GREEN FUNCTIONS ASSOCIATED TO COMPLEX REFLECTION GROUPS

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Abstract. Green functions of classical groups are determined by the data from Weyl groups and by certain combinatorial objects called symbols. Generalizing this, we define Green functions associated to complex reflection groups $G(e,1,n)$, and study their combinatorial properties. We construct Hall-Littlewood functions and Schur functions in our scheme, and show that such Green functions are obtained as a transition matrix between those two symmetric functions, as in the case of GL_n.

0. Introduction

Green polynomials $Q^{\mu}_{\lambda}(q)$ of $GL_n(\mathbf{F}_q)$, where λ, μ are partitions of n, were first introduced by J.A. Green [G] in 1955 in a combinatorial framework of symmetric functions. They are obtained as the transition matrix between power sum symmetric functions and Hall-Littlewood functions. Those Green functions play a crucial role in [G] in describing irreducible characters of $GL_n(\mathbf{F}_q)$. More generally, we consider a connected reductive group G defined over a finite field $\mathbf{F}_q$, with Frobenius map F. In 1976, P. Deligne and G. Lusztig constructed in [DL] Green functions associated to finite reductive groups $G(\mathbf{F}_q)$ in general. They defined a function Q^G_T, for each F-stable maximal torus T of G, as the restriction on unipotent elements of Deligne-Lusztig's virtual characters $R^G_T(\theta)$. The Green function Q_χ, for an irreducible character χ of the Weyl group W of G, is defined as a certain linear combination of Q^G_T for various T. In the case of $GL_n(\mathbf{F}_q)$, $Q_{\chi^\lambda}(u_\mu)$ coincides with $\widetilde{K}_{\lambda,\mu}(q)$, the modified Kostka polynomial which is defined as a linear combination of various $Q^{\nu}_{\mu}(q)$, where χ^λ is the irreducible character of $W = \mathfrak{S}_n$ corresponding to λ, and u_μ is the unipotent class corresponding to μ. The computation of $R^G_T(\theta)$ is reduced to the computation of Green functions associated to various reductive subgroups. In [L5], (see also [S2]) Lusztig showed that Green functions have a geometric interpretation in terms of intersection cohomology complexes of the closure of unipotent classes. Using this, he proved in [L3], generalizing the result in [S1], that there exists a simple algorithm of computing Green functions. Here Green functions are determined as a unique solution of a certain type of matrix equation

$$P\varLambda\,{}^tP = \Omega \qquad (*)$$

with unknown P, Λ. (The entries of the matrix P describe Green functions $Q_\chi(u)$). Note that the matrix Ω is completely determined by the property of irreducible characters of W, while the shape of P and Λ are determined by the property of the Springer correspondence between unipotent classes of G and irreducible characters of W. It is known, by the geometric interpretation of Green functions, that the entries of the matrices P and Λ are in $\mathbb{Z}[q]$.

In [GM], M. Geck and G. Malle considered an analogy of the equation (*). In their case, the matrix Ω is the same as above, but the shape of P and Λ are different, which is determined by the property of unipotent characters of $G(\mathbf{F}_q)$. They computed the solution of this equation in the case of exceptional groups, and checked that the entries of P and Λ are again in $\mathbb{Z}[q]$, and that those solutions provide the polynomials associated to special pieces discussed in [L6]. They conjecture that this will hold in general, but it is still open for the case of classical groups (see also [S3]).

In the case of classical groups, unipotent characters are parametrized by "symbols" introduced by Lusztig [L1]. While the Springer correspondence is described by "u-symbols" which are also introduced by Lusztig [L2] by modifying the notion of symbols. The remarkable thing is that the matrix equation (*) for either case is completely described by the combinatorial property of symbols or u-symbols. The notion of symbols is generalized by Malle [Ma1] so that it fits to the case of complex reflection groups $G(e,1,n)$ as a generalization of Weyl group of type C_n, where he described the unipotent degrees of $G(e,1,n)$ in terms of such (generalized) symbols. Now the ingredients used to construct the equation (*) makes sense even for this case, and one can define a matrix equation by using the combinatorics of symbols. So it would be natural to call the solution of this equation as the Green function associated to $G(e,1,n)$.

In this paper, we develop a combinatorial theory of symmetric functions which fit to the situation for $G(e,1,n)$. In this case, the role of partitions are replaced by symbols. In particular, we construct Hall-Littlewood functions associated to $G(e,1,n)$, which are parametrized by symbols. But note that for a fixed $G(e,1,n)$, there exist several different type of symbols, such as symbols or u-symbols in the case of Weyl group of type C_n. Hence one obtains several class of Hall-Littlewood functions associated to each type of symbols. We show that a large part of arguments used to construct Green polynomials of $GL_n(\mathbf{F}_q)$ can be generalized to our setting, and that the Green functions are obtained as the transition matrix between Schur functions and Hall-Littlewood functions. Thus obtained Green functions turn out to be rational functions in $\mathbb{Q}(t)$, where t is an indeterminate.

As some examples show, it is very likely that these Green functions are actually polynomials in t with positive integral coefficients. Concerning this conjecture, we can only show in section 6, that Green functions are actually polynomials for some special cases, under the condition that $e = 2$.

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1. Preliminaries

1.1. Let $W \simeq \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$ be the imprimitive complex reflection group $G(e,1,n)$, and $V = \mathbb{C}^n$ the natural reflection representation of W. Let $S(V)$ be the symmetric algebra of V, and I_+ the ideal of $S(V)$ generated by the homogeneous W-invariant vectors of strictly positive degrees. We denote by $R = \oplus R_i$ the coinvariant algebra of W, which is a graded algebra defined as the quotient of $S(V)$ by I_+. Let t be an indeterminate. The Poincaré polynomial $P_W(t)$ is defined as $P_W(t) = \sum_{i \ge 0} (\dim_{\mathbb{C}} R_i) t^i$, which is explicitly given as

$$P_W(t) = \prod_{i=1}^{n} \frac{t^{ei} - 1}{t - 1}.$$

For any class function f on W, we define $R(f)$ by

$$R(f) = (t-1)^n P_W(t) \frac{1}{|W|} \sum_{w \in W} \frac{\det_V(w) f(w)}{\det_V(t \cdot \mathrm{id}_V - w)}, \tag{1.1.1}$$

where $\det_V$ denotes the determinant on V. Note that $R(f)$ coincides with $\sum_i \langle f, R_i \rangle_W t^i$, where $\langle\, ,\, \rangle_W$ denotes the inner product of class functions on W. Hence $R(f) \in \mathbb{Z}[t]$ for a generalized character f of W.

Let N^* be the number of complex reflections in W. Then N^* is the maximal degree in R, and the W-module R_{N^*} coincides with $\overline{\det_V}$, the complex conjugate of $\det_V$.

1.2. An e-tuple of partitions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ is called an e-partition. We define the size $|\boldsymbol{\alpha}|$ of $\boldsymbol{\alpha}$ by $|\boldsymbol{\alpha}| = \sum_{k=0}^{e-1} |\alpha^{(k)}|$, where $|\alpha^{(k)}|$ is the size of a partition $\alpha^{(k)}$. Let $\mathcal{P}_{n,e}$ be the set of e-partitions of size n. The set $W^\wedge$ of irreducible characters of W is in natural bijection with the set $\mathcal{P}_{n,e}$. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. In particular, the unit character corresponds to $((n), -, \dots, -)$ and $\overline{\det_V}$ corresponds to $\boldsymbol{\alpha} = (-, \dots, -, (1^n))$.

Let $m_0, m_1, \dots, m_{e-1}$ be positive integers such that $m_k \ge n$, and put $\mathbf{m} = (m_0, \dots, m_{e-1})$. We denote by $Z_n^{0,0} = Z_n^{0,0}(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| = n$ and that each $\boldsymbol{\alpha}^{(k)}$ is regarded as an element in $\mathbb{Z}^{m_k}$, written in the form $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{m_k}^{(k)} \ge 0$. We fix integers $r \ge s \ge 0$, and consider an e-partition

$\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\mathbf{m}) = (\Lambda_0, \dots, \Lambda_{e-1})$ defined by

$$\begin{aligned} &\Lambda_0 : (m_0 - 1)r \ge \cdots \ge 2r \ge r \ge 0, \\ &\Lambda_i : s + (m_i - 1)r \ge \cdots \ge s + 2r \ge s + r \ge s \qquad \text{for } 1 \le i \le e-1. \end{aligned} \tag{1.2.1}$$

We denote by $Z_n^{r,s} = Z_n^{r,s}(\mathbf{m})$ the set of e-partitions of the form $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha} \in Z_n^{0,0}$ and the sum is taken entry-wise. We denote by $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, and call it the e-symbol of type (r, s) corresponding to $\boldsymbol{\alpha}$. We write $|\boldsymbol{\Lambda}| = n$ if $\boldsymbol{\Lambda} \in Z_n^{r,s}$.

Put $\mathbf{m}' = (m_0 + 1, \dots, m_{e-1} + 1)$, and we define a shift operation $Z_n^{r,s}(\mathbf{m}) \to Z_n^{r,s}(\mathbf{m}')$ by associating $\boldsymbol{\Lambda}' = (\Lambda'_0, \dots, \Lambda'_{e-1}) \in Z_n^{r,s}(\mathbf{m})$ to $\boldsymbol{\Lambda} = (\Lambda_0, \dots, \Lambda_{e-1}) \in Z_n^{r,s}(\mathbf{m})$, where $\Lambda'_0 = (\Lambda_0 + r) \cup \{0\}$ and $\Lambda'_k = (\Lambda_k + r) \cup \{s\}$ for $k = 1, \dots, e-1$. In other words, for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, $\boldsymbol{\Lambda}'$ is obtained as $\boldsymbol{\Lambda}' = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\mathbf{m}')$, where $\boldsymbol{\alpha}$ is regarded as an element of $Z_n^{0,0}(\mathbf{m}')$ by adding 0 in the entries of $\boldsymbol{\alpha}$. We denote by $\bar{Z}_n^{r,s}$ the classes in $\coprod_{\mathbf{m}'} Z_n^{r,s}(\mathbf{m}')$ under the equivalence relation generated by shift operations. Note that $\mathcal{P}_{n,e}$ coincides with the set $\bar{Z}_n^{0,0}$. Also note that $\boldsymbol{\Lambda}^0$ is regarded as a symbol in $Z_n^{r,s}$ with $n = 0$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $\bar{Z}_n^{r,s}$ is said to be similar if there exist representatives in $Z_n^{r,s}(\mathbf{m})$ such that all the entries of them coincide each other with multiplicities. The set of symbols which are similar to a fixed symbol is called a similarity class in $Z_n^{r,s}$.

We shall define a function $a : \bar{Z}_n^{r,s} \to \mathbb{N}$. For $\boldsymbol{\Lambda} \in Z_n^{r,s}$, we put

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda, \lambda' \in \boldsymbol{\Lambda}} \min(\lambda, \lambda') - \sum_{\mu, \mu' \in \boldsymbol{\Lambda}^0} \min(\mu, \mu'), \tag{1.2.2}$$

where in the first sum, we assume that $\lambda \ne \lambda'$ if λ and λ' are contained in the same Λ_i, and similarly for the second sum (this occurs only when $r = 0$). The function a on $Z_n^{r,s}$ is invariant under the shift operation, and it induces a function a on $\bar{Z}_n^{r,s}$. Clearly, the a-function takes a constant value on each similarity class in $Z_n^{r,s}$.

Remark 1.3. (i) The general notion of e-symbols is obtained by Malle [Ma2]. The e-symbols described above is just a part of such symbols corresponding to the set of e-partitions, which is enough for our later discussions. In [Ma1], the e-symbols are specified to the case where $\mathbf{m} = (m + 1, m, \dots, m)$ for some $m > 0$. However our arguments in subsequent sections work well for arbitrary $\mathbf{m}$.

(ii) The notion of symbols had appeared in several articles in various forms. If $e = 1$, $Z_n^{r,s}$ is in bijection with the set of partitions of n. In the case where $e = 2$, the sets $\bar{Z}_n^{1,0}$ was used in [L1] to parameterize unipotent characters of $Sp_{2n}(\mathbf{F}_q)$ or $SO_{2n+1}(\mathbf{F}_q)$. In that case, the a-function defined in (1.2.2) coincides with the original a-function, i.e., the exact power of t dividing the generic degree $D_\rho(t) \in \mathbb{Q}[t]$ of the corresponding unipotent character ρ. In the case $e = 2$, the set $\bar{Z}_n^{2,1}$ (resp. $\bar{Z}_n^{2,0}$) was used in [L2] to describe the generalized Springer correspondence for $G = Sp_{2n}(k)$ (resp. $G = SO_{2n+1}(k)$) in the case where ch $k \ne 2$. In those cases, the similarity classes are in 1 to 1 correspondence with the unipotent classes of G, and the value $a(\boldsymbol{\Lambda})$ for the symbol $\boldsymbol{\Lambda}$ corresponding to the unipotent class u coincides with $\dim \mathcal{B}_u$, where $\mathcal{B}_u$ is the variety of Borel subgroups in G containing u (see, for example [L4, 4.4]). While, the set $\bar{Z}_n^{4,2}$ was used in [LS] to describe the generalized Springer correspondence for $Sp_{2n}(k)$ with ch $k = 2$. On the other hand, for arbitrary e, the set $Z_n^{1,0}$ was used in

[Ma1] to parameterize unipotent degrees associated to the complex reflection groups $G(e,1,n)$.

1.4. Under the identification of $Z_n^{r,s}$ with $\mathcal{P}_{n,e}$, we may consider the a function on the set $\mathcal{P}_{n,e}$. Also the similarity classes in $\mathcal{P}_{n,e}$ are defined by inheriting the classes in $Z_n^{r,s}$. We denote by $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$ are in the same similarity class. We choose a total order $\succ$ on $\mathcal{P}_{n,e}$ such that $a(\boldsymbol{\alpha}) \le a(\boldsymbol{\beta})$ whenever $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$, and that each similarity class in $\mathcal{P}_{n,e}$ forms an interval. In the following, we consider the matrices of degree $|\mathcal{P}_{n,e}|$ with respect to this order. For each $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$, we define a polynomial $\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Z}[t]$ by

$$\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*} R(\chi^{\boldsymbol{\alpha}} \otimes \chi^{\boldsymbol{\beta}} \otimes \overline{\det}_V), \tag{1.4.1}$$

and put $\Omega = (\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}})$, the matrix of degree $|\mathcal{P}_{n,e}|$. Let $P = (p_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $\varLambda = (\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be matrices of degree $|\mathcal{P}_{n,e}|$. We consider the following system of equations with respect to the unknown variables $\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}, p_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

$$\begin{aligned} &\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{unless } \boldsymbol{\alpha} \sim \boldsymbol{\beta}, \\ &p_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0 \quad \text{unless either } \boldsymbol{\alpha} \succ \boldsymbol{\beta} \text{ and } \boldsymbol{\alpha} \not\sim \boldsymbol{\beta}, \text{ or } \boldsymbol{\alpha} = \boldsymbol{\beta}, \\ &p_{\boldsymbol{\alpha},\boldsymbol{\alpha}} = t^{a(\boldsymbol{\alpha})} \\ &P \varLambda\, {}^tP = \Omega \end{aligned} \tag{1.4.2}$$

In the case where $e = 2$ and $Z_n^{r,s} = Z_n^{2,1}$ (resp. $Z_n^{r,s} = Z_n^{2,0}$), it is known by [L3] that the solution of (1.4.2) produces, by the substitution $t \mapsto q$, the Green functions of $Sp_{2n}(\mathbf{F}_q)$ (resp. $SO_{2n+1}(\mathbf{F}_q)$) with ch $\mathbf{F}_q \ne 2$, which are known to be polynomials. Therefore in these cases, (1.4.2) has a solution with $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Z}[t]$. In the case where $e = 2$ and $Z_n^{r,s} = Z_n^{1,0}$ it is shown in [GM] that (1.4.2) has a unique solution with $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Q}(t)$, rational functions in t. They expect that $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are actually in $\mathbb{Z}[t]$, which is a part of their conjecture for general Weyl groups. In our setting for complex reflection groups $W = G(e,1,n)$, we can also conjecture that $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ are in $\mathbb{Z}[t]$.

In the case of GL_n, the system of equations such as (1.4.2) arises naturally in a combinatorial setting concerning Schur functions and Hall-Littlewood functions. In later discussions, we show that this holds also in the case of complex reflection groups W.

1.5. There exists a system of equations which is closely related to (1.4.2). Let $\Omega' = (\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be a matrix defined by

$$\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*} R(\chi^{\boldsymbol{\alpha}} \otimes \overline{\chi}^{\boldsymbol{\beta}} \otimes \overline{\det}_V), \tag{1.5.1}$$

where $\overline{\chi}^{\boldsymbol{\beta}}$ is the complex conjugate of $\chi^{\boldsymbol{\beta}}$. We consider a matrix equation of the form

$$P' \varLambda'\, {}^tP'' = \Omega', \tag{1.5.2}$$

where either of $P' = (p'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ and $P'' = (p''_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ satisfies similar conditions as in the second and third one in (1.4.2), and $\varLambda' = (\lambda'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ as in the first one. Let $\sigma = (\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the permutation matrix realizing the complex conjugation of irreducible characters of W, i.e., $\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$ if $\chi^{\boldsymbol{\beta}} = \overline{\chi}^{\boldsymbol{\alpha}}$ and $\sigma_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ otherwise. Then we have $\Omega' = \Omega\sigma$, and the equation $P\Lambda {}^tP = \Omega$ implies that

$$P \cdot \Lambda\sigma \cdot \sigma^{-1t}P\sigma = \Omega'.$$

Note that if $\chi^{\boldsymbol{\beta}} = \overline{\chi}^{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$, then $\boldsymbol{\beta} = (\beta^{(0)}, \dots, \beta^{(e-1)})$ is given by $\beta^{(k)} = \alpha^{(e-k)}$ for $k = 1, \dots, e-1$ and $\beta^{(0)} = \alpha^{(0)}$. It follows that the permutation of symbols in $Z_n^{r,s}$ induced from the complex conjugation $\chi^{\boldsymbol{\alpha}} \mapsto \overline{\chi}^{\boldsymbol{\alpha}}$ preserves each similarity class. In particular, the matrices $\Lambda\sigma, \sigma^{-1t}P\sigma$ have the same shape as Λ and ${}^tP$, respectively. Now it is easily checked that the equation (1.4.2) has a solution if and only if the equation (1.5.2) has a solution, and in that case, we have $P' = P, \Lambda' = \Lambda\sigma, P'' = \sigma^{-1t}P\sigma$. In the remainder of the paper, we consider the equation (1.5.2) instead of (1.4.2).

2. Symmetric functions associated to partitions

2.1. In this section we shall define several symmetric functions associated to e-partitions. We fix $\mathbf{m} = (m_0, \dots, m_{e-1})$ as in 1.2, and introduce indeterminates $x_j^{(k)}$ ($0 \le k \le e-1, 1 \le j \le m_k$). We denote by x the whole variables $(x_j^{(k)})$, and also denote by $x^{(k)}$ the variables $x_1^{(k)}, \dots, x_{m_k}^{(k)}$. Power sum symmetric functions and Schur functions are defined as in [M, Appendix B]. Let ζ be a primitive e-th root of unity in $\mathbb{C}$. For each integer $r \ge 1$ and i such that $0 \le i \le e-1$, put

$$p_r^{(i)}(x) = \sum_{j=0}^{e-1} \zeta^{ij} p_r(x^{(j)}),$$

where $p_r(x^{(j)})$ denotes the r-th power sum symmetric function with respect to the variables $x^{(j)}$. We put $p_r^{(i)}(x) = 1$ for $r = 0$. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ with $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{m_k}^{(k)}$, we define a function $p_{\boldsymbol{\alpha}}(x)$ by

$$p_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} p^{(k)}_{\alpha_j^{(k)}}(x). \tag{2.1.1}$$

(But note that our definition of the power symmetric function given in (2.1.1) is not the same as the one in [M], which coincides with the complex conjugate of ours.)

Next, we define the Schur function $s_{\boldsymbol{\alpha}}(x)$ and monomial symmetric functions $m_{\boldsymbol{\alpha}}(x)$ associated to $\boldsymbol{\alpha}$ by

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}), \qquad m_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} m_{\alpha^{(k)}}(x^{(k)}), \tag{2.1.2}$$

where $s_{\alpha^{(k)}}(x^{(k)})$ (resp. $m_{\alpha^{(k)}}(x^{(k)}$) denotes the usual Schur function (resp. monomial symmetric function) associated to the partition $\alpha^{(k)}$ with respect to the variables $x^{(k)}$.

