ON GREEN FUNCTIONS ASSOCIATED TO COMPLEX REFLECTION GROUPS

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§1. What are the Green functions

Green functions are introduced by J.A. Green[G] in 1955, in connection with the representation theory of general linear groups $GL_n(\mathbf{F}_q)$ over a finite field $\mathbf{F}_q$. Green constructed a system of polynomials (now called Green functions) by a method based on the combinatorial theory, and showed that the values of irreducible characters of $GL_n(\mathbf{F}_q)$ can be described by those functions. A discovery of Green functions provided us a uniform algorithm of computing irreducible characters of $GL_n(\mathbf{F}_q)$. In this sense, it may fairly be said that the core of the character theory of $GL_n(\mathbf{F}_q)$ lies in the Green functions.

Finite classical groups such as symplectic groups $Sp_{2n}(\mathbf{F}_q)$ or special orthogonal groups $SO_n(\mathbf{F}_q)$ have similar structures as $GL_n(\mathbf{F}_q)$. However, one has to wait for almost 20 years before finding a proper approach for the problem of generalizing Green's results and of describing irreducible characters of such finite groups, in particular the problem of how to define Green functions. In 1976, Deligne and Lusztig succeeded in constructing, for finite Chevalley groups $G(\mathbf{F}_q)$ in general, a class of representations $R_T^G(\theta)$ (called virtual representations of Deligne-Lusztig) endowed with very nice properties, by making use of l-adic cohomology groups. Furthermore they defined Green functions as the restriction of the character of $R_T^G(\theta)$ to the set of unipotent elements, and showed that the computation of characters of $R_T^G(\theta)$ is reduced to the determination of Green functions. The classification of irreducible characters of $G(\mathbf{F}_q)$ was completed by Lusztig in 1980's, and it was those virtual characters $R_T^G(\theta)$ which turned out to be a main tool for it. For example, in the case of $GL_n(\mathbf{F}_q)$, all the irreducible characters are written as a linear combination of characters of $R_T^G(\theta)$ in a simple form. Although a situation is much more complicated for other finite reductive groups, still many irreducible characters are obtained from $R_T^G(\theta)$. In view of these facts, the determination of Green functions is a very important problem in the representation theory of finite Chevalley groups.

However in general, it is a difficult problem to investigate the trace of representations constructed on the l-adic cohomology groups. It is almost impossible to compute Green functions directly from the original definition. In turn, Lusztig showed that Green functions have a geometric interpretation as characteristic functions of intersection cohomology complexes of certain kinds. (Remind that it is not a geometric realization of representations, but a realization of functions). It follows from this fact that Green functions are determined as a unique solution of a certain simple matrix equation. This gives, at the same time, an algorithm of computing Green functions explicitly.

It was more than 30 years ago that the author found an algorithm of computing Green functions, and computed the case of exceptional groups of type F_4 by making use of a computer ([S2]). I remember that it was a troublesome work since, at that time, there were no good computer softwares in computing polynomials, in contrast with the convenient today's softwares. Successively, Beynon-Spaltenstein [BS] computed Green functions of exceptional groups of type E_6, E_7 and E_8, by using a similar algorithm. (G_2 was already known). The algorithm used there was improved and developed further by Lusztig later, and was completed as the algorithm mentioned above. In its original form, Green functions of $GL_n(\mathbf{F}_q)$ had explicit shapes, and were constructible objects. Then they were generalized and changed to an abstract form by Deligne-Lusztig. And now they came back with a very concrete form that no one can expect more, after the process of further abstraction. We may see there an endless circulation of abstraction and materialization in the world of mathematics.

Anyway in the case of exceptional groups, the explicit computation might be the best solution for many problems (though this is simply thanks to the fact that the ability of the computer exceeded the huge scale of Green functions of type E_8). But this does not work at all for the case of classical groups. The algorithm has no power for the groups which belong to the infinite series. No one tries to compute the Green functions of $Sp_{2n}(\mathbf{F}_q)$ for $n = 1000$. But here we should remember that in the case of $GL_n(\mathbf{F}_q)$, the combinatorial description of Green functions provides us more informations than the algorithm does. People deduced various useful properties from there. As is well known, classical groups are something like relatives of GL_n, and many properties of them are described in a framework of the combinatorial theory of GL_n. It would be desirable to find a combinatorial construction for the case of Green functions of classical groups.

At present, it seems that there exists a big gap between the combinatorial construction of Green functions given by Green, and the geometric construction given by Lusztig. In this note, as a first step to fill in this gap, I would like to explain an approach by the author for the combinatorial construction of Green functions of classical groups. This is also a trial to extend the combinatorial theory of type A_n, dominated by the notion of "partitions of n", to the case of other groups.

§2. Why do the complex reflection groups appear on the stage

Another purpose of this note is to extend the theory of Green functions to the case of certain complex reflection groups. It is natural that Weyl groups are involved in the theory since Green functions are related to Chevalley groups. But why do the complex reflection groups must appear here suddenly? As K. Harada writes in his survey [H], finite Chevalley groups have only one genus and one species when one fixes its type and dimension, its skeleton is determined by Weyl groups, and the difference of $\mathbf{F}_q$ (spices) is only the difference of the physique, the difference of being fat or slim. This type of phenomenon appears conspicuously when we look at the character table. For example, the Weyl group of $GL_n(\mathbf{F}_q)$ coincides with the symmetric group S_n, and its conjugacy classes and irreducible characters are parametrized naturally by partitions of n. It follows that the character table of S_n is dominated completely by partitions of

n, and Green functions turn out to be polynomials in q determined by partitions of n. Therefore one may say (as a very rough idea) that the character table of $GL_n(\mathbf{F}_q)$ is determined by partitions of n and q, in the sense that the character table of $GL_n(\mathbf{F}_q)$ is described by Green functions.

We now consider the complex reflection group $W = G(e, 1, n)$. W is a semidirect product of S_n and $(\mathbf{Z}/e\mathbf{Z})^n$, and is isomorphic to S_n if $e = 1$, to Weyl groups of type B_n if $e = 2$. One may say that W posses a skeleton very similar to classical groups $Sp_{2n}(\mathbf{F}_q)$ or $SO_{2n+1}(\mathbf{F}_q)$, and is regarded as their neighboring species (genus?). However, it is a pity that there exist no finite groups $G(\mathbf{F}_q)$ (individual) having W as its Weyl groups (skeleton). Complex reflection groups look like dinosaurs who existed on the earth in ancient days, but whose fossils of the skeleton only remains nowadays. What we are trying is to reconstruct (?) the body of Weyl groups by collecting tiny little signs, as if we are reconstructing the body of dinosaurs from the blood of mosquitoes trapped inside of the amber.

It was 1954 that the complex reflection groups have obtained their citizenship, the year when Shepard-Todd [ST] completed the classification of complex reflection groups. In 1955, Green's monumental work [G] as mentioned before was published, and in the same year, Chevalley's paper, which announced proudly the uniform construction of finite Chevalley groups, is also published. All the players came in the stage at this time. However, it was only after 90's that people became to notice that complex reflection groups are neighboring species having a common ancestry with Weyl groups or real reflection groups, and because of it they posses lots of representation theoretic properties originate from $G(\mathbf{F}_q)$.

The turning point was the construction of Hecke algebras associated to complex reflection groups $G(e, 1, n)$ by Ariki-Koike [AK] (now called the Ariki-Koike algebras). Original Hecke algebras are the ones obtained by extending the group algebra of Weyl groups by parameter q, and it plays an important role in the representation theory of finite Chevalley groups. (One may regard this as a half-fleshed, half-boned state between $G(\mathbf{F}_q)$ and W). At the same time, a group of M. Broué in Paris was convinced of the existence of the Hecke algebras associated to complex reflection groups, in connection with the modular representation theory of $G(\mathbf{F}_q)$. In response to the work of Ariki-Koike, they advanced the study extensively, and it became apparent the existence of the formal "group" $G_W(q)$ associated to complex reflection groups (here, q is a parameter).

In these days, the properties of irreducible characters of $G(\mathbf{F}_q)$, which are described by Weyl groups only, are verified to hold also for $G_W(q)$ in a considerably large extent. For example, there exists an important class of irreducible characters of $G(\mathbf{F}_q)$, called unipotent characters. The classification of unipotent characters are described completely by the properties of the characters of Weyl groups (more precisely, of Hecke algebras), and their degrees (it is expressed as a polynomial in q, and is called the generic degree) are also obtained from the investigation of the Hecke algebras. Broué's group has shown that the classification theory of unipotent characters can be extended formally to a large class of complex reflection groups, and that the "unipotent characters" of $G_W(q)$ can be defined as well, and they have determined their generic degrees. Actually, this is exactly what Lusztig had carried out for the case of non-crystallographic real reflection groups H_3, H_4, on the way of the classification of irreducible representations of $G(\mathbf{F}_q)$. (Note that there does not exist $G(\mathbf{F}_q)$ having such a group as its Weyl group.) Lusztig left these results unpublished nearly for 10 years, and released them in the appendix of the book of Broue and others [BMM]. The following is one of the opening paragraphs of his paper.

