GREEN FUNCTIONS ASSOCIATED TO COMPLEX REFLECTION GROUPS, II

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Abstract. Green functions associated to complex reflection groups $G(e,1,n)$ were discussed in the author's previous paper. In this paper, we consider the case of complex reflection groups $W = G(e,p,n)$. Schur functions and Hall-Littlewood functions associated to W are introduced, and Green functions are described as the transition matrix between those two symmetric functions. Furthermore, it is shown that these Green functions are determined by means of Green functions associated to various $G(e',1,n')$. Our result involves, as a special case, a combinatorial approach to the Green functions of type D_n.

0. Introduction

This paper is a continuation of [S]. In [S], Hall-Littlewood functions associated to the complex reflection group $G(e,1,n)$ were introduced. Green functions associated to $G(e,1,n)$ are defined as a solution of a certain matrix equation arising from the combinatorics of e-symbols. It was shown that such Green functions are obtained as coefficients of the expansion of Schur functions in terms of Hall-Littlewood functions. In the case where $e = 2$, $G(e,1,n)$ coincides with the Weyl group of type B_n, and the Green function in that case coincides with the Green function associated to finite classical groups $Sp_{2n}(\mathbf{F}_q)$ or $SO_{2n+1}(\mathbf{F}_q)$ introduced by Deligne-Lusztig, in a geometric way. So our result is regarded as a first step towards the combinatorial description of such Green functions, just as in the case of Green polynomials of $GL_n(\mathbf{F}_q)$.

In this paper, we take up the complex reflection group $G(e,p,n)$, and show that a similar formalism as in the case of $G(e,1,n)$ works also for such groups. In the case of $G(e,p,n)$, symmetric functions such as Schur functions, Hall-Littlewood functions, etc. appear as p-tuples of similar functions associated to the various complex reflection groups $G(e',1,n')$. In particular, Green functions associated to $G(e,p,n)$ can be described in terms of Green functions associated to $G(e',1,n')$. In the case where $e = p = 2$, the group $G(e,p,n)$ is equal to the Weyl group of type D_n. In this case our Green functions coincide with the Green functions associated to the finite groups $SO_{2n}(\mathbf{F}_q)$ of split type or non-split type. So our result implies, in this case, that the Green functions of type D_n can be described completely in terms of various "Green functions" of type $B_{n'}$. However, note that the Green function of type $B_{n'}$ appearing in this context is not the Green function associated to $SO_{2n'+1}$. They are the functions introduced in [S], associated to different type of symbols.

In the case where $n = 2$, the group $G(e, e, n)$ is equal to the dihedral group of degree $2e$. In this case, our Green function coincides with the function obtained by Geck-Malle [GM] in connection with special pieces and unipotent characters of finite reductive groups (see [L2]).

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1. Schur-Weyl reciprocity for $G(e, p, n)$

1.1. Let $\widetilde{W} \simeq \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$ be the imprimitive complex reflection group $G(e, 1, n)$ acting on the complex vector space $\mathbb{C}^n$. Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$. Then $\widetilde{W}$ is realized as a subgroup of $GL(\mathbb{C}^n)$ consisting of w such that $w(e_i) = \xi_i e_{x(i)}$ for $i = 1, \dots, n$, where $x \in \mathfrak{S}_n$ and ξ_i is an e-th root of unity (depending on w). We fix a primitive e-th root of unity ζ. Then $\xi_i = \zeta^{a_i}$ with $a_i \in \mathbb{Z}/e\mathbb{Z}$, and $w \in \widetilde{W}$ can be written as $w = x(a_1, \dots, a_n)$ uniquely. The complex reflection group $W_0 = G(e, e, n)$ is defined as the subgroup of $\widetilde{W}$ consisting of $w \in \widetilde{W}$ such that $\sum_{i=1}^n a_i \equiv 0 \pmod{e}$. W_0 is a normal subgroup of $\widetilde{W}$ of index e. Let us define, for $k = 0, \dots, e-1$, $t_k \in \widetilde{W}$ by

$$t_k(e_i) = \begin{cases} \zeta e_i & \text{if } i = k, \\ e_i & \text{otherwise.} \end{cases}$$

Then $w = x(a_1, \dots, a_n)$ can be written as $w = x\prod_{i=1}^n t_i^{a_i}$. Hence if we put $\sigma = t_1$, $\widetilde{W}$ is a semidirect product of W_0 with the cyclic group of order e generated by σ. For each factor p of e, we put $W = W_0 \rtimes \langle \sigma^p \rangle$. Then W is the subgroup of $\widetilde{W}$ consisting of w such that $\sum_{i=1}^n a_i \equiv 0 \pmod{p}$, and is isomorphic to the complex reflection group $G(e, p, n)$.

1.2. For a factor p of e, we put $d = e/p$. Let $\mathcal{P}_{n,e}$ be the set of e-tuples $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ of partitions such that $\sum_{k=0}^{e-1} |\alpha^{(k)}| = n$. An element $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ is called an e-partition of n. We put $l(\boldsymbol{\alpha}) = \sum_{k=0}^{e-1} l(\alpha^{(k)})$, where $l(\alpha^{(k)})$ denotes the number of parts of the partition $\alpha^{(k)}$. Let us define an operator $\theta = \theta_p$ on $\mathcal{P}_{n,e}$ by $\theta(\boldsymbol{\alpha}) = (\alpha^{(k-d)})$ for $\boldsymbol{\alpha} = (\alpha^{(k)}) \in \mathcal{P}_{n,e}$. We denote by $c_{\boldsymbol{\alpha},p}$ the number of elements in the θ-orbit of $\boldsymbol{\alpha}$. Then the irreducible representations of W are described as follows. It is known that the set of isomorphism classes of irreducible representations of $\widetilde{W}$ is parametrized by $\mathcal{P}_{n,e}$. Let $Z_{\boldsymbol{\alpha}}$ be an irreducible $\widetilde{W}$-module corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. Then $Z_{\boldsymbol{\alpha}}$ is decomposed, as a W-module, as

$$Z_{\boldsymbol{\alpha}} = Z^p_{\boldsymbol{\alpha},1} \oplus \cdots \oplus Z^p_{\boldsymbol{\alpha},r}, \tag{1.2.1}$$

where $r = p/c_{\boldsymbol{\alpha},p}$, and σ permutes factors in a cyclic way. Each $Z^p_{\boldsymbol{\alpha},i}$ is an irreducible W-module, mutually non-isomorphic of the same dimension. Furthermore, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the same orbit under θ, then $Z_{\boldsymbol{\alpha}} \simeq Z_{\boldsymbol{\beta}}$ as W-modules. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character of $\widetilde{W}$ afforded by $Z_{\boldsymbol{\alpha}}$, and by $\chi_p^{\boldsymbol{\alpha},i}$ the irreducible character of W afforded by $Z^p_{\boldsymbol{\alpha},i}$ for $i = 1, \dots, p/c_{\boldsymbol{\alpha},p}$. We write $\boldsymbol{\alpha} \sim_p \boldsymbol{\beta}$ if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the same θ-orbit, and by $\mathcal{P}_{n,e}/{\sim_p}$ the set of θ-orbits in $\mathcal{P}_{n,e}$. Then the set $W^\wedge$ of irreducible characters of W is parametrized as

$$W^\wedge = \{\chi_p^{\boldsymbol{\alpha},i} \mid \boldsymbol{\alpha} \in \mathcal{P}_{n,e}/{\sim_p}, 1 \le i \le p/c_{\boldsymbol{\alpha},p}\}. \tag{1.2.2}$$

1.3. Here we recall the Schur-Weyl reciprocity between $\widetilde{W}$ and a certain Levi subgroup of a general linear group over $\mathbb{C}$. Let $V = \bigoplus_{i=0}^{e-1} V_i$ be a vector space over $\mathbb{C}$ with $\dim V_i = m_i$. We fix a basis $\mathcal{E} = \{v_j^{(k)} \mid 1 \le j \le m_k\}$ of V_k for $0 \le k \le e-1$. Then $v_1^{(0)}, \dots, v_{m_0}^{(0)}, v_1^{(1)}, \dots, v_{m_1}^{(1)}, \dots$ gives a basis of V, which we write in this order as $v_1, \dots, v_M$ with $M = \sum m_i$. Let $G = GL_{m_0} \times \cdots \times GL_{m_{e-1}}$. Here GL_{m_i} acts on V_i in a natural way. Hence we have an action of G on V, and so on the n-fold tensor space $V^{\otimes n}$. On the other hand, $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors of the tensor product. This action commutes with the action of G. We extend the action of $\mathfrak{S}_n$ to that of $\widetilde{W}$ as follows: For a basis element $v = v_{i_1} \otimes \cdots \otimes v_{i_n}$ in V, we put

$$t_k(v) = \zeta^j v$$

if $v_{i_k} \in V_j$. Then this action of $t_1, \dots, t_n$ on V gives rise to an action of $\widetilde{W}$, commuting with the action of G. It is known that the following Schur-Weyl reciprocity holds (e.g., see [SS], where the Hecke algebra version is discussed).

(1.3.1) Let $\sigma_1 : \mathbb{C}\widetilde{W} \to \operatorname{End} V^{\otimes n}$, $\rho_1 : \mathbb{C}G \to \operatorname{End} V^{\otimes n}$ be the representations of $\widetilde{W}$ and G, respectively, ($\mathbb{C}\widetilde{W}$, etc. denote the group algebra of $\widetilde{W}$, etc. over $\mathbb{C}$). Then $\sigma_1(\mathbb{C}\widetilde{W})$ and $\rho_1(\mathbb{C}G)$ are the centralizer algebras of each other in $\operatorname{End} V^{\otimes n}$. More precisely, the following holds. Put $\mathbf{m} = (m_0, \dots, m_{e-1})$, and let $\varLambda_{\mathbf{m}}$ be the set of $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ such that $l(\alpha^{(i)}) \le m_i$. Then $\widetilde{W} \times G$-module $V^{\otimes n}$ is decomposed as

$$V^{\otimes n} = \bigoplus_{\boldsymbol{\alpha} \in \varLambda_{\mathbf{m}}} Z_{\boldsymbol{\alpha}} \otimes V_{\boldsymbol{\alpha}},$$

where $Z_{\boldsymbol{\alpha}}$ is the irreducible $\widetilde{W}$-module as before, and $V_{\boldsymbol{\alpha}}$ is an irreducible G-module with highest weight $\boldsymbol{\alpha}$. In particular, if $m_i \ge n$ for any i, all the irreducible $\widetilde{W}$-modules are realized in $V^{\otimes n}$.

1.4. We shall extend the Schur-Weyl reciprocity to the case of W. Here we pose the following assumption on $\mathbf{m}$.

(1.4.1) $m_k = m_{k+d}$ for $k = 0, 1, \dots, p-1$ in $\mathbf{m} = (m_0, \dots, m_{e-1})$.

Let us define a linear automorphism $\tau = \tau_p$ on V by $\tau(v_j^{(k)}) = v_j^{(k+d)}$ for $1 \le j \le m_k$ and for integers k (here we regard $k \in \mathbb{Z}/e\mathbb{Z}$). Then τ is an element of GL_M of order p, and normalizes the subgroup G. We denote by $\widetilde{G}$ the subgroup of GL_M generated by G and τ, which is isomorphic to the semidirect product $G \rtimes \langle\tau\rangle$, where τ acts on $G = GL_{m_0} \times \cdots \times GL_{m_{e-1}}$ by $\tau(GL_{m_k}) = GL_{m_{k+d}}$. Now we have an action of $\widetilde{G}$ on $V^{\otimes n}$, where τ acts on $V^{\otimes n}$ by

$$\tau(v_{i_1}^{(k_1)} \otimes \cdots \otimes v_{i_n}^{(k_n)}) = v_{i_1}^{(k_1+d)} \otimes \cdots \otimes v_{i_n}^{(k_n+d)}.$$

We denote this action also by ρ_1. Here $\rho_1(\tau)$ is an automorphism of order p on $V^{\otimes n}$, commuting with the action of $\mathfrak{S}_n$. Moreover, we have $\tau\sigma\tau^{-1} = \zeta^{-d}\sigma$ on $V^{\otimes n}$, and so $\rho_1(\tau)$ normalizes the subalgebra $\rho_1(\mathbb{C}\widetilde{W})$ of $\operatorname{End} V^{\otimes n}$. It is easy to check, for $w \in \widetilde{W}$, that $\sigma_1(w)$ commutes with $\rho_1(\tau)$ if and only if $w \in W$. Thus we have an action of $W \times \widetilde{G}$ on $V^{\otimes n}$. We consider the decomposition of $V^{\otimes n}$ as in (1.3.1). Since $\tau(V_{\boldsymbol{\alpha}}) \xrightarrow{\sim} V_{\theta(\boldsymbol{\alpha})}$, τ maps $Z_{\boldsymbol{\alpha}} \otimes V_{\boldsymbol{\alpha}}$ onto $Z_{\theta(\boldsymbol{\alpha})} \otimes V_{\theta(\boldsymbol{\alpha})}$. Here we may choose, as a model of $Z_{\boldsymbol{\alpha}}$ in $V^{\otimes n}$, the space of highest weight vectors with highest weight $\boldsymbol{\alpha}$ in $V^{\otimes n}$. Then we have $\tau(Z_{\boldsymbol{\alpha}}) = Z_{\theta(\boldsymbol{\alpha})}$. Put $\varGamma = < \tau > \simeq \mathbb{Z}/p\mathbb{Z}$, and let $\varGamma_{\boldsymbol{\alpha}}$ be the stabilizer of $Z_{\boldsymbol{\alpha}}$ in $\varGamma$. Then $\varGamma_{\boldsymbol{\alpha}} = \langle\tau^c\rangle$ with $c = c_{\boldsymbol{\alpha},p}$, and $Z_{\boldsymbol{\alpha}}$ turns out to be a $\varGamma_{\boldsymbol{\alpha}}$-module. For each $\phi \in \varGamma_{\boldsymbol{\alpha}}^\wedge$, we denote by $Z_{\boldsymbol{\alpha},\phi}$ the ϕ-isotropic subspace of $Z_{\boldsymbol{\alpha}}$, which is stable by W. Now σ permutes factors $Z_{\boldsymbol{\alpha},\phi}$ transitively, and $|\varGamma_{\boldsymbol{\alpha}}| = p/c_{\boldsymbol{\alpha},p}$. It follows that each $Z_{\boldsymbol{\alpha},\phi}$ becomes an irreducible W-module and

$$Z_{\boldsymbol{\alpha}} = \bigoplus_{\phi \in \varGamma_{\boldsymbol{\alpha}}^\wedge} Z_{\boldsymbol{\alpha},\phi}$$

gives the decomposition of $Z_{\boldsymbol{\alpha}}$ into irreducible W-modules given in (1.2.1). On the other hand, let $V'_{\boldsymbol{\alpha},\phi}$ be a G-submodule of $V^{\otimes n}$ generated by a highest weight vector in $Z_{\boldsymbol{\alpha},\phi}$. Then $V'_{\boldsymbol{\alpha},\phi}$ is isomorphic to $V_{\boldsymbol{\alpha}}$, and is stable under the action of τ^c. We denote by $V_{\boldsymbol{\alpha},\phi}$ the $\widetilde{G}$-submodule of $V^{\otimes n}$ generated by $V'_{\boldsymbol{\alpha},\phi}$. Then $V_{\boldsymbol{\alpha},\phi}$ is isomorphic to the $\widetilde{G}$-module induced from $\varGamma_{\boldsymbol{\alpha}} \ltimes G$-module $V'_{\boldsymbol{\alpha},\phi}$, and turns out to be an irreducible $\widetilde{G}$-module. Moreover, $V_{\boldsymbol{\alpha},\phi}$ are mutually non-isomorphic for distinct pairs $(\boldsymbol{\alpha}, \phi)$. Put

$$\varLambda^p_{\mathbf{m}} = \{(\boldsymbol{\alpha}, \phi) \mid \boldsymbol{\alpha} \in \varLambda_{\mathbf{m}}/\sim_p, \phi \in \varGamma_{\boldsymbol{\alpha}}^\wedge\},$$

where $\varLambda_{\mathbf{m}}/\sim_p$ denotes the set of θ-orbits in $\varLambda_{\mathbf{m}}$. It follows from the above discussion, we have the following Schur-Weyl reciprocity between W and $\widetilde{G}$.

