MACDONALD FUNCTIONS ASSOCIATED TO COMPLEX REFLECTION GROUPS

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Abstract. Let W be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$. In the author's previous paper [S1], Hall-Littlewood functions associated to W were introduced. In the special case where W is a Weyl group of type B_n, they are closely related to Green polynomials of finite classical groups. In this paper, we introduce a two variables version of the above Hall-Littlewood functions, as a generalization of Macdonald functions associated to symmetric groups. A generalization of Macdonald operators is also constructed, and we characterize such functions by making use of Macdonald operators, assuming a certain conjecture.

0. Introduction

Macdonald functions $P_\lambda(x;q,t)$, which were introduced by I.G. Macdonald [M2] in 1987, are two variables versions of Hall-Littlewood functions $P_\lambda(x;t)$. Those Hall-Littlewood functions and Macdonald functions are parametrized by partitions λ of n. Since partitions of n parameterize irreducible characters of the symmetric group $\mathfrak{S}_n$, we may say that these functions are associated to symmetric groups. On the other hand, Hall-Littlewood functions are closely related to Green polynomials of a finite general linear group $GL_n(\mathbb{F}_q)$. In this direction, partitions λ of n occur as unipotent classes of $GL_n(\mathbb{F}_q)$. Unipotent classes of other finite classical groups such as $Sp_{2n}(\mathbb{F}_q), SO_{2n+1}(\mathbb{F}_q)$ have more complicated patterns. Lusztig introduced in [L2] (unipotent) symbols, as a generalization of the notion of partitions, to describe such unipotent classes in connection with Springer representations of Weyl groups. He also introduced in [L1] a notion of symbols to parameterize unipotent characters of finite classical groups.

In [S1], the author constructed Hall-Littlewood functions associated to complex reflection groups $W \simeq \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$. In [S2], [S3], some related topics are discussed. Our Hall-Littlewood functions are parametrized by e-tuples of partitions (which parameterize irreducible characters of W), or rather by various types of e-symbols. In the case where $e = 2$, W is the Weyl group of type B_n. In this case, Hall-Littlewood functions attached to unipotent symbols are closely related to Green functions of $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$. It is also expected that our Hall-Littlewood functions attached to symbols have some connection with unipotent characters.

This paper is an attempt to generalize Hall-Littlewood functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ (where $\boldsymbol{\Lambda}$ is an e-symbol) to the two variables version $P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$, just as in the case of $P_{\lambda}(x;q,t)$. We call such functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$ Macdonald functions associated to W. In the case of original Macdonald functions, they are characterized as simultaneous eigenfunctions of various Macdonald operators. We also construct Macdonald operators having (conjecturally) good properties. However, we note that our construction of Macdonald operators works only for a special type of symbols (in the case where $e = 2$, this is exactly the symbols used to parameterize unipotent characters), though Macdonald functions can be constructed for any type of symbols. In the case of original Macdonald operators, the representation matrix with respect to the basis of Schur functions is a triangular matrix, with distinct eigenvalues. In our case, the matrix of Macdonald operators turns out to be a block triangular matrix, where the blocks correspond to families of symbols. One can conjecture that the matrices appearing in the diagonal blocks have no common eigenvalues. Assuming this conjecture, we show that the Macdonald operator characterizes Macdonald functions, not as eigenfunction, but as a unique solution of linear systems attached to the above diagonal blocks.

The properties of Macdonald functions discussed in this paper are just a part of those established for the original Macdonald functions. We hope to discuss more about them in a subsequent paper.

1. Symmetric functions with two parameters

1.1. An e-tuple of partitions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ is called an e-partition. We define the size $|\boldsymbol{\alpha}|$ of $\boldsymbol{\alpha}$ by $|\boldsymbol{\alpha}| = \sum_{k=0}^{e-1} |\alpha^{(k)}|$, where $|\alpha^{(k)}|$ is the size of the partition $\alpha^{(k)}$. For a partition $\alpha : \alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge 0$, let $l(\alpha)$ be the number of non-zero parts α_k. We denote by $\mathcal{P}_n$ the set of partitions of n, and $\mathcal{P}_{n,e}$ the set of e-partitions of size n. Let W be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$. Then the set of conjugacy classes in W is in one to one correspondence with the set $\mathcal{P}_{n,e}$.

Let us fix a sequence of positive integers $\mathbf{m} = (m_0, \dots, m_{e-1})$, and consider indeterminates $x^{(k)}_j$ $(0 \le k \le e-1, 1 \le j \le m_k)$. We denote by $x = x_{\mathbf{m}}$ the whole variables $(x^{(k)}_j)$, and also denote by $x^{(k)}$ the variables $x^{(k)}_1, \dots, x^{(k)}_{m_k}$. Let ζ be a primitive e-th root of unity in $\mathbb{C}$. For each integer $r \ge 1$ and i such that $0 \le i \le e-1$, put

$$p^{(i)}_r(x) = \sum_{j=0}^{e-1} \zeta^{ij} p_r(x^{(j)}),$$

where $p_r(x^{(j)})$ denotes the r-th power sum symmetric function with respect to the variables $x^{(j)}$. We put $p^{(i)}_r(x) = 1$ for $r = 0$. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ with $\alpha^{(k)} : \alpha^{(k)}_1 \ge \cdots \ge \alpha^{(k)}_{m_k}$, we define a function $p_{\boldsymbol{\alpha}}(x)$ by

$$p_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} p^{(k)}_{\alpha^{(k)}_j}(x).$$

Next, we define the Schur function $s_{\boldsymbol\alpha}(x)$ and monomial symmetric functions $m_{\boldsymbol\alpha}(x)$ associated to $\boldsymbol\alpha$ by

$$s_{\boldsymbol\alpha}(x)=\prod_{k=0}^{e-1}s_{\alpha^{(k)}}(x^{(k)}),\qquad m_{\boldsymbol\alpha}(x)=\prod_{k=0}^{e-1}m_{\alpha^{(k)}}(x^{(k)}),$$

where $s_{\alpha^{(k)}}(x^{(k)})$ (resp. $m_{\alpha^{(k)}}(x^{(k)}$) denotes the usual Schur function (resp. monomial symmetric function) associated to the partition $\alpha^{(k)}$ with respect to the variables $x^{(k)}$. These are the symmetric functions associated to complex reflection groups $W\simeq\mathfrak{S}_n\ltimes(\mathbb{Z}/e\mathbb{Z})^n$, as given in [M1, Appendix B].

1.2. Put $\mathfrak{S}_{\mathbf m}=\mathfrak{S}_{m_0}\times\cdots\times\mathfrak{S}_{m_{e-1}}$. We denote by $\Xi_{\mathbf m}=\bigotimes_{k=0}^{e-1}\mathbb{Z}[x_1^{(k)},\dots,x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{\mathbf m}$) with variables $x=(x_j^{(k)})$. $\Xi_{\mathbf m}$ has a structure of a graded ring $\Xi_{\mathbf m}=\bigoplus_{i\ge0}\Xi^i_{\mathbf m}$, where $\Xi^i_{\mathbf m}$ consists of homogeneous symmetric polynomials of degree i, together with the zero polynomial. We consider the inverse limit

$$\Xi^i=\varprojlim_{\mathbf m}\ \Xi^i_{\mathbf m}$$

with respect to homomorphisms $\rho_{\mathbf m',\mathbf m}:\Xi^i_{\mathbf m'}\to\Xi^i_{\mathbf m}$, where $\mathbf m'=(m_0',\dots,m_{e-1}')$ with $m_k'=m_k+l$ for some integer $l\ge0$, and $\rho_{\mathbf m',\mathbf m}$ is induced from the homomorphism $\bigotimes_k\mathbb{Z}[x_1^{(k)},\dots,x_{m_k'}^{(k)}]\to\bigotimes_k\mathbb{Z}[x_1^{(k)},\dots,x_{m_k}^{(k)}]$ given by sending $x_i^{(k)}$ to 0 for $i>m_k$, and leaving the other $x_j^{(k)}$ invariant. $\Xi=\bigoplus_{i\ge0}\Xi^i$ is called the space of symmetric functions. Schur functions $s_{\boldsymbol\alpha}(x)$ with infinitely many variables $x_1^{(k)},x_2^{(k)}\dots$ are regarded as elements in Ξ^n with $n=|\boldsymbol\alpha|$, and the set $\{s_{\boldsymbol\alpha}(x)\mid\boldsymbol\alpha\in\mathcal{P}_{n,e}\}$ forms a $\mathbb{Z}$-basis of Ξ^n. Similarly, $\{m_{\boldsymbol\alpha}(x)\mid\boldsymbol\alpha\in\mathcal{P}_{n,e}\}$ gives a $\mathbb{Z}$-basis of Ξ^n. Put $\Xi_{\mathbb C}=\mathbb{C}\otimes\Xi$. Then $\{p_{\boldsymbol\alpha}(x)\mid\boldsymbol\alpha\in\mathcal{P}_{n,e}\}$ gives rise to a basis of $\Xi_{\mathbb C}$.

1.3. Let q,t be independent indeterminates and let $F=\mathbb{C}(q,t)$ be the field of rational functions in q,t. We consider the F-algebra of symmetric functions $\Xi_F=F\otimes_{\mathbb Z}\Xi$ with coefficients in F. Let $\boldsymbol\alpha=(\alpha^{(0)},\dots,\alpha^{(e-1)})$ be an e-partition. For each partition $\alpha^{(k)}:\alpha_1^{(k)}\ge\alpha_2^{(k)}\ge\cdots$, put

$$z_{\alpha^{(k)}}(q,t)=\prod_{j=1}^{l(\alpha^{(k)})}\frac{1-\zeta^kq^{\alpha_j^{(k)}}}{1-\zeta^kt^{\alpha_j^{(k)}}}.$$

We then define $z_{\boldsymbol\alpha}(q,t)\in F$ by

$$z_{\boldsymbol\alpha}(q,t)=z_{\boldsymbol\alpha}\prod_{k=0}^{e-1}z_{\boldsymbol\alpha^{(k)}}(q,t),\tag{1.3.1}$$

where $z_{\boldsymbol\alpha}$ is the order of the centralizer of $w_{\boldsymbol\alpha}$ in W (a representative of the conjugacy class of W corresponding to $\boldsymbol\alpha\in\mathcal{P}_{n,e}$). Explicitly, $z_{\boldsymbol\alpha}$ is given as follows. For $\boldsymbol\alpha\in\mathcal{P}_{n,e}$, put $l(\boldsymbol\alpha)=\sum_{k=0}^{e-1}l(\alpha^{(k)})$. For a partition $\alpha=(1^{n_1},2^{n_2},\dots)$, put $z_\alpha=\prod_{i\ge1}i^{n_i}n_i!$. Then $z_{\boldsymbol\alpha}=e^{l(\boldsymbol\alpha)}\prod_{k=0}^{e-1}z_{\alpha^{(k)}}$.