"It should be noted that the "results" below (which were obtained in 1980-1982, except for those in 4.2) are not theorems in the accepted sense; rather, they are an indication of a not yet discovered theory. A rigorous treatment of these matters and an explanation of their meaning remain to be found."

More than 10 years have passed since then, and it looks that Lusztig's hesitation turned out to be a period story. Today, people talk loudly about complex reflection groups, and huge data about them are being stocked. However, most of the proofs rely on the case-by-case arguments, and still it does not reach the established theory as in the case of Weyl groups.

Now, our dream is to construct the character table of $G_W(q)$. But $G_W(q)$ is just a formal object, and of course it does not have a group structure. It would not be appropriate to say this as an individual (= a body) of W. But the character table of $G_W(q)$ will have an enough qualification as an individual of W. Because, "representations are more important than groups" for the researchers working on the representation theory. Once the classification of unipotent characters of $G_W(q)$ was done, and their degrees were determined, it would be natural, as a next step, to consider the values of their characters at unipotent classes. But no reasonable method for defining unipotent classes is known for general $G_W(q)$. Even in the case of finite Chevalley groups, unipotent classes are not determined by the information from Weyl groups, and their behavior is complicated. For example, though $Sp_{2n}(\mathbf{F}_q)$ and $SO_{2n+1}(\mathbf{F}_q)$ have the same Weyl group, unipotent classes are different.

As is well-known, unipotent classes of $GL_n(\mathbf{F}_q)$ are described, as in the case of irreducible characters of S_n, by partitions λ of n. On the other hand, irreducible characters of Weyl groups of type B_n are in one to one correspondence with the pairs of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$ satisfying the condition $|\lambda^{(0)}| + |\lambda^{(1)}| = n$, but unipotent classes are not. However, a natural correspondence was constructed by Springer [Sp] in general, between irreducible characters of Weyl groups and unipotent classes, the so-called Springer correspondence. (Note: not exactly unipotent classes. See Section 4 for details). In order to describe the unipotent classes of finite classical groups, Lusztig [L2] introduced unipotent symbols as an extension of the notion of partitions, based on the description of the Springer correspondence of classical groups by the author ([S1]). Unipotent symbols depend on the groups (and the characteristic of the base field). Different unipotent symbols are used for $Sp_{2n}(\mathbf{F}_q)$ and $SO_{2n+1}(\mathbf{F}_q)$. In [L2], he established the generalized Springer correspondence as a generalization of the Springer correspondence. The generalized Springer correspondence in the case of finite classical groups are completely determined by [LS].

We now return to the Green functions. The remarkable fact is that the matrix equation determining the Green functions of classical groups is described completely by the characters of Weyl groups and some combinatorial properties of unipotent symbols. The notion of partitions in the case of type A is replaced here by the notion of unipotent symbols. By the way, the irreducible characters of complex reflection groups $W = G(e,1,n)$ are in one to one correspondence with the e-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \dots, \lambda^{(e-1)})$ such that $\sum_i |\lambda^{(i)}| = n$. And e-symbols are defined as a variant of unipotent symbols. One may regard e-symbols as the ones representing the unipotent classes of $G_W(q)$. The matrix equation can be formulated for such e-symbols as in the case of classical groups. We propose to call the solution of this equation Green functions associated to complex reflection groups $G(e,1,n)$. In what follows, we try a combinatorial approach for Green functions of $G_W(q)$ obtained in this way. In a combinatorial aspect, the groups $G(\mathbf{F}_q)$ are implicit, and only q appears in the discussion as a parameter. In such a situation, there is no reason to restrict to the case where $e = 2$ (the case of Weyl groups of type B_n), and the argument becomes more transparent if one considers the general e.

§3. Green functions of $GL_n(\mathbf{F}_q)$

From now on we enter into the main part of this paper. First, following the textbook of Macdonald [M], we review the construction of Green functions of $GL_n(\mathbf{F}_q)$. The symmetric group S_m acts on the polynomial ring $\mathbf{Z}[x_1,\dots,x_m]$ with respect to the variables $x_1,\dots,x_m$ as a permutation of variables. The set $\varLambda_m = \mathbf{Z}[x_1,\dots,x_m]^{S_m}$ of S_m-invariant polynomials (= symmetric polynomials) turns out to be a graded ring $\varLambda_m = \bigoplus_{n\ge 0}\varLambda_m^n$. Here $\varLambda_m^n$ denotes the $\mathbf{Z}$-submodule spanned by homogeneous symmetric polynomials of degree n. Let $\mathcal{P}_n(m)$ be the set of partitions of n with the length smaller than m, i.e., $\mathcal{P}_n(m)$ is the set of $\lambda = (\lambda_1,\dots,\lambda_m) \in \mathbf{Z}^m$ such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m \ge 0$, $\sum_{i=1}^m \lambda_i = n$. Here $|\lambda| = n$ is called the size of λ. (Although it is often assumed that $\lambda_i > 0$ in the usual definition, here we consider $\lambda \in \mathbf{Z}^m$ by adding 0 in the tale of the sequence $\{\lambda_i\}$.) It is known that $\varLambda_m^n$ is a free $\mathbf{Z}$-module of rank $|\mathcal{P}_n(m)|$. There exists several bases in $\varLambda_m^n$ with nice properties. First of all we shall define Schur functions. For $\alpha = (\alpha_1,\dots\alpha_m) \in \mathbf{Z}_{\ge 0}^m$, let

$$a_\alpha(x) = a_\alpha(x_1,\dots,x_m) = \sum_{w\in S_m}\varepsilon(w)w(x^\alpha),$$

where $\varepsilon(w)$ denotes the signature of $w \in S_m$ and $x^\alpha = x_1^{\alpha_1}\dots x_m^{\alpha_m}$. $a_\alpha(x)$ is an alternating polynomial of x, and in particular in the case of $\alpha = \delta = (m-1, m-2,\dots,1,0)$, $a_\delta(x) = \prod_{i<j}(x_i - x_j)$ coincides with the Vandermonde determinant. By the property of alternating polynomials, $a_\alpha(x)$ is always divisible by $a_\delta(x)$. For $\lambda \in \mathcal{P}_n(m)$, we define the **Schur functions** $s_\lambda(x)$ by $s_\lambda(x) = a_{\lambda+\delta}(x)/a_\delta(x)$. By definition, $s_\lambda(x) \in \varLambda_m^n$, and it is verified that $\{s_\lambda(x) \mid \lambda \in \mathcal{P}_n(m)\}$ forms a basis of $\varLambda_m^n$.

Next, we define a power sum symmetric function. For an integer $r \ge 1$, let $p_r(x) = \sum_{i=1}^m x_i^r$, $p_0(x) = 1$. For $\lambda \in \mathcal{P}_n(m)$, we call $p_\lambda(x) = \prod_{i=1}^m p_{\lambda_i}(x)$ the **power sum symmetric functions**. $\{p_\lambda(x) \mid \lambda \in \mathcal{P}_n(m)\}$ forms a basis of $\mathbf{Q}$ vector space $\mathbf{Q}\otimes\varLambda_m^n$.

In many aspects of the theory, it is often more convenient to work with symmetric polynomials in sufficiently many variables, without restricting them. Since Schur functions have a property that $s_\lambda(x_1,\dots,x_m,0) = s_\lambda(x_1,\dots,x_m)$, $s_\lambda(x)$ makes sense even if the number of variables is changed, where the former part of the identity denotes the one obtained by substituting $x_{m+1} = 0$ into the Schur function with $m+1$ variables, and the latter denotes the Schur function with m variables. A similar property holds also for $p_\lambda(x)$. In order to make the situation clearer, we shall introduce a notion of symmetric functions. For integers $m \ge m'$, we define a homomorphism $\mathbf{Z}[x_1,\dots,x_m] \to \mathbf{Z}[x_1,\dots,x_{m'}]$ by $x_i \mapsto 0$ $(i > m')$, $x_i \mapsto x_i$ $(i \le m')$.