Proposition 1.5. *$\sigma_1(\mathbb{C}W)$ and $\rho_1(\mathbb{C}\widetilde{G})$ are mutually the full centralizer algebras of each other in* $\operatorname{End} V^{\otimes n}$. *Moreover, $W \times \widetilde{G}$-module $V^{\otimes n}$ is decomposed as*

$$V^{\otimes n} = \bigoplus_{(\boldsymbol{\alpha},\phi) \in \varLambda^p_{\mathbf{m}}} Z_{\boldsymbol{\alpha},\phi} \otimes V_{\boldsymbol{\alpha},\phi}.$$

2. Frobenius formula for $G(e,p,n)$

2.1. In the remainder of this paper, we assume that $\mathbf{m} = (m_0, \dots, m_{e-1})$ satisfies the condition (1.4.1) and the condition that $m_i \ge n$ for $i = 0, \dots, e-1$. Then any irreducible W-module is realized as $Z_{\boldsymbol{\alpha},\phi}$ by Proposition 1.5. In this case, $\varLambda^p_{\mathbf{m}}$ coincides with the set

$$\widetilde{\mathcal{P}}_W = \{(\boldsymbol{\alpha},\phi) \mid \boldsymbol{\alpha} \in \mathcal{P}_{n,e}/\sim_p, \phi \in \varGamma^\wedge_{\boldsymbol{\alpha}}\}.$$

We denote by $\chi^{\boldsymbol{\alpha},\phi}$ the irreducible character of W corresponding to the W-module $Z_{\boldsymbol{\alpha},\phi}$. Under this notation, the parameterization of $W^\wedge$ in (1.2.2) can be modified as

$$W^\wedge = \{\chi^{\boldsymbol{\alpha},\phi} \mid (\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}_W\}. \tag{2.1.1}$$

For later use, we consider a more general situation. Assume that q is a factor of e and that $\langle\sigma^q\rangle \times \langle\sigma^p\rangle$ is a subgroup of $\langle\sigma\rangle \simeq \mathbb{Z}/e\mathbb{Z}$, i.e., e/q and e/p are prime to each other. The typical case is that $W = W_0$ and q is any factor of e. Now one can form a semidirect product $\langle\sigma^q\rangle \ltimes W$ as a subgroup of $\widetilde{W}$. We are now interested in the set of σ^q-stable characters in $W^\wedge$. It follows from the discussion in 1.4 that the irreducible W-module $Z_{\boldsymbol{\alpha},\phi}$ is σ^q-stable if and only if q is divisible by $|\varGamma_{\boldsymbol{\alpha}}| = p/c_{\boldsymbol{\alpha},p}$. So we put

$$\widetilde{\mathcal{P}}^q_W = \{(\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}_W \mid qc_{\boldsymbol{\alpha},p} \equiv 0 \pmod{p}\}.$$

We also write it as $\widetilde{\mathcal{P}}^q_{n,e,p}$ to make the dependence on n,e,p more explicit. Let $W^\wedge_{\mathrm{ex},q}$ be the set of σ^q-stable irreducible characters of W. Thus $W^\wedge_{\mathrm{ex},q}$ is given as

$$W^\wedge_{\mathrm{ex},q} = \{\chi^{\boldsymbol{\alpha},\phi} \in W^\wedge \mid (\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}^q_W\}. \tag{2.1.2}$$

2.2. As in [S], we identify $\mathcal{P}_{n,e}$ with the set $Z^{0,0}_n = Z^{0,0}_n(\mathbf{m})$. Here $Z^{0,0}_n$ is the set of e-partitions $\boldsymbol{\alpha}$ written as $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ with a partition $\alpha^{(k)} : \alpha^{(k)}_1 \ge \cdots \ge \alpha^{(k)}_{m_k} \ge 0$. We prepare the indeterminates $x^{(k)}_i$ for $0 \le k < e, 1 \le i \le m_k$. We write $x = \{x^{(k)}_i\}$, and also write as $x^{(k)} = \{x^{(k)}_1, \dots, x^{(k)}_{m_k}\}$ for a fixed k. Recall that a power sum symmetric function $p_{\boldsymbol{\alpha}}(x)$ is defined for $\boldsymbol{\alpha} \in Z^{0,0}_n$ as

$$p_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1}\prod_{j=1}^{m_k} p^{(k)}_{\alpha^{(k)}_j}(x), \tag{2.2.1}$$

where we put, for each integer $r \ge 1$,

$$p^{(i)}_r(x) = \sum_{j=0}^{e-1} \zeta^{ij} p_r(x^{(j)}), \tag{2.2.2}$$

with usual r-th power sum symmetric functions $p_r(x^{(j)})$ with variables $x^{(j)}$, and put $p^{(i)}_0(x) = 1$. Also, Schur functions $s_{\boldsymbol{\alpha}}(x)$ and monomial symmetric functions $m_{\boldsymbol{\alpha}}(x)$

are defined as

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}), \qquad m_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} m_{\alpha^{(k)}}(x^{(k)}),$$

by using usual Schur functions $s_{\alpha^{(k)}}$ and monomial symmetric functions $m_{\alpha^{(k)}}$ associated to partitions $\alpha^{(k)}$.

It is known that the set of conjugacy classes in $\widetilde{W}$ is in bijection with $\mathcal{P}_{n,e}$. The explicit correspondence will be given later in 2.3. We denote by $w_{\boldsymbol{\beta}}$ a representative of the conjugacy class in $\widetilde{W}$ corresponding to $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. Then the Frobenius formula for the irreducible characters of $\widetilde{W}$ is given as follows.

(2.2.3) (Frobenius formula for $\widetilde{W}$). Let $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. Then we have

$$p_{\boldsymbol{\beta}} = \sum_{\boldsymbol{\alpha} \in \mathcal{P}_{n,e}} \chi^{\boldsymbol{\alpha}}(w_{\boldsymbol{\beta}}) s_{\boldsymbol{\alpha}}.$$

2.3. We want to generalize (2.2.3) to the case of irreducible characters on $\sigma^q W$. First of all we describe the conjugacy classes in $\widetilde{W}$ and W more precisely. The correspondence between the conjugacy classes in $\widetilde{W}$ and $\mathcal{P}_{n,e}$ is given explicitly as follows. Assume that w maps $e_{i_1}, e_{i_2}, \dots, e_{i_m}$ in a cyclic way, up to scalar, and leaves other e_j unchanged. If $w^m(e_{i_1}) = \zeta^k e_{i_1}$, we say that w is a k-cycle $(i_1, \dots, i_m)$ of length m. Now $w \in \widetilde{W}$ can be written as a product of various disjoint cycles. If we group up, for a fixed k, the k-cycles among them, it produces a partition $\alpha^{(k)}$, and $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ turns out to be an e-partition of n. This gives the required bijection. We denote by $C_{\boldsymbol{\alpha}}$ the conjugacy class of $\widetilde{W}$ corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$.

For $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)}) \in \mathcal{P}_{n,e}$, we define an integer $\varDelta(\boldsymbol{\alpha})$ by

$$\varDelta(\boldsymbol{\alpha}) = \sum_{k=0}^{e-1} l(\alpha^{(k)})k. \tag{2.3.1}$$

It is easily checked that the conjugacy class $C_{\boldsymbol{\alpha}}$ belongs to W if and only if $\varDelta(\boldsymbol{\alpha}) \equiv 0 \pmod p$. The class $C_{\boldsymbol{\alpha}}$ is decomposed into several conjugacy classes in W.

More generally, we consider a coset $\sigma^q W$ in $\widetilde{W}$. Then $\sigma^q W$ is invariant under the adjoint action of $\widetilde{W}$, and the class $C_{\boldsymbol{\alpha}}$ lies in $\sigma^q W$ if and only if $\varDelta(\boldsymbol{\alpha}) \equiv q \pmod p$. The conjugacy class $C_{\boldsymbol{\alpha}} \subset \sigma^q W$ is also decomposed into several W-orbits. The complete description of such W-orbits will be given in Proposition 2.13.

2.4. Let D be an operator on $V^{\otimes n}$ defined by

$$D(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}) = x_{i_1} x_{i_2} \cdots x_{i_n} (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}).$$

(Although we must replace V by the scalar extension $K \otimes_{\mathbb{C}} V$ with $K = \mathbb{C}(x_j^{(k)})$, we use the same notation by abbreviation). Then D commutes with the action of $\widetilde{W}$, and the Frobenius formula (2.2.3) is obtained by computing the trace $\operatorname{Tr}(Dw, V^{\otimes n})$

for $w \in \widetilde{W}$ in two different ways. We follow this strategy to establish the Frobenius formula for the characters of W, in our case by computing the traces $\mathrm{Tr}\,(D\tau^j w, V^{\otimes n})$ for $j = 0, \dots, p-1$.

For an integer j $(0 \le j < p)$, let h_j be order of the image $\bar{j}$ of j in $\mathbb{Z}/p\mathbb{Z}$. Then p is written as $p = h_j j_1$, where j_1 is the greatest common divisor of j and p. We introduce new indeterminates $\mathcal{X}_j = \{X_i^{(k)}\}$ by

$$X_i^{(k)} = x_i^{(k)} x_i^{(k+jd)} x_i^{(k+2jd)} \cdots x_i^{(k+(h-1)jd)} \qquad (0 \le k < j_1 d, 1 \le i \le m_k), \tag{2.4.1}$$

with $h = h_j$. We write also as $\mathcal{X}_j^{(k)} = \{X_1^{(k)}, \dots, X_{m_k}^{(k)}\}$ for a fixed k. One can define a function $p_r^{(i)}(\mathcal{X}_j)$ with respect to the variables $\mathcal{X}_j = \{\mathcal{X}_j^{(0)}, \dots, \mathcal{X}_j^{(j_1 d-1)}\}$ by modifying (2.2.2) as follows.

$$p_r^{(i)}(\mathcal{X}_j) = \sum_{k=0}^{j_1 d-1} \zeta^{ik} p_r(\mathcal{X}_j^{(k)}). \tag{2.4.2}$$

Then we have the following lemma.

Lemma 2.5. *Let $w \in \widetilde{W}$ be an f-cycle of length n. Assume that w is of the form $w = t_1^a t_n^b z$, where z is a cyclic permutation $(1, 2, \dots, n) \in \mathfrak{S}_n$ and $a + b \equiv f \pmod p$. Then, for each j such that $0 \le j < p$,*

$$\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = \begin{cases} h\zeta^{-(f+b)jd} p_{n/h}^{(f)}(\mathcal{X}_j) & \text{if } h \mid n \text{ and } h \mid f, \\ 0 & \text{otherwise,} \end{cases}$$

where $h = h_j$ as before.

Proof. We have

$$\tau^j w(v_{i_1}^{(k_1)} \otimes v_{i_2}^{(k_2)} \otimes \cdots \otimes v_{i_n}^{(k_n)}) = \zeta^{ak_n + bk_{n-1}} v_{i_n}^{(k_n+jd)} \otimes v_{i_1}^{(k_1+jd)} \otimes \cdots \otimes v_{i_{n-1}}^{(k_{n-1}+jd)}.$$

Now assume that $\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) \ne 0$. Then there exists $v_{i_1}^{(k_1)}, \dots, v_{i_n}^{(k_n)} \in \mathcal{E}$ satisfying the relation

$$v_{i_s}^{(k_s)} = v_{i_{s-1}}^{(k_{s-1}+jd)} \qquad \text{for } s = 1, \dots, n.$$

(Here we regard $v_{i_0} = v_{i_n}$, and $k_0 = k_n$). This implies that

$$\begin{aligned} v_{i_1}^{(k_1)} &= v_{i_1}^{(k_1+njd)}, \\ v_{i_s}^{(k_s)} &= v_{i_1}^{(k_1+(s-1)jd)} \qquad \text{for } s = 2, \dots, n. \end{aligned} \tag{2.5.1}$$

In particular, we have $njd \equiv 0 \pmod e$, or equivalently $nj \equiv 0 \pmod p$.

Under the notation in (2.4.1), we have

$$\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = \zeta^{-(a+2b)jd} \sum_{k=0}^{j_1 d-1} \sum_{s=0}^{h-1} \zeta^{f(k+sjd)} \sum_{i=1}^{m_k} (X_i^{(k)})^{n/h}. \tag{2.5.2}$$

This implies that $\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = 0$ unless $fj \equiv 0 \pmod p$. Here note that the condition $nj \equiv 0 \pmod p$ is equivalent to $h \mid n$, and similarly for $fj \equiv 0 \pmod p$. The second assertion follows from this.

We now assume that $nj \equiv 0$ and $fj \equiv 0 \pmod p$. Then by (2.4.2) and (2.5.2), we have

$$\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = h\zeta^{-(a+2b)jd} p^{(f)}_{n/h}(\mathcal{X}_j).$$

This shows the first assertion, and the lemma is proved. □

2.6. We consider the general case where w is in the class $C_{\boldsymbol{\alpha}}$ in $\widetilde{W}$. We fix j and let $h = h_j$ and j_1 be as before. For $\boldsymbol{\alpha} = (\alpha^{(k)}_i)$, we consider the following condition (for a fixed j).

$$\begin{aligned} &h \mid \alpha^{(k)}_i \quad \text{for any } \alpha^{(k)}_i, \text{ and} \\ &h \mid k \quad \text{if } |\alpha^{(k)}| \neq 0. \end{aligned} \tag{2.6.1}$$

Assume that $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ satisfies (2.6.1). We define a $j_1 d$-partition $\boldsymbol{\beta} = (\beta^{(0)}, \dots, \beta^{(j_1 d-1)})$ of n/h by $\beta^{(k/h)}_i = \alpha^{(k)}_i/h$ for any $\alpha^{(k)}_i$ such that $h \mid k$. We write $\boldsymbol{\beta}$ as $\boldsymbol{\beta} = \boldsymbol{\alpha}[j]$. Then we have $h\varDelta(\boldsymbol{\alpha}[j]) = \varDelta(\boldsymbol{\alpha})$. Note that in this situation (2.4.2) can be regarded as a formula analogous to (2.2.2) for variables $\mathcal{X}_j$, by replacing ζ by ζ^h. Thus one can define $p_{\boldsymbol{\alpha}[j]}(\mathcal{X}_j)$ just as in (2.2.1) as a product of various $p_{\beta^{(k)}_i}(\mathcal{X}_j)$.

As in 2.3, w can be written as a product of various cycles associated to $\boldsymbol{\alpha}$. Under the conjugation in W, w is changed to an element w' of the following type; the cycle corresponding to $\alpha^{(k)}_i$ is of the form $t^k_{i_1} z$, where $z = (i_1, i_2, \dots, i_m) \in \mathfrak{S}_m$ with $m = \alpha^{(k)}_i$, except one cycle. The excepted one is of the form $t^a_{i_1} t^b_{i_m} z$, where $z = (i_1, i_2, \dots, i_m)$ with $a + b \equiv k \pmod e$. We may further assume that each cyclic permutation z occurring in the decomposition of w' is of the form $z = (i_1, i_1 + 1, \dots, i_1 + (m-1))$. We write this element as $w' = w_{\boldsymbol{\alpha}}(b)$.

Note that $\mathrm{Tr}\,(D\tau^j w, V^{\otimes n})$ remain unchanged if w is replaced by its conjugate w', since W commutes with D and τ. Then $\mathrm{Tr}\,(D\tau^j w', V^{\otimes n})$ can be computed by making use of Lemma 2.5. So, as a corollary to Lemma 2.5, we have

Proposition 2.7. *Assume that $w \in C_{\boldsymbol{\alpha}}$ in $\widetilde{W}$, and that w is conjugate under W to $w_{\boldsymbol{\alpha}}(b)$. Then $\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = 0$ unless $\boldsymbol{\alpha}$ satisfies the condition (2.6.1). If $\boldsymbol{\alpha}$ satisfies the condition (2.6.1), then*

$$\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = h^{l(\boldsymbol{\alpha}[j])}\zeta^{-(\varDelta(\boldsymbol{\alpha})+b)jd} p_{\boldsymbol{\alpha}[j]}(\mathcal{X}_j).$$

2.8. Next, we consider the decomposition $V^{\otimes n} = \bigoplus V_{\boldsymbol{\alpha},\phi} \otimes Z_{\boldsymbol{\alpha},\phi}$ as given in Proposition 1.5. Assume that $w \in \sigma^q W$. If $\chi^{\boldsymbol{\alpha},\phi} \in W^\wedge$ is not σ^q-stable, then $D\tau^j w$ maps $V_{\boldsymbol{\alpha},\phi} \otimes Z_{\boldsymbol{\alpha},\phi}$ onto a different factor. It follows that

$$\mathrm{Tr}\,(D\tau^j w, V^{\otimes n}) = \sum_{(\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}^q_W} \mathrm{Tr}\,(D\tau^j w, V_{\boldsymbol{\alpha},\phi} \otimes Z_{\boldsymbol{\alpha},\phi}). \tag{2.8.1}$$

Let $\chi^{\boldsymbol{\alpha},\phi}$ be a σ^q-stable irreducible character of W. Then $\chi^{\boldsymbol{\alpha},\phi}$ can be extended to an irreducible character of $\langle\sigma^q\rangle\ltimes W \simeq W_{e,p',n}$, where $p' = pq/e$. Let $\varGamma' \simeq \mathbb{Z}/p'\mathbb{Z}$ be the subgroup of $\varGamma \simeq \mathbb{Z}/p\mathbb{Z}$ generated by $\tau^{e/q}$, and $\varGamma'_{\boldsymbol{\alpha}}$ the stabilizer of $Z_{\boldsymbol{\alpha}}$ in $\varGamma'$. Since q is divisible by $|\varGamma_{\boldsymbol{\alpha}}|$ by (2.1.2) and since $(e/p, e/q) = 1$, we see that $\varGamma'_{\boldsymbol{\alpha}}$ coincides with $\varGamma_{\boldsymbol{\alpha}}$. Hence irreducible characters of W occurring in the decomposition of $\chi^{\boldsymbol{\alpha}}$ are also regarded as irreducible characters of $W_{e,p',n}$. In this way, we can fix an extension $\widetilde\chi^{\boldsymbol{\alpha},\phi}$ of $\chi^{\boldsymbol{\alpha},\phi}$ to $W_{e,p',n}$. Accordingly, the irreducible W-module $Z_{\boldsymbol{\alpha},\phi}$ is extended to the irreducible $W_{e,p',n}$-module corresponding to $\widetilde\chi^{\boldsymbol{\alpha},\phi}$, which we denote by $\widetilde Z_{\boldsymbol{\alpha},\phi}$.

Remember that $V_{\boldsymbol{\alpha},\phi}\otimes Z_{\boldsymbol{\alpha},\phi} \simeq \bigoplus_{\boldsymbol{\beta}\in O(\boldsymbol{\alpha})} V'_{\boldsymbol{\beta},\phi}\otimes Z_{\boldsymbol{\beta},\phi}$ as $G\times W$-module, and τ permutes the summands of the right hand side. In the case where $c_{\boldsymbol{\alpha},p} \mid j$, τ^j leaves each summand $V'_{\boldsymbol{\beta},\phi}\otimes Z_{\boldsymbol{\beta},\phi}$ stable. The last space is extended to a $W_{e,p',n}$-module $V'_{\boldsymbol{\beta},\phi}\otimes \widetilde Z_{\boldsymbol{\beta},\phi}$, and we have an action of w on $V_{\boldsymbol{\alpha},\phi}\otimes\widetilde Z_{\boldsymbol{\alpha},\phi} \simeq \bigoplus_{\boldsymbol{\beta}\in O(\boldsymbol{\alpha})} V'_{\boldsymbol{\beta},\phi}\otimes\widetilde Z_{\boldsymbol{\beta},\phi}$.

Assume that $\boldsymbol{\alpha}\in\mathcal{P}_{n,e}$ satisfies the condition that $c_{\boldsymbol{\alpha},p} \mid j$. We put $\boldsymbol{\alpha}\{j\} = (\alpha^{(0)},\dots,\alpha^{(j_1d-1)})$. Since j_1d is divisible by $c_{\boldsymbol{\alpha},p}$, $\boldsymbol{\alpha}\{j\}$ turns out to be a j_1d-partition of n/h. Also note that, $\tau^j\in\varGamma_{\boldsymbol{\alpha}}$. Under this setting, we have the following lemma.

Lemma 2.9. *Let $\boldsymbol{\alpha}\in\mathcal{P}_{n,e}$, and q be as in 2.1. Assume that $\chi^{\boldsymbol{\alpha},\phi}$ is σ^q-stable. Then for $w\in\sigma^qW$, we have*

$$\operatorname{Tr}(D\tau^jw, V_{\boldsymbol{\alpha},\phi}\otimes\widetilde Z_{\boldsymbol{\alpha},\phi}) = \begin{cases} \phi(\tau^j)\displaystyle\sum_{0\le i<c}\zeta^{qid}s_{\theta^i(\boldsymbol{\alpha})\{j\}}(\mathcal{X}_j)\widetilde\chi^{\boldsymbol{\alpha},\phi}(w) & \text{if } c \mid j,\\ 0 & \text{otherwise,}\end{cases}$$

where $c = c_{\boldsymbol{\alpha},p}$, and $s_{\boldsymbol{\alpha}\{j\}}(\mathcal{X}_j)$ etc. are Schur functions given in 2.2 with respect to the variables $\mathcal{X}_j$.

