{\alpha},\pm}(x;t) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} q^{(k)}_{\alpha_j^{(k)},\pm}(x;t). \tag{3.2.2}$$

Next, for a partition $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{l_k}^{(k)} > 0$, (here $l_k = l(\alpha^{(k)})$ is the number of parts of $\alpha^{(k)}$), we define a function $z_{\alpha^{(k)}}(t)$ by

$$z_{\alpha^{(k)}}(t) = \prod_{j=1}^{l_k} (1 - \zeta^k t^{\alpha_j^{(k)}})^{-1},$$

Then we define $z_{\boldsymbol{\alpha}}(t)$ by

$$z_{\boldsymbol{\alpha}}(t) = z_{\boldsymbol{\alpha}} \prod_{k=0}^{e-1} z_{\alpha^{(k)}}(t), \tag{3.2.3}$$

where $z_{\boldsymbol{\alpha}}$ is the order of the centralizer of $w_{\boldsymbol{\alpha}}$ in W. Explicitly, $z_{\boldsymbol{\alpha}}$ is given as follows. For $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, put $l(\boldsymbol{\alpha}) = \sum_{i=0}^{e-1} l(\alpha^{(i)})$. For a partition $\alpha = (1^{n_1}, 2^{n_2}, \dots)$, put $z_\alpha = \prod_{i \ge 1} i^{n_i} n_i!$. Then $z_{\boldsymbol{\alpha}} = e^{l(\boldsymbol{\alpha})} \prod_{k=0}^{e-1} z_{\alpha^{(k)}}$.

The importance of the quantity $z_{\boldsymbol{\alpha}}(t)$ lies in the following relations with $R(f)$ in (2.1.1);

$$\det_V(t \cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}}) = \prod_{k=0}^{e-1} \prod_{j=1}^{l_k} (t^{\alpha_j^{(k)}} - \zeta^k).$$

In particular, we have $z_{\boldsymbol{\alpha}}(t^{-1}) = z_{\boldsymbol{\alpha}} t^n \det_V(t \cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}})^{-1}$.

We now introduce infinitely many variables $x_i^{(k)}, y_i^{(k)}$ for $i = 1, 2, \dots$ and for $0 \le k \le e-1$. We may regard the above functions as functions with infinitely many variables $x_1^{(k)}, x_2^{(k)}, \dots$. Under this setting, the following proposition holds.

Proposition 3.3. *Let*

$$\Omega(x,y;t)=\prod_{k=0}^{e-1}\prod_{i,j}\frac{1-tx_i^{(k+1)}y_j^{(k)}}{1-x_i^{(k)}y_j^{(k)}}.$$

Then we have

$$\Omega(x,y;t)=\sum_{\boldsymbol{\alpha}}q_{\boldsymbol{\alpha},+}(x;t)m_{\boldsymbol{\alpha}}(y)=\sum_{\boldsymbol{\alpha}}m_{\boldsymbol{\alpha}}(x)q_{\boldsymbol{\alpha},-}(y;t), \tag{3.3.1}$$

$$\Omega(x,y;t)=\sum_{\boldsymbol{\alpha}}z_{\boldsymbol{\alpha}}(t)^{-1}p_{\boldsymbol{\alpha}}(x)\bar{p}_{\boldsymbol{\alpha}}(y), \tag{3.3.2}$$

where $\boldsymbol{\alpha}$ runs over all the e-partitions of any size. In (3.3.2), $\bar{p}_{\boldsymbol{\alpha}}(y)$ denotes the complex conjugate of $p_{\boldsymbol{\alpha}}(y)$.

3.4. We now introduce functions $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ associated to symbols $\boldsymbol{\Lambda}\in Z_n^{r,s}$, which is in analogy with $Q_\lambda(x;t)$ and $P_\lambda(x;t)$ in [M, III]. Let $\mathcal{A}$ be the subring of $\mathbb{Q}(t)$ consisting of functions which have no pole at $t=0$. Then $\mathcal{A}$ is a local ring with the unique maximal ideal $t\mathcal{A}$. Hence $\mathcal{A}^*=\mathcal{A}-t\mathcal{A}$ is the set of units in $\mathcal{A}$. In what follows, we identify the set $Z_n^{r,s}$ and $Z_n^{0,0}$ by the natural map, and we fix a total order $\prec$ on $Z_n^{0,0}$ as in 1.4. Then we have the following theorem.

