# MODIFIED ARIKI-KOIKE ALGEBRAS AND CYCLOTOMIC $q$-SCHUR ALGEBRAS

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Abstract. The modified Ariki-Koike algebra $\mathcal{H}$ is a variation of the original Ariki-Koike algebra $\widetilde{\mathcal{H}}$ over an integral domain $R$. When $R$ is a rational function field over the independent parameters, $\mathcal{H} \simeq \widetilde{\mathcal{H}}$. But for general $R$, $\mathcal{H}$ is not isomorphic to $\widetilde{\mathcal{H}}$, and has a simpler structure than $\widetilde{\mathcal{H}}$. In this paper, we construct a cellular basis of $\mathcal{H}$ which has a similar property as the cellular basis of $\widetilde{\mathcal{H}}$ introduced by Dipper-James-Mathas. By comparing these two cellular bases, we obtain some estimate on the decomposition numbers of $\widetilde{\mathcal{H}}$ in terms of the decomposition numbers of $\mathcal{H}$. We also prove the integral form of the Schur-Weyl reciprocity between a certain quantum algebra $U_q$ and $\mathcal{H}$ on the tensor space $V^{\otimes n}$.

## 0. Introduction

Let $\widetilde{\mathcal{H}}$ be the Ariki-Koike algebra over an integral domain $R$ with parameters $q, q^{-1}, Q_1, \dots, Q_r \in R$, associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$. In [S], the second author constructed an algebra $\mathcal{H}$ over $R$, in the case where $R$ is a field $K_0$ of rational functions with variables $q, Q_1, \dots, Q_r$, and showed that $\mathcal{H}$ is isomorphic to $\widetilde{\mathcal{H}}$, giving a new presentation of $\widetilde{\mathcal{H}}$ (see [M2], where the survey on $\mathcal{H}$, and related topics are found). The construction of $\mathcal{H}$ works for general $R$ as far as the parameters satisfy the condition that (*) $Q_i - Q_j$ are invertible for any $i \neq j$. $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ contain the Iwahori-Hecke algebra $\mathcal{H}_n$ of type $A_{n-1}$. The algebras $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are not isomorphic in general. However there exists a natural homomorphism $\rho_0 : \widetilde{\mathcal{H}} \to \mathcal{H}$ which is identity on $\mathcal{H}_n$. Although the map is not surjective nor injective, these two algebras seem to be closely related by it. The algebra $\mathcal{H}$ has a simple (degenerate) structure compared to $\widetilde{\mathcal{H}}$, which belong essentially to an object of type $A$. Even so, it is expected that $\mathcal{H}$ gives some informations on the representation theory of $\widetilde{\mathcal{H}}$, and we call $\mathcal{H}$ the modified Ariki-Koike algebra.

The purpose of this paper is to show that many results for $\widetilde{\mathcal{H}}$ established by Dipper-James-Mathas [DJM], and results for the Iwahori-Hecke algebra $\mathcal{H}_n$ written in the textbook of Mathas [M1], obtained by using the properties of cellular basis, holds as well for our algebra $\mathcal{H}$, in many cases in a simpler form. First we construct the cellular basis of $\mathcal{H}$, which enjoys many properties in common with the cellular basis of $\widetilde{\mathcal{H}}$ constructed in [DJM], though there is no direct relation between them. The construction of our basis is carried out by making use of the representation of $\mathcal{H}$ on the tensor space $V^{\otimes n}$, on which $\widetilde{\mathcal{H}}$ also acts. This action of $\widetilde{\mathcal{H}}$ was defined in [SS], and plays a crucial role in defining $\mathcal{H}$. In [H], Jun Hu also constructed an action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$ by making use of the cellular basis of $\widetilde{\mathcal{H}}$. We show that these two actions of $\widetilde{\mathcal{H}}$ are essentially the same. (This fact was also proved by F. Stoll independently, see [HS].) This result gives a link between the cellular basis of $\widetilde{\mathcal{H}}$ and of $\mathcal{H}$. We can deduce from this that the Specht modules of $\widetilde{\mathcal{H}}$ and that of $\mathcal{H}$ (regarded as $\widetilde{\mathcal{H}}$-modules via $\rho_0$) are isomorphic, and that there are some relations for irreducible modules also. This enables us to estimate the decomposition numbers of $\widetilde{\mathcal{H}}$ in terms of the decomposition numbers of $\mathcal{H}$ (Theorem 4.13).

Assume that $V = \bigoplus_{i=1}^{m} V_i$ where $V_i$ is a free $R$-module of rank $m_i$. We put $\mathbf{m} = (m_1, \dots, m_r)$. Then the quantum algebra $U_q(\mathfrak{g})$ (its Kostant-Lusztig integral form over $R$) acts naturally on $V^{\otimes n}$, which commutes with the action of $\widetilde{\mathcal{H}}$ (and of $\mathcal{H}$), where $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$. In the generic case, (i.e., the case where $R = K_0$), the Schur-Weyl reciprocity concerning the action of $U_q(\mathfrak{g})$ and $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$ was proved by [SS] (and independently by [Hu]). In [A], S. Ariki considered the centralizer algebra $\operatorname{End}_{\widetilde{\mathcal{H}}} V^{\otimes n}$ as an analogy of the cyclotomic $q$-Schur algebra defined in [DJM], and showed, under a certain condition on parameters (the separation condition, see (5.4.2)), that $\operatorname{End}_{\widetilde{\mathcal{H}}} V^{\otimes n}$ coincides with the image of the representation $\tau : U_q(\mathfrak{g}) \to \operatorname{End} V^{\otimes n}$, and that it is isomorphic to the direct sum of the tensor product of the $q$-Schur algebras associated to $\mathcal{H}_{n_i}$ for various $n_i$. In this paper, we define a cyclotomic $q$-Schur algebra $\mathcal{S}(\mathbf{m}, n)$ associated to $\mathcal{H}$ by $\operatorname{End}_{\mathcal{H}} V^{\otimes n}$, and show that a similar result as in the case of Ariki holds for our case. (Note that our condition (*) is weaker than the separation condition.) We can also construct a cellular basis of $\mathcal{S}(\mathbf{m}, n)$ which has similar properties as the cellular basis of the cyclotomic $q$-Schur algebra $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ defined in [DJM]. We discuss the relationship between these two algebras.