Proof. By the previous remark, $D\tau^jw$ permutes the factors $V'_{\boldsymbol{\beta},\phi}\otimes Z_{\boldsymbol{\beta},\phi}$ if j is not a multiple of c, and so $\operatorname{Tr}(D\tau^jw, V_{\boldsymbol{\alpha},\phi}\otimes\widetilde Z_{\boldsymbol{\alpha},\phi}) = 0$. Now assume that $c\mid j$. Then we can write

$$\operatorname{Tr}(D\tau^jw, V_{\boldsymbol{\alpha},\phi}\otimes\widetilde Z_{\boldsymbol{\alpha},\phi}) = \sum_{\boldsymbol{\beta}\in O(\boldsymbol{\alpha})}\operatorname{Tr}(D\tau^j, V'_{\boldsymbol{\beta},\phi})\widetilde\chi^{\boldsymbol{\beta},\phi}(w).$$

Since $V'_{\boldsymbol{\beta},\phi} = \phi\otimes V'_{\boldsymbol{\beta},1}$ as $\langle\tau^c\rangle\ltimes G$-modules, we have $\operatorname{Tr}(D\tau^j, V'_{\boldsymbol{\beta},\phi}) = \phi(\tau^j)\operatorname{Tr}(D\tau^j, V'_{\boldsymbol{\beta},1})$. We note that

$$\operatorname{Tr}(D\tau^j, V'_{\boldsymbol{\beta},1}) = s_{\boldsymbol{\beta}\{j\}}(\mathcal{X}_j). \tag{2.9.1}$$

In fact, $V'_{\boldsymbol{\beta},1}$ is an irreducible G-module isomorphic to $V_{\boldsymbol{\beta}}$, and it is known from (2.2.3) that $\operatorname{Tr}(D, V_{\boldsymbol{\beta}}) = s_{\boldsymbol{\beta}}(x)$. $V_{\boldsymbol{\beta}}$ has a basis consisting of weight vectors, on which D acts diagonally. Here for a vector $v = v^{(k_1)}_{i_1}\otimes\cdots\otimes v^{(k_n)}_{i_n}\in V^{\otimes n}$, the weight of v is given by an element $\boldsymbol{\gamma}(v) = (\gamma^{(0)},\dots,\gamma^{(e-1)})\in Z^{0,0}_n$, where $\gamma^{(k)}_i$ is the number of $v^{(k)}_i$ occurring in the expression of v. On the other hand, τ^j acts on $V_{\boldsymbol{\beta}}$, by permuting the weight vectors; if v is a weight vector with weight $\boldsymbol{\gamma}$, then $\tau^j(v)$ is also a weight vector with weight $\boldsymbol{\gamma}' = (\gamma^{(-jd)},\gamma^{(1-jd)},\dots,\gamma^{(e-1-jd)})$. Hence, in order to compute $\operatorname{Tr}(D\tau^j, V_{\boldsymbol{\beta}})$, we have only to consider the weight vectors in $V_{\boldsymbol{\beta}}$ whose weights are of the type $\boldsymbol{\gamma} = (\gamma_i^{(k)})$ such that $\gamma^{(k)} = \gamma^{(k+jd)}$ for each k. Note that monomials $\prod (x_i^{(k)})^{\gamma_i^{(k)}}$ obtained as weights of D produce the Schur function $s_{\boldsymbol{\beta}}(x)$. Then the corresponding weight for $D\tau^j$ is given by $\prod (X_i^{(k)})^{\gamma_i^{(k)}}$ for $\boldsymbol{\gamma}$ as above (the product is taken for k such that $0 \le k < j_1 d$). Hence $\operatorname{Tr}(D\tau^j, V'_{\beta,1})$ is obtained by picking up the monomials of this type from $s_{\boldsymbol{\beta}}(x) = \prod_{k=0}^{e-1} s_{\boldsymbol{\beta}^{(k)}}(x^{(k)})$, which coincides with $s_{\boldsymbol{\beta}\{j\}}(\mathcal{X}_j)$. This proves (2.9.1).

To prove the lemma, it is enough to show that

$$\widetilde{\chi}^{\theta^i(\boldsymbol{\alpha}),\phi}(w) = \zeta^{qid}\widetilde{\chi}^{\boldsymbol{\alpha},\phi}(w) \tag{2.9.2}$$

for $i = 0, \dots, c-1$. We show (2.9.2). Since $\widetilde{\chi}^{\boldsymbol{\alpha},\phi}$ and $\widetilde{\chi}^{\theta^i(\boldsymbol{\alpha}),\phi}$ are both extensions of $\chi^{\boldsymbol{\alpha},\phi}$, there exists a linear character φ of the cyclic group $\langle \sigma^q \rangle$ such that $\widetilde{\chi}^{\theta^i(\boldsymbol{\alpha}),\phi} = \varphi \otimes \widetilde{\chi}^{\boldsymbol{\alpha},\phi}$. On the other hand, since $\tau\sigma\tau^{-1} = \zeta^{-d}\sigma$ on $V^{\otimes n}$, we see that the action of σ^q on $\widetilde{Z}_{\theta^i(\boldsymbol{\alpha}),\phi} = \tau^i(\widetilde{Z}_{\boldsymbol{\alpha},\phi})$ corresponds, under the isomorphism τ^i of W-modules, to the action of σ^q on $\widetilde{Z}_{\boldsymbol{\alpha},\phi}$ multiplied by ζ^{qid}. This implies (2.9.2), and the lemma follows. □

2.10. In order to formulate the Frobenius formula for the σ^q-stable characters of W, we shall define Schur functions and power sum symmetric functions associated to $\sigma^q W$. Let $(\boldsymbol{\alpha}, \phi) \in \widetilde{\mathcal{P}}_W^q$, and put, for each $0 \le j < p$,

$$s^j_{\boldsymbol{\alpha},\phi}(x) = \begin{cases} \phi(\tau^j)\sum_{0\le i<c}\zeta^{qid}s_{\theta^i(\boldsymbol{\alpha})\{j\}}(\mathcal{X}_j) & \text{if } c \mid j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.10.1}$$

where $c = c_{\boldsymbol{\alpha},p}$. We put $\mathbf{s}_{\boldsymbol{\alpha},\phi}(x) = (s^j_{\boldsymbol{\alpha},\phi}(x))_{0\le j<p}$, and call it Schur function for $\sigma^q W$ associated to $(\boldsymbol{\alpha}, \phi)$.

Next, we consider power sum symmetric functions. For a pair $(\boldsymbol{\alpha}, b)$, where $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ and $0 \le b < e$, and for $0 \le j < p$, we put

$$p^j_{\boldsymbol{\alpha},b}(x) = \begin{cases} h^{l(\boldsymbol{\alpha}[j])}\zeta^{-(\Delta(\boldsymbol{\alpha})+b)jd}p_{\boldsymbol{\alpha}[j]}(\mathcal{X}_j) & \text{if } \boldsymbol{\alpha} \text{ satisfies (2.6.1)}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10.2}$$

(For the notation, see 2.6.) We put $\boldsymbol{p}_{\boldsymbol{\alpha},b}(x) = (p^j_{\boldsymbol{\alpha},b}(x))_{0\le j<p}$, and call it a power sum symmetric function for $\sigma^q W$ associated to the pair $(\boldsymbol{\alpha}, b)$.

We are now ready to formulate a Frobenius formula for the characters on $\sigma^q W$ as follows. The proof is immediate from Proposition 2.7, (2.8.1) and Lemma 2.9.

Proposition 2.11 (Frobenius formula for $\sigma^q W$). *Let $(\boldsymbol{\beta}, b)$ be such that $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$ and $0 \le b < e$. Put $w = w_{\boldsymbol{\beta}}(b)$, and assume that $w \in \sigma^q W$, (cf. 2.6). Then we have*

$$\boldsymbol{p}_{\boldsymbol{\beta},b}(x) = \sum_{(\boldsymbol{\alpha},\phi)\in\widetilde{\mathcal{P}}_W^q} \widetilde{\chi}^{\boldsymbol{\alpha},\phi}(w)\mathbf{s}_{\boldsymbol{\alpha},\phi}(x).$$

2.12. As in [S, 3.5], we consider the ring of symmetric polynomials $\Xi_{\mathbf{m}} = \bigotimes_{k=0}^{e-1}\mathbb{Z}[x_1^{(k)},\dots,x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ with respect to $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0}\times\cdots\times\mathfrak{S}_{m_{e-1}}$. $\Xi_{\mathbf{m}}$ has a structure of graded ring $\Xi_{\mathbf{m}} = \bigoplus_{i\ge 0}\Xi_{\mathbf{m}}^i$, where $\Xi_{\mathbf{m}}^i$ consists of homogeneous symmetric polynomials of degree i. We can define a space of symmetric functions $\Xi = \bigoplus_{i\ge 0}\Xi^i$, where Ξ^i is the inverse limit of $\Xi_{\mathbf{m}}^i$ (cf. [loc. cit.]). For $\boldsymbol{\alpha}\in\mathcal{P}_{n,e}$, the functions $s_{\boldsymbol{\alpha}}(x)$, $p_{\boldsymbol{\alpha}}(x)$ given in 2.2 make sense for infinitely many variables $x_1^{(k)}, x_2^{(k)},\dots$, and give rise to elements in Ξ^n.

Schur functions $\mathbf{s}_{\boldsymbol{\alpha},\phi}(x)$ and power sum symmetric functions $\boldsymbol{p}_{\boldsymbol{\beta},b}(x)$ defined in 2.10 are also extended to the functions with infinitely many variables, and they can be regarded as elements in the space $\bigoplus_p\Xi_{\mathbb{C}}^n$, the direct sum of p copies of $\Xi_{\mathbb{C}}^n = \mathbb{C}\otimes\Xi^n$. It is easy to see that Schur functions $\{\mathbf{s}_{\boldsymbol{\alpha},\phi} \mid (\boldsymbol{\alpha},\phi)\in\widetilde{\mathcal{P}}_W^q\}$ are linearly independent in $\bigoplus_p\Xi_{\mathbb{C}}^n$. We denote by $\Xi_{\mathbb{C}}^n(p,q)$ the subspace of $\bigoplus_p\Xi_{\mathbb{C}}^n$ generated by those $\mathbf{s}_{\boldsymbol{\alpha},\phi}$. Thus Schur functions $\{\mathbf{s}_{\boldsymbol{\alpha},\phi} \mid (\boldsymbol{\alpha},\phi)\in\widetilde{\mathcal{P}}_W^q\}$ form a basis of $\Xi_{\mathbb{C}}^n(p,q)$.

The space $\Xi_{\mathbb{C}}^n(p,q)$ is also described as follows. Let $\Xi_{p,q}^n$ be the subspace of $\Xi_{\mathbb{C}}^n$ generated by $\sum_{i=0}^{c-1}\zeta^{qid}s_{\theta^i(\boldsymbol{\alpha})}$ for various $\boldsymbol{\alpha}\in\mathcal{P}_{n,e}$ (here $c = c_{\boldsymbol{\alpha},p}$). Then

$$\Xi_{\mathbb{C}}^n(p,q) = \bigoplus_{j=0}^{p-1}\Xi_{p,q}^{n/h_j}(\mathcal{X}_j), \tag{2.12.1}$$

where $\Xi_{p,q}^{n/h_j}(\mathcal{X}_j)$ denotes the space $\Xi_{p,q}^{n/h_j}$ with respect to the variables $\mathcal{X}_j$. (2.12.1) is immediate from (2.10.1).

It follows from Proposition 2.11 that $\boldsymbol{p}_{\boldsymbol{\beta},b}\in\Xi_{\mathbb{C}}^n(p;q)$ if $w_{\boldsymbol{\beta}}(b)\in\sigma^qW$. By making use of the Frobenius formula, we can give a complete description of W-orbits of σ^qW. First we prepare some notations. Take $w = w_{\boldsymbol{\beta}}(b)\in\sigma^qW$. For $0\le j<p$, we define $f_j(w)\in\mathbb{Z}/p\mathbb{Z}$ by

$$f_j(w) = \begin{cases} bj \pmod p & \text{if } \boldsymbol{\beta} \text{ satisfies (2.6.1)},\\ 0 & \text{otherwise.}\end{cases} \tag{2.12.2}$$

By Proposition 2.7 and (2.10.2), we can associate to each W-orbit in σ^qW a well-defined function $\boldsymbol{p}_{\boldsymbol{\beta},b}(x)$, where $w = w_{\boldsymbol{\beta}}(b)$ is a representative of this orbit. We denote $\boldsymbol{p}_{\boldsymbol{\beta},b}(x)$ also by $\boldsymbol{p}_w(x)$. As a corollary to Proposition 2.11, we have the following characterization of W-orbits in σ^qW.

Proposition 2.13. (i) *Assume that $w, w'\in\sigma^qW$. Then w and w' are conjugate under W if and only if $\boldsymbol{p}_w(x) = \boldsymbol{p}_{w'}(x)$.*

(ii) *Let $w = w_{\boldsymbol{\beta}}(b)$ and $w' = w_{\boldsymbol{\beta}}(b')$ are elements in σ^qW, (i.e., $\Delta(\boldsymbol{\beta})\equiv q \pmod p$, cf. 2.3). Then w and w' are conjugate under W if and only if $f_j(w) = f_j(w')$ for $j = 0,\dots,p-1$.*

Proof. It is known that the "character table" $(\widetilde{\chi}^{\boldsymbol{\alpha},\phi}(w))$ of σ^qW is a non-singular matrix. (Here $(\boldsymbol{\alpha},\phi)$ runs over all the elements in $\widetilde{\mathcal{P}}_W^q$, and w runs over all the representatives of W-orbits in σ^qW). Since Schur functions form a basis of $\Xi_{\mathbb{C}}^n(p,q)$, we see that $\{\boldsymbol{p}_w \mid w\in\sigma^qW/\!\sim\}$ is also a basis by Proposition 2.11. In particular, they are all distinct. This proves (i). The second statement is then immediate from (2.10.2). ∎

2.14. In view of Proposition 2.13, we can determine the parameter set for the set of W-orbits in $\sigma^q W$. Put

$$\mathcal{P}_q^W = \{(\boldsymbol{\beta}, b) \mid \boldsymbol{\beta} \in \mathcal{P}_{n,e}, 0 \le b < e, \varDelta(\boldsymbol{\beta}) \equiv q \pmod{p}\}.$$

We define an equivalence relation on $\mathcal{P}_q^W$ by $(\boldsymbol{\beta}, b) \sim (\boldsymbol{\beta}', b')$ if and only if $\boldsymbol{\beta} = \boldsymbol{\beta}'$ and $\boldsymbol{p}_{\boldsymbol{\beta},b}(x) = \boldsymbol{p}_{\boldsymbol{\beta}',b'}(x)$. We denote by $\widetilde{\mathcal{P}}_q^W = \widetilde{\mathcal{P}}_q^{n,e,p}$ the set of equivalence classes in $\mathcal{P}_q^W$. Then $\widetilde{\mathcal{P}}_q^W$ parameterizes the set of W-orbits in $\sigma^q W$, and so the set $\{\boldsymbol{p}_{\boldsymbol{\beta},b} \mid (\boldsymbol{\beta}, b) \in \widetilde{\mathcal{P}}_q^W\}$ gives rise to a basis of $\Xi_{\mathbb{C}}^n(p,q)$.

2.15. In this and next subsection, we give some examples of Schur functions and power sum symmetric functions in typical cases, i.e., the case where $W = G(2,2,n)$ and the case where $W = G(e,e,2)$.

First we assume that $W = G(2,2,n)$. So W (resp. $\widetilde{W}$) is the Weyl group of type D_n (resp. type B_n). In this case, $\theta^2 = 1$, and we have $\varGamma_{\boldsymbol{\alpha}} \simeq \mathbb{Z}/2\mathbb{Z}$ if $\theta(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, and $\varGamma_{\boldsymbol{\alpha}} = 1$ otherwise. In later discussion, the case $\widetilde{\mathcal{P}}_W^0$ corresponds to the split D_n case, and the case $\widetilde{\mathcal{P}}_W^1$ corresponds to the non-split D_n case. For each $(\boldsymbol{\alpha}, \phi) \in \widetilde{\mathcal{P}}_W^0$, Schur function $\mathbf{s}_{\boldsymbol{\alpha},\phi} = (s^0_{\boldsymbol{\alpha},\phi}, s^1_{\boldsymbol{\alpha},\phi})$ is defined as follows.

$$s^0_{\boldsymbol{\alpha},\phi}(x) = \sum_{\boldsymbol{\beta} \in O(\boldsymbol{\alpha})} s_{\boldsymbol{\beta}}(x), \tag{2.15.1}$$

$$s^1_{\boldsymbol{\alpha},\phi}(x) = \begin{cases} \pm s_{\boldsymbol{\beta}}(X) & \text{if } \theta(\boldsymbol{\alpha}) = \boldsymbol{\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Here if $\theta(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$ is written as $\boldsymbol{\alpha} = (\beta;\beta)$ with a partition β of $n/2$, and $s_{\boldsymbol{\beta}}(X)$ is the Schur function with respect to the variables $X_j = x_j^{(0)} x_j^{(1)}$. (Note that $\beta = \boldsymbol{\alpha}\{1\}$ in the previous notation). The sign takes $+$ (resp. $-$) if $\phi = 1$ (resp. $\phi \ne 1$).