Theorem 3.5. (i) *For each $\boldsymbol{\Lambda}\in Z_n^{r,s}$, there exists a unique function $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ satisfying the following properties.*

(a) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed as*

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)=\sum_{\boldsymbol{\beta}\in Z^{0,0}}c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)q_{\boldsymbol{\beta},\pm}(x;t),$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)\in\mathbb{Q}(t)$ and $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)=0$ unless $\boldsymbol{\beta}\succ\boldsymbol{\alpha}$ or $\boldsymbol{\beta}\sim\boldsymbol{\alpha}$.

(b) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed as*

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)=s_{\boldsymbol{\alpha}}(x)+\sum_{\boldsymbol{\beta}\in Z^{0,0}}u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)s_{\boldsymbol{\beta}}(x),$$

where $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)\in t\mathcal{A}$, and $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)=0$ unless $\boldsymbol{\beta}\prec\boldsymbol{\alpha}$ and $\boldsymbol{\beta}\not\sim\boldsymbol{\alpha}$.

(ii) *For each $\boldsymbol{\Lambda}\in Z_n^{r,s}$, there exists a unique function $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ satisfying the following properties.*

(a) *$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed as*

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)=q_{\boldsymbol{\alpha},\pm}(x;t)+\sum_{\boldsymbol{\beta}\in Z^{0,0}}d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)q_{\boldsymbol{\beta},\pm}(x;t),$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)\in\mathbb{Q}(t)$ and $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)=0$ unless $\boldsymbol{\beta}\succ\boldsymbol{\alpha}$ and $\boldsymbol{\beta}\not\sim\boldsymbol{\alpha}$.

(b) *$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed as*

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)=\sum_{\boldsymbol{\beta}\in Z^{0,0}}w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)s_{\boldsymbol{\beta}}(x),$$

where $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathcal{A}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Moreover, $w_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in t\mathcal{A}$ if $\boldsymbol{\beta} \ne \boldsymbol{\alpha}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathcal{A}^*$.

Remark 3.6. We call $P^{\pm}_{\boldsymbol{\Lambda}}$ the Hall-Littlewood function associated to $\boldsymbol{\Lambda} \in Z_n^{r,s}$. It is likely that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. In fact, in the classical case, i.e., the case where $e = 2, r = 2, s = 1$, the assertion follows from the known properties of Green functions.

The functions $P^{\pm}_{\boldsymbol{\Lambda}}, Q^{\pm}_{\boldsymbol{\Lambda}}$ have the stability property for the shift operation of symbols. Hence they can be regarded as functions with infinitely many variables. As a corollary to Theorem 3.5, we have

Corollary 3.7. *Let $\Omega(x,y;t)$ be as in Proposition 3.3. Then we have*

$$\Omega(x,y;t) = \sum_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t)P^{+}_{\boldsymbol{\Lambda}}(x;t)P^{-}_{\boldsymbol{\Lambda}'}(y;t). \tag{3.7.1}$$

$$\Omega(x,y;t) = \sum_{\boldsymbol{\Lambda}} Q^{+}_{\boldsymbol{\Lambda}}(x;t)P^{-}_{\boldsymbol{\Lambda}}(y;t) = \sum_{\boldsymbol{\Lambda}} P^{+}_{\boldsymbol{\Lambda}}(x;t)Q^{-}_{\boldsymbol{\Lambda}}(y;t). \tag{3.7.2}$$

where in (3.7.1), $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}'$ run over all the elements in $\bigcup_{n=1}^{\infty} Z_n^{r,s}$, and $b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) = 0$ unless $|\boldsymbol{\Lambda}| = |\boldsymbol{\Lambda}'|$ and $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$. In (3.7.2), $\boldsymbol{\Lambda}$ runs over all the elements in $\bigcup_{n=1}^{\infty} Z_n^{r,s}$.