By making use of the Schur functors, the above results are converted to the case of $\widetilde{\mathcal{H}}$, i.e, we obtain some estimate of the decomposition numbers of $\widetilde{\mathcal{H}}$ in terms of the decomposition numbers of $\mathcal{H}_{n_i}$ for various $n_i$ (Corollary 6.3). The Schur functor is also used to show the integral form of the Schur-Weyl reciprocity concerning the action of $U_q(\mathfrak{g})$ and $\mathcal{H}$ on $V^{\otimes n}$.

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## 1. Preliminaries on Ariki-Koike algebras

**1.1.** Let $R$ be an integral domain, $q, Q_1, \dots, Q_r$ be elements in $R$, with $q$ invertible. The Ariki-Koike algebra, i.e., the cyclotomic Hecke algebra associated to the complex reflection group $W_{n,r} = G(r,1,n)$, is the algebra $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{n,r}$ over $R$ with generators $T_1, \dots, T_n$ subject to the following conditions,

$$\begin{aligned}
&(T_1 - Q_1)\cdots(T_1 - Q_r) = 0, && \\
&(T_i - q)(T_i + q^{-1}) = 0 && (i \geq 2), \\
&T_1T_2T_1T_2 = T_2T_1T_2T_1, && \\
&T_iT_j = T_jT_i && (|i-j| \geq 2), \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (2 \leq i \leq n-1).
\end{aligned}$$

It is known that $\widetilde{\mathcal{H}}$ is a free $R$-module of rank $n!r^n$. The subalgebra $\widetilde{\mathcal{H}}(\mathfrak{S}_n)$ of $\widetilde{\mathcal{H}}$ generated by $T_2, \dots, T_n$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_n$ of the symmetric group $\mathfrak{S}_n$.

**1.2.** We consider the following condition on parameters $Q_1, \dots, Q_r$ in $R$.

(1.2.1) $Q_i - Q_j$ are invertible in R for any $i \neq j$.

In [S], a variation $\mathcal{H}$ of the Ariki-Koike algebra $\widetilde{\mathcal{H}}$ was definend subject to the condition (1.2.1). In this paper, we call $\mathcal{H}$ the modified Ariki-Koike algebra. It is known that if $R = \mathbb{Q}(\bar{q}, \bar{Q}_1, \dots, \bar{Q}_r)$, the field of rational functions with variables $\bar{q}, \bar{Q}_1, \dots, \bar{Q}$, $\mathcal{H}$ is isomorphic to $\widetilde{\mathcal{H}}$, and it gives a new presentation of $\widetilde{\mathcal{H}}$ apart from 1.1. Following [S], we recall the definition of $\mathcal{H}$. Let $A$ be a square matrix of degree $r$ whose $ij$-entry is given by $Q_j^{i-1}$ for $1 \leq i, j \leq r$. Thus $A$ is the Vandermonde matrix, and $\varDelta = \det A = \prod_{i>j}(Q_i - Q_j)$ is invertible by (1.2.1). We express the inverse of $A$ as $A^{-1} = \varDelta^{-1}B$ with $B = (h_{ij})$, and define a polynomial $F_i(X) \in R[X]$, for $1 \leq i \leq r$, by

$$F_i(X) = \sum_{1 \leq j \leq r} h_{ij}X^{j-1}. \tag{1.2.2}$$

Then $\mathcal{H} = \mathcal{H}_{n,r}$ is generated by $T_2, \dots, T_n$ and $\xi_1, \dots, \xi_n$ with relations

$$\begin{aligned}
&(T_i - q)(T_i + q^{-1}) = 0 && (2 \leq i \leq n), \\
&(\xi_i - Q_1)\cdots(\xi_i - Q_r) = 0 && (1 \leq i \leq n), \\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (2 \leq i \leq n), \\
&T_iT_j = T_jT_i && (|i-j| \geq 2), \\
&\xi_i\xi_j = \xi_j\xi_i && (1 \leq i, j \leq n)
\end{aligned} \tag{1.2.3}$$

$$T_j\xi_j = \xi_{j-1}T_j + \varDelta^{-2}\sum_{c_1<c_2}(u_{c_2}-u_{c_1})(q-q^{-1})F_{c_1}(\xi_{j-1})F_{c_2}(\xi_j) \tag{1.2.4}$$

$$T_j\xi_{j-1} = \xi_jT_j - \varDelta^{-2}\sum_{c_1<c_2}(u_{c_2}-u_{c_1})(q-q^{-1})F_{c_1}(\xi_{j-1})F_{c_2}(\xi_j) \tag{1.2.5}$$

$$T_j\xi_k = \xi_kT_j \quad (k \ne j-1, j), \tag{1.2.6}$$

The subalgebra $\mathcal{H}(\mathfrak{S}_n)$ of $\mathcal{H}$ generated by $T_2,\dots,T_n$ is isomorphic to $\mathcal{H}_n$, hence it can be naturally identified with the corresponding subalgebra $\widetilde{\mathcal{H}}(\mathfrak{S}_n)$ of $\widetilde{\mathcal{H}}$. We denote by $s_2,\dots s_n$ the simple reflections of $\mathfrak{S}_n$ corresponding to $T_2,\dots,T_n$. The set $\{T_w \mid w \in \mathfrak{S}_n\}$ forms a basis of $\mathcal{H}_n$, where $T_w = T_{i_1}\cdots T_{i_r}$ for a reduced expression $w = s_{i_1}\cdots s_{i_r}$. Moreover, it is known by [S] that the set

$$\{\xi_1^{c_1}\cdots\xi_n^{c_n}T_w \mid w \in \mathfrak{S}_n, 0 \le c_i \le r-1\}$$

gives rise to a basis of $\mathcal{H}$.