On the other hand, $(\boldsymbol{\alpha}, \phi) \in \widetilde{\mathcal{P}}_W^1$ if and only if $\theta(\boldsymbol{\alpha}) \ne \boldsymbol{\alpha}$, and so $\varGamma_{\boldsymbol{\alpha}} = \{1\}$. In this case, Schur function $\mathbf{s}_{\boldsymbol{\alpha},\phi}(x)$ is given as follows.

$$s^0_{\boldsymbol{\alpha},\phi}(x) = s_{\boldsymbol{\alpha}}(x) - s_{\theta(\boldsymbol{\alpha})}(x), \qquad s^1_{\boldsymbol{\alpha},\phi}(x) = 0. \tag{2.15.2}$$

Next we consider the power sum symmetric function $\boldsymbol{p}_{\boldsymbol{\beta},b} = (p^0_{\boldsymbol{\beta},b}, p^1_{\boldsymbol{\beta},b})$. A conjugacy class $C_{\boldsymbol{\beta}}$ of $\widetilde{W}$ lying in W decomposes into two W-orbits if and only if $\boldsymbol{\beta} = (\beta; -)$ with a partition $\beta = (\beta_1, \beta_2, \cdots)$ such that β_i is even for each i. Let $\beta' = (\beta_1/2, \beta_2/2, \dots)$ be a partition of $n/2$. Then we have $\beta' = \boldsymbol{\beta}[1]$ in the previous notation. Such a class $C_{\boldsymbol{\beta}}$ is called a degenerate class, and other classes are said to be non-degenerate. The function $\boldsymbol{p}_{\boldsymbol{\beta},b}(x)$ $(b = 0,1)$ associated to $(\boldsymbol{\beta}, b) \in \widetilde{\mathcal{P}}_0^W$ is given as follows.

$$p^0_{\boldsymbol{\beta},b}(x) = p_{\boldsymbol{\beta}}(x), \tag{2.15.3}$$

$$p^1_{\boldsymbol{\beta},b}(x) = \begin{cases} 0 & \text{if } C_{\boldsymbol{\beta}} \text{ is non-degenerate,} \\ \pm 2^{l(\boldsymbol{\beta})} p_{\beta'}(X) & \text{if } C_{\boldsymbol{\beta}} \text{ is degenerate.} \end{cases}$$

Here $p_{\boldsymbol\beta}(x)$ is the function for $\widetilde{W}$, and $p_{\beta'}(X)$ is the power sum symmetric function for $\mathfrak{S}_{n/2}$ with variables $X_j = x_j^{(0)}x_j^{(1)}$ associated to the partition β'. The sign takes $+$ (resp. $-$) if $b = 0$ (resp. $b = 1$).

On the other hand, $(\boldsymbol\beta, b) \in \widetilde{\mathcal{P}}_1^W$ if and only if $\varDelta(\boldsymbol\beta) \equiv 1 \pmod 2$ and $b = 0$. In this case, $\boldsymbol{p}_{\boldsymbol\beta,b}(x)$ is given as

$$p^0_{\boldsymbol\beta,b}(x) = p_{\boldsymbol\beta}(x), \qquad p^1_{\boldsymbol\beta,b}(x) = 0.$$

2.16. We now consider the case where $W = G(e,e,2)$. So W is the dihedral group of order $2e$. Put

$$\begin{aligned}\boldsymbol\alpha_0 &= (2; -; \cdots; -), \\ \boldsymbol\alpha_{ij} &= (-; \cdots; 1; \cdots; 1; \cdots; -) \quad (0 \le i \le j < e),\end{aligned}$$

where in the second case, 1 appears only in the i-th and j-th entries. If $i = j$, we understand that $\boldsymbol\alpha_{ii} = (-; \cdots; 11; \cdots; -)$. Then the set of representatives of θ-orbits of $\mathcal{P}_{2,e}$ is given as $\{\boldsymbol\alpha_0, \boldsymbol\alpha_{0j} \mid 0 \le j \le e/2\}$. Here $\varGamma_{\boldsymbol\alpha} = \{1\}$ unless e is even and $j = e/2$, in which case $\varGamma_{\boldsymbol\alpha_{0j}} = \mathbb{Z}/2\mathbb{Z}$. Consequently, Schur functions $\mathbf{s}_{\boldsymbol\alpha,\phi} = (s^k_{\boldsymbol\alpha,\phi})_{0 \le k < e}$ associated to $(\boldsymbol\alpha, \phi) \in \widetilde{\mathcal{P}}^0_W$ are given as follows.

$$s^k_{\boldsymbol\alpha,\phi}(x) = \begin{cases} \sum_{\boldsymbol\beta \in O(\boldsymbol\alpha)} s_{\boldsymbol\beta}(x) & \text{if } k = 0, \\ \pm \sum_{\boldsymbol\beta' \in O(\boldsymbol\alpha')} s_{\boldsymbol\beta'}(X) & \text{if } \boldsymbol\alpha = \boldsymbol\alpha_{0,e/2} \text{ and } k = e/2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.16.1}$$

In the second case, $\boldsymbol\alpha' = (1; -; \cdots; -)$ is an $e/2$-partition of 1 and coincides with $\boldsymbol\alpha\{e/2\}$. $s_{\boldsymbol\alpha'}(X)$ is the Schur function associated to $\boldsymbol\alpha'$ with respect to the variables $X^{(0)}, \dots, X^{(e/2-1)}$ for $X^{(k)}_j = x^{(k)}_j x^{(e/2+k)}_j$. Hence $s_{\boldsymbol\alpha'}(X) = \sum_j x^{(0)}_j x^{(e/2)}_j$ and we have

$$s^{e/2}_{\boldsymbol\alpha,\phi}(x) = \pm \sum_{i=0}^{e/2-1} \sum_j x^{(i)}_j x^{(i+e/2)}_j.$$

The sign takes $+$ (resp. $-$) if $\phi = 1$ (resp. $\phi \ne 1$).

Next we consider power sum symmetric functions. A conjugacy class $C_{\boldsymbol\beta} \subset \widetilde{W}$ belongs to W in the following case;

$$\{\boldsymbol\alpha_0, \quad \boldsymbol\alpha_{i,e-i} \ (0 \le i \le e/2)\}.$$

Then $\boldsymbol{p}_{\boldsymbol\beta,b} = (p^k_{\boldsymbol\beta,b}(x))_{0 \le k < e}$ associated to $(\boldsymbol\beta, b) \in \widetilde{\mathcal{P}}^W_0$ is given as

$$p^k_{\boldsymbol\beta,b}(x) = \begin{cases} p_{\boldsymbol\beta}(x) & \text{if } k = 0, \\ (-1)^b 2p_{\boldsymbol\alpha'}(X) & \text{if } k = e/2 \text{ and } \boldsymbol\beta = \boldsymbol\alpha_0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.16.2}$$

where $\boldsymbol{\alpha}' = (1; -; \cdots; -)$ is the $e/2$-partition as given before and also coincides with $\boldsymbol{\alpha}_0[e/2]$, and $p_{\boldsymbol{\alpha}'}(X)$ is the power sum symmetric function associated to $\boldsymbol{\alpha}'$ with respect to variables $\{X^{(0)}, \dots, X^{(e/2-1)}\}$. Hence we have

$$p_{\boldsymbol{\alpha}'}(X) = \sum_{i=0}^{e/2-1}\sum_j x_j^{(i)} x_j^{(i+e/2)}.$$

3. Hall-Littlewood functions associated to $G(e,p,n)$

3.1. Here we review some results from [S] concerning symmetric functions associated to $\widetilde{W}$. In what follows we regard the variables $x_i^{(k)}$ defined for $k \in \mathbb{Z}/e\mathbb{Z} \simeq \{0, 1, \dots, e-1\}$. For each $0 \le k < e$ and an integer $r \ge 0$, we define a function $\widetilde{q}^{(k)}_{r,\pm}(x;t)$ (according to the sign $+$ or $-$) by

$$\widetilde{q}^{(k)}_{r,\pm}(x;t) = \sum_{i \ge 1}(x_i^{(k)})^{r+\delta}\frac{\prod_j x_i^{(k)} - tx_j^{(k\pm1)}}{\prod_{j \ne i} x_i^{(k)} - x_j^{(k)}} \qquad (r \ge 1), \tag{3.1.1}$$

where $\delta = m_k - 1 - m_{k\pm1}$. In the product of the denominator, $x_j^{(k)}$ runs over all the variables in $x^{(k)}$ except $x_i^{(k)}$, while in the numerator, $x_j^{(k\pm1)}$ runs over all the variables in $x^{(k\pm1)}$. Then $\widetilde{q}^{(k)}_{r,\pm}(x;t)$ is a polynomial in $\mathbb{Z}[x;t]$ if $\delta \ge 0$, and lies in $\mathbb{Z}[x, x^{-1};t]$ in general. We define $q^{(k)}_{r,\pm}$ as follows. If $\delta \ge 0$ i.e., $m_k \ge m_{k\pm1}+1$, put $q^{(k)}_{r,\pm} = \widetilde{q}^{(k)}_{r,\pm}$. If $\delta < 0$, we add $m_{k\pm1} + 1 - m_k$ variables $\mathbf{x}' = x^{(k)}_{m_k+1}, \dots, x^{(k)}_{m_{k\pm1}+1}$ to $x^{(k)}$, and consider the polynomial $\widetilde{q}^{(k)}_{r,\pm}$ for such variables with $\delta = 0$, and put $q^{(k)}_{r,\pm} = \widetilde{q}^{(k)}_{r,\pm}|_{\mathbf{x}'=0}$. Hence $q^{(k)}_{r,\pm} \in \mathbb{Z}[x;t]$ in all cases, and we have $q^{(k)}_{0,\pm} = 1$. (Note that in [S, 2.2], $q^{(k)}_{r,\pm}$ was defined by using the formula (3.1.1). However, this definition is not appropriate for the case where $\delta < 0$. The property stated in [S, Lemma 2.3] then holds for $q^{(k)}_{r,\pm}$ defined above).

For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)}) \in \mathcal{P}_{n,e}$, we define a function $q_{\boldsymbol{\alpha},\pm}(x)$ by

$$q_{\boldsymbol{\alpha},\pm}(x;t) = \prod_{k=0}^{e-1}\prod_{j=1}^{m_k} q^{(k)}_{\alpha_j^{(k)},\pm}(x;t). \tag{3.1.2}$$

For $\boldsymbol{\alpha} = (\alpha_j^{(k)}) \in \mathcal{P}_{n.e}$, we define a function $z_{\boldsymbol{\alpha}}(t)$ by

$$z_{\boldsymbol{\alpha}}(t) = z_{\boldsymbol{\alpha}}\prod_{k=0}^{e-1}\prod_{j=1}^{m_k}(1 - \zeta^k t^{\alpha_j^{(k)}})^{-1}, \tag{3.1.3}$$

where in the product, we neglect the factors such that $\alpha_j^{(k)} = 0$. $z_{\boldsymbol{\alpha}}$ is the order of the centralizer of $w_{\boldsymbol{\alpha}}$ in $\widetilde{W}$. Explicitly, $z_{\boldsymbol{\alpha}}$ is given as follows. For a partition $\beta = (1^{n_1}, 2^{n_2}, \dots)$, put $z_\beta = \prod_{i \ge 1} i^{n_i} n_i!$. Then $z_{\boldsymbol{\alpha}} = e^{l(\boldsymbol{\alpha})}\prod_{k=0}^{e-1} z_{\alpha^{(k)}}$.

For later use, we also give the order of the stabilizer $Z_W(w)$ of $w \in \sigma^q W$ in W. Assume that $C_{\boldsymbol{\alpha}} \subset \sigma^q W$ and that $C_{\boldsymbol{\alpha}}$ decomposes into r distinct W-orbits. Since these W-orbits have the same cardinality, we have for $w \in C_{\boldsymbol{\alpha}}$,

$$|Z_W(w)| = \frac{r}{p}|Z_{\widetilde{W}}(w)|. \tag{3.1.4}$$

We now introduce infinitely many variables $x_i^{(k)}, y_i^{(k)}$ for $i = 1, 2, \dots$ and for $0 \le k \le e-1$. As discussed in [S], $p_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\alpha},\pm}(x;t), m_{\boldsymbol{\alpha}}(x)$ can be regarded as functions with infinitely many variables $x_1^{(k)}, x_2^{(k)}, \dots$. Following [S], we introduce Cauchy's reproducing kernel associated to $\widetilde{W}$ by

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1 - tx_i^{(k+1)}y_j^{(k)}}{1 - x_i^{(k)}y_j^{(k)}}. \tag{3.1.5}$$

The following formula was proved in [S, Proposition 2.5].

Proposition 3.2. *We have*

$$\Omega(x,y;t) = \sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha},+}(x;t)m_{\boldsymbol{\alpha}}(y) = \sum_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}(x)q_{\boldsymbol{\alpha},-}(y;t), \tag{3.2.1}$$

$$\Omega(x,y;t) = \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}(t)^{-1}p_{\boldsymbol{\alpha}}(x)\bar{p}_{\boldsymbol{\alpha}}(y), \tag{3.2.2}$$

where $\boldsymbol{\alpha}$ runs over all the e-partitions of any size. In (3.2.2), $\bar{p}_{\boldsymbol{\alpha}}(y)$ denotes the complex conjugate of $p_{\boldsymbol{\alpha}}(y)$.

3.3. We define an isomorphism θ of $\mathbb{C}[x]$ (the polynomial ring with infinitely many variables) by $\theta(x_j^{(k)}) = x_j^{(k+d)}$. Then it follows from (2.2.2) that $\theta(p_r^{(k)}) = \zeta^{-kd}p_r^{(k)}$, and so we have

$$\theta(p_{\boldsymbol{\alpha}}) = \zeta^{-\varDelta(\boldsymbol{\alpha})d}p_{\boldsymbol{\alpha}}. \tag{3.3.1}$$

On the other hand, it can be checked easily from the definition that

$$\theta(s_{\boldsymbol{\alpha}}) = s_{\theta(\boldsymbol{\alpha})}, \qquad \theta(m_{\boldsymbol{\alpha}}) = m_{\theta(\boldsymbol{\alpha})}, \qquad \theta(q_{\boldsymbol{\alpha},\pm}) = q_{\theta(\boldsymbol{\alpha}),\pm}. \tag{3.3.2}$$

We now introduce an operator $\varTheta_q : \mathbb{C}[x] \to \mathbb{C}[x]$ for a factor q of e by

$$\varTheta_q = \sum_{i=0}^{p-1}\zeta^{qid}\theta^i. \tag{3.3.3}$$

Then it is easy to see that

$$\varTheta_q(p_{\boldsymbol{\beta}}) = \begin{cases} p \cdot p_{\boldsymbol{\beta}} & \text{if } \varDelta(\boldsymbol{\beta}) \equiv q \pmod{p}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.4}$$

Take $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ and put $c = c_{\boldsymbol{\alpha},p}$ as before. Then it follows from (3.3.2) that we have

$$\Theta_q(s_{\boldsymbol{\alpha}}) = \begin{cases} \dfrac{p}{c}\displaystyle\sum_{i=0}^{c-1}\zeta^{qid}s_{\theta^i(\boldsymbol{\alpha})} & \text{if } qc \equiv 0 \pmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.5}$$

Similar formulas as (3.3.5) hold also for $\Theta_q(m_{\boldsymbol{\alpha}})$ and $\Theta_q(q_{\boldsymbol{\alpha},\pm})$ by replacing $s_{\theta^i(\boldsymbol{\alpha})}$ in (3.3.5) by $m_{\theta^i(\boldsymbol{\alpha})}$ and $q_{\theta^i(\boldsymbol{\alpha}),\pm}$.

Let us define $m^j_{\boldsymbol{\alpha},\phi}(x)$ and $q^j_{\boldsymbol{\alpha},\phi,\pm}(x;t)$, for each $0 \le j < p$, as in the case of $\mathbf{s}_{\boldsymbol{\alpha},\phi}(x)$ by replacing $s_{\theta^i(\boldsymbol{\alpha})\{j\}}$ in (2.10.1) by $m_{\theta^i(\boldsymbol{\alpha})\{j\}}$ and $q_{\theta^i(\boldsymbol{\alpha})\{j\},\pm}$. We put

$$\boldsymbol{q}_{\boldsymbol{\alpha},\phi,\pm} = (q^j_{\boldsymbol{\alpha},\phi,\pm})_{0\le j<p}, \qquad \mathbf{m}_{\boldsymbol{\alpha},\phi} = (m^j_{\boldsymbol{\alpha},\phi})_{0\le j<p}. \tag{3.3.6}$$

Note that θ induces an action $\theta(X_i^{(k)}) = X_i^{(k+d)}$ on the variables $\mathcal{X}_j$. Thus similar formulas as (3.3.4) and (3.3.5) hold also for functions $p_{\boldsymbol{\alpha}[j]}(\mathcal{X}_j)$ and $s_{\boldsymbol{\alpha}\{j\}}(\mathcal{X}_j)$, etc. (More precisely, the formula for $s_{\boldsymbol{\alpha}\{j\}}$ is completely similar to (3.3.5). But for $p_{\boldsymbol{\beta}}(\mathcal{X}_j)$, ζ must be replaced by ζ^h. In particular, we have $\theta(p_r^{(k)}(\mathcal{X}_j)) = \zeta^{-khd}p_r^{(k)}(\mathcal{X}_j)$. Hence the condition in the first formula in (3.3.4) should be replaced by "if $h\varDelta(\boldsymbol{\beta}) \equiv q \pmod p$" for a j_1d-partition $\boldsymbol{\beta}$). Now $q_{\boldsymbol{\alpha},\pm}$ can be written as a linear combination of $s_{\boldsymbol{\beta}}$. Then by applying Θ_q, $\sum_i \zeta^{qid}q_{\theta^i(\boldsymbol{\alpha}),\pm}$ can be written as a linear combination of various $\sum_i \zeta^{qid}s_{\theta^i(\boldsymbol{\beta})}$. It follows that $q^j_{\boldsymbol{\alpha},\phi,\pm} \in \Xi^{n/h_j}_{p,q}(\mathcal{X}_j)$. Hence by (2.12.1), $\boldsymbol{q}_{\boldsymbol{\alpha},\phi,\pm}$ is contained in $\Xi^n_{\mathbb{C}(t)}(p,q) = \mathbb{C}(t) \otimes \Xi^n_{\mathbb{C}}(p,q)$. Similarly, $\mathbf{m}_{\boldsymbol{\alpha},\phi}$ is contained in $\Xi^n_{\mathbb{C}(t)}(p,q)$. Now it is easy to see that the sets $\{\boldsymbol{q}_{\boldsymbol{\alpha},\phi,\pm} \mid (\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}^q_W\}$ and $\{\mathbf{m}_{\boldsymbol{\alpha},\phi} \mid (\boldsymbol{\alpha},\phi) \in \widetilde{\mathcal{P}}^q_W\}$ form bases of $\Xi^n_{\mathbb{C}(t)}(p,q)$.

