Green functions and Macdonald functions associated to complex reflection groups

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1. Introduction

1.1. Green functions are introduced by J.A. Green [G] in 1955, in connection with the representation theory of general linear groups $GL_n(\mathbf{F}_q)$ over a finite field $\mathbf{F}_q$. Green constructed a system of polynomials (now called Green polynomials) $Q^\mu_\nu(t)$ parametrized by two partitions μ, ν of n, based on a combinatorial theory of symmetric functions as follows. There exists two classes of symmetric functions, one is a class of Schur functions $s_\lambda(x)$ with respect to the variables $x : x_1, x_2, \dots, x_m$ for $m \gg n$, and the other is a class of Hall-Littlewood functions $P_\mu(x;t)$, which are symmetric polynomials with respect to x, with a parametere t. Both of Schur functions and Hall-Littlewood functions give bases of the space of symmetric polynomials of degree n, and the Kostka polynomials $K_{\lambda,\mu}(t)$ are defined as (the coefficents of) the transition matrix between those two bases. Kostka polynomials $K_{\lambda,\mu}(t)$ play a very important role in the theory of symmetric functions, and in various representation theories related it. Green polynomials $Q^\mu_\nu(t)$ is defined as a linear combination of Kostka polynomials, with a modification of parameter $t \mapsto t^{-1}$,

$$Q^\mu_\nu(t) = \sum_\lambda \chi^\lambda(w_\nu) t^{n(\mu)} K_{\lambda\mu}(t^{-1}), \tag{1.1.1}$$

where χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to the partition λ of n, w_μ is a representative of the class of $\mathfrak{S}_n$ corresponding to μ, and $n(\mu)$ is the integer defined, for $mu = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_r > 0)$, by

$$n(\mu) = \sum_{i=1}^r (i-1)\mu_i. \tag{1.1.2}$$

Let $B(\mathbf{F}_q)$ be the subgroup of $G(\mathbf{F}_q) = GL_n(\mathbf{F}_q)$ consisting of upper triangular matrices. We consider the permutation module $V = \operatorname{Ind}_{B(\mathbf{F}_q)}^{G(\mathbf{F}_q)} 1$ over $\mathbf{C}$ of $G(\mathbf{F}_q)$. Then the irreducible constitutents in V are in one to one correspondence with irreducible characters of $\mathfrak{S}_n$, hence by partitions of n. We denote by ρ_λ the irreducible character of $G(\mathbf{F}_q)$ corresponidng to χ^λ. As is well-known, the unipotent classes in $GL_n(\mathbf{F}_q)$ are parametrized by partitions of n through Jordan's normal form. We denote by u_μ the unipotent class of $GL_n(\mathbf{F}_q)$ corresponidng to a partition μ. The following formula describes the character values of ρ_λ at unipotent elements in terms of Kostka polynomials.

$$\rho_\lambda(u_\mu) = q^{n(\mu)} K_{\lambda,\mu}(q^{-1}). \tag{1.1.3}$$

1.2. There is a wide class of finite groups containing $GL_n(\mathbf{F}_q)$, called finite reductive groups $G(\mathbf{F}_q)$, arising from the groups of $\mathbf{F}_q$-rational points of reductuve groups G defined over $\mathbf{F}_q$. Symplectic groups $Sp_{2n}(\mathbf{F}_q)$ and special orthogonal groups $SO_n(\mathbf{F}_q)$ are typical examples of finite redeuctive groups, and are called classical groups. In 1976, P. Deligne and G. Lusztig constructed a general theory on the representations of finite reductive groups $G(\mathbf{F}_q)$. In their context, they defined a certain function on the set of unipotent elements in $G(\mathbf{F}_q)$, called Green functions of $G(\mathbf{F}_q)$, which plays an important role in their theory. We breiefly its construction. Let $F : G \to G$ be the Frobenius endomorphism of G. For an F-stable maximal torus T of G, and a linear character θ of T^F, Delinge and Lusztig constructed a virtual character $R_T^G(\theta)$ of $G^F = G(\mathbf{F}_q)$ by making use of the l-adic cohomology theory. They defined a Green function Q_T^G as the restriction of $R_T^G(\theta)$ on the set of unipotent elemetns G^F_{uni} of G^F, which is independent of the choice of θ. In the case of $G(\mathbf{F}_q) = GL_n(\mathbf{F}_q)$, the set of G^F-conjugacy classed of F-stable maximal tori is parametrized by partitions of n. If we denote by T_λ the F-stable maximal torus of G^F corresponidng to λ, the Green function $Q_{T_\lambda}^{G^F}(u_\mu)$ of Deligne-Lusztig coincides with the Green polynomial $Q_\lambda^\mu(q)$. Hence the Green functions Q_T^G are regarded as a generalization of Green polynomials of GL_n to other reductive groups.