**Remark 1.3.** In this paper, all the $\mathcal{H}$-modules and $\widetilde{\mathcal{H}}$-modules are assumed to be right modules. In [SS], [S], the left $\mathcal{H}$-module structure is discussed. The left module structure is converted to the right module structure if one changes the generators $T_i, \xi_j$ by $T_i^{-1}, \xi_j^{-1}$, i.e., by replacing the parameters $q, Q_j$ by $q^{-1}, Q_j^{-1}$. However, it is also possible to exchange the left module structure and the right module structure without changing the parameters since $\mathcal{H}$ is isomorphic to the opposite algebra $\mathcal{H}^\circ$. Namely, if the left $\mathcal{H}$-module $M$ is given, we define a right action of $\mathcal{H}$ on $M$ by $vh = h^*v$ $(v \in M, h \in \mathcal{H})$, where $*$ is an anti-algebra automorphism of $\mathcal{H}$ (see 2.7 below). The situation is the same for $\widetilde{\mathcal{H}}$ also. In this paper, we adopt this right action, and use freely the results in [S] accordingly.

**1.4.** Let $V = \bigoplus_{i=1}^r V_i$ be a free $R$-module, with rank $V_i = m_i$. We put $m = \sum m_i$. Let us fix a basis $\{v_1^{(i)},\dots,v_{m_i}^{(i)}\}$ of $V_i$ for $i = 1,\dots,r$. The vectors

$$v_1^{(1)},\dots,v_{m_1}^{(1)}, v_1^{(2)},\dots,v_{m_2}^{(2)},\dots,v_1^{(r)},\dots,v_{m_r}^{(r)}$$

forms a basis of $V$, which we denote also by $v_1,\dots,v_m$ in this order. We write $v_i \sim v_j$ if $v_i$ and $v_j$ are contained in a common subspace $V_k$. Let us define linear operators $T, S$ on $V \otimes V$ and $wt : V \to V$ by

$$(v_i \otimes v_j)T = \begin{cases} qv_i \otimes v_j & \text{if } i = j, \\ v_j \otimes v_i & \text{if } i > j, \\ v_j \otimes v_i + (q-q^{-1})v_i \otimes v_j & \text{if } i < j, \end{cases} \tag{1.4.1}$$

$$(v_i \otimes v_j)S = \begin{cases} (v_i \otimes v_j)T & \text{if } v_i \sim v_j, \\ v_j \otimes v_i & \text{otherwise.} \end{cases} \tag{1.4.2}$$

(1.4.3) $$(v_i)wt = Q_j v_i \quad \text{if } v_i \in V_j.$$

Then define $T_i, S_j, \xi_j \in \operatorname{End}_R V^{\otimes n}$ $(2 \le i \le n, 1 \le j \le n)$ by

$$\begin{aligned} T_i &= \mathrm{id}_V^{\otimes(i-2)} \otimes T \otimes \mathrm{id}_V^{\otimes(n-i)}, \\ S_i &= \mathrm{id}_V^{\otimes(i-2)} \otimes S \otimes \mathrm{id}_V^{\otimes(n-i)}, \\ \xi_j &= \mathrm{id}_V^{\otimes(j-1)} \otimes wt \otimes \mathrm{id}_V^{\otimes(n-j)}. \end{aligned}$$

We define an operator $T_1$ on $V^{\otimes n}$ by

(1.4.4) $$T_1 = \xi_1 S_2 \cdots S_n T_n^{-1} \cdots T_2^{-1}.$$

It is known by [SS] that the actions of $T_i$ $(1 \le i \le n)$ define a right $\widetilde{\mathcal{H}}$-module structure on $V^{\otimes n}$. We denote this representation by $\widetilde{\rho} : \widetilde{\mathcal{H}} \to \operatorname{End} V^{\otimes n}$. Note that this construction works under the general setting in 1.1, without (1.2.1). Also it is shown in [S] that, under the assumption (1.2.1), the actions of $T_i$ $(2 \le i \le n)$, $\xi_j (1 \le j \le n)$ define a right action of $\mathcal{H}$ on $V^{\otimes n}$. We denote this representation by $\rho : \mathcal{H} \to \operatorname{End} V^{\otimes n}$. By [S, Lemma 3.5], we know that $S_i \in \operatorname{Im}\rho$. It follows that $\operatorname{Im}\widetilde{\rho} \subset \operatorname{Im}\rho$ by (1.4.4).

Assuming that $m_i \ge n$ for $i = 1, \dots, r$, we consider the $R$-submodule $\mathcal{U}$ of $V^{\otimes n}$ generated by

$$v_{w(1)}^{(c_1)} \otimes \cdots \otimes v_{w(n)}^{(c_n)} \quad (w \in \mathfrak{S}_n, 1 \le c_i \le r).$$

Then $\mathcal{U}$ turns out to be an $\mathcal{H}$-submodule of $V^{\otimes n}$. Put

$$v_0 = \big(\sum_{c_1=1}^{r} v_n^{(c_1)}\big)\otimes, \big(\sum_{c_2=1}^{r} v_{n-1}^{(c_2)}\big) \otimes \cdots \otimes \big(\sum_{c_n=1}^{r} v_1^{(c_n)}\big).$$

It is easy to see, by using the actions of $\xi_j$, that the $\mathcal{H}$-module $v_0\mathcal{H}$ contains the elements of the form $v_n^{(c_1)} \otimes \cdots \otimes v_1^{(c_1)}$. It follows, by (1.4.1), that

(1.4.5) $\mathcal{U}$ is generated by $v_0$ as an $\mathcal{H}$-module. The map $h \mapsto v_0 h$ gives an isomorphism $\mathcal{H} \simeq \mathcal{U}$ as right $\mathcal{H}$-modules. In particular, $\rho$ is a faithful representation of $\mathcal{H}$.

We have a lemma.