Note that the set of conjugacy classes of W is parametrized by $\mathcal{P}_{n,e}$ ([M, Appendix B], see also [S2]). We denote by $w_{\boldsymbol{\beta}}$ a representative of the class in W corresponding to $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. It is known by [M] that the following Frobenius type formula holds for the irreducible characters of W.

$$p_{\boldsymbol{\beta}} = \sum_{\boldsymbol{\alpha} \in \mathcal{P}_{n,e}} \chi^{\boldsymbol{\alpha}}(w_{\boldsymbol{\beta}}) s_{\boldsymbol{\alpha}}, \tag{2.1.3}$$

2.2. In what follows we regard the variables $x_i^{(k)}$ defined for $k \in \mathbb{Z}/e\mathbb{Z} \simeq \{0, 1, \dots, e-1\}$. For each $0 \le k \le e-1$ and an integer $r \ge 0$, we define a function $q_{r,\pm}^{(k)}(x;t)$ by

$$q_{r,\pm}^{(k)}(x;t) = \sum_{i \ge 1} (x_i^{(k)})^{r+\delta} \frac{\prod_j x_i^{(k)} - t x_j^{(k\pm1)}}{\prod_{j \ne i} x_i^{(k)} - x_j^{(k)}} \qquad (r \ge 1), \tag{2.2.1}$$

where $\delta = m_k - 1 - m_{k\pm1}$, and by $q_{r,\pm}^{(k)}(x;t) = 1$ for $r = 0$. In the product of the denominator, $x_j^{(k)}$ runs over all the variables in $x^{(k)}$ except $x_i^{(k)}$, while in the numerator, $x_j^{(k\pm1)}$ runs over all the variables in $x^{(k\pm1)}$. Note that if $x_i^{(k)} = x_i^{(k\pm1)} = x_i$ and $m_k = m_{k\pm1}$, $q_{r,\pm}^{(k)}(x;t)$ coincide with the function $q_r(x;t)$ given in [M, III, 2.9]. The following lemma describes the generating function for $q_{r,\pm}^{(k)}(x;t)$.

Lemma 2.3. (i) *$q_{r,\pm}^{(k)}(x;t)$ is a polynomial in $\mathbb{Z}[x;t]$, which is homogeneous of degree r with respect to the variables $x^{(k)}, x^{(k\pm1)}$.*

(ii) *$q_{r,\pm}^{(k)}(x;t)$ has the stability property for both variables $x^{(k)}, x^{(k\pm1)}$, i.e., the substitution of $x_{m_k}^{(k)} = 0$ produces the function $q_{r,\pm}^{(k)}(x,t)$ with respect to the variables $x_1^{(k)}, \dots, x_{m_k-1}^{(k)}$, and similarly for $x^{(k\pm1)}$.*

(iii) *Let u be an indeterminate. Then we have*

$$\sum_{r=0}^{\infty} q_{r,\pm}^{(k)}(x;t) u^r = \frac{\prod_i 1 - tux_i^{(k\pm1)}}{\prod_j 1 - ux_j^{(k)}}. \tag{2.3.1}$$

Proof. First we consider the case where $\delta = -1$, and put $m = m_k = m_{k\pm1}$. By Lagrange's interpolation, we have

$$\prod_{i=1}^m \frac{z - b_i}{z - a_i} = 1 + \sum_{i=1}^m \frac{a_i - b_i}{z - a_i} \prod_{j \ne i} \frac{a_i - b_j}{a_i - a_j}.$$

Substituting $a_i = zux_i^{(k)}, b_i = ztux_i^{(k\pm1)}$ into this formula, we have

$$\prod_{i=1}^m \frac{1 - tux_i^{(k\pm1)}}{1 - ux_i^{(k)}} = 1 + \sum_{i=1}^m \frac{ux_i^{(k)} - tux_i^{(k\pm1)}}{1 - ux_i^{(k)}} \prod_{j \ne i} \frac{x_i^{(k)} - tx_j^{(k\pm1)}}{x_i^{(k)} - x_j^{(k)}}.$$

The coefficient of u^r in the expansion of the right hand side coincides with $q^{(k)}_{r,\pm}(x;t)$. This shows (2.3.1) in the case where $\delta = -1$. In particular, in this case $q^{(k)}_{r,\pm}(x;t)$ is a polynomial in $x^{(k)}, x^{(k\pm1)}$ and t, with integral coefficients, which is homogeneous of degree r with respect to the variables x. Now one can check by using the definition (2.2.1), that if $q^{(k)}_{r,\pm}(x;t)$ is a polynomial in $x^{(k)}, x^{(k\pm1)}$, then it has a stability property for both variables $x^{(k)}, x^{(k\pm1)}$. It follows from this that $q^{(k)}_{r,\pm}(x;t)$ satisfies the assertions (i), (ii) in the lemma even in the general case $\delta \ne -1$. Since the right hand side of (2.3.1) also has the stability property, the formula (2.3.1) for the general case follows from the case $\delta = -1$. The lemma is proved. □

2.4. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)}) \in \mathcal{P}_{n,e}$, we define a function $q_{\boldsymbol{\alpha},\pm}(x)$ by

$$q_{\boldsymbol{\alpha},\pm}(x;t) = \prod_{k=0}^{e-1}\prod_{j=1}^{m_k} q^{(k)}_{\alpha^{(k)}_j,\pm}(x;t). \tag{2.4.1}$$

For a partition $\alpha^{(k)} : \alpha^{(k)}_1 \ge \cdots \ge \alpha^{(k)}_{m_k} \ge 0$, we define a function $z_{\alpha^{(k)}}(t)$ by

$$z_{\alpha^{(k)}}(t) = \prod_{j=1}^{m_k}(1 - \zeta^k t^{\alpha^{(k)}_j})^{-1},$$

which is determined only by parts such that $\alpha^{(k)}_j \ne 0$. Then we define $z_{\boldsymbol{\alpha}}(t)$ by

$$z_{\boldsymbol{\alpha}}(t) = z_{\boldsymbol{\alpha}}\prod_{k=0}^{e-1} z_{\alpha^{(k)}}(t), \tag{2.4.2}$$

where $z_{\boldsymbol{\alpha}}$ is the order of the centralizer of $w_{\boldsymbol{\alpha}}$ in W. Explicitly, $z_{\boldsymbol{\alpha}}$ is given as follows. For $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, put $l(\boldsymbol{\alpha}) = \sum_{k=0}^{e-1} l(\alpha^{(k)})$, where $l(\alpha^{(k)})$ is the number of parts in the partition $\alpha^{(k)}$. For a partition $\alpha = (1^{n_1}, 2^{n_2}, \dots)$, put $z_\alpha = \prod_{i\ge1} i^{n_i} n_i!$. Then $z_{\boldsymbol{\alpha}} = e^{l(\boldsymbol{\alpha})}\prod_{k=0}^{e-1} z_{\alpha^{(k)}}$.

We now introduce infinitely many variables $x^{(k)}_i, y^{(k)}_i$ for $i = 1, 2, \dots$ and for $0 \le k \le e-1$. Since $p_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\alpha},\pm}(x;t), m_{\boldsymbol{\alpha}}(x)$ have the stability property for the variables $x^{(k)}_i$, we may regard them as functions with infinitely many variables $x^{(k)}_1, x^{(k)}_2, \dots$. Under this setting, we have the following proposition, which is a variant of Lemma 7.8 in [S4].

Proposition 2.5. *Let*

$$\Omega(x, y; t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1 - t x^{(k+1)}_i y^{(k)}_j}{1 - x^{(k)}_i y^{(k)}_j}.$$

Then we have

$$\Omega(x,y;t) = \sum_{\boldsymbol\alpha} q_{\boldsymbol\alpha,+}(x;t)m_{\boldsymbol\alpha}(y) = \sum_{\boldsymbol\alpha} m_{\boldsymbol\alpha}(x)q_{\boldsymbol\alpha,-}(y;t), \tag{2.5.1}$$

$$\Omega(x,y;t) = \sum_{\boldsymbol\alpha} z_{\boldsymbol\alpha}(t)^{-1}p_{\boldsymbol\alpha}(x)\bar p_{\boldsymbol\alpha}(y), \tag{2.5.2}$$

where $\boldsymbol\alpha$ runs over all the e-partitions of any size. In (2.5.2), $\bar p_{\boldsymbol\alpha}(y)$ denotes the complex conjugate of $p_{\boldsymbol\alpha}(y)$.

Proof. We show the first equality of (2.5.1). It follows from Lemma 2.3 that we have

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_j \sum_{r_j^{(k)}=0}^{\infty} q^{(k)}_{r_j^{(k)},+}(x;t)(y_j^{(k)})^{r_j^{(k)}}.$$

By multiplying out, the right hand side turns out to be $\sum_{\boldsymbol\alpha} q_{\boldsymbol\alpha,+}(x;t)m_{\boldsymbol\alpha}(y)$, which shows the first equality. If we note the equality

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1-tx_i^{(k)}y_j^{(k-1)}}{1-x_i^{(k)}y_j^{(k)}}, \tag{2.5.3}$$

the second equality in (2.5.1) follows from the first one, by replacing the role of $x_i^{(k)}$ and $y_j^{(k)}$.

Next we show the equality (2.5.2). The proof is similar to the case where $e=1$, which is given in [M, III, 4.1]. See also [S4, Lemma 7.8]. We start from the formula (2.5.3). By a similar argument as in [M], we have

$$\log\Omega(x,y;t) = \sum_k\sum_{i,j}\sum_{m=1}^{\infty}\biggl\{\frac{1}{m}(x_i^{(k)}y_j^{(k)})^m - \frac{t^m}{m}(x_i^{(k)}y_j^{(k-1)})^m\biggr\}.$$

But since

$$\frac{1}{e}\sum_{a=0}^{e-1}(\zeta^{-k}\zeta^{k'})^a = \delta_{k,k'},$$

we have

$$\begin{aligned}
&\log\,\Omega(x,y;t) \\
&\quad= \sum_{a=0}^{e-1}\sum_{m=1}^{\infty}\sum_{k,k',i,j}\biggl\{\frac{1}{em}\zeta^{-ak}\zeta^{ak'}(x_i^{(k)}y_j^{(k')})^m - \frac{t^m}{em}\zeta^a\zeta^{-ak}\zeta^{a(k'-1)}(x_i^{(k)}y_j^{(k'-1)})^m\biggr\} \\
&\quad= \sum_{a=0}^{e-1}\sum_{m=1}^{\infty}\frac{1-\zeta^a t^m}{em}p_m^{(a)}(x)\overline{p_m^{(a)}(y)}.
\end{aligned}$$

Hence

$$\begin{aligned}\Omega(x,y;t) &= \prod_{a=0}^{e-1}\prod_{m=1}^{\infty}\exp\biggl\{\frac{1-\zeta^a t^m}{em}p_m^{(a)}(x)\overline{p_m^{(a)}(y)}\biggr\} \\ &= \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}(t)^{-1}p_{\boldsymbol{\alpha}}(x)\bar{p}_{\boldsymbol{\alpha}}(y),\end{aligned}$$

and we get (2.5.2). Thus, the proposition is proved. □

3. Symmetric functions associated to symbols

3.1. In this section, we assume that $e > 1$ and $r > 0$. In order to proceed the arguments smoothly, we need to consider a more general situation than section 2. Let $\widehat{Z}^{0,0} = \widehat{Z}^{0,0}_n(\mathbf{m})$ be the set of e-compositions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$, where $\alpha^{(k)} = (\alpha^{(k)}_1, \dots, \alpha^{(k)}_{m_i}) \in \mathbb{Z}^{m_k}_{\ge 0}$ such that $\sum_{k,j}\alpha^{(k)}_j = n$. We denote by $\widehat{Z}^{r,s}_n = \widehat{Z}^{r,s}_n(\mathbf{m})$ the set of "symbols" $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\mathbf{m})$, where $\boldsymbol{\alpha} \in \widehat{Z}^{0,0}_n$ and $\boldsymbol{\Lambda}^0(\mathbf{m}) \in Z^{r,s}_n(\mathbf{m})$ is as in 1.2. Hence $\widehat{Z}^{0,0}_n \supset Z^{0,0}_n$ and $\widehat{Z}^{r,s}_n \supset Z^{r,s}_n$. We denote by $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ the element in $\widehat{Z}^{r,s}_n$ corresponding to $\boldsymbol{\alpha} \in \widehat{Z}^{0,0}_n$ under the above bijection.

In this section, we shall define a symmetric function $R^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ associated to $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in \hat{Z}^{r,s}_n$. For this, first we prepare some notation. Take $\boldsymbol{\alpha} \in \widehat{Z}^{0,0}_n$ and write it as $\boldsymbol{\alpha} = (\alpha^{(k)}_j)$. We fix a total order $<$ on the set $\mathcal{M} = \{(i,k) \mid 1 \le i \le m_k, 0 \le k \le e-1\}$. Let $\nu_0 = (i,k) \in \mathcal{M}$ be the element such that $\alpha^{(k)}_i \ne 0$ and that $\alpha^{(h)}_j = 0$ for any (j,h) such that $(j,h) > \nu_0$.

We define a function $I^{(k)}_{i,\pm}(x;t)$ for $0 \le k \le e-1, 1 \le i \le m_k$ by

$$I^{(k)}_{i,\pm}(x;t) = \begin{cases} \displaystyle\prod_{\substack{1 \le j \le m_{k\pm1} \\ (i,k)<(j,k\pm1)}} (x^{(k)}_i - tx^{(k\pm1)}_j) & \text{if } (i,k) \le \nu_0, \\[3ex] \displaystyle\prod_{\substack{1 \le j \le m_k \\ (i,k)<(j,k)}} (x^{(k)}_i - tx^{(k)}_j) & \text{if } (i,k) > \nu_0. \end{cases} \tag{3.1.1}$$

Put

$$\delta^{(k)}_{i,\pm} = \sharp\{j \mid (i,k) < (j,k)\} - \sharp\{j \mid (i,k) < (j,k\pm1)\}$$

if $(i,k) \le \nu_0$, and put $\delta^{(k)}_{i,\pm} = 0$ if $(i,k) > \nu_0$. Now set $\delta^{(k)}_{\pm} = (\delta^{(k)}_{1,\pm}, \dots, \delta^{(k)}_{m_k,\pm})$. For each integer $r \ge 1$, let

$$v_r(t) = \prod_{i=1}^{r}\frac{1-t^i}{1-t}.$$

We define a polynomial $v_{\boldsymbol{\alpha}}(t)$ by

$$v_{\boldsymbol{\alpha}}(t) = \prod_{k=0}^{e-1} v_{\mu_k}(t),$$

where $\mu_k = \sharp\{j \mid (j,k) > \nu_0\}$. For a sequence $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{m_k})$, we write as

$$(x^{(k)})^{\beta} = (x_1^{(k)})^{\beta_1}(x_2^{(k)})^{\beta_2}\cdots(x_{m_k}^{(k)})^{\beta_{m_k}}.$$

Finally, put $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0}\times\cdots\times\mathfrak{S}_{m_{e-1}}$. We now define a function $R^{\pm}_{\boldsymbol{\alpha},<}(x;t)$ associated to $\boldsymbol{\alpha}$ and the order $<$ on $\mathcal{M}$ by
(3.1.2)

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t) = v_{\boldsymbol{\alpha}}(t)^{-1}\sum_{w\in\mathfrak{S}_{\mathbf{m}}} w\biggl\{\prod_k (x^{(k)})^{\alpha^{(k)}+\delta^{(k)}_{\pm}}\prod_{k,i} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k\prod_{(i,k)<(j,k)}(x_i^{(k)}-x_j^{(k)})\biggr\}.$$

3.2. We regard $\mathfrak{S}_{\mu_k}$ as a subgroup of $\mathfrak{S}_{m_k}$ as a permutation group with respect to the letters $\{1\le j\le m_k \mid (j,k) > \nu_0\}$. Thus $\mathfrak{S}_{\boldsymbol{\alpha}} = \mathfrak{S}_{\mu_0}\times\cdots\times\mathfrak{S}_{\mu_{e-1}}$ is regarded as a subgroup of $\mathfrak{S}_{\mathbf{m}}$. We note that (3.1.2) can also be written as
(3.2.1)

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t) = \sum_{w\in\mathfrak{S}_{\mathbf{m}}/\mathfrak{S}_{\boldsymbol{\alpha}}} w\biggl\{\prod_k (x^{(k)})^{\alpha^{(k)}+\delta^{(k)}_{\pm}}\prod_{\substack{k,i\\(i,k)\le\nu_0}} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k\prod_{\substack{(i,k)<(j,k)\\(i,k)\le\nu_0}}(x_i^{(k)}-x_j^{(k)})\biggr\}.$$

We show (3.2.1). It is known ([M, III]) that

$$\sum_{w\in\mathfrak{S}_n} w\biggl(\prod_{i<j}\frac{x_i - tx_j}{x_i - x_j}\biggr) = v_n(t), \tag{3.2.2}$$

for variables $x_1,\dots,x_n$. Since the factors $I^{(k)}_{i,\pm}(x;t), \prod_{(i,k)<(j,k)}(x_i^{(k)}-x_j^{(k)})$ for $(i,k)\le\nu_0$ and $(x^{(k)})^{\alpha^{(k)}+\delta^{(k)}_{\pm}}$ are invariant under the action of $\mathfrak{S}_{\boldsymbol{\alpha}}$, we have

$$\begin{aligned}
\sum_{w\in\mathfrak{S}_{\boldsymbol{\alpha}}} & w\biggl\{\prod_k (x^{(k)})^{\alpha^{(k)}+\delta^{(k)}_{\pm}}\prod_{k,i} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k\prod_{(i,k)<(j,k)}(x_i^{(k)}-x_j^{(k)})\biggr\} \\
&= \prod_k (x^{(k)})^{\alpha^{(k)}+\delta^{(k)}_{\pm}}\prod_{\substack{k,i\\(i,k)\le\nu_0}} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k\prod_{\substack{(i,k)<(j,k)\\(i,k)\le\nu_0}}(x_i^{(k)}-x_j^{(k)}) \\
&\qquad\times\sum_{w\in\mathfrak{S}_{\boldsymbol{\alpha}}} w\biggl\{\prod_k\prod_{\substack{(i,k)<(j,k)\\(i,k)>\nu_0}}(x_i^{(k)}-tx_j^{(k)})/(x_i^{(k)}-x_j^{(k)})\biggr\}.
\end{aligned}$$

The last sum is equal to $v_{\boldsymbol{\alpha}}(t)$ by (3.2.2). Hence (3.2.1) holds.

We consider the special case where $\nu_0 = (i,k)$ is the smallest element in $\mathcal{M}$. Then we have $\alpha_i^{(k)} = r$ for some $r > 0$, and $\alpha_j^{(h)} = 0$ for any $(j,h)\ne(i,k)$. Under this setting, we have

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t) = q^{(k)}_{r,\pm}(x;t). \tag{3.2.3}$$

In fact, by the formula (3.2.1), we have

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t) = \sum_{w\in\mathfrak{S}_{m_k}/\mathfrak{S}_{m_k-1}} w\biggl\{(x_i^{(k)})^{r+\delta^{(k)}_{i,\pm}}\prod_{j\ge 1}(x_i^{(k)}-tx_j^{(k\pm 1)})/\prod_{j\ne i}(x_i^{(k)}-x_j^{(k)})\biggr\},$$

where, $\mathfrak{S}_{m_k}$ is the symmetric group with respect to the letters $\{1,2,\dots,m_k\}$ and $\mathfrak{S}_{m_k-1}$ is the subgroup of $\mathfrak{S}_{m_k}$ fixing the letter i, and $\delta^{(k)}_{i,\pm}=(m_k-1)-m_{k\pm1}$. Thus (3.2.3) holds.

Let $\nu_1=(q,p)$ be the smallest element in $\mathcal{M}$. Put $\mathbf{m}'=(m_0,\dots,m_p-1,\dots,m_{e-1})$, and let $\boldsymbol{\alpha}'\in Z^{0,0}_{n'}(\mathbf{m}')$ be the element obtained from $\boldsymbol{\alpha}$ by removing $\alpha^{(p)}_q$, where $n'=n-\alpha^{(p)}_q$. The set $\mathcal{M}'=\mathcal{M}\setminus\{\nu_1\}$ has a total order $<$ inherited from $\mathcal{M}$. We consider the functions $R^{\pm}_{\boldsymbol{\alpha}',<}(x';t)$ with variables $x'=\{x^{(k)}_j \mid (j,k)\ne(q,p)\}$. We have the following lemma.