3.8. Let $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \cdots \times \mathfrak{S}_{m_{e-1}}$ be a parabolic subgroup of $\mathfrak{S}_n$ of type $\mathbf{m}$. We denote by $\Xi_{\mathbf{m}} = \bigotimes_{k=0}^{e-1} \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{\mathbf{m}}$) with variables $x = (x_j^{(k)})$. $\Xi_{\mathbf{m}}$ has a structure of a graded ring $\Xi_{\mathbf{m}} = \bigoplus_{i \ge 0} \Xi^i_{\mathbf{m}}$, where $\Xi^i_{\mathbf{m}}$ consists of homogeneous symmetric polynomials of degree i, together with the zero polynomial. We consider the inverse limit

$$\Xi^i = \lim_{\substack{\longleftarrow \\ \mathbf{m}}} \Xi^i_{\mathbf{m}}$$

with respect to homomorphisms $\rho_{\mathbf{m}',\mathbf{m}} : \Xi^i_{\mathbf{m}'} \to \Xi^i_{\mathbf{m}}$, where $\mathbf{m}' = (m'_0, \dots, m'_{e-1})$ with $m'_k = m_k + l$ for some integer $l \ge 0$, and $\rho_{\mathbf{m}',\mathbf{m}}$ is induced from the homomorphism $\bigoplus_k \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}] \to \bigoplus_k \mathbb{Z}[x_1^{(k)}, \dots, x_{m'_k}^{(k)}]$ given by sending $x_i^{(k)}$ to 0 for $i \ge m_k$, and leaving other $x_j^{(k)}$ invariant. $\Xi = \bigoplus_{i \ge 0} \Xi^i$ is called the space of symmetric functions. Schur function $s_{\boldsymbol{\alpha}}(x)$ with infinitely many variables $x_1^{(k)}, \dots$ is regarded as an element in Ξ^n with $n = |\boldsymbol{\alpha}|$, and the set $\{s_{\boldsymbol{\alpha}}(x)\}$ with $\boldsymbol{\alpha} \in Z_n^{0,0}$ forms a $\mathbb{Z}$-basis of Ξ^n. Put $\Xi^n_{\mathbb{Q}}[t] = \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. It can be shown that the set $\{P^{\pm}_{\boldsymbol{\Lambda}} \mid \boldsymbol{\Lambda} \in Z_n^{r,s}\}$ (resp. the set $\{Q^{\pm}_{\boldsymbol{\Lambda}} \mid \boldsymbol{\Lambda} \in Z_n^{r,s}\}$, $\{q_{\boldsymbol{\alpha},\pm} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$) forms a $\mathbb{Q}(t)$-basis of the space $\Xi^n_{\mathbb{Q}}[t]$. A similar property holds if one replaces Ξ^n by $\Xi^n_{\mathbf{m}}$

It follows from the above discussion that $\{s_{\boldsymbol{\alpha}}\}$ gives rise to a basis of the $\mathbb{Z}[t]$-module $\Xi[t] = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Xi$, hence so is $\{m_{\boldsymbol{\alpha}}\}$. Also, we see that $\{q_{\boldsymbol{\alpha},\pm}\}$, $\{Q^{\pm}_{\boldsymbol{\Lambda}}\}$, and $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$ turn out to be bases of $\Xi_{\mathbb{Q}}[t]$. Moreover, $\{p_{\boldsymbol{\alpha}}\}$ gives a basis of $\mathbb{C}(t)$-space $\Xi_{\mathbb{C}}[t] = \mathbb{C}(t) \otimes_{\mathbb{Z}} \Xi$. We now define a scalar product on $\Xi_{\mathbb{Q}}[t]$ by the condition that

$$\langle q_{\boldsymbol{\alpha},+}(x;t), m_{\boldsymbol{\beta}}(x)\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{3.8.1}$$