By restricting this to the ring of symmetric polynomials, a surjective homomorphism $\rho^n_{m,m'} : \Lambda^n_m \to \Lambda^n_{m'}$ is obtained. We denote by Λ^n the projective limit $\lim\limits_{\leftarrow} \Lambda^n_m$ with respect to $\rho^n_{m,m'}$. $\Lambda = \bigoplus_{n\ge 0} \Lambda^n$ has a natural ring structure, which is called the ring of symmetric functions. It follows from the previous discussion that $s_\lambda(x), p_\lambda(x)$ are regarded as elements of Λ^n. For a fixed n, $\mathcal{P}_n(m)$ does not depend on m if $m \ge n$, which we denote by $\mathcal{P}_n$. Then $\{s_\lambda(x) \mid \lambda \in \mathcal{P}_n\}, \{p_\lambda(x) \mid \lambda \in \mathcal{P}_n\}$ give rise to basis of $\mathbf{Z}$-module Λ^n, and of $\mathbf{Q}$-space $\mathbf{Q} \otimes \Lambda^n$, respectively.

Symmetric functions $s_\lambda(x), p_\mu(x)$ have a close relationship with the representation theory of S_n in the following sense. The set of conjugacy classes in S_n are in one to one correspondence, through the decomposition into cyclic permutations, to the set of partitions of n. We denote by w_μ a representative of the conjugacy class in S_n corresponding to $\mu \in \mathcal{P}_n$. Also we denote by χ^λ the irreducible character of S_n corresponding to $\lambda \in \mathcal{P}_n$. The unit character corresponds to the partition (n) and the sign character corresponds to (1^n) under this correspondence. Then the following Frobenius formula holds.

(Frobenius formula) For $\lambda, \mu \in \mathcal{P}_n$,

$$p_\mu(x) = \sum_{\lambda \in \mathcal{P}_n} \chi^\lambda(w_\mu) s_\lambda(x). \tag{3.1}$$

In particular, the character table of S_n can be obtained as the transition matrix between the bases $\{s_\lambda(x)\}, \{p_\mu(x)\}$ in $\mathbf{Q} \otimes \Lambda^n$.

Next, we shall define the Hall-Littlewood functions. Let $t, x_1, \dots, x_m$ be variables, and for $\lambda \in \mathcal{P}_n(m)$, let

$$P_\lambda(x;t) = v_\lambda(t)^{-1} \sum_{w \in S_m} w\left(x_1^{\lambda_1} \cdots x_m^{\lambda_m} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right), \tag{3.2}$$

where $v_\lambda(t)$ is a polynomial given by $v_\lambda(t) = \prod_{i=1}^m v_{\lambda_i}(t)$ if we put $v_0(t) = 1$ and $v_r(t) = \prod_{j=1}^r (1 - t^j)/(1 - t)$ for $r \ge 1$. One sees that $P_\lambda(x;t) \in \mathbf{Z}[t] \otimes_{\mathbf{Z}} \Lambda^n_m$. $P_\lambda(x;t)$ are called the **Hall-Littlewood functions** associated to λ. As in the case of Schur functions, $P_\lambda(x;t)$ may be regarded as functions of infinitely many variables $x_1, x_2, \dots$ with coefficients in $\mathbf{Z}[t]$. Then $\{P_\lambda(x;t) \mid \lambda \in \mathcal{P}_n\}$ forms a basis of $\mathbf{Z}[t]$-module $\mathbf{Z}[t] \otimes_{\mathbf{Z}} \Lambda^n$. Moreover, the property $P_\lambda(x;0) = s_\lambda(x)$ holds. In this sense, Hall-Littlewood functions are regarded as a t-deformation of Schur functions.

We now define a partial order $\lambda \le \mu$ on $\mathcal{P}_n$. For a sufficiently large m, put $\lambda = (\lambda_1, \dots, \lambda_m), \mu = (\mu_1, \dots, \mu_m)$, and define $\lambda \le \mu$ by the condition $\sum_{i=1}^r \lambda_i \le \sum_{i=1}^r \mu_i$ for $r = 1, \dots, m$. The set of unipotent classes of $GL_n(\overline{\mathbf{F}}_q)$ ($\overline{\mathbf{F}}_q$ is the algebraic closure of $\mathbf{F}_q$) corresponds bijectively to $\mathcal{P}_n$ through Jordan's normal form. If we denote by C_λ the unipotent class corresponding to λ, the partial order defined above coincides with the partial order obtained from the incidence relation $\overline{C}_\lambda \subseteq \overline{C}_\mu$ of the closures of unipotent classes.

$\{s_\lambda(x)\}, \{P_\mu(x;t)\}$ are both basis of $\mathbf{Z}[t]\otimes \varLambda^n$. Thus $s_\lambda(x)$ can be expressed as a linear combination of $P_\mu(x;t)$,

$$s_\lambda(x) = \sum_{\mu\in\mathcal{P}_n} K_{\lambda,\mu}(t)P_\mu(x;t). \tag{3.3}$$

It is known that $K_{\lambda,\mu}(t)\in\mathbf{Z}[t]$ and that $K_{\lambda,\mu}(t) = 0$ unless $K_{\lambda,\lambda}(t) = 1$, $\lambda\ge\mu$. $K_{\lambda,\mu}(t)$ are called the **Kostka polynomials**.

We express $\lambda\in\mathcal{P}_n$ as $\lambda = (\lambda_1\ge\cdots\ge\lambda_r > 0)$ (we only consider λ_i such that $\lambda_i > 0$), and define $n(\lambda)\in\mathbf{Z}_{\ge 0}$ by

$$n(\lambda) = \sum_{i=1}^{r}(i-1)\lambda_i. \tag{3.4}$$

We note that $n(\mu)\le n(\lambda)$ if $\lambda\le\mu$. Put $\widetilde{K}_{\lambda,\mu}(t) = t^{n(\mu)}K_{\lambda,\mu}(t^{-1})$. Since $\deg K_{\lambda,\mu}(t) = n(\mu)-n(\lambda)$, we have $\widetilde{K}_{\lambda,\mu}(t)\in\mathbf{Z}[t]$. $\widetilde{K}_{\lambda,\mu}(t)$ are called the **modified Kostka polynomials**. After replacing $s_\lambda(x)$ by $p_\rho(x)$ in (3.3), by putting

$$p_\rho(x) = \sum_{\mu\in\mathcal{P}_n} X_{\rho,\mu}(t)P_\mu(x;t), \tag{3.5}$$

we obtain $X_{\rho,\mu}(t) = \sum_{\lambda\in\mathcal{P}_n}\chi^\lambda(w_\rho)K_{\lambda,\mu}(t)$ by the Frobenius formula (3.1) and (3.3). If we put $Q^\mu_\rho(t) = t^{n(\mu)}X_{\rho,\mu}(t^{-1})$, we have $Q^\mu_\rho(t)\in\mathbf{Z}[t]$. These $Q^\mu_\rho(t)$ are the **Green polynomials** (= Green functions) introduced by Green. Let us derive the orthogonality relation which is one of the main properties of Green functions. For infinitely many variables $x_1, x_2, \dots, y_1, y_2, \dots$, we consider the infinite product

$$\Omega(x,y;t) = \prod_{i,j}(1-tx_iy_j)/(1-x_iy_j). \tag{3.6}$$

$\Omega(x,y;t)$ are called the **Cauchy's reproducing kernel**. The importance of the Cauchy's reproducing kernel relies on the following expansion formula.

$$\Omega(x,y;t) = \sum_\lambda z_\lambda(t)^{-1}p_\lambda(x)p_\lambda(y) = \sum_\lambda b_\lambda(t)P_\lambda(x;t)P_\lambda(y;t), \tag{3.7}$$

where λ runs over partitions of any size. Moreover, if we put $z_\lambda = |Z_{S_n}(w_\lambda)|$ (the order of the centralizer of w_λ in S_n) for $|\lambda| = n$, $z_\lambda(t)$ is given by

$$z_\lambda(t) = z_\lambda\prod_{i\ge 1}(1-t^{\lambda_i})^{-1}. \tag{3.8}$$

We omit the explanation for $b_\lambda(t)\in\mathbf{Z}[t]$ though they can be written explicitly.