**Lemma 1.5.** *Assume that $m_i \ge n$ for $i = 1, \dots, r$. Then there exists an $R$-algebra homomorphism $\rho_0 : \widetilde{\mathcal{H}} \to \mathcal{H}$ such that $\rho_0$ induces the identity on $\mathcal{H}_n$. (Here we regard $\mathcal{H}_n \subset \widetilde{\mathcal{H}}$, $\mathcal{H}_n \subset \mathcal{H}$ under the previous identifications.) If $\operatorname{Im}\rho = \operatorname{Im}\widetilde{\rho}$ and $R$ is a field, then $\widetilde{\mathcal{H}} \simeq \mathcal{H}$.*

*Proof.* By (1.4.5), we have an isomorphism $\mathcal{H} \simeq \operatorname{Im}\rho$. Since $\operatorname{Im}\widetilde{\rho} \subset \operatorname{Im}\rho$, we have a homomorphism $\rho_0 : \widetilde{\mathcal{H}} \to \operatorname{Im}\widetilde{\rho} \to \mathcal{H}$. If $\operatorname{Im}\rho = \operatorname{Im}\widetilde{\rho}$, then $\rho_0$ is surjective. Since $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ are both free $R$-modules of the same rank, they are isomorphic if $R$ is a field. □

**1.6.** Let $R_0 = \mathbb{Z}[\bar{q}, \bar{q}^{-1}, \bar{Q}_1, \dots, \bar{Q}_r]$ be the polynomial ring with variables $\bar{q}, \bar{Q}_j$, and $K_0$ be the quotient field of $R_0$. Let $\bar{V} = \bigoplus_{i=1}^{r} \bar{V}_i$ be the vector space over $K_0$ with $\dim \bar{V}_i = m_i$. Then the quantum algebra $U_{q,K_0}(\mathfrak{gl}_m)$ associated to the Lie algebra

$\mathfrak{gl}_m$ over $K_0$ acts from the left on the tensor space $\bar{V}^{\otimes n}$ via the natural action of $U_{q,K_0}(\mathfrak{gl}_m)$ on $\bar{V}$. Let $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ be the subalgebra of $\mathfrak{gl}_m$. Then $U_{q,K_0}(\mathfrak{g}) \simeq U_{q,K_0}(\mathfrak{gl}_{m_1}) \otimes \cdots \otimes U_{q,K_0}(\mathfrak{gl}_{m_r})$ is a subalgebra of $U_{q,K_0}(\mathfrak{gl}_m)$, and so $U_{q,K_0}(\mathfrak{g})$ acts on $\bar{V}^{\otimes n}$ as the restriction of the action of $U_{q,K_0}(\mathfrak{gl}_m)$.

Let $U_{q,R_0}(\mathfrak{g})$ be the Kostant-Lusztig form of $U_{q,K_0}(\mathfrak{g})$. We denote by $U_{q,R}(\mathfrak{g})$ the specialized algebra $R \otimes_{R_0} U_{q,R_0}(\mathfrak{g})$ by the specialization $R_0 \to R$ via $\bar{q} \mapsto q, \bar{Q}_j \mapsto Q_j$. If there is no fear of confusion, we simply write $U_{q,R}(\mathfrak{g})$ by $U_q$ or $U_q(\mathfrak{g})$. Now $U_{q,R_0}$ stabilizes the $R_0$-form $\bar{V}_{R_0}$ of $\bar{V}$, and so the space $\bar{V}_{R_0}^{\otimes n}$. Via the specialization $V = R \otimes_{R_0} \bar{V}_{R_0}$, $U_q$ acts on the $R$-module $V^{\otimes n}$. It is known by [SS] that the thus obtained action of $U_q$ on $V^{\otimes n}$ commutes with the left action of $\mathcal{H}$, and that in the generic case, i.e., the case where $R = K_0$, the Schur-Weyl reciprocity holds between $U_q$ and $\mathcal{H}$. By Remark 1.3, the left action of $U_q$ on $V^{\otimes n}$ commutes with the right action of $\mathcal{H}$ on $V^{\otimes n}$, and a similar results hold also for this case.

## 2. A cellular basis of $\mathcal{H}$

**2.1.** From now on, throughout the paper, we fix an $r$-tuple $\mathbf{m} = (m_1, \dots, m_r)$ of integers such that $m_i \geq n$.

An ordered $r$-turple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_{m_i}^{(i)}) \in \mathbb{Z}_{\geq 0}^{m_i}$ is called an $r$-composition. For an $r$-composition $\mu$, we define the type $\alpha = \alpha(\mu)$ of $\mu$ by $\alpha = (n_1, \dots, n_r)$ with $n_i = |\mu^{(i)}|$, and the size of $\mu$ by $n = \sum_{i=1}^r n_i$. We also define a sequence $\mathbf{a} = \mathbf{a}(\mu) = (a_1, \dots, a_r)$ by $a_i = \sum_{k=1}^{i-1} n_i$. If each $\mu^{(i)}$ is a partition, $\mu$ is called an $r$-partition. We denote by $\widetilde{\mathcal{P}}_{n,r} = \widetilde{\mathcal{P}}_{n,r}(\mathbf{m})$ (resp. $\mathcal{P}_{n,r} = \mathcal{P}_{n,r}(\mathbf{m})$) the set of $r$-compositions (resp. $r$-partitions) of size $n$. (Note that $\mathcal{P}_{n,r}(\mathbf{m})$ are naturally identified with each other for any $\mathbf{m}$ such that $m_i \geq n$. However, $\widetilde{\mathcal{P}}_{n,r}$ depends on the choice of $\mathbf{m}$.)

Any $\mu \in \widetilde{\mathcal{P}}_{n,r}$ may be regarded as an element in $\widetilde{\mathcal{P}}_{n,1}$ (i.e., 1-composition) of $n$ by arranging the entries of $\mu = (\mu_j^{(i)})$ in the order

$$\mu_1^{(1)}, \dots, \mu_{m_1}^{(1)}, \mu_1^{(2)}, \dots, \mu_{m_2}^{(2)}, \dots, \mu_1^{(r)}, \dots, \mu_{m_r}^{(r)},$$

which we denote by $\{\mu\}$.