For each j $(0 \le j < p)$, we also consider the variables $\mathcal{Y}_j = (Y_i^{(k)})$ defined in a similar way as $\mathcal{X}_j$ but obtained from the variables $y = (y_i^{(k)})$ instead of $x = (x_i^{(k)})$. We now define Cauchy's reproducing kernel associated to σ^qW by $\boldsymbol{\Omega}_q(x,y;t) = (\Omega^j_q(x,y;t))_{0\le j<p}$, with

$$\Omega^j_q(x,y;t) = \Theta_{q,\mathcal{X}_j}(\Omega(\mathcal{X}_j,\mathcal{Y}_j;t^{h_j})), \tag{3.3.7}$$

where $\Omega(\mathcal{X}_j,\mathcal{Y}_j;t^{h_j})$ is the function defined in a similar way as (3.1.4), with respect to the variables $\mathcal{X}_j, \mathcal{Y}_j$ and t^{h_j}. $\Theta_{q,\mathcal{X}_j}$ stands for the action of Θ_q on the variables $\mathcal{X}_j$. For $w = w_{\boldsymbol{\alpha}}(b) \in \sigma^qW$, we define $z_{\boldsymbol{\alpha},b}(t)$ by

$$z_{\boldsymbol{\alpha},b}(t) = z_{\boldsymbol{\alpha},b}\prod_{k=0}^{e-1}\prod_{j=1}^{m_k}(1-\zeta^k t^{\alpha_j^{(k)}})^{-1}. \tag{3.3.8}$$

Hence, $z_{\boldsymbol{\alpha},b}(t)$ is the one obtained from $z_{\boldsymbol{\alpha}}(t)$ by replacing $z_{\boldsymbol{\alpha}} = |Z_{\widetilde{W}}(w)|$ by $z_{\boldsymbol{\alpha},b} = |Z_W(w)|$, and so it coincides with $rp^{-1}Z_{\boldsymbol{\alpha}}(t)$ by (3.1.4).

The following proposition gives a counter part of Proposition 3.2 to the case of σ^qW.

Proposition 3.4. *$\mathbf{\Omega}_q(x,y;t)$ has the following expansions.*

$$\mathbf{\Omega}_q(x,y;t) = \sum_{(\boldsymbol{\alpha},\phi)} \boldsymbol{q}_{\boldsymbol{\alpha},\phi,+}(x;t)\overline{\mathbf{m}}_{\boldsymbol{\alpha},\phi}(y) = \sum_{(\boldsymbol{\alpha},\phi)} \mathbf{m}_{\boldsymbol{\alpha},\phi}(x)\overline{\boldsymbol{q}}_{\boldsymbol{\alpha},\phi,-}(y;t), \tag{3.4.1}$$

$$\mathbf{\Omega}_q(x,y;t) = \sum_{(\boldsymbol{\alpha},b)} z_{\boldsymbol{\alpha},b}(t)^{-1}\boldsymbol{p}_{\boldsymbol{\alpha},b}(x)\overline{\boldsymbol{p}}_{\boldsymbol{\alpha},b}(y), \tag{3.4.2}$$

where $(\boldsymbol{\alpha},\phi)$ runs over all the elements in $\bigcup_{n=1}^{\infty}\widetilde{\mathcal{P}}^q_{n,e,p}$ in (3.4.1). In (3.4.2), $(\boldsymbol{\alpha},b)$ runs over all the elements in $\bigcup_{n=1}^{\infty}\widetilde{\mathcal{P}}^{n,e,p}_q$. $\overline{\mathbf{m}}_{\boldsymbol{\alpha},\phi}$ (resp. $\overline{\boldsymbol{q}}_{\boldsymbol{\alpha},\phi,-}$, $\overline{\boldsymbol{p}}_{\boldsymbol{\alpha},b}$) denotes the complex conjugate of $\mathbf{m}_{\boldsymbol{\alpha},\phi}$ (resp. $\boldsymbol{q}_{\boldsymbol{\alpha},\phi,-}$, $\boldsymbol{p}_{\boldsymbol{\alpha},b}$), respectively.

Proof. First we show (3.4.1). We fix j and put $h = h_j$ as before. Let us consider the expansion of $\Omega(\mathcal{X}_j,\mathcal{Y}_j;t^h)$ by making use of the first equality of (3.2.1). By applying the operator $\Theta_{q,\mathcal{X}_j}$ on both sides of this expansion, we have

$$\Omega^j_q(x,y;t) = \sum_{\boldsymbol{\beta}} \Theta_q(q_{\boldsymbol{\beta},+}(\mathcal{X}_j;t^h))m_{\boldsymbol{\beta}}(\mathcal{Y}_j), \tag{3.4.3}$$

where $\boldsymbol{\beta}$ runs over j_1d-partitions of any size. Take $\boldsymbol{\beta} \in \mathcal{P}_{n/h,j_1d}$. In view of (3.3.5), we may only consider $\boldsymbol{\beta}$ such that $qc \equiv 0 \pmod p$ in the sum (3.4.3), where $c = c_{\boldsymbol{\beta},p}$. By making use of the formula for $\Theta_q(q_{\boldsymbol{\beta},+})$ which is similar to (3.3.5), we have

$$\begin{aligned}\sum_{\boldsymbol{\beta}'\in O(\boldsymbol{\beta})} \Theta_q(q_{\boldsymbol{\beta}',+})m_{\boldsymbol{\beta}'} &= \sum_{\boldsymbol{\beta}'\in O(\boldsymbol{\beta})}\biggl\{\frac{p}{c}\sum_{i=0}^{c-1}\zeta^{qid}q_{\theta^i(\boldsymbol{\beta}'),+}\biggr\}m_{\boldsymbol{\beta}'} \\ &= \frac{p}{c}\sum_{i=0}^{c-1}\zeta^{qid}q_{\theta^i(\boldsymbol{\beta}),+}\sum_{k=0}^{c-1}\zeta^{-qkd}m_{\theta^k(\boldsymbol{\beta})}.\end{aligned} \tag{3.4.4}$$

Now there exists $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha}\{j\}$. Then c coincides with $c_{\boldsymbol{\alpha},p}$. Moreover the action of θ on $\boldsymbol{\beta}$ is compatible with that on $\boldsymbol{\alpha}$. Then the last formula of (3.4.4) can be written as

$$\begin{aligned}\sum_{\phi\in\Gamma^\wedge_{\boldsymbol{\alpha}}}&\biggl\{\phi(\tau^j)\sum_{i=0}^{c-1}\zeta^{qid}q_{\theta^i(\boldsymbol{\alpha})\{j\},+}(\mathcal{X}_j;t^h)\biggr\}\biggl\{\phi(\tau^{-j})\sum_{k=0}^{c-1}\zeta^{-kid}m_{\theta^i(\boldsymbol{\alpha})\{j\}}(\mathcal{Y}_j)\biggr\} \\ &= \sum_{\phi\in\Gamma^\wedge_{\boldsymbol{\alpha}}} q^j_{\boldsymbol{\alpha},\phi,+}(x;t))\overline{m^j_{\boldsymbol{\alpha},\phi}(y)}.\end{aligned}$$

It follows that

$$\Omega^j_q(x,y;t) = \sum_{(\boldsymbol{\alpha},\phi)} q^j_{\boldsymbol{\alpha},\phi,+}(x;t)\overline{m^j_{\boldsymbol{\alpha},\phi}(y)},$$

where $(\boldsymbol{\alpha},\phi)$ runs over all the elements in $\bigcup_{n=1}^{\infty}\widetilde{\mathcal{P}}^q_{n,e,p}$. This shows the first equality of (3.4.1). The second equality is shown similarly.

Next we show (3.4.2). We consider the expansion of $\Omega(\mathcal{X}_j, \mathcal{Y}_j; t^h)$ by (3.2.2). By applying $\Theta_{q,\mathcal{X}_j}$ on this equality, together with (3.3.4), we have

$$\Omega^j_q(x,y;t) = p\sum_{\boldsymbol{\beta}} z_{\boldsymbol{\beta}}(t^h)^{-1}p_{\boldsymbol{\beta}}(\mathcal{X}_j)\overline{p_{\boldsymbol{\beta}}(\mathcal{Y}_j)}, \tag{3.4.5}$$

where $\boldsymbol{\beta}$ runs over j_1d-partitions of any size such that $h\Delta(\boldsymbol{\beta}) \equiv q \pmod p$ (see the discussion in 3.3). $z_{\boldsymbol{\beta}}(t^h)$ is the function defined similar to (2.1.3) for a j_1d-partition $\boldsymbol{\beta}$, by replacing ζ by ζ^h. We fix $\boldsymbol{\beta} \in \mathcal{P}_{n/h,j_1d}$ as above. Then there exists $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ such that $\boldsymbol{\beta} = \boldsymbol{\alpha}[j]$. Hence $\boldsymbol{\alpha}$ satisfies (2.6.1), and $\Delta(\boldsymbol{\alpha}) = h\Delta(\boldsymbol{\beta})$. Let r be the number of all the pairs $(\boldsymbol{\alpha}, b) \in \widetilde{\mathcal{P}}^{n,e,p}_q$ for a fixed $\boldsymbol{\alpha}$. Then by (2.10.2), we have

$$p_{\boldsymbol{\beta}}(\mathcal{X}_j)\overline{p_{\boldsymbol{\beta}}(\mathcal{Y}_j)} = h^{-2l(\boldsymbol{\beta})}r^{-1}\sum_b p^j_{\boldsymbol{\alpha},b}(x)\overline{p^j_{\boldsymbol{\alpha},b}(y)}, \tag{3.4.6}$$

where the sum is taken over all the pairs $(\boldsymbol{\alpha}, b)$ for a fixed $\boldsymbol{\alpha}$. Now using the explicit description of $z_{\boldsymbol{\alpha}}(t)$ in (3.1.3) and subsequent parts, one can check that

$$z_{\boldsymbol{\beta}}(t^h) = h^{-2l(\boldsymbol{\beta})}z_{\boldsymbol{\alpha}}(t). \tag{3.4.7}$$

Substituting (3.4.6) and (3.4.7) into (3.4.5), together with (3.3.8), we have

$$\Omega^j_q(x,y;t) = \sum_{(\boldsymbol{\alpha},b)} z_{\boldsymbol{\alpha},b}(t)^{-1}p^j_{\boldsymbol{\alpha},b}(x)\overline{p^j_{\boldsymbol{\alpha},b}(y)},$$

where $(\boldsymbol{\alpha}, b)$ runs over all the elements in $\bigcup_{n=1}^{\infty}\widetilde{\mathcal{P}}^{n,e,p}_q$. This implies (3.4.2), and the proposition is proved. □

3.5. As in [S], we denote by $Z^{0,0}_n = Z^{0,0}_n(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = n$ and that each $\alpha^{(k)}$ is regarded as an element in $\mathbb{Z}^{m_k}$, written in the form $\alpha^{(k)} : \alpha^{(k)}_1 \geq \cdots \geq \alpha^{(k)}_{m_k} \geq 0$. We fix an integer $r > 0$, and consider an e-partition $\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\mathbf{m}) = (\Lambda_0, \dots, \Lambda_{e-1})$ defined by

$$\Lambda_i : \ (m_i - 1)r > \cdots > 2r > r > 0 \qquad \text{for } 0 \leq i \leq e-1. \tag{3.5.1}$$

In [S], the set $Z^{r,s}_n$ of symbols are introduced. In this paper, we are only concerned with the symbol of the form $Z^{r,0}_n = Z^{r,0}_n(\mathbf{m})$, i.e., the set of e-partitions of the form $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha} \in Z^{0,0}_n$ and the sum is taken entry-wise. We denote by $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, and call it the e-symbol of type $(r, 0)$ corresponding to $\boldsymbol{\alpha}$. We write $|\boldsymbol{\Lambda}| = n$ if $\boldsymbol{\Lambda} \in Z^{r,0}_n$.

As in [S], we consider the shift operation $Z^{r,0}_n(\mathbf{m}) \to Z^{r,0}_n(\mathbf{m}')$ for $\mathbf{m}' = (m_0 + 1, \dots, m_{e-1} + 1)$ by associating $\boldsymbol{\Lambda}' = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\mathbf{m}')$ to $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\mathbf{m})$, where $\boldsymbol{\alpha}$ is regarded as an element of $Z^{0,0}_n(\mathbf{m}')$ by adding 0 in the entries of $\boldsymbol{\alpha}$. We often consider the symbols as elements in the equivalence class in the set $\coprod_{\mathbf{m}'} Z^{r,0}_n(\mathbf{m}')$ under the shift operation. We denote by $\bar{Z}^{r,0}_n$ the set of equivalence classes. Note that $\boldsymbol{\Lambda}^0$ is regarded as a symbol in $Z^{r,0}_n$ with $n = 0$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $\bar{Z}_n^{r,0}$ are said to be similar (and denoted as $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$) if there exist representatives in $Z_n^{r,0}$ such that all the entries of them coincide each other with multiplicities. The equivalence class with respect to this relation is called a similarity class in $\bar{Z}_n^{r,0}$.

We shall define a function $a : \bar{Z}_n^{r,0} \to \mathbb{N}$. For $\boldsymbol{\Lambda} \in Z_n^{r,0}$, we put

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda,\lambda' \in \boldsymbol{\Lambda}} \min(\lambda, \lambda') - \sum_{\mu,\mu' \in \boldsymbol{\Lambda}^0} \min(\mu, \mu'), \tag{3.5.2}$$

which induces a well-defined function a on $\bar{Z}_n^{r,0}$. The a-function takes a constant value on each similarity class in $Z_n^{r,0}$. By using the bijection $Z_n^{0,0} \simeq Z_n^{r,0}$, a-functions and similarity classes are defined for $Z_n^{0,0}$, for which we use the same notation as for $Z_n^{0,0}$.

Under the natural bijection $\mathcal{P}_{n,e} \simeq Z_n^{0,0} \simeq Z_n^{r,0}$, the operation θ is transfered to the action on $Z_n^{r,0}$, which we denote also by θ. It is clear that θ preserves each similarity class in $Z_n^{r,0}$.

3.6. As discussed in [S], $\{s_{\boldsymbol{\alpha}}(x)\}$ and $\{m_{\boldsymbol{\alpha}}(x)\}$ form bases of the $\mathbb{Z}[t]$-module $\mathbb{Z}[t] \otimes \Xi$, and $\{q_{\boldsymbol{\alpha},\pm}(x;t)\}$ form a basis of $\Xi_{\mathbb{Q}(t)} = \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi$. Moreover, $\{p_{\boldsymbol{\alpha}}(x)\}$ gives a basis of $\mathbb{C}(t)$-space $\Xi_{\mathbb{C}(t)} = \mathbb{C}(t) \otimes_{\mathbb{Z}} \Xi$.

Following [S], we define a scalar product on $\Xi_{\mathbb{Q}}[t]$ by the condition that

$$\langle q_{\boldsymbol{\alpha},+}(x;t), m_{\boldsymbol{\beta}}(x)\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{3.6.1}$$

and extend it to a sesquilinear form on $\Xi_{\mathbb{C}(t)}$. Then by Proposition 3.2, we also have

$$\begin{aligned} \langle m_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\beta},-}(x;t)\rangle &= \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \\ \langle p_{\boldsymbol{\alpha}}(x), p_{\boldsymbol{\beta}}(x)\rangle &= z_{\boldsymbol{\alpha}}(t)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}. \end{aligned} \tag{3.6.2}$$

The Hall-Littlewood functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ were introduced in [S]. By Corollary 4.6 in [S], they satisfy the following formulas.

$$\Omega(x,y;t) = \sum_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) P^{+}_{\boldsymbol{\Lambda}}(x;t) P^{-}_{\boldsymbol{\Lambda}'}(y;t). \tag{3.6.3}$$

$$\Omega(x,y;t) = \sum_{\boldsymbol{\Lambda}} Q^{+}_{\boldsymbol{\Lambda}}(x;t) P^{-}_{\boldsymbol{\Lambda}}(y;t) = \sum_{\boldsymbol{\Lambda}} P^{+}_{\boldsymbol{\Lambda}}(x;t) Q^{-}_{\boldsymbol{\Lambda}}(y;t), \tag{3.6.4}$$

where in (3.6.3), $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}'$ run over all the elements in $\bigcup_{n=1}^{\infty} Z_n^{r,0}$, and $b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) = 0$ unless $|\boldsymbol{\Lambda}| = |\boldsymbol{\Lambda}'|$ and $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$. In (3.6.4), $\boldsymbol{\Lambda}$ runs over all the elements in $\bigcup_{n=1}^{\infty} Z_n^{r,0}$.

We now define a total order $\prec$ on the set $Z_n^{0,0}$ satisfying the condition that each similarity class forms an interval and that $a(\boldsymbol{\alpha}) > a(\boldsymbol{\beta})$ implies that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$. The following characterization of Hall-Littlewood functions was given in Proposition 4.8 in [S].

Proposition 3.7. (i) $P^{\pm}_{\boldsymbol{\Lambda}}$ *are characterized by the following two properties.*

(a) $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ *can be expressed in terms of* $s_{\boldsymbol{\beta}}(x)$ *as*

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}}$$

with $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Q}(t)$, *where* $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, *and* $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ *unless* $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ *and* $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(b) $\langle P^{+}_{\boldsymbol{\Lambda}}, P^{-}_{\boldsymbol{\Lambda}'}\rangle = 0$ *unless* $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$.

(ii) *The functions* $\{Q^{\pm}_{\boldsymbol{\Lambda}}\}$ *are characterized as the dual basis of* $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$, *i.e.*,

$$\langle P^{+}_{\boldsymbol{\Lambda}}, Q^{-}_{\boldsymbol{\Lambda}'}\rangle = \langle Q^{+}_{\boldsymbol{\Lambda}}, P^{-}_{\boldsymbol{\Lambda}'}\rangle = \delta_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}.$$

3.8. We now define a sesquilinear form on $\Xi^n_{\mathbb{C}(t)}(p,q)$ by the condition that

$$\langle \boldsymbol{q}_{z,+}(x;t), \mathbf{m}_{z'}(x)\rangle = \delta_{z,z'} \tag{3.8.1}$$

for $z, z' \in \widetilde{\mathcal{P}}^q_{n,e,p}$. Then by Proposition 3.4, we also have

$$\begin{aligned} \langle \mathbf{m}_z(x), \boldsymbol{q}_{z'}(x;t)\rangle &= \delta_{z,z'} \qquad (z, z' \in \widetilde{\mathcal{P}}^q_{n,e,p}), \\ \langle \boldsymbol{p}_{\xi}(x), \boldsymbol{p}_{\xi'}(x)\rangle &= z_{\xi}(t)\delta_{\xi,\xi'} \qquad (\xi, \xi' \in \widetilde{\mathcal{P}}^{n,e,p}_q). \end{aligned} \tag{3.8.2}$$

Let us denote by $\widetilde{Z}^{0,0}_{n,q}$ (resp. $\widetilde{Z}^{r,0}_{n,q}$) the set of pairs $(\boldsymbol{\alpha}, \phi)$ (resp. $(\boldsymbol{\Lambda}(\boldsymbol{\alpha}), \phi)$) such that $(\boldsymbol{\alpha}, \phi) \in \widetilde{\mathcal{P}}^q_{n,e,p}$ and that $\boldsymbol{\alpha} \in Z^{0,0}_n$. We often identify $\widetilde{Z}^{0,0}_{n,q}$ and $\widetilde{Z}^{r,0}_{n,q}$ with $\widetilde{\mathcal{P}}^q_{n,e,p}$. A similarity class and an a-function on $\widetilde{Z}^{0,0}_{n,q}$ are defined as those inherited from $Z^{0,0}_n$. We define a total order $\prec$ on $\widetilde{Z}^{0,0}_{n,q}$ compatible with the order on $Z^{0,0}_n$, i.e., for $z = (\boldsymbol{\alpha}, \phi), z' = (\boldsymbol{\alpha}', \phi') \in \widetilde{Z}^{0,0}_{n,q}$, we have $z \prec z'$ if $a(\boldsymbol{\alpha}) > a(\boldsymbol{\alpha}')$, and the set $\{(\boldsymbol{\alpha}, \phi)\}$, where $\boldsymbol{\alpha}$ falls in a fixed similarity class in $Z^{0,0}_n$, form an interval in this order.