Now by comparing the terms of degree $2n$ with respect to x, y in (3.7), we have

$$\sum_{|\rho|=n} z_\rho(t)^{-1} p_\rho(x) p_\rho(y) = \sum_{|\lambda|=n} b_\lambda(t) P_\lambda(x;t) P_\lambda(y;t).$$

Then by substituting (3.5) into p_ρ, and by comparing the terms $P_\mu(x;t)P_\mu(y;t)$, we obtain

$$ {}^tX(t) Z(t)^{-1} X(t) = B(t), \tag{3.9}$$

where $X(t) = (X_{\rho,\mu}(t))$ denotes a matrix of degree $|\mathcal{P}_n|$ with entries $X_{\rho,\mu}(t)$, and ${}^tX(t)$ is its transposed matrix, $Z(t), B(t)$ denote the diagonal matrices with $\lambda\lambda$-entries $z_\lambda(t), b_\lambda(t)$, respectively. Since $X(t), Z(t), B(t)$ are non-singular, by taking the inverse on both sides, we have

$$X(t) B(t)^{-1} {}^tX(t) = Z(t). \tag{3.10}$$

The formula obtained by writing down (3.9) and (3.10) in an entry-wise, and by replacing them by $Q_{\lambda,\mu}(t)$, is nothing but the orthogonality relations for Green functions. For example, (3.10) implies that

$$\sum_{|\lambda|=n} a_\lambda(t)^{-1} Q^\lambda_\rho(t) Q^\lambda_\mu(t) = \delta_{\rho,\mu} y_\rho(t), \tag{3.11}$$

where $a_\lambda(t) = t^{|\lambda|+2n(\lambda)} b_\lambda(t^{-1})$ and $y_\rho(t) = t^{-|\rho|} z_\rho(t^{-1})$. We also note that (3.9) can be modified to the formula

$$\widetilde{K}(t) B'(t)\, {}^t\widetilde{K}(t) = Z'(t) \tag{3.12}$$

by changing $X(t)$ to the form $\widetilde{K}(t) = (\widetilde{K}_{\lambda,\mu}(t))$ by making use of the character table of S_n. If we choose a total order on $\mathcal{P}_n$ compatible with the partial order $\lambda \le \mu$, the corresponding matrices turn out to be the following shape; $\widetilde{K}(t)$ is lower triangular with $\lambda\lambda$-entry $t^{n(\lambda)}$, $B'(t)$ is diagonal, and $Z'(t)$ is the matrix obtained from $Z(t)$ by modifying it by the character table of S_n.

Before closing this section, we give the relationship with the representation theory. Let $B(\mathbf{F}_q)$ be the subgroup of $GL_n(\mathbf{F}_q)$ consisting of upper triangular matrices. The irreducible representations of $GL_n(\mathbf{F}_q)$ appearing in the decomposition of the induced representation $\mathrm{Ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)} 1$ from the unit representation of $B(\mathbf{F}_q)$ are in one to one correspondence with the irreducible characters of S_n. We denote by ρ_λ the irreducible character of $GL_n(\mathbf{F}_q)$ corresponding to χ^λ. The unipotent classes in $GL_n(\mathbf{F}_q)$ also correspond bijectively to $\mathcal{P}_n$, and we denote by u_μ a representative of the unipotent class corresponding to $\mu \in \mathcal{P}_n$. Then we have the following formula

$$\rho_\lambda(u_\mu) = \widetilde{K}_{\lambda,\mu}(q). \tag{3.13}$$

In other words, modified Kostka functions can describe the values of irreducible characters at unipotent classes.

Remark 3.1. Schur functions, power sum symmetric functions and Hall-Littlewood functions are all parametrized by the set $\mathcal{P}_n$ of partitions of n. However, the set $\mathcal{P}_n$ has a different meaning in each case, and accordingly, the meaning of the related symmetric functions is also different. As the Frobenius formula shows, Schur functions reflect the property of irreducible characters of S_n, the power sum symmetric functions reflect that of conjugacy classes of S_n. As (3.3) and (3.13) suggest, it would be natural to think that Hall-Littlewood functions reflect the unipotent classes of $GL_n(\mathbf{F}_q)$. This kind of characterization becomes important when later we consider the symmetric functions related to other classical groups.

§4. Green functions of finite Chevalley groups

In this section, we explain the Green functions of finite Chevalley groups and the algorithm of computing them. I have written a survey article [S4] before on a similar theme. Since some explanations are omitted in this section, the readers who are interested in more details are requested to refer to [S4].

Let G be a connected reductive algebraic group defined over a finite field $\mathbf{F}_q$, $F: G \to G$ the corresponding Frobenius map. The finite group $G(\mathbf{F}_q)$ consisting of $\mathbf{F}_q$-rational points of G coincides with the subgroup G^F of F-fixed points of G. Let $B_0 \supset T_0$ be a pair of an F-stable Borel subgroup and an F-stable maximal torus of G. Such a pair is determined uniquely up to G^F-conjugate. Let $W = N_G(T_0)/T_0$ be the Weyl group of G. F acts naturally on W. In the following, we consider the case where G^F is a finite Chevalley group, i.e., F acts trivially on W. For making use of l-adic cohomology theory, we assume that all the representations are over $\bar{\mathbf{Q}}_l$. For a pair (T, θ), where T is an F-stable maximal torus of G and θ is a homomorphism $T^F \to \bar{\mathbf{Q}}_l^*$, Deligne-Lusztig's virtual representation $R_T^G(\theta)$ is defined. Put $\mathrm{Tr}\,(u, R_T^G(\theta)) = Q_T^G(u)$ for a unipotent element u in G^F. Q_T^G is a G^F-invariant function on the set G^F_{uni} of unipotent elements in G^F, which is called the **Green function** of G^F. Q_T^G does not depend on the choice of θ. Moreover, $Q_T^G = Q_{T'}^G$ if T and T' are G^F-conjugate. The following character formula reduces the computation of characters of $R_T^G(\theta)$ to that of reductive subgroups $Z_G^0(s)$.

(Character formula) Let $g = su = us \in G^F$, where $s, u \in G^F$ are semisimple and unipotent element, respectively. Then

$$\mathrm{Tr}\,(g, R_T^G(\theta)) = |Z_G^0(s)^F|^{-1} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} Q_{xTx^{-1}}^{Z_G^0(s)}(u)\theta(x^{-1}sx). \tag{4.1}$$

The inner product of $R_T^G(\theta)$'s can be expressed in terms of a simple formula (orthogonality relations). By making use of it, one can see that $\pm R_T^G(\theta)$ turns out to be irreducible for many θ. Moreover, an irreducible representation of G^F appears in the decomposition of $R_T^G(\theta)$ for some pair (T, θ). It follows that one can divide the set of irreducible characters of G^F into several small classes by making use of $R_T^G(\theta)$. The irreducible characters having non-zero inner product with $R_T^G(1)$ (the case where $\theta = 1$) are called **unipotent characters**. The set of unipotent characters forms a single class, and it plays, in some sense, a similar role as unipotent classes among the set of conjugacy classes.

Now, by combining the orthogonality relations of $R_T^G(\theta)$ with the character formula, one obtains the following orthogonality relations for Green functions.

(Orthogonality relations for Green functions)
(4.2)

$$|G^F|^{-1}\sum_{u\in G^F_{\mathrm{uni}}} Q_T^G(u)Q_{T'}^G(u) = \begin{cases} |W(T)^F|/|T^F| & \text{if } T = T', \\ 0 & \text{if } T \text{ and } T' \text{ are not } G^F\text{-conjugate,}\end{cases}$$

where $W(T) = N_G(T)/T$.

In the case where G^F is a finite Chevalley group, the G^F-conjugacy classes of F-stable maximal tori are in one to one correspondence with the set of conjugacy classes of W. For each $w \in W$, we denote by T_w the corresponding F-stable maximal torus (determined up to G^F-conjugacy). Note that $T_w = T_0$ if $w = 1$. Let $W^\wedge$ be the set of irreducible characters of W. For each $\chi \in W^\wedge$, we define a function Q_χ on G^F_{uni} by

$$Q_\chi = |W|^{-1}\sum_{w\in W}\chi(w)Q^G_{T_w}.$$

The determination of Q_{T_w} for all $w \in W$ is equivalent to that of Q_χ for all $\chi \in W^\wedge$.

Remark 4.1. In the case where $G^F = GL_n(\mathbf{F}_q)$, Green functions, etc. defined here coincide with those obtained from Green polynomials, modified Kostka polynomials given in Section 3, by substituting $t = q$. Namely, we have

$$Q^G_{T_{w_\rho}}(u_\mu) = Q^\mu_\rho(q), \qquad Q_{\chi^\lambda}(u_\mu) = \widetilde{K}_{\lambda,\mu}(q).$$

Moreover, the orthogonality relations of Green polynomials (3.11) is nothing but the formula (4.2).

As mentioned in Section 1, in order to compute the Green functions it was necessary to obtain the geometric interpretation in terms of the intersection cohomology theory. Let I be the set of pairs $(C,\mathcal{E})$ of unipotent classes C of G and G-equivariant simple local systems $\mathcal{E}$ on C. The (degree shift of) intersection cohomology complex $K = \mathrm{IC}(\bar{C},\mathcal{E})[\dim C - 2d]$ is defined for $(C,\mathcal{E})$, where $d = \dim G/B$. K is a complex of l-adic sheaves on $\bar{C}$ (the closure of C in G), and satisfies the property $F^*K \simeq K$. The characteristic function $\chi_{K,\varphi} : \bar{C}^F \to \bar{\mathbf{Q}}_l$ of K is determined by fixing an isomorphism $\varphi : F^*K \to K$. We regard $\chi_{K,\varphi}$ as a function on G^F_{uni} by taking the value 0 outside of $\bar{C}^F$. By combining the result of Lusztig [L4] with Borho-MacPherson [BM], the following holds.