The virtual character $R_T^G(\theta)$ is defined through the representations of G^F on certain l-adic cohomology groups. So, it is very difficult to compute Green functions from the original definition. But later Lusztig showed that Green functions can be realized geometrically in terms of the intersection cohomology on the unipotent classes on G. By making use of this geometric interpretstion, the general algorithm of computing Green functions are found by the author, and Lusztig. In order to explain this algotitm, we preprare some notions. For each unipotent element $u \in G$, let $A_G(u) = Z_G(u)/Z_G^0(u)$ be the component group (finite group) of u. Let C_u be the class containing u. Now assume that $u \in G^F_{\text{uni}}$. Then F acts naturally on $A_G(u)$. We assume that F acts trivially on $A_G(u)$. It is known that the G^F-classes in C_u^F are in one to one correspondence with the conjugacy classes in $A_G(u)$. We denote by u_a the representative of the G^F-class in C_u^F corresponidng to $a \in A_G(u)$.

We denote by I the set of pairs (u, ρ), where u runs over the representatives of unipotent classes in G, and ρ runs over the irreducible characters of $A_G(u)$. F acts naturally on the set I. We assume that F acts trivially on I, (actually this holds if G^F is of split type, except the case of type E_8). For each $i = (u, \rho) \in I$, we define a function Y_i on G^F_{uni} by

$$Y_i(g) = \begin{cases} \rho(a) & \text{if } g \text{ is } G^F\text{-conjugate to } u_a, \\ 0 & \text{if } g \notin C_u^F. \end{cases}$$

Then it is checked that the set of Y_i gives a basis of the space $\mathcal{U}$ of G^F-invariant functions on G^F_{uni}.

Let W be the Weyl group of G, and $W^\wedge$ the set of irreducible charactes of W. It is known that there exists a natural injection $f : W^\wedge \to I$, called the Springer correspondence. In the case where $G = GL_n$, $A_G(u) = \{1\}$ for any u, and the set I is nothing but the set of unipotent classes in GL_n. W coincides with $\mathfrak{S}_n$, and the

Springer correspondence is given by the bijection $\mathfrak{S}_n^\wedge \to I, \chi^\lambda \mapsto u_\lambda$. In general, the map f is not surjective, and we denote by I_0 the image $f(W^\wedge)$ in I.

As before, let G be connectd reductive, and we assume that G^F is of split type. Then the G^F-conjugacy classes of F-stable maximal tori are in one to one correspondence with the conjugacy classes in the Weyl group W. We denote by T_w the F-stable maximal torus corresponding to $w \in W$. Let us define a function Q_χ, for each $\chi \in W^\wedge$, by

$$Q_\chi = |W|^{-1} \sum_{w \in W} \chi(w) Q^G_{T_w}. \tag{1.2.1}$$

Since $Q_\chi \in \mathcal{U}$, Q_χ can be written as a linear combination of the basis Y_j. Let $i = f(\chi) \in I_0$, and we can write