The scalar product on $\Xi_{\mathbb{C}(t)}$ given in 3.6 induces a sesquilinear form $\langle\ ,\ \rangle_j$ on the space $\mathbb{C}(t) \otimes \Xi^{n/h_j}_{p,q}(\mathcal{X}_j)$, which satisfies the formula

$$\langle q^j_{\boldsymbol{\alpha},\phi,+}(x), m^j_{\boldsymbol{\alpha}',\phi'}(x)\rangle_j = \begin{cases} \phi(\tau^j)\overline{\phi'(\tau^j)}c_{\boldsymbol{\alpha},p} & \text{if } \boldsymbol{\alpha} = \boldsymbol{\alpha}' \text{ and } c_{\boldsymbol{\alpha},p} \mid j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8.3}$$

This implies that for $\mathbf{u} = (u^j), \mathbf{v} = (v^j) \in \Xi^n_{\mathbb{C}(t)}(p,q) = \bigoplus_{j=0}^{p-1} \mathbb{C}(t) \otimes \Xi^{n/h_j}_{p,q}(\mathcal{X}_j)$, the following formula holds, since it certainly holds for $\mathbf{u} = \boldsymbol{q}_{\boldsymbol{\alpha},\phi,+}$ and $\mathbf{v} = \mathbf{m}_{\boldsymbol{\alpha}',\phi'}$ by (3.8.3).

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{p}\sum_{j=0}^{p-1} \langle u^j, v^j\rangle_j\,. \tag{3.8.4}$$

3.9. In view of (3.3.2) and (3.6.1), the automorphism θ leaves the scalar product on $\Xi_{\mathbb{Q}(t)}$ invariant. It follows by Proposition 3.7 that we have

$$\theta(P^{\pm}_{\boldsymbol{\Lambda}}) = P^{\pm}_{\theta(\boldsymbol{\Lambda})}, \qquad \theta(Q^{\pm}_{\boldsymbol{\Lambda}}) = Q^{\pm}_{\theta(\boldsymbol{\Lambda})}.$$

Hence a similar formula as (3.3.5) holds also for $P^{\pm}_{\boldsymbol{\Lambda}}$ (resp. for $Q^{\pm}_{\boldsymbol{\Lambda}}$), by replacing $s_{\theta^i(\boldsymbol{\alpha})}$ by $P^{\pm}_{\theta^i(\boldsymbol{\Lambda})}$ (resp. by $Q^{\pm}_{\theta^i(\boldsymbol{\Lambda})}$). Now for each $z = (\boldsymbol{\alpha}, \phi) \in \widetilde{Z}^{0,0}_{n,q}$, we define

$$\mathbf{P}^{\pm}_z = (P^{\pm,j}_z)_{0 \le j < p}, \qquad \mathbf{Q}^{\pm}_z = (Q^{\pm,j}_z)_{0 \le j < p}, \tag{3.9.1}$$

where $P^{\pm,j}_z$ and $Q^{\pm,j}_z$ are defined as in the case of $\mathbf{s}_{\boldsymbol{\alpha},\phi}(x)$ by replacing $s_{\theta^i(\boldsymbol{\alpha})\{j\}}$ in (2.10.1) by $P^{\pm}_{\theta^i(\boldsymbol{\Lambda})\{j\}}$ and $Q^{\pm}_{\theta^i(\boldsymbol{\Lambda})\{j\}}$. By using a similar argument as in 3.3, we see that the sets $\{\mathbf{P}^{\pm}_z\}$ and $\{\mathbf{Q}^{\pm}_z\}$ give rise to bases of $\Xi^n_{\mathbb{C}(t)}(p, q)$.

It is known that $P^{\pm}_{\boldsymbol{\Lambda}}(x; 0) = s_{\boldsymbol{\alpha}}(x)$ for $\boldsymbol{\Lambda}(\boldsymbol{\alpha}) = \boldsymbol{\Lambda}$ by [S]. Hence by (3.9.1), we have

$$\mathbf{P}^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha}),\phi}(x; 0) = \mathbf{s}_{\boldsymbol{\alpha},\phi}(x). \tag{3.9.2}$$

The functions $\mathbf{P}^{\pm}_z(x)$ and $\mathbf{Q}^{\pm}_z(x)$ are characterized by the following properties, in analogy to Proposition 3.7.

Proposition 3.10. (i) *$\mathbf{P}^{\pm}_z$ ($z \in \widetilde{Z}^{r,0}_{n,q}$) are characterized by the following two properties.*

(a) *$\mathbf{P}^{\pm}_z(x; t)$ can be expressed in terms of $\mathbf{s}_{z'}(x)$ as*

$$\mathbf{P}^{\pm}_z = \mathbf{s}_z + \sum_{z'} u^{\pm}_{z,z'} \mathbf{s}_{z'}$$

with $u^{\pm}_{z,z'} \in \mathbb{C}(t)$, and $u^{\pm}_{z,z'} = 0$ unless $z' \prec z$ and $z' \not\sim z$.

(b) *$\langle \mathbf{P}^+_z, \mathbf{P}^-_{z'} \rangle = 0$ unless $z \sim z'$.*

(ii) *The functions $\{\mathbf{Q}^{\pm}_z\}$ are characterized as the dual basis of $\{\mathbf{P}^{\pm}_z\}$, i.e.,*

$$\langle \mathbf{P}^+_z, \mathbf{Q}^-_{z'} \rangle = \langle \mathbf{Q}^+_z, \mathbf{P}^-_{z'} \rangle = \delta_{z,z'} \qquad (z, z' \in \widetilde{Z}^{r,0}_{n,q}).$$

Proof. By substituting $t = 0$ into (3.4.2), we have

$$\boldsymbol{\Omega}_q(x, y; 0) = \sum_{(\boldsymbol{\alpha},b)} z^{-1}_{\boldsymbol{\alpha},b} \boldsymbol{p}_{\boldsymbol{\alpha},b}(x) \overline{\boldsymbol{p}}_{\boldsymbol{\alpha},b}(y),$$

where $z_{\boldsymbol{\alpha},b} = |Z_W(w)|$ for $w = w_{\boldsymbol{\alpha}}(b)$. On the other hand, starting from the expansion of $\Omega(x, y; 0)$ by means of Schur functions $s_{\boldsymbol{\alpha}}(x)$ as given in Remark 4.9 in [S], by using a similar argument as in the proof of (3.4.1), one can show that

$$\boldsymbol{\Omega}_q(x, y; 0) = \sum_{(\boldsymbol{\alpha},\phi)} \mathbf{s}_{\boldsymbol{\alpha},\phi}(x) \mathbf{s}_{\boldsymbol{\alpha},\phi}(y).$$

It follows that one can define a hermitian form on $\Xi^n_{\mathbb{C}}(p,q)$ satisfying the properties that $\langle \boldsymbol{p}_\xi, \boldsymbol{p}_{\xi'}\rangle = \delta_{\xi,\xi'}$ $(\xi,\xi' \in \widetilde{\mathcal{P}}^{n,e,p}_q)$, and that $\langle \mathbf{s}_z, \mathbf{s}_{z'}\rangle = \delta_{z,z'}$ $(z,z' \in \widetilde{\mathcal{P}}^q_{n,e,p})$. Hence the sesquilinear form on $\Xi^n_{\mathbb{C}(t)}(p,q)$ is reduced to the hermitian form on $\Xi^n_{\mathbb{C}}(p,q)$ by substituting $t=0$. Now the arguments in the proof of Proposition 4.8 and Remark 4.9 in [S] can be applied to our situation, and one can show that there exist unique functions satisfying the properties (i), (ii) in the proposition, respectively. So, we have only to show that $\mathbf{P}^\pm_z, \mathbf{Q}^\pm_z$ satisfy the properties in the proposition.

First we note that $\langle P^{+,j}_{\boldsymbol{\Lambda},\phi}, P^{-,j}_{\boldsymbol{\Lambda}',\phi'}\rangle = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$ by Proposition 3.7. It follows from (3.8.4) that we have $\langle \mathbf{P}^+_z, \mathbf{P}^-_{z'}\rangle = 0$ unless $z \sim z'$. This shows the property (b) of (i). Next we shall show the property (a). Clearly the statement (a) is equivalent to the statement that

$$\mathbf{P}^\pm_z = \mathbf{s}_z + \sum_{z'} d^\pm_{z,z'}\mathbf{P}^\pm_{z'} \tag{3.10.1}$$

for any $z, z' \in \widetilde{Z}^{r,0}_{n,q}$, where $d^\pm_{z,z'} = 0$ unless $z' \prec z$ and $z \not\sim z'$.

We consider the equation (3.10.1) for a fixed z with respect to the $+$ sign. Take z' such that $z' \prec z$ and $z \not\sim z'$. Taking the scalar product with $\mathbf{P}^-_{z'}$ on both sides of (3.10.1), we have

$$\langle \mathbf{s}_z, \mathbf{P}^-_{z'}\rangle + \sum_{z'' \sim z'} d^+_{z,z''}\langle \mathbf{P}^+_{z''}, \mathbf{P}^-_{z'}\rangle = 0. \tag{3.10.2}$$

When z' runs over all the elements in a fixed similarity class, (3.10.2) can be regarded as a system of equations with unknown variables $\{d^+_{z,z''}\}$. As in the arguments in the first part of the proof (cf. Remark 4.9 in [S]), the matrix $(\langle \mathbf{P}^+_{z''}, \mathbf{P}^-_{z'}\rangle)$ is non-singular. Hence (3.10.2) has a unique solution $\{d^+_{z,z''}\}$. Put

$$F_z = \mathbf{s}_z + \sum_{z''} d^+_{z,z''}\mathbf{P}^+_{z''}$$

using thus determined $d^+_{z,z''}$ for $z'' \prec z$ and $z \not\sim z''$. Then clearly we have

$$\langle \mathbf{P}^+_z, \mathbf{P}^-_{z'}\rangle = \langle F_z, \mathbf{P}^-_{z'}\rangle \qquad \text{for } z' \prec z \text{ and } z' \not\sim z. \tag{3.10.3}$$

On the other hand, it follows from Proposition 3.7 that we have

$$\langle P^+_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}, P^-_{\boldsymbol{\Lambda}'}\rangle = \langle \mathbf{s}_{\boldsymbol{\alpha}}, P^-_{\boldsymbol{\Lambda}'}\rangle \qquad \text{for } \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \prec \boldsymbol{\Lambda}' \text{ or } \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \sim \boldsymbol{\Lambda}'.$$

This implies, in view of (3.8.4), that

$$\langle \mathbf{P}^+_z, \mathbf{P}^-_{z'}\rangle = \langle \mathbf{s}_z, \mathbf{P}^-_{z'}\rangle = \langle F_z, \mathbf{P}^-_{z'}\rangle \qquad \text{for } z \prec z' \text{ or } z \sim z'. \tag{3.10.4}$$

Now (3.10.3) and (3.10.4) implies that $\mathbf{P}^+_z = F_z$, and (3.10.1) is proved for $\mathbf{P}^+_z$. The proof for $\mathbf{P}^-_z$ is similar.

Finally, we show (ii). By Proposition 3.7 (ii), we have

$$\langle P_z^{+,j}, Q_{z'}^{-,j}\rangle_j = \begin{cases} \phi(\tau^j)\overline{\phi'(\tau^j)}c_{\boldsymbol{\alpha},p} & \text{if } c_{\boldsymbol{\alpha},p} \mid j \text{ and } z = z', \\ 0 & \text{otherwise.} \end{cases}$$

This implies, by (3.8.4), that $\langle \mathbf{P}_z^+, \mathbf{Q}_{z'}^-\rangle = \delta_{z,z'}$. The formula $\langle \mathbf{Q}_z^+, \mathbf{P}_{z'}^-\rangle = \delta_{z,z'}$ is shown similarly. □

As a corollary to Proposition 3.10, we have

Corollary 3.11. *$\mathbf{\Omega}_q(x, y; t)$ has the following expansions.*

$$\mathbf{\Omega}_q(x.y; t) = \sum_{z,z'} b_{z,z'}(t)\mathbf{P}_z^+(x; t)\bar{\mathbf{P}}_{z'}^-(y; t) \tag{3.11.1}$$

$$\mathbf{\Omega}_q(x, y; t) = \sum_z \mathbf{Q}_z^+(x; t)\bar{\mathbf{P}}_z^-(y; t) = \sum_z \mathbf{P}_z^+(x; t)\bar{\mathbf{Q}}_z^-(y; t), \tag{3.11.2}$$

where $z = (\boldsymbol{\Lambda}, \phi), z' = (\boldsymbol{\Lambda}', \phi')$ run over all the elements in $\bigcup_{n=1}^\infty \widetilde{Z}_{n,q}^{r,0}$, and $b_{z,z'} = 0$ unless $|\boldsymbol{\Lambda}| = |\boldsymbol{\Lambda}'|$ and $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$ in (3.9.1). In (3.9.2), $z = (\boldsymbol{\Lambda}, \phi)$ runs over all the elements in $\bigcup_{n=1}^\infty \widetilde{Z}_{n,q}^{r,0}$.

Proof. (3.11.2) follows from (3.4.1) and Proposition 3.10 (ii) by a standard argument, (e.g. [M, I, 4.6]). We show (3.11.1). We write $\mathbf{P}_z^- = \sum_{z'} a_{z,z'}\mathbf{Q}_{z'}^-$. Since $\langle \mathbf{P}_{z'}^+, \mathbf{P}_z^-\rangle = a_{z,z'}$, Proposition 3.10 (b) implies that the transition matrix between two bases $\{\mathbf{P}_z^-\}$ and $\{\mathbf{Q}_z^-\}$ is a block diagonal matrix with respect to the partition by similarity classes. Hence its inverse matrix is also block diagonal, i.e., $\mathbf{Q}_z^-$ is written as $\mathbf{Q}_z^- = \sum_{z'} b_{z,z'}\mathbf{P}_{z'}^-$, where $b_{z,z'} = 0$ unless $z \sim z'$. Substituting this into the second formula in (3.11.2), we obtain (3.11.1). Thus the corollary is proved. □

4. Green functions associated to $G(e, p, n)$

4.1. For functions f, h on $\sigma^q W$, we define an inner product $\langle f, h\rangle_q$ by

$$\langle f, h\rangle_q = |W|^{-1} \sum_{w\in\sigma^q W} f(w)\overline{h(w)}.$$

It is known in general that the number of σ^q-stable irreducible characters is equal to the number of W-orbits in $\sigma^q W$. An σ^q-stable irreducible character χ on W can be extended to an irreducible character of $\langle\sigma^q\rangle \ltimes W$, in e/q distinct ways. We fix an extension $\widetilde{\chi}$ of χ. Then $\widetilde{\chi}$ gives a function on $\sigma^q W$ by restriction, which is constant on each W-orbit. It is known that

$$\langle \widetilde{\chi}, \widetilde{\chi}'\rangle_q = \begin{cases} 1 & \text{if } \widetilde{\chi} = \widetilde{\chi}', \\ 0 & \text{if } \chi \neq \chi'. \end{cases} \tag{4.1.1}$$

Now the set of σ^q-stable irreducible characters is parametrized by $\widetilde{\mathcal{P}}_W^q$. Also the set of W-orbits in $\sigma^q W$ is parametrized by $\widetilde{\mathcal{P}}_q^W$. We fix a total order on $\widetilde{\mathcal{P}}_W^q \simeq \widetilde{Z}_{n,q}^{0,0}$ as in 3.6. Let $X = (\widetilde{\chi}^z(\xi))\,(z \in \widetilde{\mathcal{P}}_W^q, \xi \in \widetilde{\mathcal{P}}_q^W)$ be the character table of $\sigma^q W$. We define H as the diagonal matrix indexed by $\widetilde{\mathcal{P}}_{n,e,p}^q$ whose diagonal entry corresponding to (ξ, ξ) is $z_\xi^{-1} = |Z_W(w_{\boldsymbol{\alpha}}(b))|^{-1}$ for $\xi = (\boldsymbol{\alpha}, b)$. Then (4.1.1) can be written in a matrix form

$$
{}^tXH\bar{X} = I. \tag{4.1.2}
$$

4.2. Let $M = \mathbb{C}^n$ be the natural reflection representation of $\widetilde{W}$. We consider M as a W-module by the restriction. Let $S(M)$ be the symmetric algebra of M, and I_+ the ideal of $S(M)$ generated by W-invariant homogeneous vectors of strictly positive degree. We denote by $R = S(M)/I_+$ the coinvariant algebra of W. The Poincaré polynomial $P_W(t)$ associated to W is defined in terms of the graded algebra $R = \bigoplus R_i$. In our case, $P_W(t)$ is explicitly given as

$$
P_W(t) = \prod_{i=1}^{n-1}\frac{t^{ei}-1}{t-1}\cdot\frac{t^{dn}-1}{t-1}.
$$

Since σ normalizes W, σ acts naturally on R, stabilizing each summand R_i. We denote by $\widetilde{R}_i$ the thus obtained $\widetilde{W}$-module.

Let $\det_M$ be the linear character of $\widetilde{W}$ defined by the determinant on M. For each $\widetilde{W}$-stable function f on $\sigma^q W$, we define $R_q(f) \in \mathbb{C}(t)$ by

$$
R_q(f) = (\zeta^{qd}t^{dn}-1)\prod_{i=1}^{n-1}(t^{ei}-1)\cdot\frac{1}{|W|}\sum_{w\in\sigma^qW}\frac{\det_M(w)f(w)}{\det_M(t\cdot \mathrm{id}_M - w)}. \tag{4.2.1}
$$

Then we have

$$
R_q(f) = \sum_i \langle f, \widetilde{R}_i\rangle_q\, t^i,
$$

and, in particular, $R_q(\widetilde{\chi}^z) \in \mathbb{Z}[\zeta][t]$ for $z \in \widetilde{\mathcal{P}}_W^q$.