We define a sesquilinear form on Ξ_F by

$$\langle p_{\boldsymbol{\alpha}}, p_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} z_{\boldsymbol{\alpha}}(q,t) \tag{1.3.2}$$

for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. Let $x = (x_j^{(k)})$, $y = (y_j^{(k)})$ $(0 \le k \le e-1)$ be two sequences of infinitely many variables, and we define an infinite product of x and y by

$$\begin{aligned} \Pi(x,y;q,t) &= \prod_{k=0}^{e-1}\prod_{i,j=1}^{\infty}\prod_{r=0}^{\infty}\frac{1-tx_i^{(k-r-1)}y_j^{(k)}q^r}{1-x_i^{(k-r)}y_j^{(k)}q^r} \\ &= \prod_{k=0}^{e-1}\prod_{i,j=1}^{\infty}\prod_{r=0}^{\infty}\frac{1-tx_i^{(k)}y_j^{(k+r+1)}q^r}{1-x_i^{(k)}y_j^{(k+r)}q^r}. \end{aligned} \tag{1.3.3}$$

(Here the upper indices of the variables in the formula should be read $\mod e$). Note that in the case where $q = 0$, the product $\Pi(x,y;q,t)$ reduces to the product $\Omega(x,y;t)$ introduced in [S1, 2.5], (or rather [S2, (5.7.1)], see the remark there).

Lemma 1.4. $\Pi(x,y;q,t)$ *has the following expansion.*

$$\Pi(x,y;q,t) = \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}(q,t)^{-1}p_{\boldsymbol{\alpha}}(x)\overline{p_{\boldsymbol{\alpha}}(y)}$$

where $\boldsymbol{\alpha}$ *runs over all the e-partitions of any size.*

Proof. Taking the log on both sides of the first formula of (1.3.3),

$$\begin{aligned} &\log\Pi(x,y;q,t) \\ &= \sum_k\sum_{i,j}\sum_{r=0}^{\infty}\sum_{m=1}^{\infty}\biggl\{\frac{1}{m}(x_i^{(k-r)}y_j^{(k)}q^r)^m - \frac{t^m}{m}(x_i^{(k-r-1)}y_j^{(k)}q^r)^m\biggr\}. \end{aligned}$$

By making use of the equation

$$\frac{1}{e}\sum_{a=0}^{e-1}(\zeta^k\zeta^{-k'})^a = \delta_{k,k'},$$

the above formula can be written as

$$\begin{aligned} \log\Pi(x,y;q,t) = \sum_{a=0}^{e-1}\sum_{m\ge 1}\sum_{r\ge 0}\sum_{k,k',i,j}\biggl\{&\frac{1}{em}\zeta^{ar}\zeta^{a(k-r)}\zeta^{-ak'}(x_i^{(k-r)}y_j^{(k')}q^r)^m \\ &- \frac{t^m}{em}\zeta^{a(r+1)}\zeta^{a(k-r-1)}\zeta^{-ak'}(x_i^{(k-r-1)}y_j^{(k')}q^r)^m\biggr\}. \end{aligned}$$

It follows that

$$\begin{aligned}\log \Pi(x,y;q,t) &= \sum_{a=0}^{e-1}\sum_{m\ge 1}\sum_{r\ge 0}\sum_{k,k',i,j}\frac{1-\zeta^a t^m}{em}\zeta^{ar}q^{rm}(x_i^{(k)}y_j^{(k')})^m\zeta^{ak}\zeta^{-ak'} \\ &= \sum_{a=0}^{e-1}\sum_{m\ge 1}\sum_{k,k',i,j}\frac{1}{em}\cdot\frac{1-\zeta^a t^m}{1-\zeta^a q^m}\zeta^{ak}\zeta^{-ak'}(x_i^{(k)}y_j^{(k')})^m \\ &= \sum_{a=0}^{e-1}\sum_{m=1}^{\infty}\frac{1}{em}\cdot\frac{1-\zeta^a t^m}{1-\zeta^a q^m}p_m^{(a)}(x)\overline{p_m^{(a)}(y)}.\end{aligned}$$

Hence we have

$$\begin{aligned}\Pi(x,y;q,t) &= \prod_{a=0}^{e-1}\prod_{m=1}^{\infty}\exp\left\{\frac{1}{em}\cdot\frac{1-\zeta^a t^m}{1-\zeta^a q^m}p_m^{(a)}(x)\overline{p_m^{(a)}(y)}\right\} \\ &= \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}(q,t)^{-1}p_{\boldsymbol{\alpha}}(x)\overline{p_{\boldsymbol{\alpha}}(y)}.\end{aligned}$$

□

1.5. For a fixed $y_j^{(k)}$, we define a function $g_{m,\pm}^{(k)}(x;q,t)$ as the coefficient of $(y_j^{(k)})^m$ in

$$\prod_i\prod_{r\ge 0}\frac{1-tx_i^{(k\mp r\mp 1)}y_j^{(k)}q^r}{1-x_i^{(k\mp r)}y_j^{(k)}q^r} = \sum_{m\ge 0} g_{m,\pm}^{(k)}(x;q,t)(y_j^{(k)})^m \tag{1.5.1}$$

and put, for each $\boldsymbol{\alpha} = (\alpha_j^{(k)}) \in \mathcal{P}_{n,e}$,

$$g_{\boldsymbol{\alpha},\pm}(x;q,t) = \prod_{j,k} g^{(k)}_{\alpha_j^{(k)},\pm}.$$

Then we have

$$\Pi(x,y;q,t) = \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha},+}(x;q,t)m_{\boldsymbol{\alpha}}(y) = \sum_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}(x)g_{\boldsymbol{\alpha},-}(y;q,t). \tag{1.5.2}$$

In fact, by comparing the first formula of (1.3.3) and (1.5.1), we have

$$\begin{aligned}\Pi(x,y;q,t) &= \prod_{k,j}\sum_{\alpha_j^{(k)}\ge 0} g^{(k)}_{\alpha_j^{(k)},+}(x;q,t)(y_j^{(k)})^{\alpha_j^{(k)}} \\ &= \sum_{\boldsymbol{\alpha}} g_{\boldsymbol{\alpha},+}(x;q,t)m_{\boldsymbol{\alpha}}(y).\end{aligned}$$

This shows the first equality. If we compare the second formula of (1.3.3) and (1.5.1) by replacing x and y, the second equality is obtained in a similar way.

Now by using a similar argument as in [M1, VI, 2.7], we see that

$$\langle g_{\boldsymbol{\alpha},+}(x;q,t), m_{\boldsymbol{\beta}}(x)\rangle = \langle m_{\boldsymbol{\alpha}}(x), g_{\boldsymbol{\beta},-}(x;q,t)\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}. \tag{1.5.3}$$

In particular, the functions $g_{\boldsymbol{\alpha},\pm}(x;q,t)$ form a basis of Ξ_F dual to $m_{\boldsymbol{\alpha}}$. Hence $\{g_n(x;q,t) \mid n \geq 0\}$ are algebraically independent over F, and $\Xi_F = F[g_1, g_2, \dots]$.

The following lemma can be proved in a similar way as [M1, VI, 2.13] by using (1.5.2) and (1.5.3).

Lemma 1.6. *Let $E^{\pm} : \Xi_F \to \Xi_F$ be two F-linear operators. Then the following conditions are equivalent.*

(i) $\langle E^+ f, g\rangle = \langle f, E^- g\rangle$ *for any* $f, g \in \Xi_F$

(ii) $E^+_x \Pi(x,y;q,t) = E^-_y \Pi(x,y;q,t)$, *where the suffix x indicates the action of $E^{\pm}$ on the x variables, and similarly for y.*

1.7. We shall give an explicit form of the function $g_{\boldsymbol{\alpha},\pm}(x;q,t)$. For this, we prepare some notation. For $\mu = (\mu_0, \dots, \mu_{e-1}) \in \mathbb{Z}^e_{\geq 0}$, put

$$f^{(k,i)}_{\mu,\pm}(x;q,t) = \prod_{a=0}^{e-1}\prod_{j=1}^{\mu_a} \frac{x_i^{(k\mp a)} - tx_i^{(k\mp a\mp 1)}q^{e(j-1)}}{1-q^{ej}}.$$

Let $\mathcal{M}_m$ be the set of sequences $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \dots)$ such that $\mu^{(i)} \in \mathbb{Z}^e_{\geq 0}$ and that $\sum_i |\mu^{(i)}| = m$. For $\mu = (\mu_1, \mu_2, \dots, \mu_e) \in \mathbb{Z}^e_{\geq 0}$, put $n(\mu) = \sum_j (j-1)\mu_j$. Then

Proposition 1.8. *For each $m \geq 0$, we have*

$$g^{(k)}_{m,\pm}(x;q,t) = \sum_{\boldsymbol{\mu}\in\mathcal{M}_m}\prod_i f^{(k,i)}_{\mu^{(i)},\pm}(x;q,t)q^{n(\mu^{(i)})}. \tag{1.8.1}$$

Proof. By [M1, I, §2, Ex. 5], the following identity of formal power series is known.

$$\prod_{i=0}^{\infty}\frac{1-bq^it}{1-aq^it} = \sum_{m\geq 0}\biggl(\prod_{i=1}^{m}\frac{a-bq^{i-1}}{1-q^i}\biggr)t^m.$$

Substituting this into

$$A = \prod_{r\geq 0}\frac{1-tx_i^{(k\mp r\mp 1)}yq^r}{1-x_i^{(k\mp r)}yq^r} = \prod_{a=0}^{e-1}\prod_{r\geq 0}\frac{1-tx_i^{(k\mp a\mp 1)}yq^{re+a}}{1-x_i^{(k\mp a)}yq^{re+a}}.$$

with $a = x_i^{(k\mp a)}, b = tx_i^{(k\mp a\mp 1)}$, $t = q^a y$, $q = q^e$, we see that

$$\begin{aligned} A &= \prod_{a=0}^{e-1}\sum_{\mu_a=0}^{\infty}\prod_{j=1}^{\mu_a}\frac{x_i^{(k\mp a)} - tx_i^{(k\mp a\mp 1)}q^{e(j-1)}}{1-q^{ej}}(q^a y)^{\mu_a} \\ &= \sum_{\mu\in\mathbb{Z}^e_{\geq 0}} f^{(k,i)}_{\mu,\pm}(x;q,t)q^{n(\mu)}y^{|\mu|}. \end{aligned}$$

It follows that

$$\prod_{i\geq 1}\prod_{r\geq 0}\frac{1-tx_i^{(k\mp r\mp 1)}y_j^{(k)}q^r}{1-x_i^{(k\mp r)}y_j^{(k)}q^r}=\sum_{m\geq 0}\sum_{\boldsymbol{\mu}\in\mathcal{M}_m}\prod_{i\geq 1}f^{(k,i)}_{\mu^{(i)},\pm}(x;q,t)q^{n(\mu^{(i)})}(y_j^{(k)})^m.$$

By comparing this with (1.5.1), we obtain the required formula. □

Remark 1.9. $g^{(k)}_{m,\pm}(x;0,t)$ coincides with the function $q^{(k)}_{m,\pm}(x;t)$ introduced in [S1, 2.2]. By using a similar, but much simpler arguments as above, one obtains an alternative expression of $q^{(k)}_{m,\pm}(x;t)$ as follows.

$$q^{(k)}_{m,\pm}(x;t)=\sum_{\mu\in\mathcal{P}_m}|\mathfrak{S}_\mu|^{-1}\sum_{w\in\mathfrak{S}_\mu}w\bigg\{\prod_{i=1}^{l(\mu)}(x_i^{(k)}-tx_i^{(k\mp 1)})(x_i^{(k)})^{\mu_i-1}\bigg\},$$

where $\mathfrak{S}_\mu$ is the stabilizer of μ in $\mathfrak{S}_m$. (Here we are considering finite variables $x_i^{(k)}, x_i^{(k\mp 1)}$ $(1\leq i\leq m)$, and $\mathfrak{S}_m$ acts on both variables).