and extend it to a sesquilinear form on $\Xi_{\mathbb{C}}[t]$. By using a similar argument as in [M, Chap. I, 4], it follows from Corollary 3.7 and Proposition 3.3 that we have

$$
\begin{aligned}
&\langle m_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\beta},-}(x;t)\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \\
(3.8.2) \qquad &\langle P^+_{\boldsymbol{\Lambda}}(x;t), Q^-_{\boldsymbol{\Lambda}'}(x;t)\rangle = \langle Q^+_{\boldsymbol{\Lambda}}(x;t), P^-_{\boldsymbol{\Lambda}'}(x;t)\rangle = \delta_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}, \\
&\langle p_{\boldsymbol{\alpha}}(x), p_{\boldsymbol{\beta}}(x)\rangle = z_{\boldsymbol{\alpha}}(t)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.
\end{aligned}
$$

In the special case where $e = 2$, the function $q_{\boldsymbol{\alpha},+}$ (resp. $P^+_{\boldsymbol{\Lambda}}$, $Q^+_{\boldsymbol{\Lambda}}$) coincides with $q_{\boldsymbol{\alpha},-}$ (resp. $P^-_{\boldsymbol{\Lambda}}$, $Q^-_{\boldsymbol{\Lambda}}$). We also have $p_{\boldsymbol{\alpha}} = \bar{p}_{\boldsymbol{\alpha}}$ and $z_{\boldsymbol{\alpha}}(t) \in \mathbb{Q}(t)$. Hence in this case, the scalar product $\langle\ ,\ \rangle$ turns out to be symmetric.

The following result gives a characterization of $P^{\pm}_{\boldsymbol{\Lambda}}$ and $Q^{\pm}_{\boldsymbol{\Lambda}}$ in terms of the inner product.

Proposition 3.9. *The functions $P^{\pm}_{\boldsymbol{\Lambda}}$ are characterized by the following properties. (Then the functions $Q^{\pm}_{\boldsymbol{\Lambda}}$ is charactersized as the dual bases of $P^{\mp}_{\boldsymbol{\Lambda}}$ in the sense of (3.8.2)).*

(i) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed in terms of $m_{\boldsymbol{\beta}}(x)$ as*

$$P^{\pm}_{\boldsymbol{\Lambda}} = m_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}} c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} m_{\boldsymbol{\beta}},$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, and $c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(ii) *$\langle P^+_{\boldsymbol{\Lambda}}, P^-_{\boldsymbol{\Lambda}'}\rangle = \langle P^-_{\boldsymbol{\Lambda}}, P^+_{\boldsymbol{\Lambda}'}\rangle = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$.*

3.10. Let $K_{\pm}(t) = M(s, P^{\pm})$ be the transition matrix between Schur functions and Hall-Littlewood functions, i.e.,

$$s_{\boldsymbol{\beta}}(x) = \sum_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})} K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t).$$

We call $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ Kostka functions associated to $Z^{r,s}_n$, in analogy to the case of $GL_n(\mathbf{F}_q)$. By Theorem 3.5, $K_{\pm}(t)$ is a block lower triangular matrix with identity diagonal blocks, with entries $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ in $\mathbb{Q}(t)$. We put $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = t^{a(\boldsymbol{\beta})} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1})$ and let $\widetilde{K}_{\pm}(t) = (\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$. Then the matrix $\widetilde{K}_{\pm}(t)$ satisfies a similar condition as P given in (2.4.2). Next we consider the matrix $D(t) = (b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t))$, indexed by symbols in $Z^{r,s}_n$, where $b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t)$ is as in Corollary 3.7. We write $D(t)^{-1} = (b'_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$, and define a matrix $\widetilde{\Lambda}(t)$ by $\widetilde{\Lambda}(t) = (d(t)t^{a(\boldsymbol{\beta})-a(\boldsymbol{\alpha})} b'_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1}))$, where $d(t) = t^{N^*-n}(t-1)^n P_W(t)$. Clearly, $\widetilde{\Lambda}(t)$ is a block diagonal matrix. Under this setting, we have the following theorem.