The diagram of $\mu$ is $[\mu] = \{(i,j,s) \mid 1 \leq j \leq \mu_i^{(s)}, 1 \leq s \leq r\}$. The elements of $[\mu]$ are called nodes. A $\mu$-tableau $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$ is a bijection $\mathfrak{t} : [\mu] \to \{1, 2, \dots, n\}$, where $\mathfrak{t}^{(i)}$ is a tableau of Shape$(t^{(i)}) = \mu^{(i)}$. We write Shape$(\mathfrak{t}) = \mu$ if $\mathfrak{t}$ is a $\mu$-tableau. A $\mu$-tableau $\mathfrak{t}$ is called standard (resp. row standard) if all $\mathfrak{t}^{(i)}$ are standard (resp. row standard). For an $r$-composition $\mu$, we denote by r-Std$(\mu)$ the set of row standard tableaux of shape $\mu$. If $\lambda$ is an $r$-partition, we denote by Std$(\lambda)$ the set of standard tableaux of shape $\lambda$.

For each $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we denote by $\mathfrak{t}^\mu$ the tableau obtained from $\mu$ by associating numbers $1, 2, \dots, n$ to nodes of $[\mu]$, from left to right, and from top to bottom, regarding $\mu$ as a 1-composition $\{\mu\}$. $\mathfrak{S}_n$ acts on the set of tableaux of shape $\mu$ from the right, and we denote by $\mathfrak{S}_\mu$ the row stabilizer of $\mathfrak{t}^\mu$ in $\mathfrak{S}_n$. Hence

$$\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1^{(1)}} \times \mathfrak{S}_{\mu_2^{(1)}} \times \cdots \times \mathfrak{S}_{\mu_{m_r}^{(r)}},$$

which coincides with the Young subgroup $\mathfrak{S}_{\{\mu\}}$ of $\mathfrak{S}_n$.

**2.2.** We consider the representation of $\mathcal{H}$ on $V^{\otimes n}$ as in 1.4. From now on we always assume that $m_i \ge n$ for $i = 1, \dots, r$ whenever we consider $V^{\otimes n}$. Let us prepare some notation. We write $v_{i_1}^{(b_1)} \otimes \cdots \otimes v_{i_n}^{(b_n)} \in V^{\otimes n}$ by $v_I^b$, where $I = (i_1, \dots, i_n)$ and $b = (b_1, \dots, b_n)$. The permutation action of $w \in \mathfrak{S}_n$ on $V^{\otimes n}$ maps $v_I^b$ to $v_{Iw}^{bw}$, where $Iw = (j_{w(1)}, \dots, j_{w(n)})$ and $bw = (b_{w(1)}, \dots, b_{w(n)})$, respectively. We also write as $v_I = v_{i_1} \otimes \cdots \otimes v_{i_n}$ with $I = (i_1, \dots, i_n)$. The action of $\mathfrak{S}_n$ on $v_I$ is given similarly.

For $\alpha = (n_1, \dots, n_r) \in \mathbb{Z}_{\ge 0}^r$ such that $\sum n_i = n$, we define $v_\alpha \in V^{\otimes n}$ by $v_\alpha = v_J^c$, where

$$J = (n, n-1, \dots, 2, 1), \quad c = (\underbrace{r, \dots, r}_{n_1-\text{times}}, \underbrace{r-1, \dots, r-1}_{n_2-\text{times}}, \dots, \underbrace{1, \dots, 1}_{n_r-\text{times}}).$$

We note that $v_\alpha \in \mathcal{U}$. For $\mu \in \widetilde{\mathcal{P}}_{n,r}$ of type $\alpha$, we define $v_\mu \in \mathcal{U}$ by

$$v_\mu = v_\alpha m_{\{\mu\}}, \tag{2.2.1}$$

where $m_{\{\mu\}} = \sum_{w \in \mathfrak{S}_{\{\mu\}}} q^{l(w)} T_w \in \mathcal{H}_n$. Let $\mathcal{H}_{\{\mu\}}$ be the parabolic subalgebra of $\mathcal{H}_n$ corresponding to $\mathfrak{S}_\mu$, and $\theta_q$ the linear character of $\mathcal{H}_{\{\mu\}}$ defined by $\theta_q(T_w) = q^{l(w)}$ for $w \in \mathfrak{S}_\mu$. It is clear that $v_\mu \ne 0$, and since $m_{\{\mu\}}$ is an element affording $\theta_q$, $v_\mu$ also satisfies the same property, i.e., we have

$$v_\mu T_w = q^{l(w)} v_\mu \quad \text{for } w \in \mathfrak{S}_\mu. \tag{2.2.2}$$

We define $m_\mu \in \mathcal{H}$ as an element satisfying the formula $v_\mu = v_0 m_\mu$, which is determined uniquely by (1.4.5). We shall describe the element $m_\mu \in \mathcal{H}$ more explicitly. Let $c = (c_1, \dots, c_n)$ be such that $v_\alpha = v_J^c$. We define $F_\alpha \in \mathcal{H}$ by

$$F_\alpha = \varDelta^{-n} F_{c_1}(\xi_1) F_{c_2}(\xi_2) \cdots F_{c_n}(\xi_n). \tag{2.2.3}$$

Then as in the proof of Corollary 3.9 in [S], we see that $v_\alpha = v_0 F_\alpha$. Again by using (1.4.5), we have the following.