Lemma 3.3. *Let $R^{\pm[i]}_{\boldsymbol{\alpha}',<}(x;t)$ be the function obtained from $R^{\pm}_{\boldsymbol{\alpha}',<}(x',t)$ by replacing the variables $x^{(p)}_1,\dots,\hat{x}^{(p)}_q,\dots,x^{(p)}_{m_p}$ ($\hat{x}^{(p)}_q$ is removed) by $x^{(p)}_1,\dots,\hat{x}^{(p)}_i,\dots,x^{(p)}_{m_p}$. Then we have*

$$R^{\pm}_{\boldsymbol{\alpha},<}=\sum_{i=1}^{m_p}(x^{(p)}_i)^{\alpha^{(p)}_q+\delta^{(p)}_q}g_iR^{\pm[i]}_{\boldsymbol{\alpha}',<},$$

where g_i $(1\le i\le m_p)$ is given by

$$g_i(x;t)=\prod_j(x^{(p)}_i-tx^{(p\pm1)}_j)/\prod_{j\ne i}(x^{(p)}_i-x^{(p)}_j).$$

Proof. Since $(q,p)\le\nu_0$, we have $\mathfrak{S}_{\boldsymbol{\alpha}}=\mathfrak{S}_{\boldsymbol{\alpha}'}\subset\mathfrak{S}_{\mathbf{m}'}\subset\mathfrak{S}_{\mathbf{m}}$ and $\mathfrak{S}_{\mathbf{m}}/\mathfrak{S}_{\mathbf{m}'}\simeq\mathfrak{S}_{m_p}/\mathfrak{S}_{m_p-1}$. ($\mathfrak{S}_{m_p-1}$ is identified with the subgroup of $\mathfrak{S}_{m_p}$ fixing the letter q). Note that since ν_1 is minimum, no factor of the form $x^{(p\mp1)}_j-tx^{(p)}_q$ appears in the expression of $R^{\pm}_{\boldsymbol{\alpha},<}$. It follows, by (3.2.1), that we have

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t)=\sum_{w\in\mathfrak{S}_{\mathbf{m}}/\mathfrak{S}_{\mathbf{m}'}}w\bigl\{(x^{(p)}_q)^{\alpha^{(p)}_q+\delta^{(p)}_q}g_qR^{\pm}_{\boldsymbol{\alpha}',<}(x';t)\bigr\},$$

which implies the equality in the lemma. □

3.4. Let

$$\Psi^{(k)}_{\pm}(u)=\sum_{r=0}^{\infty}q^{(k)}_{r,\pm}(x;t)u^r,$$
$$F(u_1,u_2)=(1-u_1)/(1-tu_2).$$

We regard $F(u_1,u_2)$ as a formal power series $(1-u_1)(1+tu_2+(tu_2)^2+\cdots)$. We introduce infinitely many variables $u^{(k)}_1,u^{(k)}_2,\dots$ for $0\le k<e$. We fix a total order $<$ on the set $\mathcal{M}^*=\{(i,k)\mid 0\le k<e, i=1,2,\dots\}$, and define a function $\Phi_{\pm}$ with

multi-variables $\mathbf{u} = \{u_i^{(k)}\}$ by

$$\Phi_\pm(\mathbf{u}) = \prod_{k=0}^{e-1}\prod_{i\geq 1}\Psi_\pm^{(k)}(u_i^{(k)}) \prod_{\substack{(i,k)<(j,k)\\(i,k)<(l,k\mp 1)}} F((u_i^{(k)})^{-1}u_j^{(k)}, (u_i^{(k)})^{-1}u_l^{(k\mp 1)}).$$

Take $\boldsymbol{\alpha} \in Z_n^{0,0}$, and let $\mathcal{M}$ be as in 3.1. Then $\mathcal{M}$ can be embedded in $\mathcal{M}^*$. We say that $\mathcal{M}$ is compatible with $\mathcal{M}^*$ if the embedding preserves the order, and if the inequality $\nu < \nu'$ holds for $\nu \in \mathcal{M}$, $\nu' \in \mathcal{M}^* - \mathcal{M}$. We show the following proposition.

Proposition 3.5. *Assume that $\mathcal{M}$ is compatible with $\mathcal{M}^*$. Then $R^\pm_{\boldsymbol{\alpha},<}$ coincides with the coefficient of $\mathbf{u}^{\boldsymbol{\alpha}} = \prod_{k,i}(u_i^{(k)})^{\alpha_i^{(k)}}$ in the function $\Phi_\pm(\mathbf{u})$.*

Proof. We prove the proposition by induction on $l = \sharp\{\nu \in \mathcal{M} \mid \nu \leq \nu_0\}$. First assume that $l = 1$. Then $\nu_0 = (i,k)$ is the smallest element in $\mathcal{M}$, and so in $\mathcal{M}^*$. If $\alpha_i^{(k)} = r$, we have $R^\pm_{\boldsymbol{\alpha},<} = q^{(k)}_{r,\pm}$ by (3.2.3), and the assertion follows. Assume that $l > 1$. Let $\nu_1 = (q,p)$ be the minimum element in $\mathcal{M}$, and let $\boldsymbol{\alpha}'$ be as in 3.2. We may assume, by induction, that the assertion holds for $R^\pm_{\boldsymbol{\alpha}',<}(x';t)$. We denote by $\Phi_\pm^{[i]}$ the function obtained from $\Phi_\pm$ by setting $x_i^{(p)} = 0$. Then by (2.3.1) we have

$$\Phi_\pm^{[i]}(\mathbf{u}) = \Phi_\pm(\mathbf{u})\prod_{j,l\geq 1} F(x_i^{(p)}u_j^{(p)}, x_i^{(p)}u_l^{(p\mp 1)}). \tag{3.5.1}$$

By applying the induction hypothesis to $R^{[i]\pm}_{\boldsymbol{\alpha}',<}$, together with Lemma 3.3, we see that $R^\pm_{\boldsymbol{\alpha},<}(x;t)$ is the coefficient of $\mathbf{u}^{\boldsymbol{\alpha}}$ in

$$\sum_{a\geq 0}(u_q^{(p)})^a\sum_{i=1}^{m_p}(x_i^{(p)})^{a+\delta_1^{(p)}}g_i\Phi_\pm^{[i]}(\mathbf{u}'),$$

where $\mathbf{u}' = \mathbf{u} - \{u_q^{(p)}\}$. Now by (3.5.1), the last expression is equal to

$$\Psi_\pm(\mathbf{u}')\sum_{a\geq 0}(u_q^{(p)})^a\sum_{i=1}^{m_p}(x_i^{(p)})^{a+\delta_q^{(p)}}g_i\prod_{j\neq q} F(x_i^{(p)}u_j^{(p)}, x_i^{(p)}u_l^{(p\mp 1)}). \tag{3.5.2}$$

We expand the product in (3.5.2) as a power series in $x_i^{(p)}$;

$$\prod F(x_i^{(p)}u_j^{(p)}, x_i^{(p)}u_l^{(p\mp 1)}) = \sum_{m\geq 0} f_m(\mathbf{u}';t)(x_i^{(p)})^m,$$

where $f_m = f_m(\mathbf{u}'; t)$ is a polynomial in $\mathbf{u}', t$. Write the expression (3.5.2) as $\Phi_\pm(\mathbf{u}')X$. Then substituting the above expansion into X, we have

$$\begin{aligned} X &= \sum_{a\ge 0}(u_q^{(p)})^a \sum_{m\ge 0} f_m \sum_{i=1}^{m_p}(x_i^{(p)})^{a+\delta_q^{(p)}+m} g_i \\ &= \sum_{a,m\ge 0}(u_q^{(p)})^a q^{(p)}_{a+m,\pm} f_m \\ &= \sum_{b=0}^{\infty}(u_q^{(p)})^b q^{(p)}_{b,\pm}\sum_{m=0}^{b} f_m (u_q^{(p)})^{-m}. \end{aligned}$$

Hence the positive degree part in X with respect to $u_q^{(p)}$ coincides with

$$\begin{aligned} &\Psi_\pm^{(p)}(u_q^{(p)})\sum_{m\ge 0} f_m (u_q^{(p)})^{-m} \\ &= \Psi_\pm^{(p)}(u_q^{(p)}) \prod_{\substack{j\ne q \\ (q,p)<(l,p\mp 1)}} F((u_q^{(p)})^{-1}u_j^{(p)}, (u_q^{(p)})^{-1}u_l^{(p\mp 1)}). \end{aligned}$$

Thus $R^{\pm}_{\boldsymbol{\alpha},<}$ is the coefficient of $\mathbf{u}^{\boldsymbol{\alpha}}$ in

$$\Psi_\pm^{(p)}(u_q^{(p)}) \prod_{\substack{j\ne q \\ (q,p)<(l,p\mp 1)}} F((u_q^{(p)})^{-1}u_j^{(p)}, (u_q^{(p)})^{-1}u_l^{(p\mp 1)}) \cdot \Phi_\pm(\mathbf{u}') = \Phi_\pm(\mathbf{u}).$$

The proposition is proved. $\square$

3.6. Let $M = \sum_i m_i$. We define an operator R_{ij} on the set $\mathbb{Z}^M$ by $R_{ij}(\lambda) = \lambda'$, where if $\lambda = (\lambda_1, \dots, \lambda_M) \in \mathbb{Z}^M$, then $\lambda' \in \mathbb{Z}^M$ is given by

$$\lambda'_i = \lambda_i + 1, \qquad \lambda'_j = \lambda_j - 1$$

and $\lambda'_l = \lambda_l$ for $l \ne i, j$. A raising operator (resp. lowering operator) R on $\mathbb{Z}^M$ is defined as a product of various R_{ij} with $i < j$ (resp. $i > j$). We identify $\mathcal{M}$ with the set $\{1, 2, \dots, M\}$ via the total order on $\mathcal{M}$, and regard $\boldsymbol{\alpha} \in \widehat{Z}_n^{0,0}$ as an element in $\mathbb{Z}^M$. Under this identification, we express the operator R_{ij} as $R_{\nu,\nu'}$ for $\nu, \nu' \in \mathcal{M}$.

The definition of $q_{\boldsymbol{\alpha},\pm}$ can be applied to $\boldsymbol{\alpha} \in \widehat{Z}_n^{0,0}$ as well. We extend the definition of $q^{(k)}_{r,\pm}$ to the case $r < 0$ by setting $q^{(k)}_{r,\pm} = 0$. Accordingly, we define $q_{\boldsymbol{\alpha},\pm}$ by setting $q_{\boldsymbol{\alpha},\pm} = 0$ if some $\alpha_j^{(k)} < 0$. Then an action of of raising operators on the functions $q_{\boldsymbol{\alpha},\pm}$ is defined by $R(q_{\boldsymbol{\alpha},\pm}) = q_{\boldsymbol{\alpha}',\pm}$ with $\boldsymbol{\alpha}' = R\boldsymbol{\alpha}$.

Let us define a function $b : \mathcal{M} \to \mathbb{N}$ by $b(\nu) = k$ if $\nu = (i, k) \in \mathcal{M}$. As a corollary to Proposition 3.5, we have the following.

Corollary 3.7. *Let $\boldsymbol{\alpha} \in \widehat{Z}_n^{0,0}$. Then the function $R^{\pm}_{\boldsymbol{\alpha},<}$ is expressed in terms of raising operators as*

$$R^{\pm}_{\boldsymbol{\alpha},<} = \prod_{\nu<\nu'} \Big\{ \prod_{b(\nu')=b(\nu)} (1-R_{\nu\nu'}) \Big/ \prod_{b(\nu')=b(\nu)\mp 1} (1-tR_{\nu\nu'}) \Big\} q_{\boldsymbol{\alpha},\pm}. \tag{3.7.1}$$

In particular, $R^{\pm}_{\boldsymbol{\alpha},<}$ is a polynomial in x,t, and is expressed as

$$R^{\pm}_{\boldsymbol{\alpha},<}(x;t) = \sum_{\boldsymbol{\beta}\in Z_n^{0,0}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t) \tag{3.7.2}$$

with $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$.

Proof. We notice that $\prod_{k,i}\Psi^{(k)}_{\pm}(u_i^{(k)}) = \sum_{\boldsymbol{\beta}} q_{\boldsymbol{\beta},\pm}\mathbf{u}^{\boldsymbol{\beta}}$. Then it follows from Proposition 3.5, by using a similar argument as in [M, III, 2], that one can write $R^{\pm}_{\boldsymbol{\alpha},<}$ in a form as in (3.7.1). The formula (3.7.2) follows easily from this. $\square$

3.8. We consider $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in \widehat{Z}_n^{r,s}$, and write it as $\boldsymbol{\Lambda} = (\lambda_j^{(k)})$. We choose a total order $<$ on $\mathcal{M}$ satisfying the property that $\nu < \nu'$ if $\lambda_i^{(k)} > \lambda_j^{(l)}$ for $\nu = (i,k), \nu' = (j,l) \in \mathcal{M}$. We now define a function $R^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ by $R^{\pm}_{\boldsymbol{\Lambda}} = R^{\pm}_{\boldsymbol{\alpha},<}$. For this $R^{\pm}_{\boldsymbol{\Lambda}}$, we can obtain a more precise result than Corollary 3.7. First we show some properties of a-functions.

Let $\lambda = (\lambda_1,\dots,\lambda_M), \mu = (\mu_1,\dots,\mu_M) \in \mathbb{Z}^M$. We define a usual partial order $\lambda \ge \mu$ on the set $\mathbb{Z}^M$ by $\lambda_1 + \cdots \lambda_i \ge \mu_1 + \cdots \mu_i$ for $i = 1,\dots,M$. Then as is well-known that

(3.8.1) Let R be a raising operator on $\mathbb{Z}^M$. Then $R(\lambda) \ge \lambda$ for $\lambda \in \mathbb{Z}^M$.

For a partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_M \ge 0)$, we define $n(\lambda)$ by

$$n(\lambda) = \sum_{i\ge 1}(i-1)\lambda_i.$$

It is known (e.g., [M, III, (6.5)]) that

(3.8.2) Let λ, μ be partitions such that $\lambda \ge \mu$. Then we have $n(\lambda) \le n(\mu)$, and the equality $n(\lambda) = n(\mu)$ occurs only when $\lambda = \mu$.

A symbol $\boldsymbol{\Lambda} \in Z_n^{r,s}$ can be identified with an element in $\mathbb{Z}^M$ via the order on $\mathcal{M}$. Hence the raising operator acts on symbols $\boldsymbol{\Lambda}$ also. Note that the a-function on $Z_n^{r,s}$ can be extended to the function on $\widehat{Z}_n^{r,s}$, which is also denoted by a. We have the following lemma.

Lemma 3.9. *Let $\boldsymbol{\Lambda} \in Z_n^{r,s}$ and assume that $\boldsymbol{\Lambda}' \in \widehat{Z}_n^{r,s}$ is obtained from $\boldsymbol{\Lambda}$ by applying a raising operator. Then we have $a(\boldsymbol{\Lambda}) \ge a(\boldsymbol{\Lambda}')$, and the equality $a(\boldsymbol{\Lambda}) = a(\boldsymbol{\Lambda}')$ occurs only when $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}'$.*

Proof. Let $\xi \in \mathbb{Z}^M$ be a partition corresponding to $\boldsymbol{\Lambda}$, and $\xi' \in \mathbb{Z}^M$ the element obtained by applying a raising operator on ξ. Then by (3.8.1), we have $\xi' \ge \xi$. If ξ'' is the partition obtained from ξ' by arranging the parts of ξ' in decreasing order, then we have $\xi'' \geq \xi'$. Hence we have $n(\xi'') \leq n(\xi)$ by (3.8.2). But for a partition $\xi = (\xi_1 \geq \xi_2 \geq \cdots \geq \xi_M)$, $n(\xi)$ is also expressed as

$$n(\xi) = \sum_{i \neq j} \min\{\xi_i, \xi_j\}.$$

Hence $n(\xi'') \leq n(\xi)$ is equivalent to $a(\boldsymbol{\Lambda}') \leq a(\boldsymbol{\Lambda})$. The equality case also follows from (3.8.2). □

3.10. Let $\boldsymbol{\alpha} \in \widehat{Z}^{0,0}_n$. We define the a-function on $\widehat{Z}^{0,0}_n$ by $a(\boldsymbol{\alpha}) = a(\boldsymbol{\Lambda})$ if $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in \widehat{Z}^{r,s}_n$. Let $\boldsymbol{\alpha}'$ be the element in $Z^{0,0}_n$ obtained from $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ by arranging the entries in $\alpha^{(k)} = (\alpha^{(k)}_1, \dots, \alpha^{(k)}_{m_k})$ in decreasing order so that it gives a partition. Then clearly we have $q_{\boldsymbol{\alpha},\pm} = q_{\boldsymbol{\alpha}',\pm}$. However, $a(\boldsymbol{\alpha})$ and $a(\boldsymbol{\alpha}')$ are different in general. Concerning this, we have the following lemma.

Lemma 3.11. *Let $\boldsymbol{\alpha} \in \widehat{Z}^{0,0}_n$, and $\boldsymbol{\alpha}' \in Z^{0,0}_n$ be as above. Then we have $a(\boldsymbol{\alpha}') \leq a(\boldsymbol{\alpha})$. The equality occurs only when $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$.*

Proof. We consider a sequence $\alpha^{(k)} = (\alpha_1, \dots, \alpha_{m_k}) \in \mathbb{Z}^{m_k}_{\geq 0}$ and put $\Lambda^{(k)} = (\lambda_1, \dots, \lambda_{m_k})$, where $\lambda_i = \alpha_i + (m-i)r + s$ (for example, in the case where $k \geq 1$. The case $k = 0$ is similar). Assume that $\alpha_i < \alpha_{i+1}$ for some i. Let $\alpha' = (\alpha'_1, \dots, \alpha'_m)$ be a sequence obtained from α by putting $\alpha'_i = \alpha_{i+1}, \alpha'_{i+1} = \alpha_i$ and $\alpha'_j = \alpha_j$ for $j \neq i, i+1$. Put $\Lambda' = (\lambda'_1, \dots, \lambda'_m)$ with $\lambda'_j = \alpha'_j + (m-j)r + s$ for each j, and let $\boldsymbol{\Lambda}'$ be a symbol obtained from $\boldsymbol{\Lambda}$ by replacing $\Lambda^{(k)}$ by Λ' and by leaving other $\boldsymbol{\Lambda}^{(l)}$ unchanged. In order to prove the lemma, it is enough to show that $a(\boldsymbol{\Lambda}') < a(\boldsymbol{\Lambda})$. Now it is easy to see that $\lambda'_i = \lambda_{i+1} + r, \lambda'_{i+1} = \lambda_i - r$. If $\lambda_i \leq \lambda_{i+1}$, then $\boldsymbol{\Lambda}'$ coincides with $R^r_{i+1,i}\boldsymbol{\Lambda}$, and we have $a(\boldsymbol{\Lambda}') < a(\boldsymbol{\Lambda})$. On the other hand, if $\lambda_i > \lambda_{i+1}$, then we have $\lambda'_i = \lambda_i + c, \lambda'_{i+1} = \lambda_{i+1} - c$ with $c = \alpha_{i+1} - \alpha_i > 0$. Hence $\boldsymbol{\Lambda}'$ coincides with $R^c_{i,i+1}\boldsymbol{\Lambda}$, and again we have $a(\boldsymbol{\Lambda}') < a(\boldsymbol{\Lambda})$. So the lemma is proved. □

We have the following.

Proposition 3.12. *Let $\boldsymbol{\alpha} \in Z^{0,0}_n$ and put $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z^{r,s}_n$. Let $R^{\pm}_{\boldsymbol{\Lambda}}$ be as in 3.8. Then*

(i) *$R^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed as*

$$R^{\pm}_{\boldsymbol{\Lambda}}(x;t) = q_{\boldsymbol{\alpha},\pm}(x;t) + \sum_{\boldsymbol{\beta} \in Z^{0,0}_n} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t)$$

with $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$, where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $a(\boldsymbol{\beta}) < a(\boldsymbol{\alpha})$.