Theorem 3.11. *We have*

$$\widetilde{K}_-(t)\widetilde{\Lambda}(t)\,{}^t\widetilde{K}_+(t) = \Omega'.$$

In particular $P' = \widetilde{K}_-(t), P'' = \widetilde{K}_+(t)$ and $\Lambda' = \widetilde{\Lambda}(t)$ gives a solution for the equation (2.5.2).

Theorem 3.11 shows that the solution of (2.5.2), hence that of (2.4.2) can be described in terms of Kostka functions, just as in the case of Green polynomials of $GL_n(\mathbf{F}_q)$. In particular, the matrix P coincides with $\widetilde{K}_-(t)$.

Remarks 3.12. (i) As some examples show one can conjecture the following. $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is a polynomial in $\mathbb{Z}[t]$ with positive coefficients. Moreover, $\deg K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \leq$

$a(\boldsymbol{\beta}) - a(\boldsymbol{\alpha})$. This implies that $\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is also in $\mathbb{Z}[t]$. The conjecture also implies that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ is a polynomial in $\mathbb{Z}[x;t]$.

(ii) From the discussion in 3.8, we see that $\{q_{\boldsymbol{\alpha},\pm}(x;t)\}$ and $\{m_{\boldsymbol{\alpha}}(x)\}$, and $\{Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)\}$ and $\{P^{\mp}(x;t)\}$ are dual bases of each other. Hence we have

$$M(Q^{\mp}, q_{\mp}) = M(P^{\pm}, m)^* = (K_{\pm}(t)^{-1}K)^* = {}^tK_{\pm}(t)K^*, \tag{3.12.1}$$

where $K = M(s,m)$ is the Kostka matrix, and X^* denotes the transposed inverse of the matrix X. Since K^* is a matrix with entries in $\mathbb{Z}$, we see that $K_{\pm}(t)$ is a matrix with entries in $\mathbb{Z}[t]$ if and only if $M(Q^{\mp}, q_{\mp})$ is a matrix with entries in $\mathbb{Z}[t]$. It follows that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ lie in $\mathbb{Z}[x;t]$ if an only if the coefficients $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ in Theorem 3.5, (ii), (a) lie in $\mathbb{Z}[t]$ for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in Z_n^{0,0}$.

3.13. Concerning the above conjecture, we have some partial result in the case where $e = 2$. Now assume that $e = 2, r \geq 1$, and that $m_0 = m+1, m_1 = m$ for some $m > 0$. Let $\boldsymbol{\Lambda} = (\Lambda_0, \Lambda_1) \in Z_n^{r,s}$ be a symbol. We write $\Lambda_0 : \nu_1 > \nu_3 > \cdots > \nu_{2m+1}$ and $\Lambda_1 : \nu_2 > \nu_4 > \cdots > \nu_{2m}$. We say that $\boldsymbol{\Lambda}$ is very special if $\nu_i \geq \nu_{i+1} + r$ for $i = 1, 2, \ldots$. Then we have the following.

Proposition 3.14. *Assume that $e = 2$ and that $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z_n^{r,s}$ is very special. Then the coefficient $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ of $q_{\boldsymbol{\beta},\pm}$ in $Q^{\pm}_{\boldsymbol{\Lambda}}$ is in $\mathbb{Z}[t]$. In particular, $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) \in \mathbb{Z}[t]$ for any $\boldsymbol{\beta} \in Z_n^{r,s}$. Moreover, we have $\deg K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}} \leq a(\boldsymbol{\alpha}) - a(\boldsymbol{\beta})$. Hence $\widetilde{K}^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) \in \mathbb{Z}[t]$.*

3.15. In the remainder of this section, we shall give explicit examples of Green functions associated to various $W = G(e,1,n)$. In each case, $Z_n^{0,0}$ are written as $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_h$, and the corresponding symbols in $Z_n^{r,s}$ are written as $\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}(\boldsymbol{\alpha}_i)$. In the matrix P of Green functions, the rows and columns are arranged along the order $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots$.