1.10. The notion of symbols was introduced in [S1]. (Although a more general setting was discussed in [S2], we do not use it in the discussion below. We remark that similar symbols were also considered by G. Malle in [Ma]). Let $\mathbf{m}=(m_0,\dots,m_{e-1})$ be as before. We denote by $Z^{0,0}_n=Z^{0,0}_n(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha}=(\alpha^{(0)},\dots,\alpha^{e-1)})\in\mathcal{P}_{n,e}$ such that each $\alpha^{(k)}$ is written (as an element in $\mathbb{Z}^{m_k}$) in the form $\alpha^{(k)}:\alpha^{(k)}_1\geq\cdots\geq\alpha^{(k)}_{m_k}\geq 0$. We express $\boldsymbol{\alpha}$ as $\boldsymbol{\alpha}=(\alpha^{(k)}_j)$ in matrix form. Let us fix integers $r\geq s\geq 0$ and define an e-partition $\boldsymbol{\Lambda}^0=\boldsymbol{\Lambda}^0(\mathbf{m},s,r)=(\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ as follows.

$$\begin{aligned}\Lambda^{(0)}&:(m_0-1)r\geq\cdots 2r\geq r\geq 0,\\ \Lambda^{(i)}&:s+(m_i-1)r\geq\cdots\geq s+2r\geq s+r\geq s\end{aligned}\tag{1.10.1}$$

for $i=0,\dots,e-1$. We denote by $Z^{r,s}_n=Z^{r,s}_n(\mathbf{m})$ the set of e-partitions of the form $\boldsymbol{\Lambda}=\boldsymbol{\alpha}+\boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha}\in Z^{0,0}_n$ and the sum is taken entry-wise. We write $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda}=\boldsymbol{\alpha}+\boldsymbol{\Lambda}^0$, and call it the e-symbol of type (r,s) corresponding to $\boldsymbol{\alpha}$. We often denote the symbol $\boldsymbol{\Lambda}=(\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ in the form $\boldsymbol{\Lambda}=(\Lambda^{(k)}_j)$ with $\Lambda^{(k)}:\Lambda^{(k)}_1>\cdots>\Lambda^{(k)}_{m_k}$ for $k=0,\dots,e-1$.

Put $\mathbf{m}'=(m_0+1,\dots,m_{e-1}+1)$, and define a shift operation $Z^{r,s}_n(\mathbf{m})\to Z^{r,s}_n(\mathbf{m}')$ by associating $\boldsymbol{\Lambda}'=(\Lambda'_0,\dots,\Lambda'_{e-1})\in Z^{r,s}_n(\mathbf{m})$ to $\boldsymbol{\Lambda}=(\Lambda_0,\dots,\Lambda_{e-1})\in Z^{r,s}_n(\mathbf{m})$, where $\Lambda'_0=(\Lambda_0+r)\cup\{0\}$, and $\Lambda'_k=(\Lambda_k+r)\cup\{s\}$ for $k=0,\dots,e-1$. In other words, for $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}(\boldsymbol{\alpha})$, $\boldsymbol{\Lambda}'$ is obtained as $\boldsymbol{\Lambda}'=\boldsymbol{\alpha}+\boldsymbol{\Lambda}^0(\mathbf{m}',s,r)$, where $\boldsymbol{\alpha}$ is regarded as an element of $Z^{0,0}_n(\mathbf{m}')$ by adding 0 in the entries of $\boldsymbol{\alpha}$. We denote by $\bar{Z}^{r,\mathbf{s}}_n$ the set of classes in $\coprod_{\mathbf{m}'}Z^{r,\mathbf{s}}_n(\mathbf{m}')$ under the equivalence relation generated by shift operations. Note that $\mathcal{P}_{n,e}$ coincides with the set $\bar{Z}^{0,0}_n$. Also note that $\boldsymbol{\Lambda}^0$ is regarded as a symbol in $Z^{r,s}_n$ with $n=0$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $\bar{Z}^{r,\mathbf{s}}_n$ are said to be similar, and are written as $\boldsymbol{\Lambda}\sim\boldsymbol{\Lambda}'$, if there exist representatives in $Z^{r,\mathbf{s}}_n(\mathbf{m})$ such that all the entries of them coincide with multiplicities. The set of symbols which are similar to a fixed symbol is called a family in $Z^{r,\mathbf{s}}_n$.

We define a function $a : Z_n^{r,\mathbf{s}} \to \mathbb{Z}_{\geq 0}$, for $\boldsymbol{\Lambda} \in Z_n^{r,\mathbf{s}}$, by

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda,\lambda' \in \boldsymbol{\Lambda}} \min\{\lambda, \lambda'\} - \sum_{\mu,\mu' \in \boldsymbol{\Lambda}^0} \min\{\mu, \mu'\}. \tag{1.10.2}$$

The function a on $Z_n^{r,\mathbf{s}}$ is invariant under the shift operation, and it induces a function a on $\bar{Z}_n^{r,\mathbf{s}}$. Clearly, the a-function takes a constant value on each family in $Z_n^{r,\mathbf{s}}$. We regard the a-function as a function on $Z_n^{0,0}$ by using the bijection $Z_n^{0,0} \simeq Z_n^{r,s}$.

1.11. Hall-Littlewood functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ attached to symbols $\boldsymbol{\Lambda}$ were introduced in [S1]. We shall now construct a two parameter version of Hall-Littlewood functions. Let us introduce a total order $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ on $Z_n^{0,0}$ such that $a(\boldsymbol{\alpha}) \geq a(\boldsymbol{\beta})$ whenever $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ and that each family in $Z_n^{0,0}$ forms an interval.

The following proposition is easily obtained by a similar argument as in Remark 4.9 in [S1] (i.e., a generalization of Gram-Schmidt orthogonalization process) if one notices that $\Pi(x,y;0,t)$ coincides with $\Omega(x,y;t)$ in [S1, 2.5].

Proposition 1.12. *There exists a unique function $P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t) \in \Xi_F$ for $\boldsymbol{\Lambda} \in Z_n^{r,s}$ satisfying the following two properties.*

(i) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$ for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ can be expressed in terms of $s_{\boldsymbol{\beta}}(x)$ as*

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}},$$

with $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in F$, where $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(ii) *$\langle P^{+}_{\boldsymbol{\Lambda}}, P^{-}_{\boldsymbol{\Lambda}'} \rangle = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$.*

We then define $Q^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$ as the dual of $P^{\mp}_{\boldsymbol{\Lambda}}(x;q,t)$, i.e., by the property that

$$\langle P^{+}_{\boldsymbol{\Lambda}}, Q^{-}_{\boldsymbol{\Lambda}'} \rangle = \langle Q^{+}_{\boldsymbol{\Lambda}}, P^{-}_{\boldsymbol{\Lambda}'} \rangle = \delta_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}.$$

$P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$, $Q^{\pm}_{\boldsymbol{\Lambda}}(x;q,t)$ are called the Macdonald functions associated to complex reflection groups W (with respect to symbols in $Z_n^{r,s}$).

Remark 1.13. (i) The orthogonality relations of Macdonald functions given above imply, by [M1, VI, 2.7], that

$$\begin{aligned}\Pi(x,y;q,t) &= \sum_{\boldsymbol{\Lambda}} P^{+}_{\boldsymbol{\Lambda}}(x;q,t) Q^{-}_{\boldsymbol{\Lambda}}(y;q,t) \\ &= \sum_{\boldsymbol{\Lambda}} Q^{+}_{\boldsymbol{\Lambda}}(x;q,t) P^{-}_{\boldsymbol{\Lambda}}(y;q,t).\end{aligned} \tag{1.13.1}$$

(ii) In the case where $q = 0$, the scalar product given in (1.3.2) coincides with the one given in [S1, 4.7]. Then by Proposition 4.8 in [S1], one sees that

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;0,t) = P^{\pm}_{\boldsymbol{\Lambda}}(x,t), \quad Q^{\pm}_{\boldsymbol{\Lambda}}(x;0,t) = Q^{\pm}_{\boldsymbol{\Lambda}}(x;t), \tag{1.13.2}$$

where the right hand sides are the Hall-Littlewood functions defined in [S1].

(iii) In the case where $q = t$, the scalar product in (1.3.2) coincides with the usual scalar product on the space $\Xi_{\mathbb{Q}}$, where the Schur functions form an orthonormal basis of it. Hence Proposition 1.12 implies that

$$P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t,t) = Q^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t,t) = s_{\boldsymbol{\alpha}}(x). \tag{1.13.3}$$

2. Macdonald operators

2.1. The original Macdonald functions related to symmetric groups are characterized as the simultaneous eigenfunctions of Macdonald operators (see [M1, VI]). In this section, we shall construct certain operators which can be viewed as a generalization of Macdonald operators to the case of W. Here we restrict ourselves to the case where symbols are of the type (r,s) with $r = 1$ and $s = 0$. (We note that the arguments in this section can not be applied to other types of symbols. See Remark 2.9.) In particular, $\boldsymbol{\Lambda}^0 = \boldsymbol{\delta}$, where $\boldsymbol{\delta} = (\delta^{(0)}, \dots, \delta^{(e-1)})$ with $\delta^{(k)} = (m_k - 1, \dots, 1, 0)$. Hence, in the case where $e = 2$ (i.e., W is the Weyl group of type B_n) with $m_1 = m_0 + 1$, these symbols are exactly the ones used to parameterize unipotent characters of finite classical groups $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$ by Lusztig [L1]. Each family $\mathcal{F}$ in $Z_n^{1,0}$ contains a unique element $\boldsymbol{\Lambda}_{\mathcal{F}} = (\Lambda_j^{(k)})$ with the property

$$\Lambda_1^{(0)} \ge \Lambda_1^{(1)} \ge \cdots \ge \Lambda_1^{(e-1)} \ge \Lambda_2^{(0)} \ge \Lambda_2^{(1)} \ge \cdots . \tag{2.1.1}$$

Such an element is called a special symbol associated to the family $\mathcal{F}$. The set of families is in bijection with the set of special symbols. Special symbols are regarded as partitions of $N = \sum \Lambda_j^{(k)}$ by (2.1.1).

We shall define a partial order on the set of families in $Z_n^{1,0}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be families in $Z_n^{1,0}$ and $\boldsymbol{\Lambda}_{\mathcal{F}}, \boldsymbol{\Lambda}_{\mathcal{F}'}$ be special symbols corresponding to them. We put $\mathcal{F} < \mathcal{F}'$ if $\boldsymbol{\Lambda}_{\mathcal{F}} < \boldsymbol{\Lambda}_{\mathcal{F}'}$ with respect to the dominance order on $\mathcal{P}_N$. Recall that for $\lambda = (\lambda_i), \mu = (\mu_i) \in \mathcal{P}_N$, the dominance order $\lambda < \mu$ is defined by the condition that

$$\sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{k} \mu_i, \qquad (1 \le k \le \sum_j m_j).$$

We define a partial order on $Z_n^{1,0}$ by inheriting the partial order on the set of families. For $\lambda = (\lambda_i) \in \mathcal{P}_N$, put

$$n(\lambda) = \sum_{i \ge 1} (i-1)\lambda_i.$$

Then, for each $\boldsymbol{\Lambda}$ in a family $\mathcal{F}$, the value $a(\boldsymbol{\Lambda})$ is given as $a(\boldsymbol{\Lambda}) = n(\boldsymbol{\Lambda}_{\mathcal{F}}) - n(\boldsymbol{\Lambda}^0)$, where $\boldsymbol{\Lambda}_{\mathcal{F}}$ is regarded as an element in $\mathcal{P}_M$. In particular, we have $a(\boldsymbol{\Lambda}) > a(\boldsymbol{\Lambda}')$ if $\boldsymbol{\Lambda} < \boldsymbol{\Lambda}'$. As before, by using the bijection $Z_n^{0,0} \simeq Z_n^{1,0}$, we consider the partial order on $Z_n^{0,0}$, which will be denoted by the same symbol.