Theorem 4.2. (i) *The function Q_χ can be expressed as $Q_\chi = \chi_{K,\varphi}$ by some $(C,\mathcal{E}) \in I$ (by choosing the isomorphism $\varphi : F^*K \to K$ appropriately). In particular, Q_χ is a function vanishing outside of $\bar{C}^F$.*
(ii) *The map $f : \chi \mapsto (C,\mathcal{E})$ gives an injection from $W^\wedge$ to I.*

The map f given in (ii) is called the **Springer correspondence** of Weyl groups. In the case of GL_n, the local system on C is always the constant sheaf, and the Springer correspondence f is given by the bijection $\chi^\lambda \mapsto (C_\lambda, \bar{\mathbf{Q}}_l)$. In the case of other groups, f is not surjective, and the correspondence takes a more complicated pattern. Currently the Springer correspondence is explicitly determined for all groups.

G-equivariant simple local systems on C are in one to one correspondence with irreducible characters of $A_G(u) = Z_G(u)/Z_G^0(u)$, if we fix a representative $u \in C^F$ of the unipotent class C. Moreover, the G^F-conjugacy classes in C^F are in one to one correspondence with the conjugacy classes in $A_G(u)$. We denote by u_a a representative of the G^F-conjugacy class in C^F corresponding to $a \in A_G(u)$. Let $\mathcal{U}$ be the linear space over $\bar{\mathbf{Q}}_l$ consisting of G^F-invariant functions on G^F_{uni}. For $i = (C, \mathcal{E}) \in I$ we define $Y_i \in \mathcal{U}$ as follows; Let ρ be the irreducible character of $A_G(u)$ corresponding to $\mathcal{E}$. Put

$$Y_i(g) = \begin{cases} \rho(a) & \text{if } g = u_a, \\ 0 & \text{if } g \notin C^F. \end{cases} \tag{4.3}$$

Then $\{Y_i \mid i \in I\}$ turns out to be a basis of $\mathcal{U}$. Put $I_0 = f(W^\wedge)$. If we write $f(\chi) = i \in I_0$, it is known that Q_χ can be expressed as

$$Q_\chi = \sum_{j \in I_0} p_{ij} Y_j. \tag{4.4}$$

(In fact, since $Q_\chi \in \mathcal{U}$ and $\{Y_j\}$ is a basis of $\mathcal{U}$, it is natural that Q_χ can be expressed as a linear combination of Y_j ($j \in I$). However, in order to show $j \in I_0$, one needs a deep result on the classification of character sheaves.) Now let $P = (p_{ij})$ be the matrix of $\bar{\mathbf{Q}}_l$-coefficients with the index set I_0. It is enough to determine the matrix P for determining the Green functions. In what follows we shall explain an algorithm of determining P. First, we introduce the following scalar product $\langle\ ,\ \rangle$ on $\mathcal{U}$. For $f, h \in \mathcal{U}$, put

$$\langle f, h\rangle = \sum_{u \in G^F_{\text{uni}}} f(u)h(u).$$

Then the orthogonality relations of Green functions (4.2) can be expressed in the form that " $\langle Q^G_{T_w}, Q^G_{T_{w'}}\rangle =$ known formula ". By replacing $Q^G_{T_w}$ by Q_χ, this formula is modified to the form

$$\langle Q_\chi, Q_{\chi'}\rangle = \omega_{\chi,\chi'}. \tag{4.5}$$

Here $\omega_{\chi,\chi'}$ are of course known formulas. By putting $i = f(\chi), j = f(\chi')$ and by identifying $W^\wedge$ with I_0 via f, $\Omega = (\omega_{\chi,\chi'}) = (\omega_{ij})$ is regarded as a matrix on I_0. Rewriting Q_χ as a linear combination of Y_j by (4.4), the formula (4.5) is modified to the form of a matrix equation

$$P \varLambda\, {}^tP = \Omega, \tag{*}$$

where $\varLambda$ is a matrix indexed by I_0 whose ij-entry is given by $\langle Y_i, Y_j\rangle$. We now define an equivalence relation on I_0 as follows; For $i = (C, \mathcal{E}), j = (C', \mathcal{E}')$, $i \sim j \Leftrightarrow C = C'$. Moreover we choose a total order on I_0 so that each equivalence class form an interval, and that the induced order on the set of equivalence classes is compatible with the partial order on the set of unipotent classes obtained from the closure relations of them, namely, $i \le j$ if $\bar{C} \subseteq \bar{C}'$. We consider the matrices $P, \varLambda, \Omega$ according to this order, and regard them as block matrices with respect to the above equivalence relation. Then by Theorem 4.2 (i) and (4.4), P turns out to be lower triangular as a block matrix. Moreover, by the property of the intersection cohomology complex, the diagonal block (minor matrix) corresponding to C is a scalar matrix $q^{d_u} I_C$, where $d_u = \dim \mathcal{B}_u$ if we set $\mathcal{B}_u = \{gB \in G/B \mid g^{-1}ug \in B\}$ for $u \in C$. On the other hand, as is easily seen from the definition, $\varLambda$ is a block-wise diagonal matrix.

We now regard the equation (*) as a matrix equation with known matrix Ω and unknown matrices $P, \varLambda$. Then it is easily seen, by an elementary linear algebra, that $P, \varLambda$ are uniquely determined by the conditions on $P, \varLambda$ mentioned above. At the same time, it also gives an algorithm of determining $P, \varLambda$. Thus we see that Green functions can be computed.

Remark 4.3. Since Green functions are determined, one can compute the characters of $R_T^G(\theta)$. In general, many irreducible characters of G^F can be expressed, in an explicit way, as a linear combination of $R_T^G(\theta)$. As mentioned in Section 1, all the irreducible characters of $GL_n(\mathbf{F}_q)$ are obtained in this way. But except the case of $GL_n(\mathbf{F}_q)$, there exist a lot of irreducible characters which can not be expressed as a linear combination of $R_T^G(\theta)$. In these circumstances, extending the argument in Theorem 4.2 based on the theory of perverse sheaves, Lusztig constructed certain class functions of G^F, starting from the pair $(C, \mathcal{E})$ of a conjugacy class C of G and a local system $\mathcal{E}$ on C. He showed that the thus obtained functions form an orthonormal basis of the space of class functions of G^F, and that those functions are computable by a similar method as the algorithm of computing Green functions. This is the theory of character sheaves by Lusztig ([L3]). Lusztig also proposed a conjecture describing the decomposition of such class functions into irreducible characters in an explicit way (Lusztig's conjecture). If Lusztig's conjecture is completely solved, it provides us a unified algorithm of computing irreducible characters of G^F. In the case where the center of G is connected, Lusztig's conjecture is solved (up to the scalar multiplication) by the author ([S3]).

§5. Green functions associated to $G(e, 1, n)$

Lusztig [L2] introduced a notion of unipotent symbols to describe the Springer correspondence $f : W^\wedge \to I_0$ in the case of classical groups. The data which are needed to write down the matrix equation (*), such as the equivalence classes of I_0 corresponding to unipotent classes, the total order, and the diagonal block $q^{d_u} I_C$ of P, etc. are all described by the combinatorial property of unipotent symbols. In this section, we consider the complex reflection group $G(e, 1, n)$ as a generalization of Weyl groups of type B_n. As mentioned in Section 2, we introduce e-symbols as a generalization of unipotent symbols, and construct a formula analogous to (*) by making use of their combinatorial properties. From now on let $W = G(e, 1, n)$. We have $W \simeq S_n \ltimes (\mathbf{Z}/e\mathbf{Z})^n$. Let $\mathcal{P}_{n,e}$ be the set of e-tuples of partitions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ such that $\sum_{k=0}^{e-1} |\alpha^{(k)}| = n$. $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ is called an e-partition of size n. It is known that irreducible characters of W are parametrized by $\mathcal{P}_{n,e}$. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character of W corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. In what follows, we fix a tuple of positive integers $\boldsymbol{m} = (m_0, \dots, m_{e-1})$, write the partition $\alpha^{(k)}$ as $\alpha_1^{(k)} \ge \dots \ge \alpha_{m_k}^{(k)} \ge 0$ by m_k integers, and express the e-partition $\boldsymbol{\alpha}$ in a matrix from as $\boldsymbol{\alpha} = (\alpha_j^{(k)})$. We denote by

$Z_n^{0,0} = Z_n^{0,0}(\boldsymbol{m})$ the set of thus expressed $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. We fix integers $r \ge s \ge 0$ and define $\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\boldsymbol{m}) = (\Lambda^{(0)}, \dots, \Lambda^{(e-1)})$ by

$$\begin{aligned}
&\Lambda^{(0)} : (m_0 - 1)r \ge \cdots \ge 2r \ge r \ge 0, \\
&\Lambda^{(k)} : (m_k - 1)r + s \ge \cdots \ge 2r + s \ge r + s \ge s, \quad (k = 1, \dots, e-1).
\end{aligned}$$

Put $Z_n^{r,s} = Z_n^{0,0} + \boldsymbol{\Lambda}^0$ (entry-wise sum). An element in $Z_n^{r,s} = Z_n^{r,s}(\boldsymbol{m})$ is called an e-symbol of type (r, s). We have $Z_n^{0,0} \simeq Z_n^{r,s}$ by the map $\boldsymbol{\alpha} \mapsto \boldsymbol{\Lambda}(\boldsymbol{\alpha}) = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$.