4.3. Let us define a square matrix $\Omega_q = (\omega_{z,z'})$ indexed by $\widetilde{Z}_{n,q}^{0,0}$ by

$$
\omega_{z,z'} = t^{N^*}R_q(\widetilde{\chi}^z\otimes\widetilde{\chi}^{z'}\otimes\overline{\det}_M),
$$

where N^* is the number of reflections in W. Also we define a matrix $\Omega'_q = (\omega'_{z,z'})$ by

$$
\omega'_{z,z'} = t^{N^*}R_q(\widetilde{\chi}^z\otimes\overline{\widetilde{\chi}^{z'}}\otimes\overline{\det}_M). \tag{4.3.1}
$$

In either of the formulas, $\overline{\det}_M$ etc. denote the complex conjugates of them. Let $P = (p_{z,z'})$ and $\varLambda = (\lambda_{z,z'})$ be the matrices indexed by $\widetilde{Z}_{n,q}^{0,0}$. We consider the following system of equations with unknown variables $p_{z,z'}, \lambda_{z,z'}$;

$$
\begin{aligned}
&\lambda_{z,z'} = 0 \quad \text{unless } z \sim z', \\
&p_{z,z'} = 0 \quad \text{unless either } z \succ z' \text{ and } z \not\sim z', \text{ or } z = z', \\
&p_{z,z'} = t^{a(z)}, \\
&P \Lambda\, {}^t\! P = \Omega_q.
\end{aligned} \tag{4.3.2}
$$

We now assume that $m_0 = m_1 = \cdots = m_{e-1}$. Then as discussed in [S, 1.5], the system of equations (4.3.2) is equivalent to the system of equations

$$
P' \Lambda'\, {}^t\! P'' = \Omega'_q, \tag{4.3.3}
$$

where either of $P' = (p'_{z,z'})$ and $P'' = (p_{z,z'})$ satisfies similar conditions as in the second and third one in (4.3.2), and $\Lambda' = (\lambda_{z,z'})$ as in the first one. (Actually, it is shown that $P = P'$). In the remainder of this section, we shall show that the solution of (4.3.3) can be described in a combinatorial way.

Remark 4.4. In the case where $e = p = 2$, $G(e, e, n)$ is the Weyl group of type D_n. In this case, the equation (4.3.2) coincides with (4.3.3), and the solution for it describes the Green functions of SO_{2n} defined over a finite field of odd characteristic of split type (resp. non-split type) if $q = 0$ (resp. $q = 1$).

4.5. For two bases X, Y of $\Xi^n_{\mathbb{C}(t)}(p, q)$, we denote by $M(X, Y)$ the transition matrix between X and Y. Let $X_\pm(t)$ be the transition matrix $M(\boldsymbol{p}, \mathbf{P}^\pm)$ between the power sum symmetric functions $\{\boldsymbol{p}_\xi\}$ and the Hall-Littlewood functions $\{\mathbf{P}^\pm_z\}$, i.e.,

$$
\boldsymbol{p}_\xi(x) = \sum_z X^z_{\xi,\pm}(t) \mathbf{P}^\pm_z(x; t). \tag{4.5.1}
$$

(Here $\xi \in \widetilde{\mathcal{P}}^{n,e,p}_q$ is the row-index and $z = (\boldsymbol{\Lambda}, \phi) \in \widetilde{\mathcal{P}}^q_{n,e,p}$ is the column-index.) Then by (3.9.1) and by the properties of $P_{\boldsymbol{\Lambda}}$ shown in [S], we see that $X^z_{\xi;\pm}(t) \in \mathbb{C}(t)$ and is equal to zero unless $|z| = |\xi|$. (For $z = (\boldsymbol{\alpha}, \phi)$ and $\xi = (\boldsymbol{\beta}, b)$, we put $|z| = |\boldsymbol{\alpha}|$ and $|\xi| = |\boldsymbol{\beta}|$, respectively.) Moreover since $\mathbf{P}^\pm_z(x; 0) = \mathbf{s}_z(x)$ by (3.9.2), we have

$$
X^z_{\xi;\pm}(0) = \widetilde{\chi}^{\boldsymbol{\alpha},\phi}(w_{\boldsymbol{\beta}}(b)) \tag{4.5.2}
$$

by Proposition 2.11. It follows that $X_\pm(0) = M(\boldsymbol{p}, \mathbf{s})$ is the character table of W. In particular, $X_\pm(0)$ is independent of the sign, which we denote simply as $X(0)$. By combining (3.4.2) and (3.11.1), we have

$$
\sum_{|\xi|=n} z_\xi(t)^{-1} \boldsymbol{p}_\xi(x) \overline{\boldsymbol{p}}_\xi(y) = \sum_{z,z'} b_{z,z'}(t) \mathbf{P}^+_z(x; t) \bar{\mathbf{P}}^-_{z'}(y; t),
$$

where in the right hand side, z, z' run over all elements in $\widetilde{Z}^{r,0}_{n,q}$. We put $D(t) = D_- = (b_{z,z'}(t))$, the matrix indexed by $\widetilde{\mathcal{P}}^q_{n,e,p} \simeq \widetilde{Z}^{0,0}_{n,q}$, and denote by $Z(t)$ the diagonal matrix indexed by $\widetilde{\mathcal{P}}^{n,e,p}_q$, where the entry corresponding to (ξ, ξ) is given by $z_\xi(t)$.

Substituting (4.5.1) into the above equation, we have

$$ {}^t X_+(t) Z(t)^{-1} \bar{X}_-(t) = D(t). $$

If we put $\varLambda(t) = D(t)^{-1}$, the above formula is equivalent to

$$ \bar{X}_-(t)\varLambda(t)\, {}^t X_+(t) = Z(t). \tag{4.5.3} $$

4.6. Let $K_\pm(t) = M(\mathbf{s}, \mathbf{P}^\pm)$ be the transition matrix between Schur functions and Hall-Littlewood functions, i.e., for $z, z' \in \widetilde{\mathcal{P}}^q_{n,e,p}$,

$$ \mathbf{s}_{z'}(x) = \sum_z K^\pm_{z',z}(t) \mathbf{P}^\pm_z(x;t). \tag{4.6.1} $$

Then by Proposition 3.10, $K_\pm(t)$ is a block lower triangular matrix with identity diagonal blocks, with entries in $\mathbb{C}(t)$. Since $M(\mathbf{s}, \mathbf{P}^\pm) = M(\boldsymbol{p}, \mathbf{s})^{-1} M(\boldsymbol{p}, \mathbf{P}^\pm)$, we have $K_\pm(t) = X(0)^{-1} X_\pm(t)$. Substituting this into (4.5.3), we have

$$ K_-(t)\varLambda(t)\, {}^t K_+(t) = \bar{X}(0)^{-1} Z(t)\, {}^t X(0)^{-1}. \tag{4.6.2} $$

We now define Green functions $Q^z_{\xi;\pm}(t) \in \mathbb{C}(t)$ by

$$ Q^z_{\xi;\pm}(t) = t^{a(z)} X^z_{\xi;\pm}(t^{-1}). $$

If we put $\widetilde{K}^\pm_{z',z}(t) = t^{a(z)} K^\pm_{z',z}(t^{-1})$, $Q^z_{\xi;\pm}(t)$ can be written as

$$ Q^z_{\xi;\pm}(t) = \sum_{z''} \widetilde{\chi}^{z''}(w_{\boldsymbol{\beta}}(b)) \widetilde{K}^\pm_{z'',z}(t) $$

for $\xi = (\boldsymbol{\beta}, b)$. Let $\widetilde{K}_\pm(t) = (\widetilde{K}^\pm_{z',z}(t))$. Then $\widetilde{K}_\pm(t) = K_\pm(t^{-1})T$, where T is a diagonal matrix with diagonal entries $t^{a(z)}$. Hence (4.6.2) can be rewritten as

$$ \widetilde{K}_-(t)\varLambda'(t)\, {}^t\widetilde{K}_+(t) = \bar{X}(0)^{-1} Z(t^{-1})\, {}^t X(0)^{-1}, \tag{4.6.3} $$

where $\varLambda'(t) = T^{-1}\varLambda(t^{-1})T$. Here $\varLambda'(t)$ is still a block diagonal matrix, and $\widetilde{K}_\pm(t)$ are block lower triangular matrices, where the diagonal blocks consist of scalar matrices $t^{a(z)}I$.

Put $\widetilde{\varLambda}(t) = t^{-n}\mathbb{G}(t)\varLambda'(t)$ with

$$ \mathbb{G}(t) = t^{N^*}(\zeta^{qd} t^{dn} - 1) \prod_{i=1}^{n-1} (t^{ei} - 1). $$

The following result gives a combinatorial description of the solution of (4.3.3).

Theorem 4.7. *We have*

$$ \widetilde{K}_-(t)\widetilde{\varLambda}(t)\, {}^t\widetilde{K}_+(t) = \Omega'_q. $$

Hence $P' = \widetilde{K}_-(t), P'' = \widetilde{K}_+(t)$ and $\varLambda' = \widetilde{\varLambda}(t)$ give a solution for the equation (4.3.3).

Proof. Let Y be the right hand side of (4.6.3). We shall compute Y. Let $w_{\boldsymbol{\alpha}}(b)$ be the element in $\sigma^q W$ corresponding to $\xi = (\boldsymbol{\alpha}, b)$. Since $X(0)$ is the character table of $\sigma^q W$, we have ${}^tX(0)H\bar{X}(0) = I$ by (4.1.2), where H is the diagonal matrix with diagonal entries z_ξ^{-1}. It follows that

$$Y = {}^tX(0)HZ(t^{-1})H\bar{X}(0). \tag{4.7.1}$$

Now it is known that if $\boldsymbol{\alpha} = (\alpha_j^{(k)}) \in \mathcal{P}_{n,e}$, we have

$$\det{}_M(t \cdot \mathrm{id}_M - w_{\boldsymbol{\alpha}}(b)) = \prod_{k=0}^{e-1} \prod_{j=1}^{l(\alpha^{(k)})} (t^{\alpha_j^{(k)}} - \zeta^k).$$

Hence by (3.3.8), $z_{\boldsymbol{\alpha},b}(t^{-1}) = z_{\boldsymbol{\alpha},b} t^n \det_V(t \cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}}(b))^{-1}$. In particular, the (z, z')-entry of Y is equal to

$$t^n|W|^{-1} \sum_{w \in \sigma^q W} \frac{\widetilde{\chi}^z(w)\overline{\widetilde{\chi}^{z'}}(w)}{\det_M(t \cdot \mathrm{id}_M - w)}.$$

Therefore $\Omega'_q = t^{-n}\mathbb{G}(t)Y$ and the theorem follows. □

4.8. Recall that Kostka functions $K^\pm_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ associated to symbols in $Z_n^{r,0}(\mathbf{m})$ are defined in [S] by the formula

$$s_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} K^\pm_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) P^\pm_{\boldsymbol{\varLambda}(\boldsymbol{\beta})}(x;t). \tag{4.8.1}$$

By making use of the preceding discussions, we can describe the Kostka functions $K^\pm_{z,z'}$ associated to $\widetilde{Z}^{r,0}_{n,q}$ in terms of $K^\pm_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. For $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in Z_n^{0,0}$, put $c = c_{\boldsymbol{\alpha},p}$, $c' = c_{\boldsymbol{\alpha}',p}$. We assume that $cq \equiv 0, c'q \equiv 0 \pmod p$. For each j such that $c' \mid j$, put

$$L^{j,\pm}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = \begin{cases} \sum_{i=0}^{c'-1} \zeta^{-qid} K^\pm_{\boldsymbol{\alpha}\{j\},\theta^i(\boldsymbol{\alpha}')\{j\}} & \text{if } c \mid j, \\ 0 & \text{otherwise.} \end{cases} \tag{4.8.2}$$

We denote by $L^\pm_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$ the diagonal matrix of size $|\varGamma_{\boldsymbol{\alpha}'}|$ whose ii-entry is given by $L^{c'i,\pm}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}$. Let $\Phi' = (\phi'(\tau^j))_{\phi',j}$ be the character table of $\varGamma_{\boldsymbol{\alpha}'}$ ($\phi' \in \varGamma^\wedge_{\boldsymbol{\alpha}'}, c'|j$), and let $\Phi = (\phi(\tau^j))_{\phi,j}$ be a part of the character table of $\varGamma_{\boldsymbol{\alpha}}$, i.e., $\phi \in \varGamma^\wedge_{\boldsymbol{\alpha}}$, but j is taken under the condition that $c \mid j$ and $c' \mid j$. For fixed $\boldsymbol{\alpha}, \boldsymbol{\alpha}'$ as above, we consider the matrix $(K^\pm_{\phi,\phi'})$, indexed by $\phi \in \varGamma^\wedge_{\boldsymbol{\alpha}}, \phi' \in \varGamma^\wedge_{\boldsymbol{\alpha}'}$, defined by $K^\pm_{\phi,\phi'} = K^\pm_{z,,z'}$. We have the following proposition.

Proposition 4.9. *The matrix $(K^\pm_{\phi,\phi'})$ can be written as*

$$(K^\pm_{\phi,\phi'}) = |\varGamma_{\boldsymbol{\alpha}'}||\varGamma_{\boldsymbol{\alpha}}|^{-1}\Phi L^\pm_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\Phi'^{-1}.$$

Proof. By using the relation (4.8.1) for Schur functions with respect to the variables $\mathcal{X}_j$, we can express $s_{\boldsymbol{\alpha}\{j\}}$ in terms of a linear combination of $P^{\pm}_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \in Z^{0,0}_{n/h_j}$. Then by applying Θ_q on both sides, we have

$$\Theta_q(s_{\boldsymbol{\alpha}\{j\}}) = \sum_{\boldsymbol{\alpha}'} K^{\pm}_{\boldsymbol{\alpha}\{j\},\boldsymbol{\alpha}'\{j\}}\Theta_q(P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha}')\{j\}}),$$

where $c' = c_{\boldsymbol{\alpha}',p}$, and $\boldsymbol{\alpha}'$ in the right hand side runs over all $\boldsymbol{\alpha}' \in Z^{0,0}_n$ such that $j \mid c'$ and that $c' \equiv 0 \pmod p$. Hence if we choose a representative $\boldsymbol{\alpha}'$ from each θ-orbit, the above formula turns out to be

$$\Theta_q(s_{\boldsymbol{\alpha}\{j\}}) = \sum_{\boldsymbol{\alpha}' \in Z^{0,0}_n/\sim_p}\sum_{i=0}^{c'-1}\zeta^{-qid}K^{\pm}_{\boldsymbol{\alpha}\{j\},\theta^i(\boldsymbol{\alpha}')\{j\}}\Theta_q(P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha}')\{j\}}). \tag{4.9.1}$$

On the other hand, by using (4.6.1), we can express $\mathbf{s}_z$ in terms of a linear combination of $\mathbf{P}^{\pm}_{z'}$. In particular, we have

$$s^j_z = \sum_{z'} K^{\pm}_{z,z'}P^{\pm,j}_{z'}$$

for any j such that $0 \le j < p$. Now by making use of (3.3.5) and a similar formula for $P_{\boldsymbol{\Lambda}}$, we can rewrite the above formula as

$$|\Gamma_{\boldsymbol{\alpha}}|^{-1}\phi(\tau^j)\Theta_q(s_{\boldsymbol{\alpha}\{j\}}) = \sum_{\boldsymbol{\alpha}' \in Z^{0,0}_n/\sim_p}|\Gamma_{\boldsymbol{\alpha}'}|^{-1}\sum_{\phi' \in \Gamma^{\wedge}_{\boldsymbol{\alpha}'}}K^{\pm}_{z,z'}\phi'(\tau^j)\Theta_q(P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha}')\{j\}})$$

for any j such that $c' \mid j$. Note that the left hand side is understood to be 0 in the case where j is not a multiple of c. Since $\Theta_q(P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha}'\{j\}})$ are linearly independent in $\Xi^{n/h_j}_{p,q}(\mathcal{X}_j)$, by comparing with (4.9.1), we have

$$|\Gamma_{\boldsymbol{\alpha}}|^{-1}\phi(\tau^j)\sum_{i=0}^{c'-1}\zeta^{-qid}K^{\pm}_{\boldsymbol{\alpha}\{j\},\theta^i(\boldsymbol{\alpha}')\{j\}} = |\Gamma_{\boldsymbol{\alpha}'}|^{-1}\sum_{\phi'}K^{\pm}_{\phi,\phi'}\phi'(\tau^j) \tag{4.9.2}$$

for each $\boldsymbol{\alpha}'$. Now, (4.9.2) holds for any $\phi \in \Gamma^{\wedge}_{\boldsymbol{\alpha}}$ and any j such that $c' \mid j$. Hence it can be translated to a matrix equation

$$|\Gamma_{\boldsymbol{\alpha}}|^{-1}\Phi L^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'} = |\Gamma_{\boldsymbol{\alpha}'}|^{-1}(K^{\pm}_{\phi,\phi'})\Phi'.$$

This proves the proposition. $\square$

The following special case would be worth mentioning.

Corollary 4.10. *Assume that $\Gamma_{\boldsymbol{\alpha}} = \Gamma_{\boldsymbol{\alpha}'} = \{1\}$. Then we have*

$$K^{\pm}_{(\boldsymbol{\alpha},1),(\boldsymbol{\alpha}',1)} = \sum_{i=0}^{p-1}\zeta^{-qid}K^{\pm}_{\boldsymbol{\alpha},\theta^i(\boldsymbol{\alpha}')}.$$

As in the case of Green functions associated to $G(e,1,n)$, one can expect that, for $z, z' \in Z^{r,0}_{n,q}$, $K^{\pm}_{z,z'}(t)$ is a polynomial in t with positive integral coefficients. In view of Corollary 4.10, the conjecture for $G(e,1,n)$, i.e., $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}(t) \in \mathbb{Z}[t]$, implies that $K^{\pm}_{z,z'}(t) \in K[t]$ with $K = \mathbb{Q}(\zeta)$.

Remarks 4.11. (i) Preposition 4.9 asserts that Green functions associated to $G(e,p,n)$ are completely described in terms of Green functions associated to $G(e',1,n')$ with various $e' \mid e, n' \mid n$. In the classical situation, this means that Green functions of type D_n can be described in terms of Green functions associated to Weyl groups of type B_n, and $\mathfrak{S}_{n/2}$ for even n. However, note that this does not mean such Green functions are described in terms of Green functions of Sp_{2n} or SO_{2n+1}. In fact, the Green functions of Sp_{2n} or SO_{2n+1} are defined in terms of symbols of the type $Z^{r,s}_n(\mathbf{m})$ with $\mathbf{m} = (m+1, m)$, while our Green functions are related to the symbols of the type $Z^{r,0}_n(\mathbf{m})$ with $\mathbf{m} = (m,m)$, and the structure of similarity classes is different.