(ii) *The polynomial $R^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ has the stability property in the following sense; let $\mathbf{m}' = (m'_0, \dots, m'_{e-1})$ with $m'_k = m_k + 1$. Let $\boldsymbol{\Lambda}' \in Z^{r,s}_n(\mathbf{m}')$ be the symbol obtained from $\boldsymbol{\Lambda} \in Z^{r,s}_n(\mathbf{m})$ by the shift operation. Then the polynomial $R^{\pm}_{\boldsymbol{\Lambda}'}(x;t)$ with $x^{(k)}_{m_k+1} = 0$ $(0 \leq k \leq e-1)$ coincides with $R^{\pm}_{\boldsymbol{\Lambda}}(x;t)$.*

Proof. We show (i). By Corollary 3.7, $R^{\pm}_{\boldsymbol{\Lambda}}$ is expressed as a linear combination of $q_{\boldsymbol{\beta},\pm}$ with coefficient $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Z}[t]$, where $\boldsymbol{\beta} \in \widehat{Z}^{0,0}_n$. Since the formula (3.7.1) involves raising operators only, we see that $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $\boldsymbol{\Lambda}(\boldsymbol{\beta})$ is obtained from $\boldsymbol{\Lambda}$ by applying the raising operators. This implies, in view of Lemma 3.9, that $c_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $a(\boldsymbol{\beta}) \le a(\boldsymbol{\alpha})$, and that the case $a(\boldsymbol{\beta}) = a(\boldsymbol{\alpha})$ occurs only when $\boldsymbol{\beta} = \boldsymbol{\alpha}$. Now by virtue of Lemma 3.11, we may replace $\boldsymbol{\beta} \in \widehat{Z}_n^{0,0}$ by $Z_n^{0,0}$. Hence the assertion follows.

Next we show (ii). Let $\mathcal{M}'$ be a set associated to $Z_n^{r,s}(\mathbf{m}')$ similar to $\mathcal{M}$. Then $\mathcal{M} \subset \mathcal{M}'$ and we may choose a total order on $\mathcal{M}'$ compatible with the order on $\mathcal{M}$ given in 3.8. Then in the expression of $R^{\pm}_{\boldsymbol{\Lambda}'}$ given in (3.7.1), the factors corresponding to $R_{\nu\nu'}$ with $\nu' \le \nu_0$ have exactly the same form as the factors for $R^{\pm}_{\boldsymbol{\Lambda}}$. The factors corresponding to $R_{\nu\nu'}$ with $\nu' > \nu_0$ are negligible since $R_{\nu\nu'}\boldsymbol{\alpha}$ contains a negative part and so $R_{\nu\nu'}q_{\boldsymbol{\alpha},\pm} = 0$. Hence (ii) follows from the stability property for $q_{\boldsymbol{\beta},\pm}$ (see Lemma 2.3). The proposition is proved. □

3.13. Let $\widetilde{Z}_n^{0,0}$ be the set of $\boldsymbol{\alpha} = (\alpha_j^{(k)})$ with $\alpha_j^{(k)} \in \mathbb{Z}$. The definition of Schur functions $s_{\boldsymbol{\alpha}}$ given in (2.1.2) can be extended to the case where $\boldsymbol{\alpha} \in \widetilde{Z}^{0,0}$. In that case, $s_{\boldsymbol{\alpha}}$ is a Laurent polynomial with respect to the variables $x_j^{(k)}$. If $\alpha_j^{(k)} + (m_k - j)$ are positive and all distinct for $j = 1, \dots, m_k$ (for a fixed k), then $s_{\boldsymbol{\alpha}}$ coincides with the usual Schur function up to sign. If $\alpha_j^{(k)} + (m_k - j)$ are not all distinct for a fixed k, then $s_{\boldsymbol{\alpha}} = 0$. The operator $R = \prod R_{\nu\nu'}$ acts on Schur functions by $R(s_{\boldsymbol{\alpha}}) = s_{\boldsymbol{\alpha}'}$ if $R(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'$. We now consider $\boldsymbol{\alpha} \in Z_n^{0,0}$ and let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$. Then we have the following formula.

$$R^{\pm}_{\boldsymbol{\Lambda}} = v_{\boldsymbol{\alpha}}(t)^{-1} \prod_{\substack{\nu<\nu', \nu\le\nu_0 \\ b(\nu')=b(\nu\pm 1)}} (1 - tR_{\nu'\nu}) \prod_{\substack{\nu<\nu', \nu>\nu_0 \\ b(\nu')=b(\nu)}} (1 - tR_{\nu'\nu}) s_{\boldsymbol{\alpha}}. \tag{3.13.1}$$

First we note that

$$R^{\pm}_{\boldsymbol{\Lambda}}(x;0) = s_{\boldsymbol{\alpha}}(x). \tag{3.13.2}$$

In fact, if we substitute $t = 0$ to $R^{\pm}_{\boldsymbol{\Lambda}}$ in the formula (3.1.2), we have

$$R^{\pm}_{\boldsymbol{\Lambda}}(x;0) = \sum_{w\in\mathfrak{S}_{\mathbf{m}}} w\Big\{\prod_k (x^{(k)})^{\alpha^{(k)}+\varepsilon^{(k)}} \Big/ \prod_k \prod_{(i,k)<(j,k)} (x_i^{(k)} - x_j^{(k)})\Big\},$$

where $\varepsilon^{(k)} = (\varepsilon_1^{(k)}, \dots, \varepsilon_{m_k}^{(k)})$ is defined by $\varepsilon_i^{(k)} = \sharp\{j \mid (i,k) < (j,k)\}$. But since $\boldsymbol{\Lambda} \in Z_n^{r,s}$, $(i,k) < (j,k)$ is equivalent to $i < j$ by the definition of the order on $\mathcal{M}$. Hence we obtain (3.13.2). Now (3.13.1) is immediate from the definition of $R^{\pm}_{\boldsymbol{\alpha},<}$ in (3.1.2) together with (3.13.2).

Note that Schur functions $s_{\boldsymbol{\alpha}}$ (in a general sense) are linearly independent if $\boldsymbol{\beta} = (\beta^{(0)}, \dots, \beta^{(e-1)})$ are all distinct, where $\beta^{(k)}$ is obtained from $\alpha^{(k)}$ by arranging the entries $(\alpha_j^{(k)} + (m_k - j))$ in decreasing order. By Corollary 3.7, we know that $R^{\pm}_{\boldsymbol{\Lambda}}$ is a polynomial in x, t. Hence it is written as a linear combination of Schur functions $s_{\boldsymbol{\beta}}(x)$ for $\boldsymbol{\beta} \in Z_n^{0,0}$. It follows that in the expansion of (3.13.1) as a sum of Schur functions, we may only pick up $s_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \in Z_n^{0,0}$. We have the following proposition.

Proposition 3.14. *Assume that $\boldsymbol{\Lambda} \in Z_n^{r,s}$. Then $R^{\pm}_{\boldsymbol{\Lambda}}$ can be written as*

$$R^{\pm}_{\boldsymbol{\Lambda}}(x;t) = \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x)$$

with $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$, where $u_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in t\mathbb{Z}[t]$ for $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$, and $u_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \notin t\mathbb{Z}[t]$.

Proof. Since both of $R^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $s_{\boldsymbol{\beta}}(x)$ have the stability property, we may assume that $\nu_0 = (m_k, k)$ for some k. It follows that $v_{\boldsymbol{\alpha}}(t) = 1$ and we see that (3.13.1) can be written, in this case, as

$$R^{\pm}_{\boldsymbol{\Lambda}} = \prod_{\substack{\nu < \nu', \nu \le \nu_0 \\ b(\nu') = b(\nu \pm 1)}} (1 - tR_{\nu'\nu}) s_{\boldsymbol{\alpha}}. \tag{3.14.1}$$

Therefore $R^{\pm}_{\boldsymbol{\Lambda}}$ can be written as a linear combination of $s_{\boldsymbol{\beta}}$ with coefficient $u_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Z}[t]$. Now the last assertion follows from (3.14.1) together with the fact that $R^{\pm}_{\boldsymbol{\Lambda}}(x;0) = s_{\boldsymbol{\alpha}}(x)$. $\square$

3.15. Let $\mathbf{m}$ be as in 2.1. We denote by $\Xi_{\mathbf{m}} = \bigotimes_{k=0}^{e-1} \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{\mathbf{m}}$) with variables $x = (x_j^{(k)})$. $\Xi_{\mathbf{m}}$ has a structure of a graded ring $\Xi_{\mathbf{m}} = \bigoplus_{i \ge 0} \Xi^i_{\mathbf{m}}$, where $\Xi^i_{\mathbf{m}}$ consists of homogeneous symmetric polynomials of degree i, together with the zero polynomial. We consider the inverse limit

$$\Xi^i = \varprojlim_{\mathbf{m}} \Xi^i_{\mathbf{m}}$$

with respect to homomorphisms $\rho_{\mathbf{m}',\mathbf{m}} : \Xi^i_{\mathbf{m}'} \to \Xi^i_{\mathbf{m}}$, where $\mathbf{m}' = (m'_0, \dots, m'_{e-1})$ with $m'_k = m_k + l$ for some integer $l \ge 0$, and $\rho_{\mathbf{m}',\mathbf{m}}$ is induced from the homomorphism $\bigoplus_k \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}] \to \bigoplus_k \mathbb{Z}[x_1^{(k)}, \dots, x_{m'_k}^{(k)}]$ given by sending $x_i^{(k)}$ to 0 for $i \ge m_k$, and leaving other $x_j^{(k)}$ invariant. $\Xi = \bigoplus_{i \ge 0} \Xi^i$ is called the space of symmetric functions. Schur function $s_{\boldsymbol{\alpha}}(x)$ with infinitely many variables $x_1^{(k)}, \dots$ is regarded as an element in Ξ^n with $n = |\boldsymbol{\alpha}|$, and the set $\{s_{\boldsymbol{\alpha}}(x)\}$ with $\boldsymbol{\alpha} \in Z_n^{0,0}$ forms a $\mathbb{Z}$-basis of Ξ^n. Put $\Xi[t] = \mathbb{Z}[t] \otimes_{\mathbb{Z}} \Xi$. For $\boldsymbol{\Lambda} \in Z_n^{r,s}$, the function $R^{\pm}_{\boldsymbol{\Lambda}}$ is regarded as an element in $\Xi[t]$, and by Proposition 3.14, the set $\{R^{\pm}_{\boldsymbol{\Lambda}} \mid \boldsymbol{\Lambda} \in Z_n^{r,s}\}$ forms a basis of the $\mathbb{Q}(t)$-space $\Xi^n_{\mathbb{Q}}[t] = \mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. This implies, by Proposition 3.12 (ii), that $\{q_{\boldsymbol{\alpha},\pm} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$ also gives rise to a basis of $\Xi^n_{\mathbb{Q}}[t]$. A similar property holds if one replace Ξ^n by $\Xi^n_{\mathbf{m}}$.

4. Hall-Littlewood functions

4.1. As described in Proposition 3.14, $R^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed in terms of Schur functions. However, the transition matrix between $R^{\pm}_{\boldsymbol{\Lambda}}$ and $s_{\boldsymbol{\alpha}}$ is not necessarily a triangular shape as a block matrix. In order to recover this property, we shall define a new function $Q^{\pm}_{\boldsymbol{\Lambda}}$ by modifying $R^{\pm}_{\boldsymbol{\Lambda}}$. In this section we are only concerned with the set $Z_n^{0,0}$ and $Z_n^{r,s}$. Let $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z^{r,s}$ and $\mathcal{M} = \mathcal{M}_{\boldsymbol{\Lambda}}$ be as in 3.8. First we note the following lemma.

Lemma 4.2. *For each $0 \le i < e$, let $\boldsymbol{\alpha}_i \in Z_n^{0,0}$ be an element such that $\alpha_1^{(i)} = n$ and that $\alpha_j^{(k)} = 0$ for $(j,k) \ne (1,i)$. Assume that $\boldsymbol{\alpha} \in Z_n^{0,0}$ is such that $a(\boldsymbol{\alpha})$ is minimum among $Z_n^{0,0}$. Then $\boldsymbol{\alpha}$ coincides with some $\boldsymbol{\alpha}_i$.*

Proof. Let $\boldsymbol{\Lambda}' \in \widehat{Z}_n^{r,s}$ and take λ, λ' in the entry of $\boldsymbol{\Lambda}'$ such that $0 < \lambda' < \lambda$. Let $\boldsymbol{\Lambda}'' \in \widehat{Z}_n^{r,s}$ be the element obtained from $\boldsymbol{\Lambda}'$ by replacing λ by $\lambda + 1$, and λ' by $\lambda' - 1$. Then clearly we have $a(\boldsymbol{\Lambda}'') < a(\boldsymbol{\Lambda}')$. Hence, repeating this operation, we reach to the minimum element with respect to a, which must have the form $\boldsymbol{\Lambda}(\boldsymbol{\alpha}_i)$. □

4.3. Let $\mathcal{A}$ be the subring of $\mathbb{Q}(t)$ consisting of functions which have no pole at $t = 0$. Then $\mathcal{A}$ is a local ring with the unique maximal ideal $t\mathcal{A}$. Hence $\mathcal{A}^* = \mathcal{A} - t\mathcal{A}$ is the set of units in $\mathcal{A}$. In what follows, we identify the set $\mathcal{P}_{n,e}$ with $Z_n^{0,0}$, and fix a total order $\prec$ on $Z_n^{0,0}$ as in 1.4. We consider a square matrix indexed by the set $Z_n^{0,0}$, and regard it as a block matrix, where each block corresponds to a similarity class.

Let $X = \{X_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$ and $Y = \{Y_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$ be two bases of $\mathbb{Q}(t) \otimes \Xi_{\mathbf{m}}^n$, indexed by $Z_n^{0,0}$. We denote by $M(X,Y)$ the transition matrix between two bases, i.e., $M(X,Y) = (m_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ if $X_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}} m_{\boldsymbol{\alpha},\boldsymbol{\beta}} Y_{\boldsymbol{\beta}}$. We also use $Z_n^{r,s}$ as the index set under the identification $Z_n^{r,s} \simeq Z_n^{0,0}$. We shall construct two types of functions $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ and $Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ as linear combinations of $R_{\boldsymbol{\Lambda}}^{\pm}(x;t)$.

Theorem 4.4. (i) *For each $\boldsymbol{\Lambda} \in Z_n^{r,s}$, there exists a unique function $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ satisfying the following properties.*

(a) *$P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed as*

$$P_{\boldsymbol{\Lambda}}^{\pm}(x;t) = \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t),$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ and $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$.

(b) *$P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed as*

$$P_{\boldsymbol{\Lambda}}^{\pm}(x;t) = s_{\boldsymbol{\alpha}}(x) + \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$, and $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(ii) *For each $\boldsymbol{\Lambda} \in Z_n^{r,s}$, there exists a unique function $Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ satisfying the following properties.*

(a) *$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed as*

$$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t) = q_{\boldsymbol{\alpha},\pm}(x;t) + \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t),$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ and $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(b) *$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed as*

$$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t) = \sum_{\boldsymbol{\beta} \in Z_n^{0,0}} w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathcal{A}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Moreover, $w_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in t\mathcal{A}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathcal{A}^*$.

(In the above formula, the coefficients $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$, etc. depend on the sign $\pm$.)

Proof. We construct the functions $P^{\pm}_{\boldsymbol{\Lambda}}$ and $Q^{\pm}_{\boldsymbol{\Lambda}}$ by backward induction on the total order $\prec$ on $Z^{r,s}_n$. We assume that $P^{\pm}_{\boldsymbol{\Lambda}'}$, $Q^{\pm}_{\boldsymbol{\Lambda}'}$ have been constructed for any $\boldsymbol{\Lambda}'$ such that $\boldsymbol{\Lambda}' \succ \boldsymbol{\Lambda}$ and that $\boldsymbol{\Lambda}' \not\sim \boldsymbol{\Lambda}$. Let Z be the similarity class containing $\boldsymbol{\Lambda}$. We shall construct $P^{\pm}_{\boldsymbol{\Lambda}''}$, $Q^{\pm}_{\boldsymbol{\Lambda}''}$ simultaneously for $\boldsymbol{\Lambda}'' \in Z$.

By Proposition 3.14, the function $R^{\pm}_{\boldsymbol{\Lambda}}$ can be written as $R^{\pm}_{\boldsymbol{\Lambda}} = \sum_{\boldsymbol{\beta}} u'_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}}$ with $u'_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathbb{Z}[t]$ if $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Let Z_1 be the set of $\boldsymbol{\beta} \in Z^{0,0}$ such that $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ and that $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$. Then, by using the property (i) (b) in the theorem, one can find a function $Q^{\pm}_{\boldsymbol{\Lambda}}$ of the form $Q^{\pm}_{\boldsymbol{\Lambda}} = R^{\pm}_{\boldsymbol{\Lambda}} - \sum_{\boldsymbol{\beta} \in Z_1} u''_{\boldsymbol{\alpha},\boldsymbol{\beta}} P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}$ with $u''_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$ satisfying the following property;

$$Q^{\pm}_{\boldsymbol{\Lambda}} = \sum_{\boldsymbol{\beta} \sim \boldsymbol{\alpha}} w_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}} + X_{\boldsymbol{\alpha}}, \tag{4.4.1}$$

where $X_{\boldsymbol{\alpha}}$ is a sum of $s_{\boldsymbol{\gamma}}$ such that $\boldsymbol{\gamma} \prec \boldsymbol{\alpha}$ and that $\boldsymbol{\gamma} \not\sim \boldsymbol{\alpha}$ with coefficients in $t\mathcal{A}$. Hence we see that $w_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in t\mathcal{A}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$, and that $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathcal{A}^*$, i.e., $Q^{\pm}_{\boldsymbol{\Lambda}}$ satisfies the property (ii), (b). On the other hand, it follows from Proposition 3.12, together with the property (i),(a) in the theorem, that $Q^{\pm}_{\boldsymbol{\Lambda}}$ satisfies the property (ii), (a). Thus we have constructed $Q^{\pm}_{\boldsymbol{\Lambda}}$.

Next, we shall construct the function $P^{\pm}_{\boldsymbol{\Lambda}}$. We now regard (4.4.1) as a system of equations for all $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ in the similarity class Z. Let $C = (w_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the matrix with entries in $\mathcal{A}$, indexed by Z, which is the coefficient matrix of this system of equations. Since $\det C \in \mathcal{A}^*$, the inverse matrix C^{-1} exists, with entries in $\mathcal{A}$. Then by applying C^{-1} to the system of equations (4.4.1), one can find functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ for each $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z$, which can be expressed in terms of Schur functions as

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}} u_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}},$$

where $\boldsymbol{\beta} \in Z^{0,0}_n$ runs over all the elements such that $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and that $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$, and $u_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in t\mathcal{A}$. Moreover, since $P^{\pm}_{\boldsymbol{\Lambda}}$ is a linear combination of various $Q^{\pm}_{\boldsymbol{\Lambda}'}$ with $\boldsymbol{\Lambda}' \sim \boldsymbol{\Lambda}$, it is expressed as a linear combination of $q_{\boldsymbol{\beta},\pm}$ with $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Hence $P^{\pm}_{\boldsymbol{\Lambda}}$ satisfies the required properties, (a), (b) of (i).

It remains to construct $P^{\pm}_{\boldsymbol{\Lambda}}$ in the case where $a(\boldsymbol{\Lambda})$ is minimal. If $a(\boldsymbol{\Lambda})$ is minimal, then $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}_i)$ for some i by Lemma 4.2. Then $R^{\pm}_{\boldsymbol{\Lambda}}$ coincides with $q_{\boldsymbol{\alpha}_i,\pm}$ by (3.2.3). We consider the similarity class Z containing $\boldsymbol{\Lambda}$ which consists of certain $\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}(\boldsymbol{\alpha}_i)$. Then the functions $R^{\pm}_{\boldsymbol{\Lambda}_i}$ already satisfy the properties required for $Q^{\pm}_{\boldsymbol{\Lambda}_i}$. So we may put $Q^{\pm}_{\boldsymbol{\Lambda}_i} = R^{\pm}_{\boldsymbol{\Lambda}_i}$. Then by using a similar argument as above, one can find $P^{\pm}_{\boldsymbol{\Lambda}_i}$ satisfying (a), (b) of (i) for each $\boldsymbol{\Lambda}_i \in Z$. Thus $P^{\pm}_{\boldsymbol{\Lambda}}$ can be constructed.