2.2. In order to construct Macdonald operators, we shall start with finitely many variables $x = x_{\mathbf{m}}$. We consider the expansion of $\Pi(x, y; q, t)$ in the case of finitely many variables. Assume that $x = (x_j^{(k)}) = x_{\mathbf{m}}$ is as in 1.2 and put $\Xi_{\mathbf{m},F} = F \otimes_{\mathbb{Z}} \Xi_{\mathbf{m}}$. Let us denote by $\mathcal{P}_{\mathbf{m}}$ the set of e-partitions $\boldsymbol{\alpha} = (\alpha_j^{(k)})$ such that $l(\alpha^{(k)}) \le m_k$. Then $m_{\boldsymbol{\alpha}}(x) = 0$ unless $\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}$, and those non-zero $m_{\boldsymbol{\alpha}}(x)$ form a basis of $\Xi_{\mathbf{m},F}$. The same is true for Schur functions. Also by a similar argument as in [M1] we see that $g_{\boldsymbol{\alpha}}(x; q, t)$ such that $\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}$ form an F-basis of $\Xi_{\mathbf{m},F}$.

Now by substituting $x_j^{(k)} = y_j^{(k)} = 0$ for $j > m_k$, we have a finite version of (1.5.2), i.e., for $x = x_{\mathbf{m}}, y = y_{\mathbf{m}}$, we have

$$\Pi(x, y; q, t) = \sum_{\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}} g_{\boldsymbol{\alpha},+}(x; q, t) m_{\boldsymbol{\alpha}}(y) = \sum_{\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}} m_{\boldsymbol{\alpha}}(x) g_{\boldsymbol{\alpha},-}(y; q, t). \tag{2.2.1}$$

This enables us to define a scalar product on $\Xi_{\mathbf{m},F}$ by

$$\langle g_{\boldsymbol{\alpha},+}, m_{\boldsymbol{\beta}} \rangle = \langle m_{\boldsymbol{\alpha}}, g_{\boldsymbol{\beta},-} \rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}. \tag{2.2.2}$$

We now consider the restriction of the functions $P^{\pm}_{\boldsymbol{\Lambda}}$ to $\Xi_{\mathbf{m},F}$. By Proposition 1.12, one sees that $\{P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})} \mid \boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}\}$ form a basis of $\Xi_{\mathbf{m},F}$. Moreover, the finite variables version of Proposition 1.12 holds, and $P^{\pm}_{\boldsymbol{\Lambda}} \in \Xi_F$ is obtained as the limit of $P^{\pm}_{\boldsymbol{\Lambda}} \in \Xi_{\mathbf{m},F}$ determined by these properties.

2.3. Let

$$\mathcal{I} = \mathcal{I}(\mathbf{m}) = \{\mathbf{i} = \begin{pmatrix} i_0 \\ \vdots \\ i_{e-1} \end{pmatrix} \in \mathbb{Z}^e \mid 1 \le i_k \le m_k\}.$$

For each $\mathbf{i} \in \mathcal{I}$ and $u \in F$, we define an F-linear operator $T^{\pm}_{u,\mathbf{i}} : F[x] \to F[x]$ by $T^{\pm}_{u,\mathbf{i}} f = f'$, where f' is a polynomial obtained from f by replacing the variables $x^{(k)}_{i_k}$ by $ux^{(k\mp 1)}_{i_{k\mp 1}}$ for $k = 0, \dots, e-1$. (Here we understand that $i_e = i_o$). More generally, for each r such that $1 \le r \le M_1 = \min_k\{m_k\}$, we define $\mathcal{I}_r$ as the set of $J = \{\mathbf{i}_1, \dots, \mathbf{i}_r\}$ consisting of $\mathbf{i}_k \in \mathcal{I}$ such that any two $\mathbf{i}_k$ have no common entries (i.e., $\mathbf{i}_j - \mathbf{i}_k$ does not contain 0 entries for each $j \ne k$ as vectors in $\mathbb{Z}^e$). Then we define, for each $J \in \mathcal{I}_r$, an operator $T^{\pm}_{u,J} : F[x] \to F[x]$ by $T^{\pm}_{u,J} = \prod_{k=1}^r T^{\pm}_{u,\mathbf{i}_k}$. Note that $T^{\pm}_{u,\mathbf{i}_k}$ in the product commute with each other, and so $T^{\pm}_{u,J}$ does not depend on the order of the product.

Let $Z = Z(\mathbf{m})$ be the set of sequences $\boldsymbol{\beta} = (\beta_j^{(k)})$ $(0 \le k \le e-1, 1 \le j \le m_k)$ with $\beta_i^{(k)} \in \mathbb{Z}_{\ge 0}$. For each $\boldsymbol{\beta} \in Z$, we denote by $[\boldsymbol{\beta}]$ the element in Z obtained from $\boldsymbol{\beta}$ by permuting the entries inside each row, so that each row is arranged in decreasing order. We often regard $\boldsymbol{\beta}$ as a matrix, and denote its k-th row by $\beta^{(k)}$. Then the set J is regarded as a subset of the set of indices $\{(k, j) \mid 0 \le k \le e-1, 1 \le j \le m_k\}$ of $(\beta_j^{(k)})$

We put, for each $\boldsymbol{\beta} \in Z$ and $J \in \mathcal{I}_r$,

$$\langle \boldsymbol{\beta}, J \rangle = \sum_{(k,j) \in J} \beta_j^{(k)}.$$

$\mathfrak{S}_{\mathbf{m}}$ acts naturally on Z and on $\mathcal{I}_r$, respectively, and this pairing is $\mathfrak{S}_{\mathbf{m}}$-invariant. We also note that the action of $\mathfrak{S}_{\mathbf{m}}$ on $\mathcal{I}_r$ is transitive.

The operation of $T^{\pm}_{q,J}$ on $x = (x_i^{(k)})$ also induces an action $\boldsymbol{\beta} \mapsto \boldsymbol{\beta}_{J\pm}$ on Z by permuting the entries of $\boldsymbol{\beta}$ so that $T^{\pm}_{q,J} x^{\boldsymbol{\beta}} = q^{\langle \boldsymbol{\beta}, J\rangle} x^{\boldsymbol{\beta}_{J\pm}}$, where, as usual, $x^{\boldsymbol{\beta}}$ denotes the monomial $\prod_{i,k} (x_i^{(k)})^{\beta_i^{(k)}}$. For each $\boldsymbol{\beta} \in Z$, we define a function $a_{\boldsymbol{\beta}}(x)$ by

$$a_{\boldsymbol{\beta}}(x) = \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) w(x^{\boldsymbol{\beta}})$$

We now define, for each $1 \le r \le M_1$, an F-linear operator $D^r_{\pm}(q,t)$ on $F[x]$ by

$$D^r_{\pm}(q,t) = a_{\boldsymbol{\delta}}(x)^{-1} \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) \sum_{J \in \mathcal{I}_r} x^{w(\boldsymbol{\delta})_{J\pm}} t^{\langle w(\boldsymbol{\delta}), J\rangle} T^{\pm}_{q,J}. \tag{2.3.1}$$

Then $D^r_{\pm}(q,t)$ can be written as

$$D^r_{\pm}(q,t) = \sum_{J \in \mathcal{I}_r} A^{\pm}_J(x;t) T^{\pm}_{q,J}, \tag{2.3.2}$$

with

$$\begin{aligned} A^{\pm}_J(x;t) &= a_{\boldsymbol{\delta}}(x)^{-1} \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) x^{w(\boldsymbol{\delta})_{J\pm}} t^{\langle w(\boldsymbol{\delta}), J\rangle} \\ &= a_{\boldsymbol{\delta}}(x)^{-1} T^{\pm}_{t,J} a_{\boldsymbol{\delta}}(x) \\ &= t^{r(r-1)/2} \prod_{k=0}^{e-1} \prod_{\substack{(k,i)\notin J \\ (k,j)\in J}} \frac{x_i^{(k)} - t x_{j'}^{(k\mp 1)}}{x_i^{(k)} - x_j^{(k)}}, \end{aligned} \tag{2.3.3}$$

where we write $J = \{\mathbf{i}_1, \dots, \mathbf{i}_r\}$, and take $(k \mp 1, j') \in \mathbf{i}_a$ if $(k,j) \in \mathbf{i}_a$ for $a = 1, \dots, r$. The formulas (2.3.2) and (2.3.3) are the analogue of $(3.4)_r$ and $(3.5)_r$ in [M, VI, 3].

First we show that

Lemma 2.4. (i) *For $\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}$, we have*

$$D^r_{\pm}(q,t) m_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta}} \sum_{J \in \mathcal{I}_r} t^{\langle \boldsymbol{\delta}, J\rangle} q^{\langle \boldsymbol{\beta}, J\rangle} s_{(\boldsymbol{\beta}+\boldsymbol{\delta})_{J\pm} - \boldsymbol{\delta}}(x),$$

where $\boldsymbol{\beta} \in Z$ runs over all the row permutations of $\boldsymbol{\alpha}$.

(ii) *For $\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}$, we have,*

$$D^r_{\pm}(q,t) m_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) m_{\boldsymbol{\beta}}(x)$$

with $b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) \in F$, where $b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) = 0$ unless $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$ or $\boldsymbol{\beta} < \boldsymbol{\alpha}$. In particular, $D^r_{\pm}$ is an operator on the space $\Xi_{\mathbf{m},F}$.

Proof. For each $\boldsymbol{\beta} \in Z_n^{0,0}$, we have

$$D^r_{\pm}(q,t)x^{\boldsymbol{\beta}} = a_{\boldsymbol{\delta}}(x)^{-1} \sum_{w_1 \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w_1) \sum_{J \in \mathcal{I}_r} t^{\langle w_1(\boldsymbol{\delta}), J\rangle} q^{\langle \boldsymbol{\beta}, J\rangle} x^{(\boldsymbol{\beta}+w_1(\boldsymbol{\delta}))_{J\pm}}.$$

If we replace $\boldsymbol{\beta}$ by $w_2(\boldsymbol{\beta})$ for $w_2 \in \mathfrak{S}_{\mathbf{m}}$ and put $w_2 = w_1 w$, the term $(\boldsymbol{\beta} + w_1(\boldsymbol{\delta}))_{J\pm}$ (resp. $\langle \boldsymbol{\beta}, J\rangle$) is replaced by $w_1((w(\boldsymbol{\beta}) + \boldsymbol{\delta})_{J'\pm})$ (resp. $\langle w(\boldsymbol{\beta}), J'\rangle$), respectively, with $J' = w_1^{-1}(J)$. It follows that for $\boldsymbol{\alpha} \in Z_n^{0,0}$, we have

$$\begin{aligned}
D^r_{\pm}(q,t)m_{\boldsymbol{\alpha}}(x) &= |\mathfrak{S}_{\boldsymbol{\alpha}}|^{-1} a_{\boldsymbol{\delta}}(x)^{-1} \sum_{w, w_1 \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w_1) \\
&\qquad \times \sum_{J' \in \mathcal{I}_r} t^{\langle \boldsymbol{\delta}, J'\rangle} q^{\langle w(\boldsymbol{\alpha}), J'\rangle} x^{w_1((w(\boldsymbol{\alpha})+\boldsymbol{\delta})_{J'\pm})} \\
&= |\mathfrak{S}_{\boldsymbol{\alpha}}|^{-1} \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \sum_{J \in \mathcal{I}_r} t^{\langle \boldsymbol{\delta}, J\rangle} q^{\langle w(\boldsymbol{\alpha}), J\rangle} s_{(w(\boldsymbol{\alpha})+\boldsymbol{\delta})_{J\pm} - \boldsymbol{\delta}}(x).
\end{aligned}$$

This proves (i).