(ii) The group $G(e,e,2)$ coincides with the dihedral group of degree $2e$. In particular, it coincides with the Weyl group of type A_2, B_2 or G_2 for $e = 3, 4$ or 6, respectively. The case where $e = 3$, our Green function coincides with the Green function associated to $GL_3(\mathbf{F}_q)$. However, as Lemma 5.4 in the next section shows, the Green function associated to $G(e,e,2)$ for $e = 4$ or $e = 6$ (for any r) does not coincide with the Green function of type B_2 or G_2.

5. Examples

5.1. In this section, we give some examples of the matrices $P = \widetilde{K}_-(t)$ associated to the group $G(e,e,n)$. But before doing it, we shall give a general remark in connection with fake degrees. Assume that $W = G(e,p,n)$, and we consider the case where $q = 0$. We write $P = (p_{z,z'})$ for $z, z' \in \widetilde{\mathcal{P}}^0_W$. For $j = 0, \dots, e-1$, put $\boldsymbol{\beta}_j = (\beta^{(0)}, \dots, \beta^{(e-1)})$, where $\beta^{(j)} = (1^n)$ and $\beta^i = \emptyset$ for $i \neq j$. In this case, $\varGamma_{\boldsymbol{\beta}_j} = \{1\}$, and we identify $z = (\boldsymbol{\beta}_j, 1)$ with $\boldsymbol{\beta}_j$. Then among $\widetilde{\mathcal{P}}^0_W$, $\mathcal{F} = \{\boldsymbol{\Lambda}(\boldsymbol{\beta}_0), \dots, \boldsymbol{\Lambda}(\boldsymbol{\beta}_{d-1})\}$ gives rise to a similarity class in $\widetilde{Z}^{r,0}_{n,0}$, which has the maximum a-value.

For each $z \in \widetilde{\mathcal{P}}^0_W$, $R_0(\chi^z)$ coincides with the fake degree $\sum_{i \ge 0} \langle \chi^z, R_i \rangle t^i$ (cf. 4.2). The following formula shows that the fake degree can be read from the matrix P, which is an analogue of Lemma 7.2 in [S]. The proof is completely similar to it.

Lemma 5.2. *Let $W = G(e,p,n)$ and assume that $q = 0$. Put $b = a(\boldsymbol{\beta}_0)$. Then we have*

$$\sum_{j=0}^{d-1} t^{-b} R(\chi^{\boldsymbol{\beta}_j}) p_{z,\boldsymbol{\beta}_j} = R(\chi^z),$$

where $R(\chi^{\boldsymbol{\beta}_j}) = t^{n(n-1)/2 + jn}$.

In particular, in the case where $W = G(e,e,n)$, we have $\mathcal{F} = \{\boldsymbol{\Lambda}(\boldsymbol{\beta}_0)\}$, and $p_{z,\boldsymbol{\beta}_0} = t^{n(n-1)/2 - b} R(\chi^z)$.

5.3. Assume that $W = G(e,e,2)$, the dihedral group of order $2e$. We shall determine the matrix P in the case where $q = 0$. Let $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_{ij}$ be e-partitions defined in 2.16. The set $\widetilde{\mathcal{P}}^0_W$ is given as follows.

$$\boldsymbol{\alpha}_{0,0}, \quad \boldsymbol{\alpha}_{0,j} \quad (1 \le j < e/2), \quad \boldsymbol{\alpha}_{0,e/2}, \quad \boldsymbol{\alpha}'_{0,e/2}, \quad \boldsymbol{\alpha}_0,$$

where in the case of $\varGamma_{\boldsymbol{\alpha}} = \{1\}$, we write as $\boldsymbol{\alpha}$ to denote the element $(\boldsymbol{\alpha}, 1) \in \widetilde{\mathcal{P}}^0_W$. Moreover, we have $\varGamma_{\boldsymbol{\alpha}} = \mathbb{Z}/2\mathbb{Z}$ for $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{0,e/2}$ (the case where e is even), and in this case, we write as $\boldsymbol{\alpha}_{0,e/2}, \boldsymbol{\alpha}'_{0,e/2}$ instead of $(\boldsymbol{\alpha}_{0,e/2}, 1), (\boldsymbol{\alpha}_{0.e/2}, -1)$, respectively. The similarity classes in $\widetilde{Z}^{r,0}_{n,0}$ are given as follows.

$$\mathcal{F}_1 = \{\boldsymbol{\varLambda}(\boldsymbol{\alpha}_{0,0})\}, \quad \mathcal{F}_2 = \{\boldsymbol{\varLambda}(\boldsymbol{\alpha}_{0,j}) \mid 1 \le j \le e/2\}, \quad \mathcal{F}_3 = \{\boldsymbol{\varLambda}(\boldsymbol{\alpha}_0)\},$$

where in $\mathcal{F}_2$, $\boldsymbol{\varLambda}(\boldsymbol{\alpha}_{0,j})$ for $j = e/2$ is understood as $\boldsymbol{\varLambda}(\boldsymbol{\alpha}_{0,e/2}), \boldsymbol{\varLambda}(\boldsymbol{\alpha}'_{0,e/2})$. The a-values on similarity classes are given as follows.

$$a(\mathcal{F}_1) = e, \quad a(\mathcal{F}_2) = 1, \quad a(\mathcal{F}_3) = 0.$$

Note that the similarity classes and a-values of them are independent from the choice of r.

Next we describe irreducible characters χ^z associated to $z \in \widetilde{\mathcal{P}}^0_W$. If $z = \boldsymbol{\alpha}_{0,0}, \boldsymbol{\alpha}_0$, χ^z coincide with the sign character ε and the identity character 1 of W, respectively. If e is even, $\boldsymbol{\alpha}_{0,e/2}$ and $\boldsymbol{\alpha}'_{0,e/2}$ correspond two linear characters (long sign character η and short sigh character η') of W. $z = \boldsymbol{\alpha}_{0,1}$ corresponds to the reflection character ρ_1 of W (hence $\deg \rho_1 = 2$). Other characters corresponding to $\boldsymbol{\alpha}_{0,j}$ have all degree 2. We remark that the above partition of $W^\wedge$ into similarity classes together with a-values coincide with the partition of $W^\wedge$ into families and their a-values introduced by Lusztig [L1] in connection with the classification of unipotent characters of finite reductive groups, where $1, \varepsilon, \rho_1$ are the special characters corresponding to families.

The fake degrees $R(\chi^z)$ of W are given as follows.

$$\begin{aligned}
&R(1) = 1, \quad R(\rho_j) = t^j + t^{e-j}\,(1 \le j < e/2) \\
&R(\eta) = t^{e/2}, \quad R(\eta') = t^{e/2}, \\
&R(\varepsilon) = t^e.
\end{aligned} \tag{5.3.1}$$

We now define an order on the set $W^\wedge$ by

$$\begin{aligned}
&\varepsilon, \rho_1, \rho_2, \dots, \rho_m, 1 && \text{if } e = 2m+1, \\
&\varepsilon, \rho_1, \rho_2, \dots, \rho_{m-1}, \eta, \eta', 1 && \text{if } e = 2m,
\end{aligned}$$

and define the order on $\widetilde{\mathcal{P}}^0_W$ accordingly. We consider the matrix equation $P\varLambda\,{}^tP = \Omega$ indexed by $\widetilde{\mathcal{P}}^0_W$. Following the partition of $\widetilde{\mathcal{P}}^0_W$ into similarity classes, we regard $P, \varLambda, \Omega$ as block matrices. By our assumption, P and $\varLambda$ have the following shape.

$$P = \begin{pmatrix} t^e & 0 & 0 \\ P_{21} & tI_k & 0 \\ p_{31} & P_{32} & 1 \end{pmatrix}, \qquad \varLambda = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \varLambda_{22} & 0 \\ 0 & 0 & \lambda_{33} \end{pmatrix}$$

where k is equal to $e/2-1$ (resp. $e/2+1$) if e is odd (resp. even), Λ_{22} is a matrix of degree k. P_{21}, P_{32} are matrices of the shape $(k,1), (1,k)$, respectively.

Since the matrix equation $P\Lambda\,{}^tP = \Omega$ is completely the same as the matrix equation arising from the partition into families studied by Geck and Malle [GM], where the case of dihedral groups is explicitly computed, the matrix P in our case is described as follows. Note that in view of Lemma 5.2, the first column of P coincides with the fake degrees of W.

Lemma 5.4 ([GM]). *Let $W = G(e,e,2)$ be the dihedral group of degree $2e$. Assume that $q = 0$. Then the matrix P associated to $\widetilde{Z}^{r,0}_{n,0}$ is given as follows.*

$$
{}^tP_{21} = \begin{cases} (t + t^{e-1}, t^2 + t^{e-2}, \dots, t^m + t^{m+1}) & \text{if } e = 2m+1, \\ (t + t^{e-1}, t^2 + t^{e-2}, \dots, t^{m-1} + t^{m+1}, t^m, t^m) & \text{if } e = 2m. \end{cases}
$$

and $p_{31} = 1, \quad P_{32} = (1, 0, \dots, 0)$.

Moreover, a part of the matrix Λ is given as follows.

$$
\lambda_{11} = 1, \quad (\Lambda_{22})_1 = t^{-1}(t^e - 1)\cdot {}^tP_{21}, \quad \lambda_{33} = t^{e-2}(t^2-1)(t^e-1),
$$

where $(\Lambda_{22})_1$ denotes the first row of the matrix Λ_{22}.

5.5. In the remainder of this section, we shall give some more examples of Green functions associated to $G(e,e,n)$. Throughout those examples, we always assume that $r = 2$ and $q = 0$.

First assume that $W = G(3,3,3)$. Then $\widetilde{\mathcal{P}}^0_W$ is given as follows.

$$
\begin{aligned}
&\boldsymbol{\alpha}_1 = (1^3; -; -), \boldsymbol{\alpha}_2 = (1^2; 1; -), \boldsymbol{\alpha}_3 = (1; 1^2; -), \\
&\boldsymbol{\alpha}_4 = (1;1;1), \boldsymbol{\alpha}_4' = (1;1;1)', \boldsymbol{\alpha}_4'' = (1;1;1)'', \\
&\boldsymbol{\alpha}_5 = (21; -; -), \boldsymbol{\alpha}_6 = (2; 1; -), \boldsymbol{\alpha}_7 = (1; 2; -), \boldsymbol{\alpha}_8 = (3; -; -).
\end{aligned}
$$

Note that $|\Gamma_{\boldsymbol{\alpha}_4}| = 3$, and we write $\boldsymbol{\alpha}_4, \boldsymbol{\alpha}_4', \boldsymbol{\alpha}_4''$ instead of $(\boldsymbol{\alpha}, \phi)$ with $\phi \in \Gamma_{\boldsymbol{\alpha}_4}$. $\Gamma_{\boldsymbol{\alpha}} = \{1\}$ for other $\boldsymbol{\alpha}$. Then the symbols and similarity classes in $\widetilde{Z}^{2,0}_{n,0}$ are given as

$$
\begin{aligned}
&\mathcal{F}_1 = \{\boldsymbol{\Lambda}_1 = (531; 420; 420)\}, \\
&\mathcal{F}_2 = \{\boldsymbol{\Lambda}_2 = (31; 30; 20), \boldsymbol{\Lambda}_3 = (30; 31; 20)\}, \\
&\mathcal{F}_3 = \{\boldsymbol{\Lambda}_4 = (1;1;1), \boldsymbol{\Lambda}_4' = (1;1;1)', \boldsymbol{\Lambda}_4'' = (1;1;1)''\}, \\
&\mathcal{F}_4 = \{\boldsymbol{\Lambda}_5 = (41; 20; 20)\}, \quad \mathcal{F}_5 = \{\boldsymbol{\Lambda}_6 = (2;1;0), \boldsymbol{\Lambda}_7 = (1;2;0)\}, \\
&\mathcal{F}_6 = \{\boldsymbol{\Lambda}_8 = (3;0;0)\}.
\end{aligned}
$$

The matrices $\varLambda(\mathcal{F}_i)$ are given as follows;

$$
\begin{aligned}
&\varLambda(\mathcal{F}_1) = (1), \qquad \varLambda(\mathcal{F}_2) = (t^6-1)\begin{pmatrix} 2t^3+1 & t(t^3+2) \\ t(t^3+2) & 3t^2 \end{pmatrix} \\
&\varLambda(\mathcal{F}_3) = t^3(t^3-1)(t^6-1)I_3, \qquad \varLambda(\mathcal{F}_4) = (t^3(t^3-1)(t^6-1)), \\
&\varLambda(\mathcal{F}_5) = t^3(t^3-1)^2(t^6-1)\begin{pmatrix} 2 & t \\ t & 0 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_6) = (t^6(t^3-1)^2(t^6-1)),
\end{aligned}
$$

where I_3 denotes the identity matrix of degree 3.

The matrix P of Green functions is given in Table 1.

TABLE 1. $G(3,3,3) \quad (r=2)$

$(1^3;-;-)$	t^9									
$(1^2;1;-)$	$2t^7+t^4$	t^4								
$(1;1^2;-)$	t^8+2t^5		t^4							
$(1;1;1)$	t^6+t^3	t^3		t^3						
$(1;1;1)'$	t^6+t^3	t^3			t^3					
$(1;1;1)''$	t^6+t^3	t^3				t^3				
$(21;-;-)$	t^6+t^3	t^3					t^3			
$(2;1;-)$	$2t^4+t$	t	t^3	t	t	t	t	t		
$(1;2;-)$	t^5+2t^2	$2t^2$		t^2	t^2	t^2	t^2		t	
$(3;-;-)$	1	1		1	1	1	1	1		1

5.6. Assume that $W = G(4,4,3)$. Then $\widetilde{\mathcal{P}}_W^0$ is given as

$$
\begin{aligned}
&\boldsymbol{\alpha}_1 = (1^3;-;-;-), \quad \boldsymbol{\alpha}_2 = (1^2;1;-;-), \quad \boldsymbol{\alpha}_3 = (1^2;-;1;-), \\
&\boldsymbol{\alpha}_4 = (1;1^2;-;-), \quad \boldsymbol{\alpha}_5 = (21;-;-;-), \quad \boldsymbol{\alpha}_6 = (1;1;1;-), \\
&\boldsymbol{\alpha}_7 = (2;1;-;-), \quad \boldsymbol{\alpha}_8 = (2;-;1;-), \quad \boldsymbol{\alpha}_9 = (1;2;-;-), \\
&\boldsymbol{\alpha}_{10} = (3;-;-;-).
\end{aligned}
$$

The symbols and similarity classes in $\widetilde{Z}_{n,0}^{2,0}$ are given as

$$
\begin{aligned}
&\mathcal{F}_1 = \{\varLambda_1 = (531;420;420;420)\}, \\
&\mathcal{F}_2 = \{\varLambda_2 = (31;30;20;20), \varLambda_3 = (31;20;30;20), \varLambda_4 = (30;31;20;20)\}, \\
&\mathcal{F}_3 = \{\varLambda_5 = (41;20;20;20)\}, \qquad \mathcal{F}_4 = \{\varLambda_6 = (1;1;1;0)\}, \\
&\mathcal{F}_5 = \{\varLambda_7 = (2;1;0;0), \varLambda_8 = (2;0;1;0), \varLambda_9 = (1;2;0;0)\}, \\
&\mathcal{F}_6 = \{\varLambda_{10} = (3;0;0;0)\}.
\end{aligned}
$$

The matrices $\Lambda(\mathcal{F}_i)$ are given as follows.

$$\begin{aligned}
\Lambda(\mathcal{F}_1) &= (1),\\
\Lambda(\mathcal{F}_2) &= (t^8-1)\begin{pmatrix} (t^5+t^4+1) & t(t^5+t+1) & t(t^4+t^2+1)\\ t(t^5+t+1) & t^2(t^2+t+1) & t^2(t^2+t+1)\\ t(t^4+t^2+1) & t^2(t^2+t+1) & t^2(t^4+t^2+1)\end{pmatrix},\\
\Lambda(\mathcal{F}_3) &= (t^5(t^3-1)(t^8-1)), \quad \Lambda(\mathcal{F}_4) = (t^4(t^2+t+1)(t^4-1)(t^8-1)),\\
\Lambda(\mathcal{F}_5) &= t^5(t^3-1)(t^4-1)(t^8-1)\begin{pmatrix} t+1 & t^2 & t\\ t^2 & 0 & 0\\ t & 0 & t^2\end{pmatrix},\\
\Lambda(\mathcal{F}_6) &= (t^9(t^3-1)(t^4-1)(t^8-1)).
\end{aligned}$$

The matrix of P of Green functions is given in Table 2.

TABLE 2. $G(4,4,3) \quad (r=2)$

$(1^3;-;-;-)$	t^{12}									
$(1^2;1;-;-)$	$t^{10}+t^9+t^5$	t^5								
$(1^2;1;-;1)$	$t^{11}+t^7+t^6$		t^5							
$(1;1^2;-;-)$	$t^{10}+t^8+t^6$			t^5						
$(21;-;-;-)$	t^8+t^4	t^4			t^4					
$(1;1;1;-)$	$t^9+t^8+t^7+t^5+t^4+t^3$	t^4+t^3		t^4		t^3				
$(2;1;-;-)$	t^6+t^5	t	t^4		t	t	t			
$(2;-;1;-)$	$t^7+t^3+t^2$	t^3+t^2			t^3	t^2		t		
$(1;2;-;-)$	$t^6+t^4+t^2$	t^2		t^3	t^2	t^2			t	
$(3;-;-;-)$	1	1			1	1	1			1

5.7. In this occasion, we shall correct some errors in [S]. The formula (2.5.2) in [S] does not hold in this form. If one wants to get this formula, the definition of powersum functions in 2.1 has to be replaced by

$$p_r^{(i)}(x) = \sum_{j=0}^{e-1} \zeta^{-ij} p_r(x^{(j)}).$$

Then this leads to a different parametrization of irreducible characters of W in terms of e-partitions, i.e., the one obtained by taking the complex conjugate of the original characters. For example, $(-;\cdots,-;1^n)$ corresponds to the character $\det_V$ instead of $\overline{\det_V}$.

However, it seems more econvenient to use the usual parametrization of irreducible characters of W as in 1.2 in [S]. In this case, the corrected form of the right hand side of (2.5.2) is obtained by replacing x and y from the old one. This change of variables will force some modifications to subsequent formulas. Instead of doing this, we argue as follows.

We define a new $\Omega(x,y;t)$ by exchanging the variables x and y from the old one,

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1 - tx_i^{(k)}y_j^{(k+1)}}{1 - x_i^{(k)}y_j^{(k)}}, \tag{5.7.1}$$

and modify the definition of $q_{r,\pm}^{(k)}$ as follows.

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