We show the uniqueness of $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$. The uniqueness of $\{Q^{\pm}_{\boldsymbol{\Lambda}}\}$ is proved similarly. Suppose that the functions $\{P'_{\boldsymbol{\Lambda}}\}$ satisfy the conditions (a) and (b) of (i). By the property (b), the sets $\{P_{\boldsymbol{\Lambda}}\}$ and $\{P'_{\boldsymbol{\Lambda}}\}$ form bases of the space $\mathbb{Q}(t) \otimes \Xi^n_{\mathbf{m}}$. Let $A = M(P,s)$ be the transition matrix between the functions $\{P^{\pm}_{\boldsymbol{\Lambda}} \mid \boldsymbol{\Lambda} \in Z^{r,s}_n\}$ and $\{s_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in Z^{0,0}_n\}$, and similarly define the transition matrix $A' = M(P', s)$ between $\{P'_{\boldsymbol{\Lambda}}\}$ and

$\{s_{\boldsymbol{\alpha}}\}$. Then by our assumption, both of A and A' are block lower triangular matrices with identity diagonal blocks. Hence $A(A')^{-1} = M(P, P')$ is also a lower triangular block matrix with identity diagonal blocks. On the other hand, let $B = M(P, q)$ be the transition matrix between $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$ and $\{q_{\boldsymbol{\alpha},\pm}\}$, and similarly define $B' = M(P', q)$. Then by our assumption, both of B and B' are block upper triangular matrices, and so is $B(B')^{-1}$. But since $B(B')^{-1} = M(P, P') = A(A')^{-1}$, we see that $A(A')^{-1}$ is simultaneously lower triangular and upper triangular with identity diagonal blocks. Hence $A = A'$ and we conclude that $P^{\pm}_{\boldsymbol{\Lambda}} = P'_{\boldsymbol{\Lambda}}$ as asserted. Thus the theorem is proved. □

Remarks 4.5. (i) We call $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ the Hall-Littlewood function associated to the symbol $\boldsymbol{\Lambda}$. It is likely that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. In some cases, including the classical case, where $e = 2, r = 2$, these statements are verified to be true. (See Remarks 5.5 (i)).

(ii) The previous construction of $P^{\pm}_{\boldsymbol{\Lambda}}$ depend on the choice of the total order $\succ$ compatible with the similarity classes. However, the construction still works if one replaces the relation $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ by the equivalence relation defined by the condition that $a(\boldsymbol{\beta}) = a(\boldsymbol{\alpha})$. In the above known cases, these two types of Hall-Littlewood functions coincide with each other. It would be interesting to know whether this is true in general since it will imply the indenpendence of $P^{\pm}_{\boldsymbol{\lambda}}$ from the chocie of the total order.

We can regard $P^{\pm}_{\boldsymbol{\Lambda}}$ as an element in $\mathbb{Q}(t) \otimes \Xi^n$ associated to the symbol class $\boldsymbol{\Lambda}$. As a corollary to Theorem 4.4, we have the following.

Corollary 4.6. *Let $\Omega(x, y; t)$ be as in Proposition 2.5. Then we have*

$$\Omega(x, y; t) = \sum_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) P^{+}_{\boldsymbol{\Lambda}}(x;t) P^{-}_{\boldsymbol{\Lambda}'}(y;t). \tag{4.6.1}$$

$$\Omega(x, y; t) = \sum_{\boldsymbol{\Lambda}} Q^{+}_{\boldsymbol{\Lambda}}(x;t) P^{-}_{\boldsymbol{\Lambda}}(y;t) = \sum_{\boldsymbol{\Lambda}} P^{+}_{\boldsymbol{\Lambda}}(x;t) Q^{-}_{\boldsymbol{\Lambda}}(y;t). \tag{4.6.2}$$

where in (4.6.1), *$\boldsymbol{\Lambda}, \boldsymbol{\Lambda}'$ run over all the elements in $\bigcup_{n=1}^{\infty} Z^{r,s}_n$, and $b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t) = 0$ unless $|\boldsymbol{\Lambda}| = |\boldsymbol{\Lambda}'|$ and $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$. In* (4.6.2), *$\boldsymbol{\Lambda}$ runs over all the elements in $\bigcup_{n=1}^{\infty} Z^{r,s}_n$.*

Proof. Let

$$A_{\pm} = M(q_{\pm}, P^{\pm}), \qquad B_{\pm} = M(m, P^{\pm}), \qquad C_{\pm} = M(q_{\pm}, m)$$

be the transition matrices between $\{q_{\boldsymbol{\alpha},\pm}\}$ and $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$, $\{m_{\boldsymbol{\alpha}}\}$ and $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$, and $\{q_{\boldsymbol{\alpha},\pm}\}$ and $\{m_{\boldsymbol{\alpha}}\}$, respectively. Put $D_{\pm} = {}^tB_{\mp}A_{\pm}$. Then by Theorem 4.4 (i),(a), $(A_{\pm})^{-1}$ is a block upper triangular matrix, hence so is $A_{\pm}$. Now $B_{\pm}^{-1}$ can be written as

$$B_{\pm}^{-1} = M(P^{\pm}, s) M(s, m).$$

By Theorem 4.4 (i),(b), $M(P^{\pm}, s)$ is block lower triangular. On the other hand, $M(s, m)$ is also lower triangular. In fact, for each partition λ, s_{λ} is written as a linear combination of m_{μ} with $\mu \le \lambda$. It follows that $s_{\boldsymbol{\alpha}}$ is a linear combination of $m_{\boldsymbol{\beta}}$, where $\boldsymbol{\beta}$ satisfies the following property. If we write $\boldsymbol{\Lambda}(\boldsymbol{\alpha}) = (\Lambda_0, \dots, \Lambda_{e-1})$ and $\boldsymbol{\Lambda}(\boldsymbol{\beta}) = (\Lambda'_0, \dots, \Lambda'_{e-1})$, then the partitions Λ_i and Λ'_i have the same size, and

$\varLambda'_i \le \varLambda_i$ (as partitions). This implies that $a(\boldsymbol{\varLambda}') \ge a(\boldsymbol{\varLambda})$ and the equality holds only when $\boldsymbol{\varLambda} = \boldsymbol{\varLambda}'$. Hence $M(s,m)$ is lower triangular with identity diagonal blocks. As a conclusion, we see that $B_{\pm}^{-1}$ is block lower triangular, and so $D_{\pm}$ is a block upper triangular matrix.

On the other hand, by (2.5.1), the matrices $C_{\pm}$ satisfy the relation ${}^tC_+ = C_-$. Since $D_{\pm} = {}^tB_{\mp}C_{\pm}B_{\pm}$, we see that ${}^tD_+ = D_-$. This implies that $D_{\pm}$ is a block diagonal matrix. If we put $A_+ = (A^+_{\boldsymbol{\alpha},\boldsymbol{\varLambda}})$, $B_- = (B^-_{\boldsymbol{\alpha},\boldsymbol{\varLambda}'})$, we have

$$\begin{aligned}\sum_{\boldsymbol{\alpha}} q_{\boldsymbol{\alpha},+}(x;t)m_{\boldsymbol{\alpha}}(y) &= \sum_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}',\boldsymbol{\alpha}} A^+_{\boldsymbol{\alpha},\boldsymbol{\varLambda}}B^-_{\boldsymbol{\alpha},\boldsymbol{\varLambda}'}P^+_{\boldsymbol{\varLambda}}(x;t)P^-_{\boldsymbol{\varLambda}'}(y;t) \\ &= \sum_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'} b^+_{\boldsymbol{\varLambda}',\boldsymbol{\varLambda}}(t)P^+_{\boldsymbol{\varLambda}}(x;t)P^-_{\boldsymbol{\varLambda}'}(y;t),\end{aligned}$$

where the matrix $(b^+_{\boldsymbol{\varLambda}',\boldsymbol{\varLambda}}(t))$ coincides with D_+. Thanks to (2.5.1), by putting $b_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'}(t) = b^-_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'}(t) = b^+_{\boldsymbol{\varLambda}',\boldsymbol{\varLambda}}$ (here $D_- = (b^-_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'})$), we obtain the formula (4.6.1).

We now look at the previous argument more precisely. In the argument below, we use a notation X_D to denote the diagonal part of the block matrix X. Let us consider the matrix $D_{\pm} = {}^tB_{\mp}A_{\pm}$. Since we know that $D_{\pm}$ is block-wisely diagonal, and ${}^tB_{\mp}, A_{\pm}$ are block upper triangular, we see that $D_{\pm} = ({}^tB_{\mp})_D(A_{\pm})_D$. But $M(P^{\pm},s)_D$ is the identity matrix by Theorem 4.4, (i), (b). Also it is easy to check that $M(s,m)_D$ is the identity matrix, and so we have $D_{\pm} = (A_{\pm})_D$. Let $M(Q^{\pm},P^{\pm})$ be the transition matrix between $Q^{\pm}_{\boldsymbol{\varLambda}}$ and $P^{\pm}_{\boldsymbol{\varLambda}}$. Then we have

$$M(Q^{\pm},P^{\pm}) = M(Q^{\pm},P^{\pm})_D = M(Q^{\pm},q_{\pm})_DM(q_{\pm},P^{\pm})_D.$$

Since $M(Q^{\pm},q_{\pm})_D = I$ by Theorem 4.4 (ii), (a), and $M(q_{\pm},P^{\pm}) = A_{\pm}$ by definition, we have $M(Q^{\pm},P^{\pm}) = D_{\pm}$. Now the formula (4.6.2) follows easily from (4.6.1), if we notice that $b_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'} = b^-_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'} = b^+_{\boldsymbol{\varLambda}',\boldsymbol{\varLambda}}$ with $D_{\pm} = (b^{\pm}_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'})$. □

4.7. As discussed in 3.15, $\{s_{\boldsymbol{\alpha}}(x)\}$ gives rise to a basis of the $\mathbb{Z}[t]$-module $\Xi[t]$, hence so is $\{m_{\boldsymbol{\alpha}}(x)\}$. Also, we see that $\{q_{\boldsymbol{\alpha},\pm}(x;t)\}, \{Q^{\pm}_{\boldsymbol{\varLambda}}(x;t)\}, \{P^{\pm}_{\boldsymbol{\varLambda}}(x;t)\}$ turn out to be bases of $\Xi_{\mathbb{Q}}[t]$. Moreover, $\{p_{\boldsymbol{\alpha}}(x)\}$ gives a basis of $\mathbb{C}(t)$-space $\Xi_{\mathbb{C}}[t] = \mathbb{C}(t)\otimes_{\mathbb{Z}}\Xi$. We now define a scalar product on $\Xi_{\mathbb{Q}}[t]$ by the condition that

$$\langle q_{\boldsymbol{\alpha},+}(x;t), m_{\boldsymbol{\beta}}(x)\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{4.7.1}$$

and extend it to a sesquilinear form on $\Xi_{\mathbb{C}}[t]$. By using a similar argument as in [M, Chap. I, 4], it follows from Corollary 4.6 and Proposition 2.5 that we have

$$\begin{aligned}\langle m_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\beta},-}(x;t)\rangle &= \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \\ \langle P^+_{\boldsymbol{\varLambda}}(x;t), Q^-_{\boldsymbol{\varLambda}'}(x;t)\rangle = \langle Q^+_{\boldsymbol{\varLambda}}(x;t), P^-_{\boldsymbol{\varLambda}'}(x;t)\rangle &= \delta_{\boldsymbol{\varLambda},\boldsymbol{\varLambda}'}, \\ \langle p_{\boldsymbol{\alpha}}(x), p_{\boldsymbol{\beta}}(x)\rangle &= z_{\boldsymbol{\alpha}}(t)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.\end{aligned} \tag{4.7.2}$$

In the special case where $e = 2$, the function $q_{\boldsymbol{\alpha},+}$ (resp. $P^+_{\boldsymbol{\Lambda}}$, $Q^+_{\boldsymbol{\Lambda}}$) coincides with $q_{\boldsymbol{\alpha},-}$ (resp. $P^-_{\boldsymbol{\Lambda}}$, $Q^-_{\boldsymbol{\Lambda}}$). We also have $p_{\boldsymbol{\alpha}} = \bar{p}_{\boldsymbol{\alpha}}$ and $z_{\boldsymbol{\alpha}}(t) \in \mathbb{Q}(t)$. Hence in this case, the scalar product $\langle\ ,\ \rangle$ turns out to be symmetric.

Proposition 4.8. *The functions $P^{\pm}_{\boldsymbol{\Lambda}}$ are characterized by the following two properties. (Then the functions $Q^{\pm}_{\boldsymbol{\Lambda}}$ are characterized as dual bases of $P^{\mp}_{\boldsymbol{\Lambda}}$ in the sense of (4.7.2)).*

(i) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ can be expressed in terms of $s_{\boldsymbol{\beta}}(x)$ as*

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} s_{\boldsymbol{\beta}}$$

with $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \in \mathbb{Q}(t)$, where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, and $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

(ii) *$\langle P^+_{\boldsymbol{\Lambda}}, P^-_{\boldsymbol{\Lambda}'}\rangle = 0$ unless $\boldsymbol{\Lambda} \sim \boldsymbol{\Lambda}'$.*

Proof. The fact that $P^{\pm}_{\boldsymbol{\Lambda}}$ satisfy the properties (i) and (ii) follows from the previous discussion. We show, by induction on the order $\prec$, that (i) $\sim$ (iii) determine $P^{\pm}_{\boldsymbol{\Lambda}}$ uniquely. First we note that $\{P^{\pm}_{\boldsymbol{\Lambda}} \mid \boldsymbol{\Lambda} \in Z^{r,s}_n\}$ give bases of $\Xi^n_{\mathbb{Q}}[t]$ by (i). Then the matrix $(\langle P^+_{\boldsymbol{\Lambda}}, P^-_{\boldsymbol{\Lambda}'}\rangle)_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}$ is non-singular. It follows, by (ii), that the diagonal submatrix corresponding to each similarity class is also non-singular. Now take $\boldsymbol{\Lambda} \in Z^{r,s}_n$, and assume that $P^{\pm}_{\boldsymbol{\Lambda}'}$ are determined for $\boldsymbol{\Lambda}'$ such that $\boldsymbol{\Lambda}' \prec \boldsymbol{\Lambda}$ and that $\boldsymbol{\Lambda}' \not\sim \boldsymbol{\Lambda}$. Note that the expression in (i) is equivalent to the expression

$$P^{\pm}_{\boldsymbol{\Lambda}} = s_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\Lambda}'} d^{\pm}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} P^{\pm}_{\boldsymbol{\Lambda}'}, \tag{4.8.1}$$

where $d^{\pm}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'} = 0$ unless $\boldsymbol{\Lambda}' \prec \boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}' \not\sim \boldsymbol{\Lambda}$. Take such $\boldsymbol{\Lambda}'$. Then we have

$$\langle s_{\boldsymbol{\alpha}}, P^-_{\boldsymbol{\Lambda}'}\rangle + \sum_{\boldsymbol{\Lambda}'' \sim \boldsymbol{\Lambda}'} d^+_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}''} \langle P^+_{\boldsymbol{\Lambda}''}, P^-_{\boldsymbol{\Lambda}'}\rangle = 0. \tag{4.8.2}$$

When $\boldsymbol{\Lambda}'$ runs over symbols contained in a fixed similarity class, (4.8.2) may be regarded as a system of linear equations, with unknown variables $\{d^+_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}''}\}$. Since the matrix $(\langle P^+_{\boldsymbol{\Lambda}'}, P^-_{\boldsymbol{\Lambda}''}\rangle)$ ($\boldsymbol{\Lambda}', \boldsymbol{\Lambda}''$ are in a fixed similarity class) is non-singular by the above remark, we see that (4.8.2) determines the coefficients $d^+_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}''}$ uniquely. Hence $P^+_{\boldsymbol{\Lambda}}$ is determined uniquely. A similar argument, by using the equality $\langle P^+_{\boldsymbol{\Lambda}'}, P^-_{\boldsymbol{\Lambda}}\rangle = 0$ instead of (4.8.2), implies that $P^-_{\boldsymbol{\Lambda}}$ is also unique. Thus the proposition is proved. □

Remark 4.9. By making use of the arguments in Proposition 4.8, one can give an alternate construction of $P^{\pm}_{\boldsymbol{\Lambda}}$. In fact, by substituting $t = 0$ into (2.5.2), we have

$$\Omega(x,y;0) = \sum_{\boldsymbol{\alpha}} z^{-1}_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}}(x)\bar{p}_{\boldsymbol{\alpha}}(y).$$

On the other hand, a similar argument as in [M, I, 4] implies that

$$\Omega(x,y;0) = \sum_{\boldsymbol{\alpha}} s_{\boldsymbol{\alpha}}(x)s_{\boldsymbol{\alpha}}(y).$$

It follows that one can define a hermitian form on $\mathbb{C}\otimes_{\mathbb{Z}}\Xi$ satisfying the properties that $\langle p_{\boldsymbol{\alpha}}, p_{\boldsymbol{\beta}}\rangle = z_{\boldsymbol{\alpha}}\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and that $\langle s_{\boldsymbol{\alpha}}, s_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. In particular, the sesquilinear form on $\mathbb{C}[t]\otimes_{\mathbb{Z}}\Xi$ defined in 4.7 is transfered to the hermitian form on $\mathbb{C}\otimes_{\mathbb{Z}}\Xi$ by substituting $t=0$.

We now construct $P^{\pm}_{\boldsymbol{\Lambda}}$ satisfying two properties in Propostion 4.8 as follows. Take $\boldsymbol{\Lambda}\in Z^{r,s}_n$. We assume that $P^{\pm}_{\boldsymbol{\Lambda}'}$ are already constructed for $\boldsymbol{\Lambda}'$ such that $\boldsymbol{\Lambda}\prec\boldsymbol{\Lambda}$ and that $\boldsymbol{\Lambda}'\not\sim\boldsymbol{\Lambda}$. We assume further that $P^{\pm}_{\boldsymbol{\Lambda}'}(x;t)\in\mathcal{A}_{\mathbb{C}}\otimes_{\mathbb{Z}}\Xi$ and that $P^{\pm}_{\boldsymbol{\Lambda}'}(x;0)=s_{\boldsymbol{\alpha}'}(x)$ for $\boldsymbol{\Lambda}'=\boldsymbol{\Lambda}(\boldsymbol{\alpha}')$. (Here $\mathcal{A}_{\mathbb{C}}=\mathbb{C}\otimes\mathcal{A}$ is the subring of $\mathbb{C}(t)$ consisting of functions which have no pole at $t=0$). We put $P^{\pm}_{\boldsymbol{\Lambda}}$ as in (4.8.1), and determine the coefficients $d^{\pm}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}$ so that it satisfies the condition (ii) in the proposition. We consider the system of equations (4.8.2). Note that the coefficient matrix $(\langle P^{+}_{\boldsymbol{\Lambda}''}, P^{-}_{\boldsymbol{\Lambda}'}\rangle)$ is non-degenerate since it gives rise to a matrix $(\langle s_{\boldsymbol{\alpha}''}, s_{\boldsymbol{\alpha}'}\rangle)$ by substituting $t=0$, which is the identity matrix by the above remark. In particular, the determinant of this matrix lies in $\mathcal{A}_{\mathbb{C}}$. Thus the equation (4.8.2) has a unique solution $d_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}\in\mathcal{A}_{\mathbb{C}}$ and we obtain $P^{+}_{\boldsymbol{\Lambda}}\in\mathcal{A}_{\mathbb{C}}\otimes_{\mathbb{Z}}\Xi$. Since $\langle s_{\boldsymbol{\alpha}}, s_{\boldsymbol{\alpha}'}\rangle=0$, we have $d^{+}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}\in t\mathcal{A}_{\mathbb{C}}$. This implies that $P^{+}_{\boldsymbol{\Lambda}}(x;0)=s_{\boldsymbol{\alpha}}(x)$. The case for $P^{-}_{\boldsymbol{\Lambda}}$ is done similarly, and $P^{\pm}_{\boldsymbol{\Lambda}}$ is constructed inductively.