Next we show (ii). Since $\boldsymbol{\delta} = \boldsymbol{\Lambda}^0$ by our assumption, $\boldsymbol{\alpha} + \boldsymbol{\delta}$ coincides with the symbol $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in Z_n^{0,0}$. We note that if $w \ne 1$, then the symbol $w(\boldsymbol{\alpha}) + \boldsymbol{\delta}$ belongs to a family strictly smaller than the family containing $\boldsymbol{\alpha}$. On the other hand, if $w = 1$, $(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J\pm}$ is obtained from the symbol $\boldsymbol{\alpha}+\boldsymbol{\delta}$ by permuting some entries, and $s_{(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J\pm}-\boldsymbol{\delta}}$ coincides with $\pm s_{\boldsymbol{\gamma}}$, where $\boldsymbol{\gamma} + \boldsymbol{\delta}$ is obtained from $(\boldsymbol{\alpha} + \boldsymbol{\delta})_{J\pm}$ by rearranging the rows in decreasing order. It follows that $\boldsymbol{\gamma} + \boldsymbol{\delta}$ is contained in the same family as $\boldsymbol{\alpha} + \boldsymbol{\delta}$. Hence, for $\boldsymbol{\beta} \in Z_n^{0,0}$ in the expression of (ii), we see that $\boldsymbol{\beta} < \boldsymbol{\alpha}$ if $w \ne 1$ and $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$ if $w = 1$. □

Next we show

Lemma 2.5. *The operators $D^r_{\pm}$ are adjoint each other, i.e., we have*

$$\langle D^r_+ f, g\rangle = \langle f, D^r_- g\rangle, \quad (f, g \in \Xi_{\mathbf{m},F}). \tag{2.5.1}$$

Proof. By Lemma 1.6, (2.5.1) is equivalent to the formula

$$\Pi^{-1}(D^r_+)_x \Pi = \Pi^{-1}(D^r_-)_y \Pi. \tag{2.5.2}$$

But for $J \in \mathcal{I}_r$, we have

$$\begin{aligned}
\Pi^{-1}(T^+_{q,J})_x \Pi &= \prod_{k=0}^{e-1} \prod_{j \ge 1} \prod_{(k,i) \in J} \frac{1 - x_i^{(k)} y_j^{(k)}}{1 - t x_i^{(k)} y_j^{(k+1)}}, \\
\Pi^{-1}(T^-_{q,J})_y \Pi &= \prod_{k=0}^{e-1} \prod_{i \ge 1} \prod_{(k,j) \in J} \frac{1 - x_i^{(k)} y_j^{(k)}}{1 - t x_i^{(k-1)} y_j^{(k)}}.
\end{aligned}$$

It follows that both of $\Pi^{-1}(D^r_+)_x\Pi$ and $\Pi^{-1}(D^r_-)_y\Pi$ are independent of q. Hence in the proof of (2.5.2), we may assume that $q = t$. In other words, we have only to prove (2.5.1) under the assumption that $q = t$.

Now assume that $q = t$. Since

$$T^{\pm}_{t,J}(x^{w(\boldsymbol{\delta})}f) = x^{w(\boldsymbol{\delta})_{J\pm}}t^{\langle w(\boldsymbol{\delta}),J\rangle}T^{\pm}_{t,J}f$$

for any polynomial $f \in F[x]$, we have

$$D^r_{\pm}(t,t)f = a_{\boldsymbol{\delta}}^{-1}\sum_{J\in\mathcal{I}_r}T^{\pm}_{t,J}(a_{\boldsymbol{\delta}}f).$$

It follows that for any $\boldsymbol{\alpha} \in Z^{0,0}_n$,

$$\begin{aligned} D^r_{\pm}(t,t)s_{\boldsymbol{\alpha}} &= a_{\boldsymbol{\delta}}^{-1}\sum_{J\in\mathcal{I}_r}T^{\pm}_{t,J}(a_{\boldsymbol{\alpha}+\boldsymbol{\delta}}) \\ &= \sum_{J\in\mathcal{I}_r}t^{\langle\boldsymbol{\alpha}+\boldsymbol{\delta},J\rangle}s_{(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J\pm}-\boldsymbol{\delta}}. \end{aligned}$$

As before, $s_{(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J\pm}-\boldsymbol{\delta}}$ coincides with $\pm s_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}+\boldsymbol{\delta} = [(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J\pm}]$ (under the notation in 2.3). Now in the case where $q = t$, $\{s_{\boldsymbol{\alpha}}(x) \mid \boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}\}$ is an orthonormal basis of $\Xi_{\mathbf{m},F}$. It follows that

$$\langle D^r_+(t,t)s_{\boldsymbol{\alpha}}, s_{\boldsymbol{\beta}}\rangle = \sum_J \varepsilon_{J+}t^{\langle\boldsymbol{\alpha}+\boldsymbol{\delta},J\rangle}, \tag{2.5.3}$$

where J runs over all the elements in $\mathcal{I}_r$ such that $[(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J+}]$ coincides with $\boldsymbol{\beta}+\boldsymbol{\delta}$, and $\varepsilon_J = (-1)^{l(w)}$ with $w \in \mathfrak{S}_{\mathbf{m}}$ such that $[(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J+}] = w((\boldsymbol{\alpha}+\boldsymbol{\delta})_{J+})$. But if $\boldsymbol{\beta}+\boldsymbol{\delta} = w((\boldsymbol{\alpha}+\boldsymbol{\delta})_{J+})$, we have $\boldsymbol{\alpha}+\boldsymbol{\delta} = w^{-1}((\boldsymbol{\beta}+\boldsymbol{\delta})_{J'-}) = [(\boldsymbol{\beta}+\boldsymbol{\delta})_{J'-}]$ with $J' = w(J)$. Also in this case,

$$\langle\boldsymbol{\beta}+\boldsymbol{\delta}, J'\rangle = \langle\, w(\boldsymbol{\alpha}+\boldsymbol{\delta})_{J'+}, J'\rangle = \langle w(\boldsymbol{\alpha}+\boldsymbol{\delta}), J'\rangle = \langle\boldsymbol{\alpha}+\boldsymbol{\delta}, J\rangle\,.$$

Since $\varepsilon_{J+} = \varepsilon_{J'-}$, we see that the right hand side of (2.5.3) is equal to

$$\sum_{J'}\varepsilon_{J'-}t^{\langle\boldsymbol{\beta}+\boldsymbol{\delta},J'\rangle},$$

where $J' \in \mathcal{I}_r$ runs over all the elements such that $[(\boldsymbol{\beta}+\boldsymbol{\delta})_{J'-}] = \boldsymbol{\alpha}+\boldsymbol{\delta}$. Clearly this coincides with $\langle s_{\boldsymbol{\alpha}}, D^r_-(t,t)s_{\boldsymbol{\beta}}\rangle$. So the lemma is proved. □

2.6. We fix a total order $\prec$ on $Z^{0,0}_n$ as in 1.11, so that it is compatible with the partial order $<$. Let $B^r_{\pm} = (b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the matrix consisting of the coefficients in the formula in Lemma 2.4 (ii) with respect to the total order $\prec$. We consider $B^r_{\pm}$ as a block matrix with respect to the equivalence relation $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$, and denote it as $(B^{r,\pm}_{\mathcal{F},\mathcal{F}'})$, where $B^{r,\pm}_{\mathcal{F},\mathcal{F}'}$ is the submatrix of $B^r_{\pm}$ corresponding to the families $\mathcal{F}, \mathcal{F}'$. Then Lemma 2.4 (ii) implies that $B^r_{\pm}$ is lower triangular as a block matrix. We consider the

Macdonald functions $P^{\pm}_{\boldsymbol{\Lambda}}$ constructed via $\prec$. The following result shows that the set of Macdonald functions attached to symbols in a fixed family behaves as an eigenfunction for Macdonald operators, where the eigenvalues should be replaced by the diagonal blocks $B^{r;\pm}_{\mathcal{F},\mathcal{F}}$ of $B_{\pm}$.

Proposition 2.7. *Let $P^{\pm}_{\boldsymbol{\Lambda}}(x;q,t) \in \Xi_{\mathbf{m},F}$ be Macdonald functions attached to $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$. Then we have*

$$D^r_{\pm}P^{\pm}_{\boldsymbol{\Lambda}} = \sum_{\boldsymbol{\beta}\sim\boldsymbol{\alpha}} b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})},$$

where the coefficients $b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are the same as in Lemma 2.4 (ii).

Proof. By Proposition 1.12, $P^{\pm}_{\boldsymbol{\Lambda}}$ can be written as

$$P^{\pm}_{\boldsymbol{\Lambda}} = m_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}\prec\boldsymbol{\alpha},\boldsymbol{\beta}\not\sim\boldsymbol{\alpha}} u'_{\boldsymbol{\alpha},\boldsymbol{\beta}} m_{\boldsymbol{\beta}}$$

with $u'_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in F$. It follows, by Lemma 2.4 (ii), that one can write as

$$D^r_{\pm}P^{\pm}_{\boldsymbol{\Lambda}} = \sum_{\boldsymbol{\beta}\sim\boldsymbol{\alpha}} b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})} + \sum_{\boldsymbol{\Lambda}'\prec\boldsymbol{\Lambda},\boldsymbol{\Lambda}'\not\sim\boldsymbol{\Lambda}} c^{\pm}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} P^{\pm}_{\boldsymbol{\Lambda}'}.$$

Hence, for each $\boldsymbol{\Lambda}'$ such that $\boldsymbol{\Lambda}' \prec \boldsymbol{\Lambda}$ and that $\boldsymbol{\Lambda}' \not\sim \boldsymbol{\Lambda}$, we have

$$\langle D^r_+P^+_{\boldsymbol{\Lambda}}, P^-_{\boldsymbol{\Lambda}'}\rangle = c^+_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}.$$

On the other hand thanks to Lemma 2.5, we have

$$\langle D^r_+P^+_{\boldsymbol{\Lambda}}, P^-_{\boldsymbol{\Lambda}'}\rangle = \langle P^+_{\boldsymbol{\Lambda}}, D^r_-P^-_{\boldsymbol{\Lambda}'}\rangle = 0$$

since $D^r_-P^-_{\boldsymbol{\Lambda}'}$ is a linear combination of $P^-_{\boldsymbol{\Lambda}''}$, where $\boldsymbol{\Lambda}'' \sim \boldsymbol{\Lambda}'$ or $\boldsymbol{\Lambda}'' \prec \boldsymbol{\Lambda}'$. It follows that $c^+_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} = 0$ and the proposition holds for the $+$ case. The $-$ case is similar. □

In view of Proposition 2.7, it is important to know the diagonal part of $B_{\pm}$. By lemma 2.4, the matrix $B^{\pm}_{\mathcal{F},\mathcal{F}}$ is described as follows.