$$Q_\chi = \sum_{j \in I_0} p_{ij} Y_j. \tag{1.2.2}$$

Note that, apriori, the sum is taken for all $j \in I$. But it is a deep result by Lusztig that actually $p_{ij} = 0$ unless $j \in I_0$. We also note, by (1.2.1) and (1.2.2) that the determination of Green functions is equivalent to the determination of the coefficients p_{ij}, i.e., the determination of the square matrix $P = (p_{ij})$ indexed by I_0. Let us define an equivalenc erelation $\sim$ on I_0 in such a way that $i \sim i'$ if u and u' are conjugate under G for $i = (u, \rho), i' = (u', \rho')$. We choose a total order on I_0 satisfying the condition that $i < i'$ if $\bar{C}_u \subset \bar{C}_{u'}$ for $i = (u, \rho), i' = (u', \rho')$, where $\bar{C}_u$ denotes the closure of the conjugacy class C_u in G.

We consider the matrix P with respect to this order on I_0. Then P can be reagared as a block matrix, where each block corresponds to an equivalence class in I_0. It follows from the geometric interpretaition of Green functions that the matrix P satisfies the following matrix equations

$$P \varLambda\, {}^t P = \Omega. \tag{1.2.3}$$

Here Ω is the matrix described completely by the data on the irreducible characters of W, while the matrix $\varLambda$ is a non-singular blockwise diagonal matirx, and the matrix P is lower triangular as a block matrix, and the diagonal block is a scalar matrix $q^{d_u} I$, where, if the block corresponds to the class C_u, $d_u = \dim\{gB \in G/B \mid g^{-1}ug \in B\}$. We regard (1.2.3) as a matrix equation with respect to unknown matrices $P, \varLambda$. We may regard Ω as a known matirx. Then the above properties for $P, \varLambda$ determines $P, \varLambda$ uniquely. A simple argument from the linear algebra gives a practical way of finidng P and $\varLambda$. As a concusion, one obtains an algorithm of computing the Green functions Q^G_T by means of the equation (1.2.3).

1.3. In the case of exceptional groups (split type or non-split type), Green polynomials are computed explicitly in terms of the above algorithm. by an aide of computers.

In the case of $G = GL_n$, the set $I = I_0$ is identified with the set of unipotent classes in G^F, hence is given by the set $\mathcal{P}_n$ of partitions of n. Since $A_G(u) = \{1\}$ for any $u \in G$, the equivalence relation $\sim$ in I is a trivial one. The function Y_i

on G^F_{uni} is a characteristic funtion on the unipotent class C^F_λ if $i \in I$ corrsponds to the class C_λ. Thee the coefficient p_{ij} coincides with the modified Kostka polynomilal $\widetilde{K}_{\lambda,\mu}(q) = q^{n(\mu)}K_{\lambda,\mu}(q^{-1})$ if $i, j \in I$ correspond to the partitions λ, μ, respectively. $P = (p_{ij})$ is a lower triangular marix, with diagonal entires $p_{ii} = q^{n(\lambda)}$ for $i \leftrightarrow \lambda$, and $\varLambda = (\lambda_{ij})$ is the diagonal matrix with diagonal entiries $\lambda_{ii} = |C^F_\lambda|$ if $i \in I$ corresponds to a partition $\lambda \in \mathcal{P}_n$.

In the case of claasical groups G, such as Sp_{2n}, SO_{2n+1} or SO_{2n}, the description of uniptent classes in G^F, and of the Springer correspondence becomes much more complicated than the case of GL_n. In order to describe the Springer correspondence for this case in a combinatorial way, Lusztig introduced new combinatorial objects called symbols, as a generalization of the notion of partitions. The properties which are used to define the matrix equation (1.2.3), such as the set I, I_0 and their equivalence class, the quantity $d_u = \dim \mathcal{B}_u$ are all described by the combinatorial proeprties of symbols. Note that the matrix Ω on the right hand side is determined only by the data from the characters of the Weyl group W of G. It follows that the Green functions of classical groups are completely determined by the combinatorics of symbols and by the Weyl group W. Remember that in the case of GL_n, Green functions are constructed in a combinatorial way, in terms of the theory of symmetric functions, such as Schur functions and Hall-Littlewood functions parametrized by partitions. In this article, we shall explain that a similar combinatorial construction of Green functions is available for the case of classical groups.