5. Green functions

5.1. Let $X_{\pm}(t)$ be the transition matrix $M(p,P^{\pm})$ between the power sum symmetric functions $\{p_{\boldsymbol{\alpha}}\}$ and the Hall-Littlewood functions $\{P^{\pm}_{\boldsymbol{\Lambda}}\}$, i.e.,

$$p_{\boldsymbol{\alpha}}(x)=\sum_{\boldsymbol{\Lambda}}X^{\boldsymbol{\Lambda}}_{\boldsymbol{\alpha},\pm}(t)P^{\pm}_{\boldsymbol{\Lambda}}(x;t). \tag{5.1.1}$$

(Here $\boldsymbol{\alpha}$ is the row-index and $\boldsymbol{\Lambda}$ is the column-index.) Then by Theorem 4.4, $X^{\boldsymbol{\Lambda}}_{\boldsymbol{\alpha},\pm}(t)\in\mathbb{C}(t)$ and is equal to zero unless $|\boldsymbol{\alpha}|=|\boldsymbol{\Lambda}|$. Moreover since $P^{\pm}_{\boldsymbol{\Lambda}}(x;0)=s_{\boldsymbol{\alpha}}(x)$ with $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ by Theorem 4.4 (i) (b), we have

$$X^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(0)=\chi^{\boldsymbol{\alpha}}(w_{\boldsymbol{\beta}}) \tag{5.1.2}$$

by (2.1.3). It follows that $X_{\pm}(0)=M(p,s)$ is the character table of W. In particular, $X_{\pm}(0)$ is independent of the sign, and we denote it simply as $X(0)$. By combining Corollary 4.6 with (2.5.2), we have

$$\sum_{|\boldsymbol{\beta}|=n}z_{\boldsymbol{\beta}}(t)^{-1}p_{\boldsymbol{\beta}}(x)\bar{p}_{\boldsymbol{\beta}}(y)=\sum_{|\boldsymbol{\Lambda}|=|\boldsymbol{\Lambda}'|=n}b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t)P^{+}_{\boldsymbol{\Lambda}}(x;t)P^{-}_{\boldsymbol{\Lambda}'}(y;t).$$

We put $D(t)=D_{-}=(b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(t))$, and denote by $Z(t)$ the diagonal matrix with entries $z_{\boldsymbol{\beta}}(t)$. (Both are indexed by the set $Z^{r,s}_n\simeq Z^{0,0}_n$.) Substituting (5.1.1) into the above equation, we have

$${}^tX_{+}(t)Z(t)^{-1}\bar{X}_{-}(t)=D(t), \tag{5.1.3}$$

where $\bar{X}_-(t)$ is the complex conjugate of the matrix $X_-(t)$. (Note that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Q}(t)[x]$.) We put $\Lambda(t) = D(t)^{-1}$. Then the formula (5.1.3) is equivalent to

$$\bar{X}_-(t)\Lambda(t)\,{}^t\!X_+(t) = Z(t). \tag{5.1.4}$$

5.2. Let $K_{\pm}(t) = M(s, P^{\pm})$ be the transition matrix between Schur functions and Hall-Littlewood functions, i.e.,

$$s_{\boldsymbol{\beta}}(x) = \sum_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})} K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t).$$

Then by Theorem 4.4 (i) (b), , $K_{\pm}(t)$ is a block lower triangular matrix with identity diagonal blocks, with entries $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ in $\mathbb{Q}(t)$. Moreover, $K_{\pm}(0)$ is the identity matrix. However, on the contrary to the classical case, $K_{\pm}(1)$ is not necessarily equal to the Kostka matrix $M(s,m)$. Since $M(s, P^{\pm}) = M(p,s)^{-1}M(p, P^{\pm})$, we have $K_{\pm}(t) = X(0)^{-1}X_{\pm}(t)$. Substituting this into (5.1.4), we have

$$K_-(t)\Lambda(t)\,{}^tK_+(t) = \bar{X}(0)^{-1}Z(t)\,{}^tX(0)^{-1}. \tag{5.2.1}$$

5.3. We now define Green functions $Q^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(t) \in \mathbb{C}(t)$, with $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, by

$$Q^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(t) = t^{a(\boldsymbol{\Lambda})}X^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(t^{-1}).$$

If we put $\widetilde{K}^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) = t^{a(\boldsymbol{\alpha})}K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t^{-1})$, $Q^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(t)$ can be written as

$$Q^{\boldsymbol{\Lambda}}_{\boldsymbol{\beta},\pm}(t) = \sum_{\boldsymbol{\gamma}} \chi^{\boldsymbol{\gamma}}(w_{\boldsymbol{\beta}})\widetilde{K}^{\pm}_{\boldsymbol{\gamma},\boldsymbol{\alpha}}(t).$$

Let $\widetilde{K}_{\pm}(t) = (\widetilde{K}^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t))$. Then $\widetilde{K}_{\pm}(t) = K_{\pm}(t^{-1})T$ for a diagonal matrix T with diagonal entries $t^{a(\boldsymbol{\alpha})}$. Hence (5.2.1) can be rewritten as

$$\widetilde{K}_-(t)\Lambda'(t)\,{}^t\widetilde{K}_+(t) = \bar{X}(0)^{-1}Z(t^{-1})\,{}^tX(0)^{-1}, \tag{5.3.1}$$

where $\Lambda'(t) = T^{-1}\Lambda(t^{-1})T$. Note that $\Lambda'(t)$ is still a block diagonal matrix, and $\widetilde{K}_{\pm}(t)$ are block lower triangular matrices, where the diagonal blocks consist of scalar matrices $t^{a(\boldsymbol{\alpha})}I$.

Put $\mathbb{G}(t) = (t-1)^n t^{N^*}P_W(t) \in \mathbb{Z}[t]$. If W is a Weyl group of type B_n, $\mathbb{G}(q)$ is the order of $SO_{2n+1}(\mathbf{F}_q)$ or $Sp_{2n}(\mathbf{F}_q)$. Even in the case where $W = G(e,1,n)$, $\mathbb{G}(t)$ is regarded as an order of a formal group (see [BMM]). We put $\widetilde{\Lambda}(t) = t^{-n}\mathbb{G}(t)\Lambda'(t)$. Recall that $\Omega' = (\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ is the matrix defined in (1.5.1). Then the following result gives a combinatorial description of the solution for the equation (1.5.2).

Theorem 5.4. *We have*

$$\widetilde{K}_-(t)\widetilde{\Lambda}(t)\,{}^t\widetilde{K}_+(t) = \Omega'.$$

Hence $P' = \widetilde{K}_-(t), P'' = \widetilde{K}_+(t)$ and $\Lambda' = \widetilde{\Lambda}(t)$ gives a solution for the equation (1.5.2).

Proof. Let M be the right hand side of (5.3.1). We shall compute M. Let $z_{\boldsymbol{\alpha}}$ be as in 2.4, and H the diagonal matrix with diagonal entries $z_{\boldsymbol{\alpha}}^{-1}$. Since $X(0)$ is the character table of W, we have ${}^tX(0)H\bar{X}(0) = I$. It follows that

$$M = {}^tX(0)HZ(t^{-1})H\bar{X}(0). \tag{5.4.1}$$

Now it is easy to see that if $\boldsymbol{\alpha} = (\alpha_j^{(k)}) \in \mathcal{P}_{n,e}$ and $w_{\boldsymbol{\alpha}}$ is the corresponding element in W, then we have

$$\det{}_V(t\cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}}) = \prod_{k=0}^{e-1}\prod_{j=1}^{l(\alpha^{(k)})}(t^{\alpha_j^{(k)}} - \zeta^k).$$

Hence by (2.4.2), $z_{\boldsymbol{\alpha}}(t^{-1}) = z_{\boldsymbol{\alpha}}t^n\det{}_V(t\cdot \mathrm{id}_V - w_{\boldsymbol{\alpha}})^{-1}$. In particular, the $\boldsymbol{\alpha\beta}$-entry of M is equal to

$$t^n|W|^{-1}\sum_{w\in W}\frac{\chi^{\boldsymbol{\alpha}}(w)\bar{\chi}^{\boldsymbol{\beta}}(w)}{\det{}_V(t\cdot \mathrm{id}_V - w)}.$$

Therefore $\Omega' = t^{N^*-n}(t-1)^nP_W(t)M$ and the theorem follows. □

Concerning the Kostka functions, we can expect that

Conjecture 5.5 The Kostka function $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is a polynomial in t with positive integral coefficients, with $\deg K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \le a(\boldsymbol{\beta}) - a(\boldsymbol{\alpha})$.

Remarks 5.6. (i) The conjecture implies that $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is also a polynomial with positive integral coefficients. In particular, we see that $p_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t), \lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$ (cf. (1.4.2)). The conjecture also implies that $P^{\pm}_{\boldsymbol{\Lambda}} \in \mathbb{Z}[x;t]$, and $Q^{\pm}_{\boldsymbol{\Lambda}} \in \mathbb{Z}[x;t]$ also. In fact, from the discussion in 4.7, we see that $\{q_{\boldsymbol{\alpha},\pm}(x;t)\}$ and $\{m_{\boldsymbol{\alpha}}(x)\}$, and $\{Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)\}$ and $\{P^{\mp}(x;t)\}$ are dual bases of each other. Hence we have

$$M(Q^{\mp}, q_{\mp}) = M(P^{\pm}, m)^* = (K_{\pm}(t)^{-1}K)^* = {}^tK_{\pm}(t)K^*, \tag{5.6.1}$$

where $K = M(s,m)$ is the Kostka matrix, and X^* denotes the transposed inverse of the matrix X. Since K^* is a matrix with entries in $\mathbb{Z}$, we see that $K_{\pm}(t)$ is a matrix with entries in $\mathbb{Z}[t]$ if and only if $M(Q^{\mp}, q_{\mp})$ is a matrix with entries in $\mathbb{Z}[t]$. It follows that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ lie in $\mathbb{Z}[x;t]$ if an only if the coefficients $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ in Theorem 4.4, (ii), (a) lie in $\mathbb{Z}[t]$ for any $\boldsymbol{\alpha},\boldsymbol{\beta} \in Z_n^{0,0}$.

(ii) The condition that Hall-Littlewood functions $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ lie in $\mathbb{Z}[t;x]$ is equivalent to that $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t,t^{-1}]$ for any $\boldsymbol{\alpha},\boldsymbol{\beta} \in Z_n^{0,0}$. In fact, if $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t,t^{-1}]$, then $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t,t^{-1}]$ also. But by Theorem 4.4, the coefficients of $M(P^{\pm},s)$ lie in $\mathcal{A}$ with $\det M(P^{\pm},s) = 1$. Hence the entries of the matrix $M(s,P^{\pm})$, i.e., the Kostka functions $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathcal{A}$. It follows that $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$. Therefore the entries of $M(P^{\pm},s)$ lie in $\mathbb{Z}[t]$, and we have $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. The other implication is obvious.

It is known that Green functions of classical groups are polynomials in $\mathbb{Z}[t]$. It follows that if $e=2, r=2$, then Kostka functions $\widetilde{K}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = \widetilde{K}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ lie in $\mathbb{Z}[t]$. Hence in this case, we see that $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. Also, it is verified by Geck and Malle that for small n, e with $r=1$ or 2, the equation (1.5.2) has a solution with coefficients in $\mathbb{Z}[t]$. Thus in these cases, also we have $P^{\pm}_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$.

6. Some special cases

6.1. In this section, we shall derive some explicit results for $Q^{\pm}_{\boldsymbol{\Lambda}}$ in the case where $e = 2$ and the symbols $\boldsymbol{\Lambda}$ satisfy a certain special condition. So, from now on, except 6.7, we assume that $e = 2, r \ge 1$, and that $m_0 = m + 1, m_1 = m$ for some $m > 0$. Let $\boldsymbol{\Lambda} \in Z^{r,s}_n$ be a symbol. We arrange the entries in $\boldsymbol{\Lambda}$ in decreasing order $\nu_1 \ge \nu_2 \ge \cdots \ge \nu_M$, where $M = 2m+1$. We say that $\boldsymbol{\Lambda}$ is very special if $\nu_i \ge \nu_{i+1} + r$ for $i = 1, 2, \dots$. We show the following.

Proposition 6.2. *Assume that $e = 2$ and that $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z^{r,s}_n$ is very special. Then $Q^{\pm}_{\boldsymbol{\Lambda}}$ coincides with $R^{\pm}_{\boldsymbol{\Lambda}}$. In particular, $Q^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed as*

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t) = q_{\boldsymbol{\alpha}}(x;t) + \sum_{\boldsymbol{\beta} \in Z^{0,0}_n} d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta}}(x;t),$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$ and $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = 0$ unless $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \not\sim \boldsymbol{\alpha}$.

6.3. The proposition will be proved in 6.6 after some preliminaries. In view of Theorem 4.4 (ii), together with Propositions 3.12, 3.14, it is enough to show that $R^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed as a linear combination of $s_{\boldsymbol{\beta}}(x)$ with $\boldsymbol{\beta} \in Z^{0,0}_n$ such that $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Now we may assume that $\nu_0 = (m_k, k)$ for some k. Then $R^{\pm}_{\boldsymbol{\Lambda}}$ can be expressed as in (3.14.1) by using lowering operators, and so it is written as a linear combination of $s_{\boldsymbol{\alpha}'}(x)$ with $\boldsymbol{\alpha}' \in \widehat{Z}^{0,0}_n$. Since $s_{\boldsymbol{\alpha}'}$ with $\boldsymbol{\alpha}' \in \widehat{Z}^{0,0}_n$ is rewritten as $\pm s_{\boldsymbol{\alpha}''}$ with $\boldsymbol{\alpha}'' \in Z^{0,0}_n$, $R^{\pm}_{\boldsymbol{\Lambda}}$ can be written as a linear combination of $s_{\boldsymbol{\alpha}''}$ ($\boldsymbol{\alpha}'' \in Z^{0,0}_n$) with coefficients in $\mathbb{Z}[t]$.

In order to see the values $a(\boldsymbol{\alpha}'')$ for $\boldsymbol{\alpha}'' \in Z^{0,0}_n$ appearing in the expansion of $R^{\pm}_{\boldsymbol{\Lambda}}$ in terms of Schur functions, one needs to consider the symbol $\boldsymbol{\Lambda}'' = \boldsymbol{\Lambda}(\boldsymbol{\alpha}'')$. The process for obtaining $\boldsymbol{\Lambda}''$ from $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ is described as follows; for a partition $\alpha : \alpha_1 \ge \cdots \ge \alpha_{m'} \ge 0$, let $\lambda : \lambda_1 > \cdots > \lambda_{m'} \ge 0$ (resp. $\mu : \mu_1 > \cdots > \mu_{m'} \ge 0$) be a sequence defined by $\lambda_i = \alpha_i + (m'-i)r + s'$ (resp. $\mu_i = \alpha_i + (m'-i)$), respectively. Let $\mu' = (\mu'_1, \dots, \mu'_{m'})$ be a sequence obtained by applying R to μ. We assume that $\mu'_i \ge 0$. Let μ'' be a partition $\mu''_1 \ge \cdots \ge \mu''_{m'} \ge 0$ obtained from μ' by rearranging the entries in decreasing order. We define a partition $\lambda'' : \lambda''_1 \ge \cdots \ge \lambda''_{m'} \ge 0$ by $\lambda''_i = \mu''_i + (m'-i)(r-1) + s'$, and put $\lambda'' = \tilde{R}\lambda$. Now $\boldsymbol{\Lambda}'' = (\Lambda''_0, \Lambda''_1)$ is obtained from $\boldsymbol{\Lambda} = (\Lambda_0, \Lambda_1)$ by $\Lambda''_k = \tilde{R}\Lambda_k$, where $s' = s, m' = m+1$ (resp. $s' = 0, m' = m$) if $k = 0$ (resp. $k = 1$), respectively.

The procedure for obtaining $\boldsymbol{\Lambda}''$ from $\boldsymbol{\Lambda}$ has an alternate description. Let $\Lambda = (\lambda_1, \dots, \lambda_{m_1})$ be a sequence of integers and assume that $\lambda_i < \lambda_{i+1} + (r-1)$ for some i. We define $\Lambda' = R^{\vee}_i \Lambda = (\lambda'_1, \dots, \lambda'_{m_1})$ by

$$\begin{aligned} \lambda'_i &= \lambda_{i+1} + (r-1), \\ \lambda'_{i+1} &= \lambda_i - (r-1), \\ \lambda'_j &= \lambda_j \qquad (j \ne i, i+1). \end{aligned} \tag{6.3.1}$$

Then we have $\lambda'_i \ge \lambda'_{i+1} + r$. Let $\boldsymbol{\Lambda}' = (\Lambda'_0, \Lambda'_1)$ be the pair of compositions obtained from $\boldsymbol{\Lambda}$ by putting $\Lambda'_k = R\Lambda_k$. We rearrange the entries in Λ'_k by the repeated use of the operators $R^{\vee}_i$. (Note that in each step of the operation of $R^{\vee}_i$, the resulting sequence again belongs in $\mathbb{Z}^{m'}_{\geq 0}$.) Finally we obtain the sequence whose entries are arranged in decreasing order, which coincides with $\varLambda''_k$.

6.4. Let $\boldsymbol{\varLambda} = (\varLambda_0, \varLambda_1)$ be a very special symbol in $Z^{r,s}_n$. We arrange the entries of $\boldsymbol{\varLambda}$ as $\nu_1 > \nu_2 > \cdots > \nu_M$. Hence $\varLambda_0 = (\nu_1 > \nu_3 > \cdots > \nu_{2m+1})$ and $\varLambda_1 = (\nu_2 > \nu_4 > \cdots > \nu_{2m})$. Let $R = \prod R_{\nu',\nu}$ be a lowering operator occurring in the expansion of (3.14.1), and put $\boldsymbol{\alpha}' = R(\boldsymbol{\alpha})$. We consider $\boldsymbol{\alpha}''$ and $\boldsymbol{\varLambda}''$ as in 6.3. For each $u \geq 1$, let $\nu''_1 \geq \nu''_2 \geq \cdots \geq \nu''_u$ be the largest u elements in the entries of $\boldsymbol{\varLambda}''$. Let $\nu'_{2a_1-1}, \nu'_{2a_2-1}, \dots, \nu'_{2a_p-1}, \nu'_{2b_1}, \nu'_{2b_2}, \dots, \nu'_{2b_q}$ be the entries in $\boldsymbol{\varLambda}'$ which produce $\nu''_1, \dots, \nu''_u$, where $a_1 < a_2 < \cdots < a_p$, $b_1 < b_2 < \cdots < b_q$ with $p + q = u$. (Hence $\nu_{2a_1-1}, \nu_{2a_2-1}, \dots$ are entries in $\varLambda_0$, and $\nu_{2b_1}, \nu_{2b_2}, \dots$ are in $\varLambda_1$). First we show the following lemma.

Lemma 6.5. *Let the notations be as above. Assume that $p \geq q$. Then we have*

$$\sum_{i=1}^{u} \nu''_i \leq \sum_{i=1}^{p} \nu_{2a_i-1} + \sum_{j=1}^{q} \nu_{2b_j} + r\Big\{\sum_{i=1}^{p}(a_i - (q+1)) + \sum_{j=1}^{q} b_j\Big\}.$$

Proof. Put $X = \sum_{i=1}^{p}(a_i - (q+1)) + \sum_{j=1}^{q} b_j$. First we show the following formula.

$$\sum_{i=1}^{u} \nu''_i \leq \sum_{i=1}^{p} \nu'_{2a_i-1} + \sum_{j=1}^{q} \nu'_{2b_j} + (r-1)X. \tag{6.5.1}$$

We write $\boldsymbol{\varLambda}' = (\varLambda'_0, \varLambda'_1)$ (resp. $\boldsymbol{\varLambda}'' = (\varLambda''_0, \varLambda''_1)$) as $\varLambda'_0 = (\lambda'_i), \varLambda'_1 = (\mu'_j)$ (resp. $\varLambda''_0 = (\lambda''_i), \varLambda''_1 = (\mu''_j)$). Let λ'_{i_1} be the element which produces λ''_1 by the operations $R^\vee_i$ given in (6.3.1). Then we have $\lambda''_1 \leq \lambda'_{i_1} + (r-1)(i_1 - 1)$ since it is obtained by at most $(i_1 - 1)$ iteration of $R^\vee_i$. If λ'_{i_2} is the element which produces λ''_2, then we have $\lambda''_2 \leq \lambda'_{i_2} + (r-1)(i_2 - 2)$. In fact, this is clear if $i_1 < i_2$. If $i_1 > i_2$, then λ'_{i_2} is modified to $\lambda^*_{i_2} = \lambda'_{i_2} - (r-1)$ under the process of obtaining λ''_1 from λ'_{i_1} (cf. (6.3.1)). After that λ''_2 is obtained from $\lambda^*_{i_2}$ by at most $(i_2 - 1)$ iteration of $R^\vee_i$, and we get the required formula. A similar argument implies that, in general, if λ'_{i_a} produces λ''_a, then $\lambda''_a \leq \lambda'_{i_a} + (r-1)(i_a - a)$. The same is true for $\varLambda'_1$.