Lemma 2.8. *For $\boldsymbol{\alpha},\boldsymbol{\beta} \in Z^{0,0}_n$ such that $\boldsymbol{\beta}\sim\boldsymbol{\alpha}$, we have*

$$b^{r,\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) = \sum_{\substack{J\in\mathcal{I}_r \\ [\boldsymbol{\Lambda}_{J\pm}]=\boldsymbol{\Lambda}(\boldsymbol{\beta})}} \varepsilon_{\boldsymbol{\Lambda},J\pm}(tq^{-1})^{\langle\boldsymbol{\delta}.J\rangle} q^{\langle\boldsymbol{\Lambda},J\rangle},$$

where $\boldsymbol{\Lambda} = \boldsymbol{\alpha}+\boldsymbol{\delta} \in Z^{1,0}_n$ and $\varepsilon_{\boldsymbol{\Lambda},J\pm} = (-1)^{l(w)}$ for $w \in \mathfrak{S}_{\mathbf{m}}$ such that $[\boldsymbol{\Lambda}_{J\pm}] = w(\boldsymbol{\Lambda}_{J\pm})$.

Remark 2.9. The results in this section work only for a special type of symbols. It seems to be difficult to extend the definition of Macdonald operators in (2.3.1) directly to a more general case. A naive idea for the general situation is to replace $\boldsymbol{\delta}$ by $\boldsymbol{\Lambda}^0$ in the definition (2.3.1). Then one gets some operator related to the symbols associated with $\boldsymbol{\Lambda}^0$. However, the thus obtained operator does not preserve the set of polynomials in x, since it involves the factor $a_{\boldsymbol{\Lambda}^0}(x)^{-1}$. If one leaves the denominator $a_{\boldsymbol{\delta}}(x)$ unchanged, and replaces $\boldsymbol{\delta}$ in all other places, then the operator preserves polynomials, but it does not preserve the degree of them, and is not so useful.

3. A characterization of Macdonald functions

3.1. We write the operator $D^r_{\pm}$ as $D^r_{\mathbf{m},\pm}$ to indicate the dependence on $\mathbf{m}$. The operators $D^r_{\mathbf{m},\pm}$ are not compatible with the restriction homomorphisms $\rho_{\mathbf{m}',\mathbf{m}}$. In the case where $r = 1$, one can modify $D^1_{\mathbf{m},\pm}$ as discussed in [M1], so that they are compatible with $\rho_{\mathbf{m}',\mathbf{m}}$. Let us define an operator $E^{\pm}_{\mathbf{m}} = E^{\pm}_{\mathbf{m}}(q,t)$ on $\Xi_{\mathbf{m},F}$ by

$$E^{\pm}_{\mathbf{m}} = t^{-M}D^1_{\mathbf{m},\pm} - \sum_{\mathbf{i}\in\mathcal{I}} t^{\langle\boldsymbol{\delta},\mathbf{i}\rangle - M}. \tag{3.1.1}$$

We show the following lemma.

Lemma 3.2. *The operators $E^{\pm}_{\mathbf{m}}$ are compatible with $\rho_{\mathbf{m}',\mathbf{m}}$, i.e., we have*

$$\rho_{\mathbf{m}',\mathbf{m}} \circ E^{\pm}_{\mathbf{m}'} = E^{\pm}_{\mathbf{m}} \circ \rho_{\mathbf{m}',\mathbf{m}}$$

for $\mathbf{m}' = (m_0 + 1, \dots, m_{e-1} + 1)$.

Proof. Let us define another operator $\widetilde{E}^{\pm}_{\mathbf{m}} : \Xi_{\mathbf{m},F} \to \Xi_{\mathbf{m},F}$ by

$$\widetilde{E}^{\pm}_{\mathbf{m}} = t^{-M}D^1_{\mathbf{m},\pm} - \sum_{\mathbf{i}\in\mathcal{I}} t^{\langle\boldsymbol{\delta},\mathbf{i}\rangle - M} a_{\boldsymbol{\delta}}^{-1} T^{\pm}_{1,\mathbf{i}} a_{\boldsymbol{\delta}}.$$

Then for each $\boldsymbol{\alpha} \in \mathcal{P}_{\mathbf{m}}$, we have

$$\widetilde{E}^{\pm}_{\mathbf{m}} m_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}}\sum_{\mathbf{i}\in\mathcal{I}} (q^{\langle\boldsymbol{\beta},\mathbf{i}\rangle} - 1)t^{-\sum i_k} s_{(\boldsymbol{\beta}+\boldsymbol{\delta})_{\mathbf{i}\pm} - \boldsymbol{\delta}}, \tag{3.2.1}$$

where $\boldsymbol{\beta} \in Z$ runs over all the row permutations of $\boldsymbol{\alpha}$, and $\mathbf{i} = (i_0, \dots, i_{e-1})$. In fact, by applying [M1, VI, 4] one can write $m_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} s_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta}$ as above. Since $a_{\boldsymbol{\delta}}^{-1} T^{\pm}_{1,\mathbf{i}} a_{\boldsymbol{\delta}} s_{\boldsymbol{\beta}} = s_{(\boldsymbol{\beta}+\boldsymbol{\delta})_{\mathbf{i}\pm} - \boldsymbol{\delta}}$, we obtain (3.2.1).

We claim that $\widetilde{E}^{\pm}_{\mathbf{m}}$ is compatible with $\rho_{\mathbf{m}',\mathbf{m}}$, i.e.,

$$\rho_{\mathbf{m}',\mathbf{m}} \circ \widetilde{E}^{\pm}_{\mathbf{m}'} = \widetilde{E}^{\pm}_{\mathbf{m}} \circ \rho_{\mathbf{m}',\mathbf{m}}. \tag{3.2.2}$$

Recall that the map $\rho_{\mathbf{m}',\mathbf{m}} : \Xi_{\mathbf{m}',F} \to \Xi_{\mathbf{m},F}$ is defined by substituting $x^{(k)}_{m_k+1} = 0$ for $k = 0, \dots, e-1$. Take $\boldsymbol{\alpha} \in Z^{0,0}_n(\mathbf{m})$ and let $\boldsymbol{\alpha}' \in Z^{0,0}_n(\mathbf{m}')$ be the element obtained by adding 0's to the last part of $\boldsymbol{\alpha}$. We consider the expression of $\widetilde{E}^{\pm}_{\mathbf{m}'} m_{\boldsymbol{\alpha}'}$ as given in (3.2.1). Let $\boldsymbol{\delta}'$ be the element for $\mathbf{m}'$ corresponding to $\boldsymbol{\delta}$ for $\mathbf{m}$, and take $\boldsymbol{\beta}' \in Z(\mathbf{m}')$. We note that $s_{(\boldsymbol{\beta}'+\boldsymbol{\delta}')_{\mathbf{i}\pm} - \boldsymbol{\delta}'}$ goes to zero under $\rho_{\mathbf{m}',\mathbf{m}}$ if there exists some k such that $\beta^{(k)}_{m_k+1} \neq 0$ for $\boldsymbol{\beta}' = (\beta^{(k)}_j)$. In fact, if we write $\boldsymbol{\beta}' + \boldsymbol{\delta}' = (\gamma^{(k)}_j)$, then $\gamma^{(k)}_j > 0$ for $j \le m_k$. Hence if $\beta^{(k)}_{m_k+1} \neq 0$ for some k, then $(\boldsymbol{\beta}' + \boldsymbol{\delta}')_{\mathbf{i}\pm} = ({\gamma'}^{(k)}_j)$ contains a row ${\gamma'}^{(k')}$ whose entries are all non-zero, and so $s_{(\boldsymbol{\beta}'+\boldsymbol{\delta}')_{\mathbf{i}\pm} - \boldsymbol{\delta}'}$ goes to zero.

It follows that in the expression of $\widetilde{E}^{\pm}_{\mathbf{m}'} m_{\boldsymbol{\alpha}'}$ in (3.2.1), we may only consider $\boldsymbol{\beta}'$ (a row permutation of $\boldsymbol{\alpha}'$) such that the last column consists of zeros. One can embed $Z(\mathbf{m})$ into $Z(\mathbf{m}')$ by adding 0 to the last part of $\boldsymbol{\beta} \in Z(\mathbf{m})$. Then those $\boldsymbol{\beta}'$ are identified, under the embedding $Z^{0,0}_n(\mathbf{m}) \hookrightarrow Z^{0,0}_n(\mathbf{m}')$, with a row permutation $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$. Now $\mathcal{I}(\mathbf{m})$ is also embedded into $\mathcal{I}(\mathbf{m}')$ in the same way. Take $\mathbf{i} \in \mathcal{I}(\mathbf{m}')$ such that $\mathbf{i} \notin \mathcal{I}(\mathbf{m})$. If $\mathbf{i}$ is not equal to $\mathbf{i}_1 = {}^t(m_0+1, \dots, m_{e-1}+1)$, then $s_{(\boldsymbol{\beta}'+\boldsymbol{\delta}')_{\mathbf{i}\pm} - \boldsymbol{\delta}'}$ goes to zero under $\rho_{\mathbf{m}',\mathbf{m}}$ by the same reason as before. If $\mathbf{i} = \mathbf{i}_1$, then $\langle \boldsymbol{\beta}', \mathbf{i}\rangle = 0$. In any case, such $\mathbf{i}$ does not give a contribution, and we may only consider $\mathbf{i} \in \mathcal{I}(\mathbf{m})$ in (3.2.1). Hence (3.2.2) holds.

Let $\widetilde{H}_{\pm}$ be the coefficient matrix of $\widetilde{E}^{\pm}_{\mathbf{m}} m_{\boldsymbol{\alpha}}$ in terms of $m_{\boldsymbol{\beta}}$. Then by Lemma 2.4 (ii), $\widetilde{H}_{\pm}$ can be expressed as a block matrix $\widetilde{H}_{\pm} = (\widetilde{H}^{\pm}_{\mathcal{F},\mathcal{F}'})$, where $\widetilde{H}^{\pm}_{\mathcal{F},\mathcal{F}'} = 0$ unless $\mathcal{F} = \mathcal{F}'$ or $\mathcal{F}' < \mathcal{F}$. Then by comparing (3.2.1) with Lemma 2.4 (i) on the diagonal parts, we see that

$$\widetilde{H}^{\pm}_{\mathcal{F},\mathcal{F}} = t^{-M} B^{\pm}_{\mathcal{F},\mathcal{F}} - \sum_{\mathbf{i}\in\mathcal{I}} t^{\langle \boldsymbol{\delta},\mathbf{i}\rangle - M}.$$

On the other hand, if we write $\widetilde{H}^{\pm}_{\mathcal{F},\mathcal{F}} = (h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}})_{\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{F}}$, Proposition 2.7 together with (3.1.1) implies that

$$E^{\pm}_{\mathbf{m}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})} = \sum_{\boldsymbol{\beta}\sim\boldsymbol{\alpha}} h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}.$$

But by (3.2.2), the matrix $\widetilde{H}^{\pm}_{\mathcal{F},\mathcal{F}}$ does not depend on the shift operation under $\mathbf{m}' \to \mathbf{m}$. This shows that the operator $E^{\pm}_{\mathbf{m}}$ is compatible with $\rho_{\mathbf{m}',\mathbf{m}}$. The lemma is proved. □

3.3. By Lemma 3.2, one can define an operator $E^{\pm} = E^{\pm}(q,t)$ on Ξ_F as the limit of $E^{\pm}_{\mathbf{m}}$. Since $E^{\pm}_{\mathbf{m}}$ satisfies a similar formula as given in Lemma 2.5, the operator $E^{\pm}$ also satisfies the adjointness property, i.e., we have

$$\langle E^+ f, g\rangle = \langle f, E^- g\rangle, \quad (f, g \in \Xi_F). \tag{3.3.1}$$

By Lemma 2.4 (ii), one can write, for each $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$,

$$E^{\pm}(q,t) m_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta}\in Z^{0,0}_n} h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) m_{\boldsymbol{\beta}}(x) \tag{3.3.2}$$

with $h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) \in F$, where $h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q,t) = 0$ unless $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$ or $\boldsymbol{\beta} < \boldsymbol{\alpha}$. Let $H_{\pm} = (h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, and write it as a block matrix $H_{\pm} = (H^{\pm}_{\mathcal{F},\mathcal{F}'})$ as in 2.6.