We give an example of e-symbols. Let $W = G(2, 1, 2)$, i.e., the Weyl group of type B_2. Hence $e = 2$. If $m_0 = 3, m_1 = 2$, we have

$$Z_n^{0,0} = \Big\{ \begin{pmatrix} 000 \\ 11 \end{pmatrix}, \begin{pmatrix} 110 \\ 00 \end{pmatrix}, \begin{pmatrix} 100 \\ 10 \end{pmatrix}, \begin{pmatrix} 000 \\ 20 \end{pmatrix}, \begin{pmatrix} 200 \\ 00 \end{pmatrix} \Big\}. \tag{5.1}$$

Then in the case where $r = 2, s = 1$, we have $\boldsymbol{\Lambda}^0 = \begin{pmatrix} 420 \\ 31 \end{pmatrix}$ and

$$Z_n^{r,s} = \Big\{ \begin{pmatrix} 420 \\ 42 \end{pmatrix}, \begin{pmatrix} 530 \\ 31 \end{pmatrix}, \begin{pmatrix} 520 \\ 41 \end{pmatrix}, \begin{pmatrix} 420 \\ 51 \end{pmatrix}, \begin{pmatrix} 620 \\ 31 \end{pmatrix} \Big\}. \tag{5.2}$$

We now introduce some notion concerning $Z_n^{r,s}$. If we put $\boldsymbol{m}' = (m_0+1, \dots, m_{e-1}+1)$, we obtain a map $\boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\boldsymbol{m}) \mapsto \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\boldsymbol{m}')$. This is called a shift operation of e-symbols. For example, $\begin{pmatrix} 31 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 530 \\ 31 \end{pmatrix}$ is a shift operation in connection with the above example.

Next, we define an equivalence relation $\sim$ on $Z_n^{r,s}$ by requiring that $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$ if $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}' \in Z_n^{r,s}$ have the same entries with multiplicities. The equivalence class with respect to $\sim$ is called a family. Moreover, we define a function $a : Z_n^{r,s} \to Z_{\ge 0}$ by

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda, \lambda' \in \boldsymbol{\Lambda}} \min\{\lambda, \lambda'\} - \sum_{\mu, \mu' \in \boldsymbol{\Lambda}^0} \min\{\mu, \mu'\}.$$

In other words, if we denote by ν, ν^0 the partitions obtained from $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^0$ by rearranging their entries in the decreasing order, $a(\boldsymbol{\Lambda})$ is given by $a(\boldsymbol{\Lambda}) = n(\nu) - n(\nu^0)$ in terms of the function $n(\lambda)$ in (3.4). By definition, $a(\boldsymbol{\Lambda})$ takes a constant value on each family. We also note that the families, $a(\boldsymbol{\Lambda})$ are invariant under the shift operations. Under the identification $\mathcal{P}_{n,e} \simeq Z_n^{0,0} \simeq Z_n^{r,s}$, we use freely the terminologies such as families, a-functions for $\mathcal{P}_{n,e}, Z_n^{0,0}$.

In the previous example, $\Big\{ \begin{pmatrix} 520 \\ 41 \end{pmatrix}, \begin{pmatrix} 420 \\ 51 \end{pmatrix} \Big\}$ forms a family consisting of two elements, and others are all 1-element families. $a(\boldsymbol{\Lambda})$ are given in (5.2) as 4, 2, 1, 1, 0 along the order from the left.

We now define the following total order $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ on the set $\mathcal{P}_{n,e} \simeq Z_n^{r,s}$ by making use of families. First, define a total order on the set $\{\mathcal{F}_i\}$ of families satisfying the property $\mathcal{F}_1 > \mathcal{F}_2$ if $a(\mathcal{F}_1) < a(\mathcal{F}_2)$, and then extend it to $\mathcal{P}_{n,e}$ compatible with it (i.e., so that each family forms an interval).

Up to now, we didn't touch with the explicit form of the matrix Ω. Here we define Ω for each W. W acts on $V = \mathbf{C}^n$ as the reflection representation. Let $S(V)$ be the symmetric algebra of V, and $R = S(V)/I$ the coinvariant algebra of W, where I is the ideal of $S(V)$ generated by W-invariant homogeneous vectors (of degree > 0). The coinvariant algebra R has a structure of the graded ring $R = \bigoplus_{i\ge 0} R_i$. Moreover, W acts on R, leaving each R_i invariant. The W-module R is isomorphic to the regular representation of W. In particular, $\dim R = |W|$. We define, for a class function f on W, a polynomial $R(f)$ with a variable t by

$$R(f) = \sum_{i\ge 0}\langle f, R_i\rangle_W\, t^i.$$

(Here $\langle\ ,\ \rangle_W$ denotes the inner product on class functions of W). If f is a generalized character of W, we have $R(f) \in \mathbf{Z}[t]$. It is known that $R(f)$ has the following expression.

$$R(f) = (t-1)^n P_W(t)\frac{1}{|W|}\sum_{w\in W}\frac{\det_V(w)f(w)}{\det_V(t\cdot \mathrm{id}_V - w)}, \tag{5.3}$$

where $P_W(t) = \sum_{i\ge 0}\dim R_i t^i$ is the Poincaré polynomial of W, and in this case it is given explicitly as $P_W(t) = \prod_{i=1}^n (t^{ei}-1)/(t-1)$. Moreover, $\det_V(x)$ denotes the determinant of the linear transformation x on V.

For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$, put

$$\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*}R(\chi^{\boldsymbol{\alpha}}\otimes\overline{\chi^{\boldsymbol{\beta}}}\otimes\overline{\det}_V),$$

where $\overline{\chi^{\boldsymbol{\beta}}}, \overline{\det}_V$ denote the complex conjugate of the characters $\chi^{\boldsymbol{\beta}}, \det_V$, respectively. N^* is the number of complex reflections in W. Then matrix $\Omega = (\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is the one we wanted.

Under these preparations, the matrix equation (*) is generalized in the following way. We regard the matrix with the index set $\mathcal{P}_{n,e}$ (with respect to the order defined before) as a block matrix via the partition corresponding to families. We assume that the block matrices $P' = (p'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, $P'' = (p''_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, $\varLambda = (\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ satisfy the following conditions.

$$P'\varLambda\,{}^tP'' = \Omega. \tag{**}$$

Here P', P'' are lower triangular as block matrices, where the diagonal block corresponding to the family $\mathcal{F}$ is given as a scalar matrix $t^{a(\boldsymbol{\alpha})}I_{\mathcal{F}}$ by $\boldsymbol{\alpha}\in\mathcal{F}$. Moreover, $\varLambda$ is a block-wise diagonal matrix.

It can be verified, by following the argument in [GM], that there exist unique matrices $P', P'', \varLambda$ satisfying the equation (**), and that the coefficients of the matrices are contained in $\mathbf{Q}(t)$. Therefore it would be natural to call these matrices P', P'' as the **Green functions** associated to W.

Remarks 5.1. (i) Let σ be the permutation matrix of $\mathcal{P}_{n,e} \simeq W^\wedge$ obtained by the complex conjugate $\chi^{\boldsymbol{\alpha}} \leftrightarrow \overline{\chi^{\boldsymbol{\alpha}}}$. Then we have $P'' = \sigma P'\sigma^{-1}$. Hence, essentially we have only to consider P'. We also note that if $e = 2$, then $\sigma = I$ since irreducible characters take integral values. It follows that $P' = P''$.