Applying this to our situation, we see that

$$\sum_{i=1}^{u} \nu''_i \leq \sum_{i=1}^{p} \nu'_{2a_i-1} + \sum_{j=1}^{q} \nu'_{2b_j} + (r-1)Y,$$

where

$$Y = \sum_{i=1}^{p} a_i + \sum_{j=1}^{q} b_j - \frac{1}{2}p(p+1) - \frac{1}{2}q(q+1).$$

But since $p \geq q$, we have $p(p+1)/2 + q(q+1)/2 \geq p(q+1)$. Hence $Y \leq X$, and (6.5.1) holds.

Next we show that

$$\sum_{i=1}^{p}\nu'_{2a_i-1}+\sum_{j=1}^{q}\nu'_{2b_j}\le\sum_{i=1}^{p}\nu_{2a_i-1}+\sum_{j=1}^{q}\nu_{2b_j}+X. \tag{6.5.2}$$

For $i=1,\dots,q+1$, let us define a subset I_i of $\{1,2,\dots,p\}$ by

$$I_i=\{j\mid b_{i-1}<a_j\le b_i\},$$

where by convention, $b_0=0, b_{q+1}=\infty$. Now take ν_{2a_i-1} for $i\in I_1$. Then by the effect of the lowering operator, we have $\nu'_{2a_i-1}\le\nu_{2a_i-1}+(a_i-1)$. (Note that $\nu_{2a_i-1}>\nu_{2a_i}$ by our assumption). A similar argument may work also for ν_{2b_1}. But in this case, if the pair $(\nu_{2a_i-1},\nu_{2b_1})$ with $a_i\in I_1$ occurs in the expression of lowering operator $R=\prod R_{\nu',\nu}$, ν_{2a_i-1} must be replaced by $\nu_{2a_i-1}-1$. Hence, we have

$$\sum_{i\in I_1}\nu'_{2a_i-1}+\nu'_{2b_1}\le\sum_{i\in I_1}\nu_{2a_i-1}+\nu_{2b_1}+\sum_{i\in I_1}(a_i-1)+(b_1-|I_1|).$$

Extending this argument to the general situation, we see that

$$\sum_{i=1}^{p}\nu'_{2a_i-1}+\sum_{j=1}^{q}\nu'_{2b_j}\le\sum_{i=1}^{p}\nu_{2a_i-1}+\sum_{j=1}^{q}\nu_{2b_j}+Z,$$

where

$$\begin{aligned}Z=&\sum_{i\in I_1}(a_i-1)+\sum_{i\in I_2}(a_i-2)+\cdots+\sum_{i\in I_{q+1}}(a_i-(q+1))\\&+(b_1-|I_1|)+(b_2-(|I_1|+|I_2|))+\cdots+(b_q-(|I_1|+\cdots+|I_q|)).\end{aligned}$$

But then

$$\begin{aligned}Z&=\sum_{i=1}^{p}a_i+\sum_{j=1}^{q}b_j-\sum_{i=1}^{q+1}i|I_i|-\sum_{j=1}^{q}(q-j+1)|I_j|\\&=\sum_{i=1}^{p}a_i+\sum_{j=1}^{q}b_j-(q+1)\sum_{i=1}^{q+1}|I_i|\\&=\sum_{i=1}^{p}a_i+\sum_{j=1}^{q}b_j-(q+1)p.\end{aligned}$$

Hence we have $Z=X$, and (6.5.2) holds. Now the lemma follows immediately from (6.5.1) and (6.5.2). $\square$

6.6. We now prove the proposition. In order to prove it, it is enough to show the following. For $u = 1, \dots, M$, we have

$$\nu''_1 + \nu''_2 + \cdots + \nu''_u \le \nu_1 + \nu_2 + \cdots + \nu_u. \tag{6.6.1}$$

In fact, (6.6.1) implies that $a(\boldsymbol{\Lambda}'') \ge a(\boldsymbol{\Lambda})$, and that $\boldsymbol{\Lambda}'' \sim \boldsymbol{\Lambda}$ if $a(\boldsymbol{\Lambda}'') = a(\boldsymbol{\Lambda})$. It follows that $\boldsymbol{\alpha}'' \prec \boldsymbol{\alpha}$ or $\boldsymbol{\alpha}'' \sim \boldsymbol{\alpha}$.

Let p and q are as in 6.4. We prove (6.6.1) under the assumption that $p \ge q$. The case where $p < q$ is done in a similar way, by using an appropriate modification of Lemma 6.5. Let l be the largest integer such that $a_l \le q$. Then since $\boldsymbol{\Lambda}$ is very special, we see that

$$\nu_{2a_i-1} + r(a_i - (q+1)) \le \nu_{a_i+q} \tag{6.6.2}$$

for $i > l$. Moreover in this case, $a_i + q \ge 2q+1$. Next, let k be the largest integer such that $a_k \le q$ and that $2a_k - k \le q$. (Since $a_1 < a_2 < \cdots$, the sequence $\{2a_i - i\}$ increases along i). Then we have

$$\begin{aligned}
A &= \sum_{j=1}^{q} \nu_{2b_j} + r\sum_{j=1}^{q} b_j \\
&\le \sum_{j_1=1}^{2a_1-2} \nu_{b_{j_1}} + \sum_{j_2=2a_1-1}^{2a_2-3} \nu_{b_{j_2}+1} + \cdots + \sum_{j_{k+1}=2a_k-k}^{q} \nu_{b_{j_{k+1}}+k},
\end{aligned}$$

since $b_{j_2} \ge j_2 \ge 1, \dots, b_{j_{k+1}} \ge j_{k+1} = 2a_k - k \ge k$. Hence

$$A \le \sum_{j_1=1}^{2a_1-2} \nu_{j_1} + \sum_{j_2=2a_1-1}^{2a_2-3} \nu_{j_2+1} + \cdots + \sum_{j_{k+1}=2a_k-k}^{q} \nu_{j_{k+1}+k}.$$

It follows that

$$A + \sum_{i=1}^{k} \nu_{2a_i-1} \le \sum_{j=1}^{q+k} \nu_j. \tag{6.6.3}$$

Suppose that $k < i \le l$. Then $2a_i - i > q$, and so $2a_i - 1 \ge q + i$. It follows that $\nu_{2a_i-1} - r((q+1) - a_i) \le \nu_{2a_i-1} \le \nu_{q+i}$. By substituting this into (6.6.3), we have

$$A + \sum_{i=1}^{l} \nu_{2a_i-1} - r\sum_{i=1}^{l}((q+1) - a_i) \le \sum_{j=1}^{q+l} \nu_j.$$

Now combining the last formula with (6.6.2), we have

$$\sum_{i=1}^{p} \nu_{2a_i-1} + \sum_{j=1}^{q} \nu_{2b_j} + rX \le \sum_{j=1}^{q+l} \nu_j + \sum_{i=l+1}^{p} \nu_{a_i+q} \le \sum_{i=1}^{u} \nu_i.$$

This formula implies (6.6.1) in view of Lemma 6.5. Hence Proposition 6.2 is proved.

6.7. In this subsection, we assume that e is arbitrary, and deduce some formula for the Kostka matrix $K = M(s,m)$ and K^*. For each $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)}) \in Z_n^{0,0}$, we define a function $h_{\boldsymbol{\alpha}}(x)$ by

$$h_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} h_{\alpha^{(k)}}(x^{(k)}),$$

where $h_{\alpha^{(k)}}(x^{(k)})$ denotes the complete symmetric function associated to a partition $\alpha^{(k)}$. As discussed in Remark 4.9, we have

$$\Omega(x,y;0) = \prod_{k=0}^{e-1}\prod_{i,j}(1 - x_i^{(k)}y_j^{(k)})^{-1} = \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}^{-1} p_{\boldsymbol{\alpha}}(x)\bar{p}_{\boldsymbol{\alpha}}(y), \tag{6.7.1}$$

Moreover, by a similar argument as in [M, I, 4], we have

$$\Omega(x,y;0) = \sum_{\boldsymbol{\alpha}} h_{\boldsymbol{\alpha}}(x)m_{\boldsymbol{\alpha}}(y) = \sum_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}}(x)h_{\boldsymbol{\alpha}}(y), \tag{6.7.2}$$
$$\Omega(x,y;0) = \sum_{\boldsymbol{\alpha}} s_{\boldsymbol{\alpha}}(x)s_{\boldsymbol{\alpha}}(y).$$

Now one can define a scalar product $\langle\, , \rangle$ on Ξ, by requiring that $\langle h_{\boldsymbol{\alpha}}, m_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. Then, by (6.7.2), we have

$$\langle m_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}},$$
$$\langle s_{\boldsymbol{\alpha}}, s_{\boldsymbol{\beta}}\rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

In particular, the scalar product is symmetric. It can be extended to a hermitian form on $\mathbb{C}\otimes_{\mathbb{Z}}\Xi$, which satisfies the property, by (6.7.1), that

$$\langle p_{\boldsymbol{\alpha}}, p_{\boldsymbol{\beta}}\rangle = z_{\boldsymbol{\alpha}}^{-1}\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

It follows from this that K^* coincides with the transition matrix $M(s,h)$ between Schur functions $\{s_{\boldsymbol{\alpha}}(x)\}$ and complete symmetric functions $\{h_{\boldsymbol{\alpha}}(x)\}$. Then in view of [M, I, 3.4″], we have the following.

(6.7.3) K^* coincides with the matrix of the operator $\prod_{\nu<\nu'}\prod_{b(\nu')=b(\nu)}(1-R_{\nu,\nu'})$.

As a corollary to Proposition 6.2, we have the following result for certain type of Kostka functions and Green functions. (Since $e=2$, the sign $\pm$ is ignored in the subsequent discussion.)

Corollary 6.8. *Assume that $e = 2$ and that $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ is very special. Then for any $\boldsymbol{\beta} \in Z_n^{r,s}$, we have $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) \in \mathbb{Z}[t]$ with $\deg K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}} \le a(\boldsymbol{\alpha}) - a(\boldsymbol{\beta})$. (The equality holds only when $\boldsymbol{\Lambda}(\boldsymbol{\beta})$ is a special symbol.) In particular, $\widetilde{K}^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ is a polynomial in $\mathbb{Z}[t]$.*

Proof. Since K is a matrix with entries in $\mathbb{Z}$, it follows from the formula (5.5.1), combined with Proposition 6.2 and Proposition 3.12, we see that $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t) \in \mathbb{Z}[t]$. Hence It is enough to estimate the degree of the polynomial $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$. A similar argument as in [M, III, (6.3)] implies, by making use of (3.7.1) and (6.7.3), the following statement.

(6.8.1) $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ is the coefficient of $q_{\boldsymbol{\beta},\mp}$ in $\prod_{\nu<\nu'}\prod_{b(\nu')=b(\nu)\pm 1}(1-tR_{\nu,\nu'})^{-1}q_{\boldsymbol{\alpha},\mp}$.

We now estimate the degree of $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ following the idea in [M, III, 6, Ex. 4]. Let $\varepsilon_1,\dots,\varepsilon_M$ be the standard basis of $\mathbb{Z}^M$. We denote by R^+ the set of positive roots of type A_{M-1}, i.e., $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le M\}$. For any $\xi = (\xi_1,\dots,\xi_M) \in \mathbb{Z}^M$ such that $\sum \xi_i = 0$, we define a polynomial $P(\xi;t)$ in t by

$$P(\xi;t) = \sum_{(m_\gamma)} t^{\sum m_\gamma},$$

where (m_γ) runs over all the choices such that $\xi = \sum_{\gamma\in R^+} m_\gamma\gamma$ with $m_\gamma \in \mathbb{Z}_{\ge 0}$. We also define $P^*(\xi;t)$ by a similar formula as $P(\xi;t)$, but this time, (m_γ) runs over only the expression such that $\xi = \sum_\gamma m_\gamma\gamma$ and that $\gamma = \varepsilon_i - \varepsilon_j$ corresponds to the raising operator R_{ν_i,ν_j} occurring in the expression in (6.8.1). Then $P(\xi;t)$ is non-zero only when $\xi = \sum \eta_i(\varepsilon_i - \varepsilon_{i+1})$ with $\eta_i \ge 0$, and in that case, $P(\xi;t)$ is a monic of degree $\sum \eta_i = \langle \xi, \delta\rangle$. (See [loc. cit.]. Here $\delta = (M,\dots,1,0) \in \mathbb{Z}^M$ and $\langle\, ,\rangle$ denotes the standard inner product in $\mathbb{Z}^M$.) Clearly $\deg P^*(\xi;t) \le \deg P(\xi;t)$. By our assumption that $\boldsymbol{\Lambda}$ is very special, the choice $(m_\gamma) = (\eta_i)$ is allowed in the expression of $P^*(\xi;t)$. Hence $P^*(\xi;t)$ is also a monic of degree $\langle \xi, \delta\rangle$.

Now it follows from (6.8.1) that $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ is also expressed as the coefficient of $s_{\boldsymbol{\beta}}$ in

$$\prod_{\nu<\nu'}\prod_{b(\nu')=b(\nu)\pm 1}(1-tR_{\nu,\nu'})^{-1}s_{\boldsymbol{\alpha}} = \prod_{\nu<\nu'}\prod_{b(\nu')=b(\nu)\pm 1}(1+tR_{\nu,\nu'}+t^2R_{\nu,\nu'}+\cdots)s_{\boldsymbol{\alpha}}.$$

Hence, by a similar argument as in [loc. cit.], we see that the degree of $K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(t)$ is equal to the degree of

$$\sum_{w\in\mathfrak{S}_{\mathbf{m}}} \varepsilon(w)P^*(w^{-1}(\boldsymbol{\beta}+\delta)-(\boldsymbol{\alpha}+\delta);t),$$

where $\boldsymbol{\alpha},\boldsymbol{\beta}$ are regarded as elements in $\mathbb{Z}^M$ in an obvious way. Furthermore, if we put $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}), \boldsymbol{\Lambda}' = \boldsymbol{\Lambda}(\boldsymbol{\beta})$, we have

$$\begin{aligned}\langle w^{-1}(\boldsymbol{\beta}+\delta)-(\boldsymbol{\alpha}+\delta),\delta\rangle &= \langle \boldsymbol{\beta}+\delta, w(\delta)\rangle - \langle \boldsymbol{\alpha}+\delta,\delta\rangle \\ &\le \langle \boldsymbol{\beta}+\delta,\delta\rangle - \langle \boldsymbol{\alpha}+\delta,\delta\rangle \\ &= \langle \boldsymbol{\Lambda}',\delta\rangle - \langle \boldsymbol{\Lambda},\delta\rangle,\end{aligned}$$

since $\boldsymbol{\Lambda} = \boldsymbol{\alpha}+\delta', \boldsymbol{\Lambda}' = \boldsymbol{\beta}+\delta'$ with $\delta' = (mr,(m-1)r+s,(m-1)r,(m-2)r+s,\dots)$ as elements in $\mathbb{Z}^M$. The equality holds only when $w=1$. Now let $\boldsymbol{\Lambda}'_1$ be the elements in $\mathbb{Z}^M$ obtained from $\boldsymbol{\Lambda}'$ by rearranging the entries in decreasing order. Then we have

$\langle \boldsymbol{\Lambda}', \delta\rangle \le \langle \boldsymbol{\Lambda}'_1, \delta\rangle$ and so

$$\begin{aligned}\langle \boldsymbol{\Lambda}', \delta\rangle - \langle \boldsymbol{\Lambda}, \delta\rangle &\le \langle \boldsymbol{\Lambda}'_1, \delta\rangle - \langle \boldsymbol{\Lambda}, \delta\rangle \\ &= n(\boldsymbol{\Lambda}) - n(\boldsymbol{\Lambda}'_1) \\ &= a(\boldsymbol{\Lambda}) - a(\boldsymbol{\Lambda}').\end{aligned}$$

Thus we see that $\deg K^{\pm}_{\boldsymbol{\beta},\boldsymbol{\alpha}} \le a(\boldsymbol{\alpha}) - a(\boldsymbol{\beta})$. The equality holds only when $\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}'_1$, i.e., $\boldsymbol{\Lambda}'$ is a special symbol. This proves the corollary.

□

7. Examples

7.1. In this section, we give some examples of Green functions, i.e., the matrix $P = \widetilde{K}_-(t)$ for the case where e and n are small. But before it, we shall discuss some general results on the similarity class having the maximum a-value.

We assume that $\mathbf{m} = (m_0, \dots, m_{e-1})$ is such that $m_0 = m+1, m_1 = \dots = m_{e-1}$ for some $m > 0$. For $i = 0, \dots, e-1$, put $\boldsymbol{\beta}_i = (\beta^{(0)}, \dots, \beta^{(e-1)})$ with $\beta^{(i)} = (1^n)$ and $\beta^{(j)} = (-)$ for $j \ne i$. Note that $\chi^{\boldsymbol{\beta}_i} = (\det_V)^i$ for $i = 1, \dots e-1$. In particular, $\chi^{\boldsymbol{\beta}_1} = \det_V$ and $\chi^{\boldsymbol{\beta}_{e-1}} = \overline{\det}_V$. Also we put $\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}(\boldsymbol{\beta}_i)$. Then it is clear that $\boldsymbol{\Lambda}_1, \dots, \boldsymbol{\Lambda}_{e-1}$ fall in the same class. More precisely, it is easy to see that $\boldsymbol{\Lambda}_0 \sim \boldsymbol{\Lambda}_1$ if $r = s$ and $\boldsymbol{\Lambda}_0 \not\sim \boldsymbol{\Lambda}_1$ if $r > s$. (The latter follows from the fact that the maximal entry in $\boldsymbol{\Lambda}_0$ is equal to $mr+1$, while that in $\boldsymbol{\Lambda}_1$ is $\le mr$ and we have $a(\boldsymbol{\Lambda}_0) < a(\boldsymbol{\Lambda}_1)$.) By using a similar argument as in Lemma 4,2, one can show that $\boldsymbol{\alpha} \in Z_n^{0,0}$ such that $a(\boldsymbol{\alpha})$ is maximum coincides with one of $\boldsymbol{\beta}_i$. Hence the class $\mathcal{F}$ having the maximum a-value is given as follows

$$\mathcal{F} = \begin{cases} \{\boldsymbol{\Lambda}_1, \dots, \boldsymbol{\Lambda}_{e-1}\} & \text{if } r > s, \\ \{\boldsymbol{\Lambda}_0, \boldsymbol{\Lambda}_1, \dots, \boldsymbol{\Lambda}_{e-1}\} & \text{if } r = s. \end{cases}$$

We identify the class $\mathcal{F}$ also as the subset of $Z_n^{0,0}$ by the natural bijection $\boldsymbol{\alpha} \to \boldsymbol{\Lambda}(\boldsymbol{\alpha})$.

In the case of Green polynomials of classical groups, (i.e., e = 2, r = 2, s = 1), the family of maximum a-value corresponds to the identity unipotent class under the Springer correspondence, hence consists of one element $\{\boldsymbol{\Lambda}_1\}$, where $\boldsymbol{\beta}_1$ corresponds to the sign character of W. Let $P = \widetilde{K}_-(t)$ be the matrix as given in 1.4. Then it is known that the first column $\{p_{\boldsymbol{\alpha},\boldsymbol{\beta}_1}\}$ of P gives rise to the fake degrees $\{R(\chi^{\boldsymbol{\alpha}})\}$ of $\chi^{\boldsymbol{\alpha}}$.

The following result, which is due to Malle [Ma2], is the counterpart of the above result to the general setting.

Lemma 7.2. *Let the notations be as above. Put $b = a(\boldsymbol{\beta}_1)$. Then we have*

$$\sum_{\boldsymbol{\beta}_j \in \mathcal{F}} t^{-b} R(\chi^{\boldsymbol{\beta}_j}) p_{\boldsymbol{\alpha},\boldsymbol{\beta}_j} = R(\chi^{\boldsymbol{\alpha}}). \tag{7.2.1}$$

Moreover $R(\chi^{\boldsymbol{\beta}_j}) = t^{n(n-1)/2 + jn}$ for $j = 0, \dots, e-1$.