In the case of symmetric groups, the matrix $H_{\pm}$ is a triangular matrix, with distinct eigenvalues. This property was used to characterize Macdonald functions as eigenfunctions of Macdonald operators. As an analogy, it is likely that the following property holds for the diagonal parts of the block matrix $B_{\pm}$.

Conjecture A. Let $\mathcal{F}, \mathcal{F}'$ be any distinct families in $Z^{0,0}_n$. Then the matrices $H^{\pm}_{\mathcal{F},\mathcal{F}}$ and $H^{\pm}_{\mathcal{F}',\mathcal{F}'}$ have no common eigenvalues (according to the sign + or −, respectively).

We have verified the conjecture in the case where $e = 2$, and $n \le 5$.

3.4. Before giving a characterization of Macdonald functions in terms of Macdonald operators, we prepare an easy lemma. Let $A = (a_{ij})$ (resp. $B = (b_{ij})$) be a square matrix of degree m (resp. n), and let $C = (C_{\boldsymbol{\alpha},\boldsymbol{\beta}})_{1 \le \alpha,\beta \le n}$ be a block matrix of size mn, consisting of blocks $C_{\alpha,\beta}$ of size m, defined by

$$C_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \begin{cases} A - b_{\alpha,\alpha} I_m & \text{if } \beta = \alpha, \\ -b_{\alpha,\beta} I_m & \text{otherwise.} \end{cases}$$

We consider a matrix equation $AX = XB$, where $X = (x_{ij})$ is a $m \times n$ matrix of unknown variables. Then this equation can be regarded as a system of linear equations with respect to the mn variables $\{x_{ij}\}$, whose coefficient matrix is given by the matrix C. Moreover, if B is a triangular matrix, then C is block wise triangular, and so $\det C \ne 0$ if and only if $\det(A - b_{\alpha,\alpha} I_m) \ne 0$ for $1 \le \alpha \le n$. Hence we have the following lemma.

Lemma 3.5. *Under the above notation, the following are equivalent.*

(i) *The matrix equation $AX = XB$ has a unique solution $X = 0$.*
(ii) $\det C \ne 0$.
(iii) *The matrices A and B have no common eigenvalues.*

We now show the following.

Theorem 3.6. *Suppose that Conjecture A holds. Then the Macdonald functions $P^{\pm}_{\boldsymbol{\Lambda}} \in \Xi_F$ are characterized by the following two properties.*

$$P^{\pm}_{\boldsymbol{\Lambda}} = m_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta} < \boldsymbol{\alpha}} w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} m_{\boldsymbol{\beta}}, \tag{3.6.1}$$

$$E^{\pm} P^{\pm}_{\boldsymbol{\Lambda}} = \sum_{\boldsymbol{\beta} \sim \boldsymbol{\alpha}} h^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}. \tag{3.6.2}$$

In particular, $P^{\pm}_{\boldsymbol{\Lambda}}$ are determined independently from the choice of the total order $\prec$.

Proof. Let $\{\mathcal{F}_i \mid i \in I\}$ be the set of families in $Z_n^{0,0}$. We give a total order $\prec$ on the index set I, according to the total order $\prec$ on $Z_n^{0,0}$, and write $H_{\pm}$ as $H_{\pm} = (H_{ij})_{i,j \in I}$, where $H_{ij} = H^{\pm}_{\mathcal{F}_i,\mathcal{F}_j}$. (Since the following discussion is independent of the sign $\{\pm\}$, we omit them). Let $X = (w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the transition matrix between basis $\{m_{\boldsymbol{\alpha}}\}$ and $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$ of Ξ_F, and write it as $X = (X_{ij})$. By Proposition 1.12, X is block wise lower triangular, with identity diagonal blocks. Moreover, by Proposition 2.7, we see that $XHX^{-1} = G$, where $G = (G_{ij})$ is a block diagonal matrix with diagonal blocks $G_{ii} = H_{ii}$. In order to prove the theorem, we have only to show the following.

(3.6.3) Let $X = (X_{ij})$ be a block wise lower triangular matrix, with identity diagonal blocks, such that $XHX^{-1} = G$. Then X is determined uniquely, and $X_{ij} = 0$ unless $i = j$ or $\mathcal{F}_j < \mathcal{F}_i$.

We show (3.6.3). The equation $XHX^{-1} = G$ can be written as

$$H_{ii} X_{ij} - X_{ij} H_{jj} = \sum_{j \prec k \prec i} X_{ik} H_{kj} \tag{3.6.4}$$

for any pair $j \prec i$. By backwards induction on j, we may assume that X_{ik} are already determined for $j \prec k \prec i$. Then (3.6.4) determines X_{ij} uniquely, by Conjecture A and

Lemma 3.5. Now suppose that $\mathcal{F}_j \not< \mathcal{F}_i$. Again by induction, we may assume that $X_{ik} = 0$ unless $\mathcal{F}_k < \mathcal{F}_i$. Since $H_{kj} = 0$ unless $\mathcal{F}_j < \mathcal{F}_k$ by (3.3.2), we must have $H_{ii}X_{ij} - X_{ij}H_{jj} = 0$. This implies that $X_{ij} = 0$ by Lemma 3.5, and we obtain (3.6.3). Thus the theorem is proved. □

Remark 3.7. The Hall-Littlewood function $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ given in [S1] coincides with $P^{\pm}_{\boldsymbol{\Lambda}}(x;0,t)$. Hence it satisfies similar formulas as (3.6.1), (3.6.2). In particular, $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$, and so the Kostka functions $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(x;t)$ do not depend on the choice of the total order. This answers the questions posed in [S1, Remark 4.5, (ii)] and in [GM, Remark 2.4], modulo the truth of the conjecture.

3.8. We give here some examples of the matrices $B^{\pm}_{\mathcal{F},\mathcal{F}}$ and $H^{\pm}_{\mathcal{F},\mathcal{F}}$ for some small rank cases with $e = 2$. Assume that W is the Weyl group of type C_2. Then the symbols and families are given as follows.

$$\begin{aligned}
&\mathcal{F}_1 = \left\{\begin{pmatrix} 3\ 0 \\ 0 \end{pmatrix}\right\}, \quad \mathcal{F}_2 = \left\{\begin{pmatrix} 2\ 1\ 0 \\ 2\ 1 \end{pmatrix}\right\}, \\
&\mathcal{F}_3 = \left\{\begin{pmatrix} 2\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1\ 0 \\ 2 \end{pmatrix}\right\},
\end{aligned}$$

which correspond, in this order, to the double partitions of 2,

$$(2;-), \quad (-;11), \quad (1;1), \quad (11;-), \quad (-;2)$$

respectively. In the tables below, we express the matrices $B^{\pm}_{\mathcal{F},\mathcal{F}}, H^{\pm}_{\mathcal{F},\mathcal{F}}$ as $B_{\mathcal{F}}, H_{\mathcal{F}}$, since they are independent of the sign $\pm$. We have

$$B_{\mathcal{F}_1} = (1), \quad B_{\mathcal{F}_2} = (t^3q + tq), \quad B_{\mathcal{F}_3} = \begin{pmatrix} 0 & q & tq^2 \\ q & 0 & -tq \\ tq^2 & -q^2 & 0 \end{pmatrix}.$$

Up to C_5, only 3-element or 1-element families occur. The Weyl group of type C_6 contains a unique 10-element family, which is given as follows,

$$\begin{aligned}
\mathcal{F}_4 = \Bigl\{ &\begin{pmatrix} 4\ 2\ 0 \\ 3\ 1 \end{pmatrix}, \begin{pmatrix} 4\ 2\ 1 \\ 3\ 0 \end{pmatrix}, \begin{pmatrix} 4\ 3\ 0 \\ 2\ 1 \end{pmatrix}, \begin{pmatrix} 4\ 3\ 1 \\ 2\ 0 \end{pmatrix}, \begin{pmatrix} 4\ 3\ 2 \\ 1\ 0 \end{pmatrix}, \\
&\begin{pmatrix} 4\ 1\ 0 \\ 3\ 2 \end{pmatrix}, \begin{pmatrix} 3\ 2\ 0 \\ 4\ 1 \end{pmatrix}, \begin{pmatrix} 3\ 2\ 1 \\ 4\ 0 \end{pmatrix}, \begin{pmatrix} 3\ 1\ 0 \\ 4\ 2 \end{pmatrix}, \begin{pmatrix} 2\ 1\ 0 \\ 4\ 3 \end{pmatrix}\Bigr\}.
\end{aligned}$$

The corresponding double partitions of 6 are, in this order, as follows.

$$\begin{aligned}
&(21;21), \quad (211;2), \quad (22;11), \quad (221;1), \quad (222;-), \\
&(2;22), \quad (11;31), \quad (111;3), \quad (1;32), \quad (-;33).
\end{aligned}$$

The matrix $B_{\mathcal{F}}$ is given by

$$B_{\mathcal{F}} = \begin{pmatrix} 0 & q & t^2q^3 & 0 & tq^2 & tq^2 & t^3q^4 & 0 & 0 & t^2q^3 \\ q & 0 & 0 & t^2q^3 & -tq^3 & -tq & 0 & t^3q^4 & 0 & -t^2q^2 \\ t^2q^3 & 0 & 0 & q & -tq & -tq^3 & -t^3q^3 & 0 & t^2q^3 & 0 \\ 0 & t^2q^3 & q & 0 & tq^2 & tq^2 & 0 & -t^3q^3 & -t^2q^2 & 0 \\ tq^2 & -t^2q^2 & -q^2 & tq^2 & 0 & 0 & -t^2q^2 & t^3q^2 & 0 & 0 \\ tq^2 & -q^2 & -t^2q^2 & tq^2 & 0 & 0 & 0 & 0 & t^3q^4 & -t^2q^4 \\ t^3q^4 & 0 & -t^2q^4 & 0 & -tq^3 & 0 & 0 & q & tq^2 & -t^2q^2 \\ 0 & t^3q^4 & 0 & -t^2q^4 & tq^4 & 0 & q & 0 & -tq & t^2q \\ 0 & 0 & t^2q^3 & -tq^3 & 0 & t^3q^4 & tq^2 & -q^2 & 0 & t^2q^3 \\ t^2q^3 & -tq^3 & 0 & 0 & 0 & -t^3q^3 & -tq^3 & q^3 & t^2q^3 & 0 \end{pmatrix}.$$

Throughout the above examples, $H_{\mathcal{F}}$ is given by

$$H_{\mathcal{F}} = t^{-(2m+1)}B_{\mathcal{F}} - t^{-(2m+1)}\frac{(1-t^m)(1-t^{m+1})}{(1-t)^2}$$

for symbols of the shape $\mathbf{m} = (m+1, m)$, (so, $m = 1$ for $\mathcal{F}_1, \mathcal{F}_3$, and $m = 2$ for $\mathcal{F}_2, \mathcal{F}_4$, respectively). Note that $B_{\mathcal{F}}$ is not necessarily symmetric. However, if we put $q = t$, it turns out to be symmetric since D^1 is a self adjoint operator, and the representation matrix with respect to the orthonormal basis of Schur functions coincides with the diagonal blocks $(B_{\mathcal{F}})$.