(ii) In the case where $e = 2$, e-symbols $Z_n^{r,s}$ give unipotent symbols corresponding to each classical group G, by choosing r, s accordingly. In fact, $Z_n^{2,1}$ (resp. $Z_n^{2,0}$, $Z_n^{4,2}$) is nothing but the unipotent symbol of Sp_{2n} ($p \neq 2$) (resp. SO_{2n+1} ($p \neq 2$), Sp_{2n} ($p = 2$)), respectively. In these cases, the equation (**) coincides with the equation (*) determining the original Green functions. In particular, for $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, we have $a(\boldsymbol{\alpha}) = d_u$ if u is a representative of the corresponding unipotent class C.

(iii) Geck-Malle [GM] formulated a similar equation as (*) for unipotent characters of finite Chevalley groups and their families, and conjectured that the coefficients of the matrix $\varLambda$ produce polynomials giving the number of rational points $\widehat{C}(\mathbf{F}_q)$ of the special piece $\widehat{C}$ of the unipotent class C. They verify that the conjecture holds for the case of exceptional groups, by using the computer. A similar computation works even for the non-crystallographic Coxeter group H_3, H_4 or certain complex reflection groups, and one obtains certain reasonable polynomials as coefficients of $\varLambda$ (though of course, unipotent classes and special pieces do not make sense in these cases). It is expected that these polynomials coincide with the polynomials, whose existence was conjectured in [L6], expressing the number of points of the virtual $\widehat{C}(\mathbf{F}_q)$ in the case of H_3, H_4.

On the other hand, in the case of classical groups Lusztig introduced a notion of symbols in order to describe unipotent characters. (Chronologically, this is earlier than the case of unipotent symbols). $Z_n^{1,0}$ is nothing but the set of symbols, and the equation (**) in that case coincides with the equation considered by Geck-Malle (by choosing a total order on $\mathcal{P}_{n,e}$ appropriately). In this situation, $a(\boldsymbol{\alpha})$ coincides with the value $a(\chi^{\boldsymbol{\alpha}})$ of the original a-function of $\chi^{\boldsymbol{\alpha}} \in W^\wedge$.

§6. A combinatorial construction of Green functions

In this section, we try to construct Green functions associated to W in a combinatorial way, following the case of GL_n. Let us introduce variables $x = (x_j^{(k)})$ corresponding to $\boldsymbol{\alpha} = (\alpha_j^{(k)}) \in Z_n^{0,0}(\boldsymbol{m})$. For a fixed k, we denote by $x^{(k)}$ the variables $x_1^{(k)}, \dots, x_{m_k}^{(k)}$. We fix a primitive e-th root of unity ζ. Following [M], we introduce the Schur function $s_{\boldsymbol{\alpha}}(x)$ and the power sum symmetric function $p_{\boldsymbol{\alpha}}(x)$ with respect to $\boldsymbol{\alpha} \in Z_n^{0,0}$. As for the Schur function, put

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}),$$

where $s_{\alpha^{(k)}}$ is the original Schur function (§1) with respect to the partition $\alpha^{(k)}$. Next, for integers $1 \le r, 0 \le k \le e-1$, put

$$p_r^{(k)}(x) = \sum_{j=0}^{e-1} \zeta^{-kj} p_r(x^{(j)}),$$

where p_r is the one defined in §1), and put $p_0^{(k)}(x) = 1$. For $\boldsymbol{\alpha} = (\alpha_j^{(k)})$, we define $p_{\boldsymbol{\alpha}}(x)$ by the following formula.

$$p_{\boldsymbol{\alpha}}(x) = \prod_{j,k} p^{(k)}_{\alpha_j^{(k)}}(x).$$

Let $\Xi_{\boldsymbol{m}} = \bigotimes_{k=0}^{e-1}\mathbf{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{S_{m_k}}$ be the ring of symmetric polynomials with respect to $S_{\boldsymbol{m}} = S_{m_0} \times \cdots \times S_{m_{e-1}}$. $\Xi_{\boldsymbol{m}}$ is a graded ring and is decomposed as $\Xi_{\boldsymbol{m}} = \bigoplus_{n\ge 0}\Xi^n_{\boldsymbol{m}}$. If we put $\boldsymbol{m}' = (m_0+1, \dots, m_{e-1}+1)$, a homomorphism $\Xi^n_{\boldsymbol{m}'} \to \Xi^n_{\boldsymbol{m}}$ is defined by $x^{(k)}_{m_k+1} \mapsto 0, x_i^{(k)} \mapsto x_i^{(k)} (i \le m_k)$, and $\{\Xi^n_{\boldsymbol{m}}\}_{\boldsymbol{m}}$ turns out to be a projective system. We denote by Ξ^n the projective limit $\varprojlim \Xi^n_{\boldsymbol{m}}$, and define a ring $\Xi = \bigoplus_{n\ge 0}\Xi^n$ of symmetric functions. $\{s_{\boldsymbol{\alpha}}(x) \mid \boldsymbol{\alpha} \in \mathcal{P}_{n,e}\}$ is a $\mathbf{Z}$-basis of Ξ^n, and $\{p_{\boldsymbol{\alpha}}(x) \mid \boldsymbol{\alpha} \in \mathcal{P}_{n,e}\}$ is a basis of $\mathbf{C} \otimes \Xi^n$.

Under this setting, the **Frobenius formula** holds as in the case of S_n. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$,

$$p_{\boldsymbol{\beta}}(x) = \sum_{\boldsymbol{\alpha}\in\mathcal{P}_{n,e}} \chi^{\boldsymbol{\alpha}}(w_{\boldsymbol{\beta}})s_{\boldsymbol{\alpha}}(x), \tag{6.1}$$

where $w_{\boldsymbol{\beta}}$ denotes a representative of the conjugacy class in W corresponding to $\boldsymbol{\beta}$.

Next for $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, put

$$z_{\boldsymbol{\alpha}}(t) = z_{\boldsymbol{\alpha}} \prod_{\substack{k,j\\ \alpha_j^{(k)}\ne 0}} (1 - \zeta^k t^{\alpha_j^{(k)}})^{-1}, \tag{6.2}$$

where $z_{\boldsymbol{\alpha}} = |Z_W(w_{\boldsymbol{\alpha}})|$. We remark that the formula (6.2) gives a link between the symmetric functions and the matrix Ω. In fact, the formula appearing in the denominator in (5.3) is related to (6.2) in the following sense.

$$\det\nolimits_V(t\cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}}) = \prod_{\substack{j,k\\ \alpha_j^{(k)}\ne 0}} (t^{\alpha_j^{(k)}} - \zeta^k).$$

We now define the **Cauchy's reproducing kernel** $\Omega(x, y; t)$ with respect to W as follows. For $k = 0, \dots, e-1$, we prepare infinitely many variables $x_1^{(k)}, x_2^{(k)}, \dots, y_1^{(k)}, y_2^{(k)}, \dots$. For $x = (x_j^{(k)}), y = (y_j^{(k)})$, put

$$\Omega(x, y; t) = \prod_{k=0}^{e-1}\prod_{i,j}(1 - tx_i^{(k+1)}y_j^{(k)})/(1 - x_i^{(k)}y_j^{(k)}).$$

We note that the formula of the right hand side can not be separated to the product form, contrast to the case of Schur functions, since the variables $x^{(k+1)}$ and $y^{(k)}$ are contained there in a mixed way. As in the case of GL_n, $\Omega(x, y; t)$ can be expanded in terms of $p_{\boldsymbol{\alpha}}(x)$ as follows.

$$\Omega(x,y;t) = \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}(t)^{-1} p_{\boldsymbol{\alpha}}(x)\overline{p_{\boldsymbol{\alpha}}(y)}, \tag{6.3}$$

where $\boldsymbol{\alpha}$ runs over the e-partitions of any size. $\overline{p_{\boldsymbol{\alpha}}(y)}$ is the complex conjugate of $p_{\boldsymbol{\alpha}}(y)$.

Next, in oder to define the Hall-Littlewood functions we introduce a sesquilinear form $\langle\, , \,\rangle$ on $\mathbf{C}(t)\otimes_{\mathbf{Z}} \Xi^n$ by

$$\langle p_{\boldsymbol{\alpha}}, p_{\boldsymbol{\beta}}\rangle = z_{\boldsymbol{\alpha}}(t)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

In the following, we assume that the order $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ defined in Section 5 is given on $\mathcal{P}_{n,e} \simeq Z_n^{r,s}$. The following proposition holds.