**Lemma 2.3.** *Assume that $\mu$ is of type $\alpha$. Then we have $m_\mu = F_\alpha m_{\{\mu\}}$.*

**2.4.** For a 1-composition $\nu = (\nu_1, \dots, \nu_k)$, we set

$$\mathcal{H}_{\nu,r} = \mathcal{H}_{\nu_1,r} \otimes \mathcal{H}_{\nu_2,r} \otimes \cdots \otimes \mathcal{H}_{\nu_k,r}.$$

For an $r$-composition $\mu$ of $n$ with $\alpha(\mu) = (n_1, \dots, n_r)$, let $\mathcal{H}_\mu$ be the subalgebra of $\mathcal{H}$ generated by $\mathcal{H}_{\{\mu\}} \subset \mathcal{H}_n$ and $\xi_1, \dots, \xi_n$. Then by the discussion in [S,4], $\mathcal{H}_\mu$ is isomorphic to the algebra

$$\mathcal{H}_{\mu^{(1)},r} \otimes \mathcal{H}_{\mu^{(2)},r} \otimes \cdots \otimes \mathcal{H}_{\mu^{(r)},r}$$

according to the natural embedding $W_{\mu^{(1)}} \times W_{\mu^{(2)}} \times \cdots \times W_{\mu^{(r)}} \subset W_{n,r}$, where $W_{\mu^{(i)}} \simeq \mathfrak{S}_{\mu^{(i)}} \ltimes (\mathbb{Z}/r\mathbb{Z})^{n_i}$. Let $\theta^{(s)}_{q,\mu}$ $(1 \le s \le r)$ be the linear character of $\mathcal{H}_{\mu,r}$ given by

$$\begin{aligned} \theta^{(s)}_{q,\mu}(T_j) &= q \qquad \text{for } T_j \in \mathcal{H}_{\{\mu\}}, \\ \theta^{(s)}_{q,\mu}(\xi_j) &= Q_s \qquad (1 \le j \le n) \end{aligned}$$

(cf. [S, 5.2]), which is an extension of $\theta_q$ on $\mathcal{H}_{\{\mu\}}$. We define a linear character $\theta^\mu_q$ of $\mathcal{H}_\mu$ by

$$\theta^\mu_q = \theta^{(r)}_{q,\mu^{(1)}} \otimes \theta^{(r-1)}_{q,\mu^{(2)}} \otimes \cdots \otimes \theta^{(1)}_{q,\mu^{(r)}}.$$

Then we have

(2.4.1) $v_\mu$ spans a one-dimensional $\mathcal{H}_\mu$-module affording the linear character $\theta^\mu_q$.

In fact it is enough to show that

$$\begin{aligned} v_\mu T_w &= q^{l(w)} v_\mu \qquad \text{for } w \in \mathfrak{S}_\mu \\ v_\mu \xi_j &= Q_{r-s+1} v_\mu \qquad \text{if } a_s + 1 \le j \le a_{s+1}, \end{aligned}$$

where $a_s = \sum_{k=1}^{s-1} n_k$. But the first formula follows from (2.2.2). The second one follows easily, if we notice that $v_\mu$ is a linear combination of the vectors $v^c_I$, where $c$ is the same one appeared in $v_\alpha = v^c_J$. Hence (2.4.1) holds.

For a row standard $\mu$-tableau $\mathfrak{t}$, we denote by $d(\mathfrak{t})$ the unique element in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\mu d(\mathfrak{t})$. Then the map $\mathfrak{t} \mapsto d(\mathfrak{t})$ gives a bijection from the set r-Std$(\mu)$ of row standard $\mu$-tableaux to the set $\mathcal{D}_\mu$ of distinguished coset representatives of $\mathfrak{S}_\mu \backslash \mathfrak{S}_n$.

Let $\mathcal{M}^\mu = v_\mu \mathcal{H}$ be the $\mathcal{H}$-submodule of $V^{\otimes n}$ generated by $v_\mu$. We have the following lemma.

**Lemma 2.5.** (i) *$\mathcal{M}^\mu$ is isomorphic to to $\operatorname{Ind}^{\mathcal{H}}_{\mathcal{H}_\mu} \theta^\mu_q$ (resp to $\operatorname{Ind}^{\mathcal{H}_n}_{\mathcal{H}_{\{\mu\}}} \theta_q$ ) as an $\mathcal{H}$-module (resp. an $\mathcal{H}_n$-module).*

(ii) *$\mathcal{M}^\mu$ is a free $R$-module with basis $\{v_\mu T_{d(\mathfrak{t})} \mid \mathfrak{t} \in \text{r-Std}(\mu)\}$.*

*Proof.* By using the commutation relation between $T_i$ and $\xi_j$, together with (2.4.1), we see that $\mathcal{M}^\mu = v_\mu \mathcal{H}_n$. It is easy to see that $v_\mu \mathcal{H}_n$ is spanned by $\{v_\mu T_d \mid d \in \mathcal{D}_\mu\}$. Now $v_\mu$ can be written as a linear combination of $v^c_I$ with the common $c$ as before. Then one can check that $v_\mu T_d$ are linearly independent, and so $\{v_\mu T_d\}$ gives a basis of $v_\mu \mathcal{H}_n = \mathcal{M}^\mu$. Thus (ii) holds. Also we see that $v_\mu \mathcal{H}_n \simeq \operatorname{Ind}^{\mathcal{H}_n}_{\mathcal{H}_{\{\mu\}}} \theta_q$. as an $\mathcal{H}_n$-module.

Since $\mathcal{H}_\mu$ is a free $R$-module with basis $\{\xi_1^{c_1} \cdots \xi_n^{c_n} T_\sigma \mid \sigma \in \mathfrak{S}_\mu\}$, $\mathcal{H}$ is a free $\mathcal{H}_\mu$-module with basis $\{T_d \mid d \in \mathcal{D}_\mu\}$. Now $\mathcal{M}^\mu$ is generated by a vector $v_\mu$, and $v_\mu T_d$ corresponds to a basis of $\theta^\mu_q \otimes T_d \subset \theta^\mu_q \otimes_{\mathcal{H}_\lambda} \mathcal{H}$ with the action of $\mathcal{H}$. Thus we obtain (i). □

**Lemma 2.6.** *Let $\mathfrak{t} \in$ r-Std$(\mu)$.*

(i) *If $\mathfrak{s} = \mathfrak{t} s_i$ for some $i$ $(2 \le i \le n)$, we have*

$$v_\mu T_{d(\mathfrak{t})} T_i = \begin{cases} q v_\mu T_{d(\mathfrak{t})} & \text{if } \mathfrak{s} \notin \text{r-Std}(\mu), \\ v_\mu T_{d(\mathfrak{t})} & \text{if } \mathfrak{s} \in \text{r-Std}(\mu) \text{ and } l(d(\mathfrak{s})) > l(d(\mathfrak{t})), \\ v_\mu T_{d(\mathfrak{s})} + (q - q^{-1}) v_\mu T_{d(\mathfrak{t})} & \text{if } \mathfrak{s} \in \text{r-Std}(\mu) \text{ and } l(d(\mathfrak{s})) < l(d(\mathfrak{t})). \end{cases}$$