(A) $W = G(2,1,2)$. In this case W is the Weyl group of type B_2, and we give the table just for the sake of reference. $Z_n^{0,0}$ is given as

$$\boldsymbol{\alpha}_1 = (-;1^2), \boldsymbol{\alpha}_2 = (1^2;-), \boldsymbol{\alpha}_3 = (1;1), \boldsymbol{\alpha}_4 = (-;2), \boldsymbol{\alpha}_5 = (2;-).$$

Assume that $r = 2$ and $s = 1$. Then the corresponding symbols and similarity classes are given as

$$\{\boldsymbol{\Lambda}_1 = (420;42)\}, \quad \{\boldsymbol{\Lambda}_2 = (31;1)\},$$
$$\{\boldsymbol{\Lambda}_3 = (30;2), \boldsymbol{\Lambda}_4 = (20;3)\}, \quad \{\boldsymbol{\Lambda}_5 = (2;-)\}.$$

The matrix P of Green functions is given in Table 1.

TABLE 1. $G(2,1,2) \quad (r = 2, s = 1)$

$(-;1^2)$	q^4				
$(1^2;-)$	q^2	q^3			
$(1;1)$	q^3+q	q	q		
$(-;2)$	q^2			q	
$(2;-)$	1	1	1		1

(B) $W = G(3,1,2)$. Then $Z_n^{0,0}$ is given as

$$\boldsymbol{\alpha}_1 = (-;1^2;-), \boldsymbol{\alpha}_2 = (-;-;1^2), \boldsymbol{\alpha}_3 = (1^2;-;-), \boldsymbol{\alpha}_4 = (-;1;1),$$
$$\boldsymbol{\alpha}_5 = (1;1;-), \boldsymbol{\alpha}_6 = (1;-;1), \boldsymbol{\alpha}_7 = (-;2;-), \boldsymbol{\alpha}_8 = (-;-;2), \boldsymbol{\alpha}_9 = (2;-;-).$$

Assume that $r = 2, s = 1$. Then the corresponding symbols and similarity classes are given as

$$\{\boldsymbol{\Lambda}_1 = (420;42;31), \boldsymbol{\Lambda}_2 = (420;31;42)\}, \quad \{\boldsymbol{\Lambda}_3 = (31;1;1)\}, \quad \{\boldsymbol{\Lambda}_4 = (20;2;2)\}$$
$$\{\boldsymbol{\Lambda}_5 = (30;2;1), \boldsymbol{\Lambda}_6 = (30;1;2), \boldsymbol{\Lambda}_7 = (20;3;1)\boldsymbol{\Lambda}_8 = (20;1;3)\}, \quad \{\boldsymbol{\Lambda}_9 = (2;-;-)\}.$$

The matrix P of Green functions is given in Table 2.

Table 2. $G(3,1,2) \quad (r = 2, s = 1)$

$(-;1^2;-)$	q^5								
$(-;-;1^2)$		q^5							
$(1^2;-;-)$	q^3	q^4	q^3						
$(-;1;1)$	q^3			q^3					
$(1;1;-)$	q^4+q		q	q	q				
$(1;-;1)$	q^2	q^3	q^2	q^2		q			
$(-;2;-)$	q^2		q^2				q		
$(-;-;2)$		q^2	q					q	
$(2;-;-)$	1		1	1	1				1

(C) $W = G(4,1,2)$. Then $Z_n^{0,0}$ is given as

$$\boldsymbol{\alpha}_1 = (-;1^2;-;-), \boldsymbol{\alpha}_2 = (-;-;1^2;-;-), \boldsymbol{\alpha}_3 = (-;-;-;1^2),$$
$$\boldsymbol{\alpha}_4 = (1^2;-;-;-), \boldsymbol{\alpha}_5 = (-;1;1;-), \boldsymbol{\alpha}_6 = (-;1;-;1), \boldsymbol{\alpha}_7 = (-;-;1;1),$$
$$\boldsymbol{\alpha}_8 = (1;1;-;-), \boldsymbol{\alpha}_9 = (1;-;1;-), \boldsymbol{\alpha}_{10} = (-;2;-;-), \boldsymbol{\alpha}_{11} = (1;-;-;1),$$
$$\boldsymbol{\alpha}_{12} = (-;-;2;-), \boldsymbol{\alpha}_{13} = (-;-;-;2), \boldsymbol{\alpha}_{14} = (2;-;-;-).$$