Proposition 6.1. *For each $\boldsymbol{\alpha} \in Z_n^{0,0}$, $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z_n^{r,s}$, there exist unique functions (one for each sign $\pm$) $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ satisfying the following properties.*

(i) *$P^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed by Schur functions $s_{\boldsymbol{\beta}}$ as,*

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}\in Z_n^{0,0}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)s_{\boldsymbol{\beta}},$$

where $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbf{Q}(t)$, and $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $\boldsymbol{\beta} < \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(ii) *$\langle P^{+}_{\boldsymbol{\Lambda}}, P^{-}_{\boldsymbol{\Lambda}'}\rangle = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$.*

$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are called the **Hall-Littlewood functions** corresponding to the e-symbol $\boldsymbol{\Lambda}$. Note that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are determined, not by e-partitions $\mathcal{P}_{n,e}$, but by e-symbols $Z_n^{r,s}$. Hence several types of Hall-Littlewood functions are constructed from the same W by a choice of r,s. Also the above definition depends on the choice of the order $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ on $\mathcal{P}_{n,e} \simeq Z_n^{r,s}$. However, one may expect that $P^{\pm}_{\boldsymbol{\Lambda}}$ does not depend on the choice of the order, as far as it is defined as in Section 5. Furthermore, though apriori $P^{\pm}_{\boldsymbol{\Lambda}}$ is an element of $\mathbf{Q}(t)\otimes_{\mathbf{Z}} \Xi^n$ from its construction, it is conjectured that $P^{\pm}_{\boldsymbol{\Lambda}} \in \mathbf{Z}[t]\otimes_{\mathbf{Z}} \Xi^n$.

Although we didn't explain in Section 3, $P_\lambda(x;t)$ in the case of GL_n can be characterized in a similar way, in terms of the inner product on $\mathbf{Q}(t)\otimes_{\mathbf{Z}} \Lambda^n$. However, the explicit description of Hall-Littlewood functions as in (3.2) is not known for $P^{\pm}_{\boldsymbol{\Lambda}}$.

As a corollary to Proposition 6.1, the expansion of $\Omega(x,y;t)$ in terms of Hall-Littlewood functions is obtained.

Corollary 6.2.

$$\Omega(x,y;t) = \sum_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t)P^{+}_{\boldsymbol{\Lambda}}(x;t)P^{-}_{\boldsymbol{\Lambda}'}(y;t),$$

where $b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$. $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}'$ run over all the elements in $\bigcup_{n=0}^{\infty} Z_n^{r,s}$.

By Proposition 6.1, we see that $\{P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \mid \boldsymbol{\Lambda} \in Z_n^{r,s}\}$ turns out to be a $\mathbf{Q}(t)$-basis of $\mathbf{Q}(t)\otimes_{\mathbf{Z}} \Xi^n$. Hence, as in the case of GL_n, the **Kostka function** $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is defined by

$$s_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta}\in\mathcal{P}_{n,e}} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}(x;t). \tag{6.4}$$

Then we define the **modified Kostka function** by $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = t^{a(\boldsymbol{\beta})}K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1})$. Let $\widetilde{K}_{\pm}(t) = (\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$ be the corresponding matrix on $\mathcal{P}_{n,e}$. On the other hand, if we consider the power sum symmetric function $p_{\boldsymbol{\alpha}}(x)$ in (6.4) instead of the Schur function $s_{\boldsymbol{\alpha}}(x)$, we obtain $X^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ as the coefficient of $P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}(x;t)$ as in the formula (3.5). If we put $\widetilde{X}_{\pm}(t) = (t^{a(\boldsymbol{\beta})}X^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1}))$, the matrix $\widetilde{X}_{\pm}(t)$ is obtained, by the Frobenius formula, by multiplying to $\widetilde{K}_{\pm}(t)$ the character table of W from the left. It follows from this, by a similar argument as in Section 3, that the formula corresponding to the orthogonality relations for Green functions is obtained. In order to explain this, we prepare some notations. Put $\varLambda(t) = (b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t))$ $(\boldsymbol{\Lambda},\boldsymbol{\Lambda}' \in Z_n^{r,s})$. $\varLambda(t)$ is non-singular by Corollary 6.2, and becomes a block-wise diagonal matrix. Put $\varLambda(t)^{-1} = (b'_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t))$, and we define the matrix $\widetilde{\varLambda}(t)$ by

$$\widetilde{\varLambda}(t) = (d(t)t^{a(\boldsymbol{\Lambda}')-a(\boldsymbol{\Lambda})}b'_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t^{-1})),$$

where $d(t) = t^{N^*-n}(t-1)^nP_W(t)$. We note that $\widetilde{\varLambda}(t)$ is also a block-wise diagonal matrix. By rewriting the above formula concerning Green functions by using $\widetilde{K}_{\pm}(t)$, in a similar way as (3.11) is obtained from (3.10), we obtain the following theorem.

Theorem 6.3. *The matrices $\widetilde{K}_{\pm}(t), \widetilde{\varLambda}(t)$ satisfy the following relation.*

$$\widetilde{K}_{-}(t)\widetilde{\varLambda}(t)\,{}^t\widetilde{K}_{+}(t) = \Omega$$

It follows that $P' = \widetilde{K}_{-}(t), P'' = \widetilde{K}_{+}(t), \varLambda = \widetilde{\varLambda}(t)$ give a solution of (**).

Theorem 6.3 is nothing but the combinatorial construction of Green functions. Concerning Kostka functions, we have the following conjecture.

Conjecture 6.4. $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is a polynomial of degree $\deg K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \le a(\boldsymbol{\beta}) - a(\boldsymbol{\alpha})$. Furthermore the coefficeints are all non-negative integers.

If the conjecture is true, it implies that $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbf{Z}[t], P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbf{Z}[x;t]$. Moreover, it also implies that $\lambda_{ij}(t) \in \mathbf{Z}[t]$ for the matrix $\varLambda = (\lambda_{ij})$. The conjecture holds in the case of original Green functions. In fact, in this case, the coefficient of $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is interpreted as the multiplicity of the irreducible character $\chi^{\boldsymbol{\alpha}}$ of W in a certain cohomology group (the Springer representation of W determined by the unipotent class corresponding to $\boldsymbol{\beta}$), and so a non-negative integer. The basic reason why such an expression is possible relies on the geometric interpretation of Green functions (Threorem 4.2) explained in Section 4. (In the case of type A_n, this is also obtained from a combinatorial description of Kostka functions by Lascoux-Schützenberger ([M, III, 6.5])). For the case of our Kostka functions also, finding a geoemtric interpretation will give a path for the proof of the conjecture.

As a generalization of the group $G(e,1,n)$, a class of complex reflection groups $G(e,p,n)$ can be considered. Here p is a factor of e, and $G(e,p,n)$ is a normal subgroup of $G(e,q,n)$ of index p. In particular, in the case where $e = p = 2$, $G(2,2,n)$ is ismorphic to the Weyl group of type D_n. $G(e,e,n)$ is regarded as a generalization of the Weyl group of D_n, as $G(e,1,n)$ is the generalization of the Weyl group of type B_n. We have $G(e,e,r) \subset G(e,p,n) \subset G(e,1,n)$. Furthermore $G(e,e,2)$ is isomorphic to the dihedral group of order $2e$. All obtained for $G(e,1,n)$ up to now can be generalized to the case of $G(e,p,n)$. As a generalization of unipotent symbols of type D_n, (modified)

e-symbols are defined, and Green functions are defined by making use of the matrix equation analogous to (**). Power sum symmetric funtions, Schur functions and Hall-Littlewood functions are formulated appropriately, and a similar result as Theorem 6.3 holds. In particular, we obtain a combinatorial approach for the Green functions of finite Chevalley groups of type D_n. This gives, at the same time, a construction of Green functions associated to dihedral groups. However, it is an interesting thing (it is a pity ?) that the thus obtained Green functions do not coincide with the original Green functions, in the case of type B_2 or type G_2. Probably, a different type of Hall-Littlewood functions would be required for treating Green functions of dihedral groups as a genralizaiton of Weyl groups.

In any way, the above results suggest, as the author's opinion, the possibility that the combinatorial theory of type A in connection with Green functions can be applied also to the case of exceptional groups, beyond the framework of classical groups. It is known that Shcur functions and Hall-Littlewood functions can be generalized to any type of root systems, by regarding the partitions of n as weights of the Lie algebra $\mathfrak{sl}_n$. However we remark that the content in this note (a generalization to the case of type B_n and D_n) is a generalizaiton to a different direction from that point of view. Although at present I don't have any idea, I think it would not be a mere fantastic story to construct Green functions of type E_8 or type H_3, H_4 in a combinatorial way. So far, we have only discussed the Green functions associated to $G(e,p,n)$. This might be somewhat unfair in view of a general title, "Green functions of complex reflection groups". It is my hope related to this title that I want to discuss such Green functions in general in a future.

The details of Green funtions of $G(e,1,n), G(e,p,n)$ are written in [S5], [S6]. There is a survey on the Green functions of type D_n in [S7]. The reader who is interested in it is refferd to them.

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