(ii) *For* $\xi_j$ $(1 \le j \le n)$, *we have*

$$v_\mu T_{d(\mathfrak{t})}\xi_j = Q_{\varepsilon(j)} v_\mu T_{d(\mathfrak{t})},$$

*where* $\varepsilon(j) = \varepsilon(j;\mathfrak{t}) \in \mathbb{Z}_{>0}$ *is determined by the condition that*

$$v_\alpha T_{d(\mathfrak{t})} = v_{j_1}^{(\varepsilon(1))} \otimes v_{j_2}^{(\varepsilon(2))} \otimes \cdots \otimes v_{j_n}^{(\varepsilon(n))}.$$

*Proof.* We note that $d(\mathfrak{t}) \in \mathfrak{S}_n$ coincides with the corresponding element in $\mathfrak{S}_n$ obtained from $\mathfrak{t}$ and $\mathfrak{t}^{\{\mu\}}$, where $\mathfrak{t}$ is regarded as a row standard tableau of shape $\{\mu\}$. Now Lemma 2.5 shows that $\mathcal{M}^\mu \simeq m_{\{\mu\}}\mathcal{H}_n$ as $\mathcal{H}_n$-modules, via $v_\mu h \leftrightarrow m_{\{\mu\}}h$ for $h \in \mathcal{H}_n$. Hence (i) follows from the known formula for $m_{\{\mu\}}\mathcal{H}_n$ (see, e.g., [M1, Corollary 3.4]) with respect to the action of $T_i$ on $m_{\{\mu\}}T_{d(\mathfrak{t})}$.

Next we show (ii). First we note the following general fact.

(2.6.1) Let $v_I = v_{j_1} \otimes \cdots \otimes v_{j_n}$ with $j_1 > j_2 > \cdots > j_n$. Then for $w \in \mathfrak{S}_n$, $v_I T_w$ coincides with $v_{Iw}$.

In fact, for $w = w's_i$ with $l(w) > l(w')$, we have $w'(i-1) < w'(i)$ and $w(i-1) > w(i)$. Assume that $v_I T_{w'}$ is of the form $v_{j_{w'(1)}} \otimes \cdots \otimes v_{j_{w'(n)}}$. Then we have $j_{w'(i-1)} > j_{w'(i)}$, and by (1.4.1), $v_I T_w$ is obtained from $v_I T_{w'}$ by permuting $i-1$ and $i$ factors. This shows (2.6.1) by induction on $l(w)$.

Now $v_\mu T_{d(\mathfrak{t})} = \sum_{\sigma \in \mathfrak{S}_n} q^{l(\sigma)} v_\alpha T_{\sigma d(\mathfrak{t})}$. By applying (2.6.1) for $v_I = v_\alpha$, we see that $v_\alpha T_{wd(\mathfrak{t})} = v_{J'}^{c'}$, where $J' = Jwd(\mathfrak{t})$ and $c' = cwd(\mathfrak{t})$. Since $cw = c$ for $w \in \mathfrak{S}_\alpha$, we see that $v_\mu T_{d(\mathfrak{t})}$ is a linear combination of $v_I^{cd(\mathfrak{t})}$. Since $cd(\mathfrak{t}) = (\varepsilon(1), \dots, \varepsilon(n))$, the assertion (ii) follows. □

**2.7.** Let $M^\mu = m_\mu \mathcal{H}$. Then the map in (1.3.3) gives an isomorphism $M^\mu \simeq \mathcal{M}^\mu$ as $\mathcal{H}$-modules, which maps $m_\mu$ to $v_\mu$. Thus the statements in Lemma 2.5 and Lemma 2.6 hold without change for $M^\mu$ by replacing $v_\mu$ by $m_\mu$.

We now define an $R$-linear anti-automorphism $h \mapsto h^*$ on $\mathcal{H}$ by the condition that $*$ fixes the generators $T_i$ $(2 \le i \le n)$ and $\xi_j$ $(1 \le j \le n)$. We note that this condition induces a well-defined anti-automorphism on $\mathcal{H}$ since the map $*$ permutes (1.2.4) and (1.2.5) in the defining relations in 1.2, and leaves other formulas fixed.

Before going further, we remark the following property.

**Lemma 2.8.** *The algebra* $R[\xi_1, \dots, \xi_n]^{\mathfrak{S}_n}$ *of symmetric polynomials of* $\xi_1, \dots, \xi_n$ *is contained in the center of* $\mathcal{H}$.

*Proof.* It is enough to show that the elementary symmetric function $e_i(\xi_1, \dots, \xi_n)$ commutes with $T_j$ for $1 \le i \le n$ and $2 \le j \le n$. Since $T_j\xi_k = \xi_k T_j$ for $k \ne j-1, j$, we have only to verify the formulas

$$\begin{aligned}(\xi_{j-1} + \xi_j)T_j &= T_j(\xi_{j-1} + \xi_j),\\ (\xi_{j-1}\xi_j)T_j &= T_j(\xi_{j-1}\xi_j).\end{aligned}$$

Now the first one follows by adding (1.2.4) and (1.2.5) in the defining relation in 1.2. The second one follows by multiplying $\xi_{j-1}$ on (1.2.4) from the right, on (1.2.5) from the left, and then by adding these two formulas. □

By making use of Lemma 2.8, we show

**Lemma 2.9.** *Let $\mu$ be an $r$-composition of type $\alpha = (n_1, \dots, n_r)$. Then*

(i) *$F_\alpha$ is contained in the center of $\mathcal{H}_{n_1,r} \otimes \cdots \otimes \mathcal{H}_{n_r,r}$. In particular, we have $F_\alpha \in Z(\mathcal{H}_\mu)$.*
(ii) *We have $m^*_\mu = m_\mu$.*