Assume that $r = 2$ and $s = 1$. Then the corresponding symbols and similarity classes are given as

$$\{\boldsymbol{\Lambda}_1 = (420;42;31;31), \boldsymbol{\Lambda}_2 = (420;31;42;31), \boldsymbol{\Lambda}_3 = (420;31;31;42)\},$$
$$\{\boldsymbol{\Lambda}_4 = (31;1;1;1)\}, \quad \{\boldsymbol{\Lambda}_5 = (20;2;2;1), \boldsymbol{\Lambda}_6 = (20;2;1;2), \boldsymbol{\Lambda}_7 = (20;1;2;2)\},$$
$$\{\boldsymbol{\Lambda}_8 = (30;2;1;1), \boldsymbol{\Lambda}_9 = (30;1;2;1), \boldsymbol{\Lambda}_{10} = (20;3;1;1),$$
$$\boldsymbol{\Lambda}_{11} = (30;1;1;2), \boldsymbol{\Lambda}_{12} = (20;1;3;1), \boldsymbol{\Lambda}_{13} = (20;1;1;3)\}$$
$$\{\boldsymbol{\Lambda}_{14} = (2;-;-;-)\}.$$

The matrix P of Green functions is given in Table 3.

Table 3. $G(4,1,2) \quad (r=2, s=1)$

$(-;1^2;-;-)$	q^6													
$(-;-;1^2;-)$		q^6												
$(-;-;-;1^2)$			q^6											
$(1^2;-;-;-)$	q^4			q^4										
$(-;1;1;-)$	q^3	q^5			q^3									
$(-;1;-;1)$	q^4		q^4			q^3								
$(-;-;1;1)$		q^3	q^5				q^3							
$(1;1;-;-)$	q^5+q			q	q			q						
$(1;-;1;-)$	q^2	q^4		q^2	q^2				q					
$(-;2;-;-)$	q^2				q^2					q				
$(1;-;-;1)$	q^3		q^3	q^3		q^2					q			
$(-;-;2;-)$		q^2					q^2					q		
$(-;-;-;2)$			q^2			q							q	
$(2;-;-;-)$	1			1	1			1						1

(D) $W = G(3,1,3)$. Then $Z_n^{0,0}$ is given as

$$\begin{aligned}
&\boldsymbol{\alpha}_1 = (-;1^3;-), \boldsymbol{\alpha}_2 = (-;-;1^3), \boldsymbol{\alpha}_3 = (1^3;-;-), \boldsymbol{\alpha}_4 = (-;1^2;1),\\
&\boldsymbol{\alpha}_5 = (-;1;1^2), \boldsymbol{\alpha}_6 = (1;1^2;-), \boldsymbol{\alpha}_7 = (-;21;-), \boldsymbol{\alpha}_8 = (1;-;1^2),\\
&\boldsymbol{\alpha}_9 = (-;-;21), \boldsymbol{\alpha}_{10} = (1^2;1;-), \boldsymbol{\alpha}_{11} = (1^2;-;1), \boldsymbol{\alpha}_{12} = (21;-;-),\\
&\boldsymbol{\alpha}_{13} = (1;1;1), \boldsymbol{\alpha}_{14} = (-;2;1), \boldsymbol{\alpha}_{15} = (-;1;2), \boldsymbol{\alpha}_{16} = (1;2;-),\\
&\boldsymbol{\alpha}_{17} = (1;-;2), \boldsymbol{\alpha}_{18} = (2;1;-), \boldsymbol{\alpha}_{19} = (2;-;1), \boldsymbol{\alpha}_{20} = (-;3;-),\\
&\boldsymbol{\alpha}_{21} = (-;-;3), \boldsymbol{\alpha}_{22} = (3;-;-).
\end{aligned}$$