Proof. Let us consider the equation $P\Lambda{}^tP = \Omega$ in (1.4.2). We write the matrices P, Λ, Ω in a block matrix form, where the first block corresponds the family $\mathcal{F}$.

$$P = \begin{pmatrix} t^bI & 0 \\ P_1 & P_2 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_1 & {}^t\Omega_2 \\ \Omega_2 & \Omega_3 \end{pmatrix}.$$

Then we have $D_1 = t^{-2b}\Omega_1$ and $P_1 = t^b\Omega_2\Omega_1^{-1}$. Let us denote the entries of Ω_1^{-1} by $(\tilde{\omega}_{jk})$ with j, k such that $\boldsymbol{\beta}_j, \boldsymbol{\beta}_k \in \mathcal{F}$. Then we have

$$p_{\boldsymbol{\alpha},\boldsymbol{\beta}_j} = t^b \sum_{\boldsymbol{\beta}_k \in \mathcal{F}} \omega_{\boldsymbol{\alpha},\boldsymbol{\beta}_k}\tilde{w}_{kj},$$

and so

$$\begin{aligned}
\sum_{\boldsymbol{\beta}_j \in \mathcal{F}} t^{-b}R(\chi^{\boldsymbol{\beta}_j})p_{\boldsymbol{\alpha},\boldsymbol{\beta}_j} &= \sum_{\boldsymbol{\beta}_k \in \mathcal{F}} \omega_{\boldsymbol{\alpha},\boldsymbol{\beta}_k} \sum_{\boldsymbol{\beta}_j \in \mathcal{F}} R(\chi^{\boldsymbol{\beta}_j})\tilde{w}_{kj} \\
&= \sum_{\boldsymbol{\beta}_k \in \mathcal{F}} \omega_{\boldsymbol{\alpha},\boldsymbol{\beta}_k} \sum_{\boldsymbol{\beta}_j \in \mathcal{F}} t^{-N^*}\tilde{w}_{kj}\omega_{\boldsymbol{\beta}_j,\boldsymbol{\beta}_1},
\end{aligned}$$

since $R(\chi^{\boldsymbol{\beta}_j}) = R(\chi^{\boldsymbol{\beta}_j} \otimes \chi^{\boldsymbol{\beta}_1} \otimes \overline{\det}_V) = t^{-N^*}\omega_{\boldsymbol{\beta}_j,\boldsymbol{\beta}_1}$. Now the last formula is equal to

$$t^{-N^*} \sum_{\boldsymbol{\beta}_k \in \mathcal{F}} \omega_{\boldsymbol{\alpha},\boldsymbol{\beta}_k}\delta_{k,1} = t^{-N^*}\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}_1} = R(\chi^{\boldsymbol{\alpha}}).$$

This prove (7.2.1). The last assertion follows easily from the property of coinvariant algebra R of W since $\deg \chi^{\boldsymbol{\beta}_j} = 1$. □

7.3. In the remainder of this section, we shall give explicit examples of the matrices P and Λ associated to various $W = G(e, 1, n)$. These examples support the conjecture 5.5. It would be worthwhile to point out that the entries $\lambda_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ of Λ always factorize the polynomial $\mathbb{G}(t)$.

In the following examples, $Z_n^{0,0} = Z_n^{0,0}(\mathbf{m})$ are written as $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_h$, and the corresponding symbols in $Z_n^{r,s}$ are written as $\boldsymbol{\Lambda}_i = \boldsymbol{\Lambda}(\boldsymbol{\alpha}_i)$, with $\mathbf{m}$ as in 7.1. In the matrix P of Green functions, the rows and columns are arranged along the order $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \dots$. The matrix Λ is given by the collection of $\Lambda(\mathcal{F})$, where $\Lambda(\mathcal{F})$ denotes the diagonal block in Λ corresponding the class $\mathcal{F}$.

First we consider the case where $W = G(2, 1, 2)$, i.e., W is the Weyl group of type B_2 for the sake of reference. Then $Z_n^{0,0}$ is given as

$$\boldsymbol{\alpha}_1 = (-;1^2), \boldsymbol{\alpha}_2 = (1^2;-), \boldsymbol{\alpha}_3 = (1;1), \boldsymbol{\alpha}_4 = (-;2), \boldsymbol{\alpha}_5 = (2;-).$$

Assume that $r = 2$ and $s = 1$. Then the corresponding symbols and similarity classes are

$$\begin{aligned}
&\mathcal{F}_1 = \{\boldsymbol{\Lambda}_1 = (420;42)\}, \quad \mathcal{F}_2 = \{\boldsymbol{\Lambda}_2 = (31;1)\}, \\
&\mathcal{F}_3 = \{\boldsymbol{\Lambda}_3 = (30;2), \boldsymbol{\Lambda}_4 = (20;3)\}, \quad \mathcal{F}_4 = \{\boldsymbol{\Lambda}_5 = (2;-)\}.
\end{aligned}$$

The matrices $(\Lambda(\mathcal{F}_i))$ are given as follows.

$$\Lambda(\mathcal{F}_1) = (1), \qquad \Lambda(\mathcal{F}_2) = ((t^4-1)),$$
$$\Lambda(\mathcal{F}_3) = (t^4-1)\begin{pmatrix} t^2 & t \\ t & t^2 \end{pmatrix}, \qquad \Lambda(\mathcal{F}_4) = (t^2(t^2-1)(t^4-1)).$$

The matrix P of Green functions is given in Table 1.

TABLE 1. $G(2,1,2) \quad (r=2, s=1)$

$(-;1^2)$	t^4				
$(1^2;-)$	t^2	t^2			
$(1;1)$	t^3+t	t	t		
$(-;2)$	t^2			t	
$(2;-)$	1	1	1		1

7.4. Assume that $W = G(3,1,2)$. Then $Z_n^{0,0}$ is given as

$$\boldsymbol{\alpha}_1 = (-;1^2;-), \boldsymbol{\alpha}_2 = (-;-;1^2), \boldsymbol{\alpha}_3 = (1^2;-;-), \boldsymbol{\alpha}_4 = (-;1;1),$$
$$\boldsymbol{\alpha}_5 = (1;1;-), \boldsymbol{\alpha}_6 = (1;-;1), \boldsymbol{\alpha}_7 = (-;2;-), \boldsymbol{\alpha}_8 = (-;-;2), \boldsymbol{\alpha}_9 = (2;-;-).$$

Assume that $r = 2, s = 1$. Then the corresponding symbols and similarity classes are given as

$$\mathcal{F}_1 = \{\boldsymbol{\Lambda}_1 = (420;42;31), \boldsymbol{\Lambda}_2 = (420;31;42)\},$$
$$\mathcal{F}_2 = \{\boldsymbol{\Lambda}_3 = (31;1;1)\}, \quad \mathcal{F}_3 = \{\boldsymbol{\Lambda}_4 = (20;2;2)\},$$
$$\mathcal{F}_4 = \{\boldsymbol{\Lambda}_5 = (30;2;1), \boldsymbol{\Lambda}_6 = (30;1;2), \boldsymbol{\Lambda}_7 = (20;3;1)\boldsymbol{\Lambda}_8 = (20;1;3)\},$$
$$\mathcal{F}_5 = \{\boldsymbol{\Lambda}_9 = (2;-;-)\}.$$

The matrices $(\Lambda(\mathcal{F}_i))$ are given as follows.

$$\Lambda(\mathcal{F}_1) = \begin{pmatrix} t^2 & t^4 \\ t^4 & 1 \end{pmatrix}, \qquad \Lambda(\mathcal{F}_2) = (t^2(t^6-1)), \qquad \Lambda(\mathcal{F}_3) = (t^2(t^6-1)),$$
$$\Lambda(\mathcal{F}_4) = (t^3-1)(t^6-1)\begin{pmatrix} t^2 & t^3 & 0 & t^2 \\ t^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^3 \\ t^2 & 0 & t^3 & 0 \end{pmatrix}, \qquad \Lambda(\mathcal{F}_5) = (t^5(t^3-1)(t^6-1)).$$

The matrix P of Green functions is given in Table 2.

TABLE 2. $G(3,1,2) \quad (r=2, s=1)$

$(-;1^2;-)$	t^5								
$(-;-;1^2)$		t^5							
$(1^2;-;-)$	t^3	t^4	t^3						
$(-;1;1)$	t^3			t^3					
$(1;1;-)$	t^4+t		t	t	t				
$(1;-;1)$	t^2	t^3	t^2	t^2		t			
$(-;2;-)$	t^2		t^2				t		
$(-;-;2)$		t^2	t					t	
$(2;-;-)$	1		1	1	1				1

7.5. Assume that $W = G(4,1,2)$. Then $Z_n^{0,0}$ is given as

$$\begin{aligned}
&\boldsymbol{\alpha}_1 = (-;1^2;-;-), \boldsymbol{\alpha}_2 = (-;-;1^2;-;-), \boldsymbol{\alpha}_3 = (-;-;-;1^2),\\
&\boldsymbol{\alpha}_4 = (1^2;-;-;-), \boldsymbol{\alpha}_5 = (-;1;1;-), \boldsymbol{\alpha}_6 = (-;1;-;1), \boldsymbol{\alpha}_7 = (-;-;1;1),\\
&\boldsymbol{\alpha}_8 = (1;1;-;-), \boldsymbol{\alpha}_9 = (1;-;1;-), \boldsymbol{\alpha}_{10} = (-;2;-;-), \boldsymbol{\alpha}_{11} = (1;-;-;1),\\
&\boldsymbol{\alpha}_{12} = (-;-;2;-), \boldsymbol{\alpha}_{13} = (-;-;-;2), \boldsymbol{\alpha}_{14} = (2;-;-;-).
\end{aligned}$$

Assume that $r=2$ and $s=1$. Then the corresponding symbols and similarity classes are given as

$$\begin{aligned}
&\mathcal{F}_1 = \{\boldsymbol{\Lambda}_1 = (420;42;31;31), \boldsymbol{\Lambda}_2 = (420;31;42;31), \boldsymbol{\Lambda}_3 = (420;31;31;42)\},\\
&\mathcal{F}_2 = \{\boldsymbol{\Lambda}_4 = (31;1;1;1)\},\\
&\mathcal{F}_3 = \{\boldsymbol{\Lambda}_5 = (20;2;2;1), \boldsymbol{\Lambda}_6 = (20;2;1;2), \boldsymbol{\Lambda}_7 = (20;1;2;2)\},\\
&\mathcal{F}_4 = \{\boldsymbol{\Lambda}_8 = (30;2;1;1), \boldsymbol{\Lambda}_9 = (30;1;2;1), \boldsymbol{\Lambda}_{10} = (20;3;1;1),\\
&\qquad \boldsymbol{\Lambda}_{11} = (30;1;1;2), \boldsymbol{\Lambda}_{12} = (20;1;3;1), \boldsymbol{\Lambda}_{13} = (20;1;1;3)\}\\
&\mathcal{F}_5 = \{\boldsymbol{\Lambda}_{14} = (2;-;-;-)\}.
\end{aligned}$$

The matrices $(\Lambda(\mathcal{F}_i))$ are given as follows.

$$\Lambda(\mathcal{F}_1) = \begin{pmatrix} t^4 & t^6 & t^8 \\ t^6 & t^8 & t^2 \\ t^8 & t^2 & t^4 \end{pmatrix}, \quad \Lambda(\mathcal{F}_2) = (t^4(t^8-1)), \quad \Lambda(\mathcal{F}_3) = (t^8-1)\begin{pmatrix} t^4 & t^5 & t^6 \\ t^5 & t^6 & t^3 \\ t^6 & t^3 & t^4 \end{pmatrix},$$

$$\Lambda(\mathcal{F}_4) = (t^4-1)(t^8-1)\begin{pmatrix} t^4 & t^5 & 0 & t^6 & 0 & t^5 \\ t^5 & t^6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^6 & 0 \\ t^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^6 & 0 \\ t^5 & 0 & t^6 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda(\mathcal{F}_5) = (t^8(t^4-1)(t^8-1)).$$

The matrix P of Green functions is given in Table 3.

TABLE 3. $G(4,1,2) \quad (r = 2, s = 1)$

$(-;1^2;-;-)$	t^6													
$(-;-;1^2;-)$		t^6												
$(-;-;-;1^2)$			t^6											
$(1^2;-;-;-)$	t^4			t^4										
$(-;1;1;-)$	t^3	t^5			t^3									
$(-;1;-;1)$	t^4		t^4			t^3								
$(-;-;1;1)$		t^3	t^5				t^3							
$(1;1;-;-)$	t^5+t			t	t			t						
$(1;-;1;-)$	t^2	t^4		t^2	t^2				t					
$(-;2;-;-)$	t^2				t^2					t				
$(1;-;-;1)$	t^3		t^3	t^3		t^2					t			
$(-;-;2;-)$		t^2					t^2					t		
$(-;-;-;2)$			t^2			t							t	
$(2;-;-;-)$	1			1	1			1						1

7.6. Assume that $W = G(3,1,3)$. Then $Z_n^{0,0}$ is given as

$$\begin{aligned}
&\boldsymbol{\alpha}_1 = (-;1^3;-), \boldsymbol{\alpha}_2 = (-;-;1^3), \boldsymbol{\alpha}_3 = (1^3;-;-), \boldsymbol{\alpha}_4 = (-;1^2;1),\\
&\boldsymbol{\alpha}_5 = (-;1;1^2), \boldsymbol{\alpha}_6 = (1;1^2;-), \boldsymbol{\alpha}_7 = (-;21;-), \boldsymbol{\alpha}_8 = (1;-;1^2),\\
&\boldsymbol{\alpha}_9 = (-;-;21), \boldsymbol{\alpha}_{10} = (1^2;1;-), \boldsymbol{\alpha}_{11} = (1^2;-;1), \boldsymbol{\alpha}_{12} = (21;-;-),\\
&\boldsymbol{\alpha}_{13} = (1;1;1), \boldsymbol{\alpha}_{14} = (-;2;1), \boldsymbol{\alpha}_{15} = (-;1;2), \boldsymbol{\alpha}_{16} = (1;2;-),\\
&\boldsymbol{\alpha}_{17} = (1;-;2), \boldsymbol{\alpha}_{18} = (2;1;-), \boldsymbol{\alpha}_{19} = (2;-;1), \boldsymbol{\alpha}_{20} = (-;3;-),\\
&\boldsymbol{\alpha}_{21} = (-;-;3), \boldsymbol{\alpha}_{22} = (3;-;-).
\end{aligned}$$

Assume that $r = 2$ and $s = 1$. Then the corresponding symbols and similarity classes are given as

$$\begin{aligned}
&\mathcal{F}_1 = \{\boldsymbol{\Lambda}_1 = (6420;642;531), \boldsymbol{\Lambda}_2 = (6420;531;642)\},\\
&\mathcal{F}_2 = \{\boldsymbol{\Lambda}_3 = (531;31;31)\}, \quad \mathcal{F}_3 = \{\boldsymbol{\Lambda}_4 = (420;42;41), \boldsymbol{\Lambda}_5 = (420;41;42)\},\\
&\mathcal{F}_4 = \{\boldsymbol{\Lambda}_6 = (520;42;31), \boldsymbol{\Lambda}_7 = (420;52;31), \boldsymbol{\Lambda}_8 = (520;31;42), \boldsymbol{\Lambda}_9 = (420;31;52)\},\\
&\mathcal{F}_5 = \{\boldsymbol{\Lambda}_{10} = (31;2;1), \boldsymbol{\Lambda}_{11} = (31;1;2)\}, \qquad \mathcal{F}_6 = \{\boldsymbol{\Lambda}_{12} = (41;1;1)\},\\
&\mathcal{F}_7 = \{\boldsymbol{\Lambda}_{13} = (30;2;2), \boldsymbol{\Lambda}_{14} = (20;3;2), \boldsymbol{\Lambda}_{15} = (20;2;3)\},\\
&\mathcal{F}_8 = \{\boldsymbol{\Lambda}_{16} = (30;3;1), \boldsymbol{\Lambda}_{17} = (30;1;3)\},\\
&\mathcal{F}_9 = \{\boldsymbol{\Lambda}_{18} = 40;2;1), \boldsymbol{\Lambda}_{19} = (40;1;2), \boldsymbol{\Lambda}_{20} = (20;4;1), \boldsymbol{\Lambda}_{21} = (20;1;4)\},\\
&\mathcal{F}_{10} = \{\boldsymbol{\Lambda}_{22} = (3;-;-)\}.
\end{aligned}$$

The matrices $(\varLambda(\mathcal{F}_i))$ are given as follows.

$$
\begin{aligned}
&\varLambda(\mathcal{F}_1) = \begin{pmatrix} t^3 & t^6 \\ t^6 & 1 \end{pmatrix}, \quad \varLambda(\mathcal{F}_2) = (t^3(t^9-1)), \quad \varLambda(\mathcal{F}_3) = (t^3+1)(t^9-1)\begin{pmatrix} t^3 & t^4 \\ t^4 & t^2 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_4) = (t^6-1)(t^9-1)\begin{pmatrix} t^3 & 0 & t^5 & t^4 \\ 0 & 0 & 0 & t^5 \\ t^5 & 0 & 0 & 0 \\ t^4 & t^5 & 0 & 0 \end{pmatrix}, \quad \varLambda(\mathcal{F}_5) = (t^6-1)(t^9-1)\begin{pmatrix} t^6 & t^7 \\ t^7 & t^5 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_6) = ((t^9(t^6-1)(t^9-1)), \quad \varLambda(\mathcal{F}_7) = (t^6-1)(t^9-1)\begin{pmatrix} t^9 & t^7 & t^8 \\ t^7 & t^8 & t^9 \\ t^8 & t^9 & t^7 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_8) = (t^3-1)(t^6-1)(t^9-1)\begin{pmatrix} 0 & t^8 \\ t^8 & 0 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_9) = (t^3-1)(t^6-1)(t^9-1)\begin{pmatrix} t^9 & t^{10} & 0 & 0 \\ t^{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & t^{10} \\ 0 & 0 & t^{10} & 0 \end{pmatrix}, \\
&\varLambda(\mathcal{F}_{10}) = t^{12}(t^3-1)(t^6-1)(t^9-1).
\end{aligned}
$$

The matrix P of Green functions is given in Table 4.

Table 4. $G(3,1,3) \quad (r=2, s=1)$

$(-;1^3;-)$	t^{12}																					
$(-;-;1^3)$		t^{12}																				
$(1^3;-;-)$	t^9		t^9																			
$(-;1^2;1)$	$t^{10}+t^7$	t^{10}		t^7																		
$(-;1;1^2)$	t^8	$t^{11}+t^8$			t^7																	
$(1;1^2;-)$	$t^{11}+t^8+t^5$		t^5	t^5		t^5																
$(-;21;-)$	t^9+t^6			t^6			t^5															
$(1;-;1^2)$	t^7	$t^{10}+t^7$	t^7		t^6			t^5														
$(-;-;21)$		t^9+t^6			t^5				t^5													
$(1^2;1;-)$	$t^{10}+t^7+t^4$		t^7+t^4	t^4		t^4				t^4												
$(1^2;-;1)$	t^8+t^5	t^8	t^8+t^5	t^5				t^3			t^4											
$(21;-;-)$	t^6+t^3		t^6+t^3	t^3		t^3				t^3		t^3										
$(1;1;1)$	$t^9+2t^6+t^3$	t^9+t^6	t^6+t^3	t^6+t^3	t^5	t^3		t^4		t^3			t^3									
$(-;2;1)$	t^7+t^4	t^7		t^4	t^6		t^3							t^3								
$(-;1;2)$	t^5	t^8+t^5		t^5	t^4				t^4						t^3							
$(1;2;-)$	$t^8+t^5+t^2$		t^2	t^5+t^2		t^2	t^4			t^2			t^2			t^2						
$(1;-;2)$	t^4	t^7+t^4	t^4	t^4	t^3			t^2	t^3		t^3				t^2		t^2					
$(2;1;-)$	t^7+t^4+t		t^4+t	t^4+t		t^4+t				t		t	t			t		t				
$(2;-;1)$	t^5+t^2	t^5	t^5+t^2	t^2	t^4	t^2		t^3		t^2		t^2	t^2						t			
$(-;3;-)$	t^3			t^3			t^2							t^2						t		
$(-;-;3)$		t^3			t^2				t^2						t		t				t	
$(3;-;-)$	1		1	1		1				1		1	1			1		1				1

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