3.9. In the case of symmetric groups, Macdonald operators are commuting with each other since they are simultaneously diagonalizable. In our case, Proposition 2.7 shows that $D^r_{\pm}$ are simultaneously diagonalizable in the sense of block matrices. So they are commuting with each other if and only if the matrices $B^{r,\pm}_{\mathcal{F},\mathcal{F}}$ are commuting with for $r = 1, \dots, M_1$ (for a fixed $\mathcal{F}$). As the following examples show, one might expect that $B^{r,\pm}_{\mathcal{F},\mathcal{F}}$ are commuting with each other in general.

First consider a simple example. Let $\mathcal{F}_3 = \{\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \boldsymbol{\Lambda}_3\}$ be the 3-element family of C_2 as in 3.8. We consider its m times shifts $\mathcal{F}_3^{(m)} = \{\boldsymbol{\Lambda}'_1, \boldsymbol{\Lambda}'_2, \boldsymbol{\Lambda}'_3\}$, where

$$\boldsymbol{\Lambda}'_i = \begin{pmatrix} x+m, \ y+m, \ m-1, \ \dots, \ 0 \\ z+m, \ m-1, \ \dots, \ 0 \end{pmatrix}$$

for $\boldsymbol{\Lambda}_i = \binom{x\ y}{z}$. Then the matrices $B^r_{\mathcal{F}_3}$ are given as

$$B^r_{\mathcal{F}_3} = aB^1_{\mathcal{F}_3} + b$$

with

$$a = t^{2m}\sum_{J'} t^{\langle \boldsymbol{\delta}, J'\rangle}, \quad b = \sum_{J} t^{\langle \boldsymbol{\delta}, J\rangle},$$

where $J \in \mathcal{I}_r$ runs over the elements of the form $J = \{\mathbf{i}_{j_1}, \dots, \mathbf{i}_{j_r}\}$ for $\mathbf{i}_j = \binom{j+1}{j}$ with $2 \le j \le m+1$, and $J' \in \mathcal{I}_{r-1}$ runs over the elements having similar properties. This implies that $B^r_{\mathcal{F}_3}$ are all commuting with for $1 \le r \le m$.

In the following, we discuss some related results, i.e., we show that when $e = 2$, and $q = t$, then the operators D^r are commuting with each other. First we prepare a lemma. (Since we deal with the case where $e = 2$, we omit the sign $\pm$ in the discussion below.)

Lemma 3.10. *Assume that $e = 2$. Then, for each r, r', there exists a bijective map $\varphi : \mathcal{I}_r \times \mathcal{I}_{r'} \to \mathcal{I}_r \times \mathcal{I}_{r'}$ satisfying the following properties: let $J \in \mathcal{I}_r$, $J' \in \mathcal{I}_{r'}$ and put $\varphi(J, J') = (K, K')$. For each $\boldsymbol{\alpha} \in Z$, we have*

(i) $\boldsymbol{\alpha}_{JJ'} = \boldsymbol{\alpha}_{K'K}$,
(ii) $\langle \boldsymbol{\alpha}, J\rangle + \langle \boldsymbol{\alpha}_J, J'\rangle = \langle \boldsymbol{\alpha}, K'\rangle + \langle \boldsymbol{\alpha}_{K'}, K\rangle$.

Proof. Take $J \in \mathcal{I}_r, J' \in \mathcal{I}_{r'}$. Let J_0 be the subset of J consisting of $\mathbf{i} = \binom{a}{b}$ such that there exists $\mathbf{i}' = \binom{x}{y} \in J'$ with $x = a$ or $y = b$, and let J_0' be the subset of J' having similar properties. We define an equivalence relation on J_0 by connecting $\mathbf{i} = \binom{a}{b}, \mathbf{i}' = \binom{c}{d} \in J_0$ when there exists $\binom{x}{y} \in J'$ such that $(x, y) = (a, d)$ or $(x, y) = (c, b)$. We denote by $\{J_C \mid C \in \mathcal{C}\}$ the set of equivalence classes in J_0. Then the class J_C has the following form.

$$J_C = \Bigl\{ \begin{pmatrix} a_1 \\ c \end{pmatrix}, \begin{pmatrix} a_2 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{k-1} \\ b_{k-2} \end{pmatrix}, \begin{pmatrix} a_k \\ b_{k-1} \end{pmatrix}, \begin{pmatrix} d \\ b_k \end{pmatrix} \Bigr\},$$

where $\binom{a_i}{b_i} \in J'$ for $1 \le i \le k$. We put

$$J_C' = \Bigl\{ \begin{pmatrix} a_i \\ b_i \end{pmatrix} \mid p \le i \le q \Bigr\},$$

where $p = 0$ (resp. $q = k+1$) if there exists $\binom{a_0}{b_0} \in J'$ (resp. $\binom{a_{k+1}}{b_{k+1}} \in J'$) such that $b_0 = c$ (resp. $a_{k+1} = d$), and $p = 1, q = k$ otherwise.

For each $C \in \mathcal{C}$, we define the sets K_C, K_C' as follows:

$$K_C = \begin{cases} J_C & \text{if } p = 1, q = k, \\ J_C' & \text{if } p = 1, q = k+1, \\ J_C' & \text{if } p = 0, q = k, \\ \Bigl\{ \begin{pmatrix} a_0 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_k \\ b_{k+1} \end{pmatrix} \Bigr\} & \text{if } p = 0, q = k+1. \end{cases}$$

$$K_C' = \begin{cases} \Bigl\{ \begin{pmatrix} a_2 \\ c \end{pmatrix}, \begin{pmatrix} a_3 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_k \\ b_{k-2} \end{pmatrix}, \begin{pmatrix} d \\ b_{k-1} \end{pmatrix} \Bigr\} & \text{if } p = 1, q = k, \\ J_C & \text{if } p = 1, q = k+1, \\ J_C & \text{if } p = 0, q = k, \\ J_C' & \text{if } p = 0, q = k+1. \end{cases}$$

Since J, J' are subsets of the index set of elements in Z, J_C, etc. induce permutations on the entries of elements in Z. We denote by x_C, x_C' (resp. y_C, y_C') the permutations in $\mathfrak{S}_M$ corresponding to J_C, J_C' (resp. K_C, K_C'), respectively. Then it is easy to check that

$$x_C' \circ x_C = y_C \circ y_C'. \tag{3.10.1}$$

We put

$$K = (J - J_0) \cup \bigcup_{C \in \mathcal{C}} K_C, \quad K' = (J' - J_0') \cup \bigcup_{C \in \mathcal{C}} K_C'.$$

Components of K (resp. K') are mutually disjoint, and we see that $K \in \mathcal{I}_r, K' \in \mathcal{I}_{r'}$. We now define the map $\varphi : \mathcal{I}_r \times \mathcal{I}_{r'} \to \mathcal{I}_r \times \mathcal{I}_{r'}$ by $\varphi(J, J') = (K, K')$. Then one can check that $\varphi^2 = \mathrm{id}$, and so φ is a bijection. The assertion (i) follows from (3.10.1). To show (ii), it is enough to verify the formula in the case where $J = J_C, J' =$

$J'_C, K = K_C, K' = K'_C$. The assertion is clear when $K_C = J'_C, K'_C = J_C$. Assume that $p = 1, q = k$ or $p = 0, q = k+1$. Then one can check by a direct computation that

$$\langle \boldsymbol{\alpha}, J_C \rangle = \langle \boldsymbol{\alpha}_{K'_C}, K_C \rangle, \quad \langle \boldsymbol{\alpha}_{J_C}, J'_C \rangle = \langle \boldsymbol{\alpha}, K'_C \rangle .$$

Hence the formula holds in these cases also, and the lemma follows. □

Proposition 3.11. *Assume that $e = 2$, and $q = t$. Then the operators $D^r(t,t)$ are commuting with each other for $r = 1, \dots, M_1$.*

Proof. As remarked in 3.9, it is enough to show that the matrices $B^r_{\mathcal{F}} = B^{r,\pm}_{\mathcal{F},\mathcal{F}}$ are commuting with each other for $r = 1, \dots, M_1$. By Lemma 2.4 (i), the part corresponding to the diagonal block in the expression of $D^r(t,t)s_{\boldsymbol{\alpha}}$ is given as

$$\sum_{J \in \mathcal{I}_r} t^{\langle \boldsymbol{\Lambda}, J \rangle} s_{\boldsymbol{\Lambda}_J - \boldsymbol{\delta}},$$

where $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\delta}$. Let $\mathcal{X}$ be the part of $D^{r'}(t,t)D^r(t,t)s_{\boldsymbol{\alpha}}$ corresponding to the diagonal blocks. Then we have

$$\mathcal{X} = \sum_{J \in \mathcal{I}_r} \varepsilon_J t^{\langle \boldsymbol{\Lambda}, J \rangle} \sum_{J'' \in \mathcal{I}_{r'}} t^{\langle [\boldsymbol{\Lambda}_J], J'' \rangle} s_{[\boldsymbol{\Lambda}_J]_{J''} - \boldsymbol{\delta}}.$$

For each J, there exists $w \in \mathfrak{S}_{\mathbf{m}}$ such that $[\boldsymbol{\Lambda}_J] = w(\boldsymbol{\Lambda}_J)$ and ε_J is given by $\varepsilon_J = (-1)^{l(w)}$. Then if we put $J' = w^{-1}(J'')$, we have $\langle [\boldsymbol{\Lambda}_J], J'' \rangle = \langle \boldsymbol{\Lambda}_J, J' \rangle$, and $[\boldsymbol{\Lambda}_J]_{J''} = w(\boldsymbol{\Lambda}_{JJ'})$. Hence one can write

$$\mathcal{X} = \sum_{J \in \mathcal{I}_r} \sum_{J' \in \mathcal{I}_{r'}} \varepsilon_{J,J'} t^{\langle \boldsymbol{\Lambda}, J \rangle + \langle \boldsymbol{\Lambda}_J, J' \rangle} s_{[\boldsymbol{\Lambda}_{JJ'}] - \boldsymbol{\delta}},$$

where $\varepsilon_{J,J'} = (-1)^{l(w')}$ for $w' \in \mathfrak{S}_{\mathbf{m}}$ such that $[\boldsymbol{\Lambda}_{JJ'}] = w'(\boldsymbol{\Lambda}_{JJ'})$. Now by applying Lemma 3.10 for $\boldsymbol{\alpha} = \boldsymbol{\Lambda}$, we see that $\mathcal{X}$ coincides with the diagonal part for $D^r(t,t)D^{r'}(t,t)s_{\boldsymbol{\alpha}}$. The proposition is proved. □

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