Assume that $r=2$ and $s=1$. Then the corresponding symbols and families are given as

$$\begin{aligned}
&\{\boldsymbol{\Lambda}_1 = (6420;642;531), \boldsymbol{\Lambda}_2 = (6420;531;642)\},\\
&\{\boldsymbol{\Lambda}_3 = (531;31;31)\}, \quad \{\boldsymbol{\Lambda}_4 = (420;42;41), \boldsymbol{\Lambda}_5 = (420;41;42)\},\\
&\{\boldsymbol{\Lambda}_6 = (520;42;31), \boldsymbol{\Lambda}_7 = (420;52;31), \boldsymbol{\Lambda}_8 = (520;31;42), \boldsymbol{\Lambda}_9 = (420;31;52)\},\\
&\{\boldsymbol{\Lambda}_{10} = (31;2;1), \boldsymbol{\Lambda}_{11} = (31;1;2)\}, \qquad \{\boldsymbol{\Lambda}_{12} = (41;1;1)\},\\
&\{\boldsymbol{\Lambda}_{13} = (30;2;2), \boldsymbol{\Lambda}_{14} = (20;3;2), \boldsymbol{\Lambda}_{15} = (20;2;3)\},\\
&\{\boldsymbol{\Lambda}_{16} = (30;3;1), \boldsymbol{\Lambda}_{17} = (30;1;3)\},\\
&\{\boldsymbol{\Lambda}_{18} = 40;2;1), \boldsymbol{\Lambda}_{19} = (40;1;2), \boldsymbol{\Lambda}_{20} = (20;4;1), \boldsymbol{\Lambda}_{21} = (20;1;4)\},\\
&\{\boldsymbol{\Lambda}_{22} = (3;-;-)\}.
\end{aligned}$$

The matrix P of Green functions is given in Table 4.

Table 4. $G(3,1,3) \quad (r=2, s=1)$

$(-;1^3;-)$	q^{12}																					
$(-;-;1^3)$		q^{12}																				
$(1^3;-;-)$	q^9		q^9																			
$(-;1^2;1)$	$q^{10}+q^7$	q^{10}		q^7																		
$(-;1;1^2)$	q^8	$q^{11}+q^8$			q^7																	
$(1;1^2;-)$	$q^{11}+q^8+q^5$		q^5	q^5		q^5																
$(-;21;-)$	q^9+q^6			q^6			q^5															
$(1;-;1^2)$	q^7	$q^{10}+q^7$	q^7		q^6			q^5														
$(-;-;21)$		q^9+q^6			q^5				q^5													
$(1^2;1;-)$	$q^{10}+q^7+q^4$		q^7+q^4	q^4		q^4				q^4												
$(1^2;-;1)$	q^8+q^5	q^8	q^8+q^5	q^5				q^3			q^4											
$(21;-;-)$	q^6+q^3		q^6+q^3	q^3		q^3				q^3		q^3										
$(1;1;1)$	$q^9+2q^6+q^3$	q^9+q^6	q^6+q^3	q^6+q^3	q^5	q^3		q^4		q^3			q^3									
$(-;2;1)$	q^7+q^4	q^7		q^4	q^6		q^3							q^3								
$(-;1;2)$	q^5	q^8+q^5		q^5	q^4				q^4						q^3							
$(1;2;-)$	$q^8+q^5+q^2$		q^2	q^5+q^2		q^2	q^4			q^2			q^2			q^2						
$(1;-;2)$	q^4	q^7+q^4	q^4	q^4	q^3			q^2	q^3		q^3				q^2		q^2					
$(2;1;-)$	q^7+q^4+q		q^4+q	q^4+q		q^4+q				q		q	q			q		q				
$(2;-;1)$	q^5+q^2	q^5	q^5+q^2	q^2	q^4	q^2		q^3		q^2		q^2	q^2						q			
$(-;3;-)$	q^3			q^3			q^2							q^2						q		
$(-;-;3)$		q^3			q^2				q^2						q		q				q	
$(3;-;-)$	1		1	1		1				1		1	1			1		1				1

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