*Proof.* By (2.2.3), $F_\alpha$ is a polynomial in $\xi_1, \dots, \xi_n$, and is written as

$$F_\alpha = \varDelta^{-n} F_r(\xi_1) \cdots F_r(\xi_{n_1}) F_{r-1}(\xi_{n_1+1}) \cdots F_{r-1}(\xi_{n_1+n_2}) F_{r-2}(\xi_{n_1+n_2+1}) \cdots$$

It follows that $F_\alpha \in R[\xi_1, \dots, \xi_n]^{\mathfrak{S}_\alpha}$ with $\mathfrak{S}_\alpha = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$. By applying Lemma 2.8 to $\mathcal{H}_{\alpha,r} = \mathcal{H}_{n_1,r} \otimes \cdots \otimes \mathcal{H}_{n_r,r}$, we see that $F_\alpha$ is contained in the center of $\mathcal{H}_{\alpha,r}$. Since $\mathcal{H}_\mu \subset \mathcal{H}_{\alpha,r}$, we have $F_\alpha \in Z(\mathcal{H}_\mu)$. Hence (i) holds.

Since $m_\mu = F_\alpha m_{\{\mu\}}$ by Lemma 2.3, we have $m^*_\mu = m^*_{\{\mu\}} F^*_\alpha$. The map $*$ induces an $R$-linear anti-automorphism on $\mathcal{H}_n$ which coincides with the standard map $*$ defined by $T_i \mapsto T_i$ for $i = 2, \dots, n$. Then it is easy to see that $m^*_{\{\mu\}} = m_{\{\mu\}}$. Clearly we have $F^*_\alpha = F_\alpha$. Since $m_{\{\mu\}} \in \mathcal{H}_\mu$ and $F_\alpha \in Z(\mathcal{H}_\mu)$ by (i), we have $m_{\{\mu\}} F_\alpha = F_\alpha m_{\{\mu\}} = m_\mu$. This shows (ii). □

**2.10.** For $\mathfrak{s}, \mathfrak{t} \in \text{r-Std}(\mu)$, we define an element $m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{H}$ by $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\mu T_{d(\mathfrak{t})}$. By Lemma 2.9, we have $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$. We want to show that such $m_{\mathfrak{s}\mathfrak{t}}$ for various $\mu \in \widetilde{\mathcal{P}}_{n,r}$ span the space $\mathcal{H}$. First, we note the following lemmas.

**Lemma 2.11.** *Let $F_\alpha = \varDelta^{-n} F_{c_1}(\xi_1) F_{c_2}(\xi_2) \cdots F_{c_n}(\xi_n)$ with $c = (c_1, \dots, c_n)$. Then for any vector $v^b_I \in V^{\otimes n}$, we have*

$$v^b_I F_\alpha = \begin{cases} v^c_I & \textit{if } b = c, \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* Let $b = (b_1, \dots, b_n)$. Since $v^b_I \xi_k = Q_{b_k} \cdot v^b_I$, we have

$$v^b_I F_\alpha = \varDelta^{-n} F_{c_1}(Q_{b_1}) F_{c_2}(Q_{b_2}) \cdots F_{c_n}(Q_{b_n}) v^b_I.$$

The lemma then follows from the formula $F_i(Q_j) = \varDelta \delta_{ij}$ (cf. [S, (3.3.3)]). □

**Lemma 2.12.** *For $d = (d_1, \dots, d_n) \in [1, r]^n$, let $w = w(d) \in \mathfrak{S}_n$ be the unique element such that $d_{w(1)} \ge d_{w(2)} \ge \cdots \ge d_{w(n)}$ and that $l(w)$ is minimal among those elements. Let $\alpha = \alpha(d) = (n_1, \dots, n_r)$ with $n_i = \sharp\{j \mid d_j = r - i + 1\}$. Then for $v^b_J$ with $J = (n, n-1, \dots, 1)$ we have*

$$v^b_J T_w F_\alpha T_{w^{-1}} = \begin{cases} v^d_J & \textit{if } b = d, \\ v^b_J h \ (h \in \mathcal{H}_n) & \textit{if } \alpha(d) = \alpha(b), l(w(d)) > l(w(b)), \\ 0 & \textit{otherwise.} \end{cases}$$

*Proof.* By (1.4.1), we see easily describe the effect of $T_w$ on $v_J^b$ as

$$v_J^b T_w = v_{Jw}^{bw} + \sum_{x<w} r_x v_{Jx}^{bx} \quad (r_x \in R), \tag{2.12.1}$$

where $x < w$ means the Bruhat order on $\mathfrak{S}_n$. First we consider the case where $b = d$, and we apply $F_\alpha$ on both sides of (2.12.1). Note that $c = (c_1, \dots, c_n)$ associated to $F_\alpha$ coincides with $dw$ in this case. Also note that $dx \neq dw$ for any $x < w$ by our assumption on $w$. It follows from Lemma 2.11 that

$$v_J^d T_w F_\alpha = v_{Jw}^{dw}. \tag{2.12.2}$$

Here we note that

(2.12.3) Write $v_{Jw}^{dw}$ as $v_{k_1} \otimes \cdots \otimes v_{k_n}$. Then we have $k_1 > k_2 > \cdots > k_n$.

In fact, let $w_0$ be the longest element of $\mathfrak{S}_n$. Then $v_{Jw}^{dw}$ can be written as

$$v_{Jw}^{dw} = v_{w_0w(1)}^{d_{w(1)}} \otimes \cdots \otimes v_{w_0w(n)}^{d_{w(n)}}.$$

Hence we have only to show that if $d_{w(i-1)} = d_{w(i)}$, then $w_0w(i-1) > w_0w(i)$, in other words, $w(i-1) < w(i)$. Suppose that this fails for some $i$, i.e., $w(i-1) > w(i)$. Then we have $l(ws_i) < l(w)$. Since $d_{ws_i(1)} \geq d_{ws_i(2)} \geq \cdots \geq d_{ws_i(n)}$, this contradicts the choice of $w$. So (2.12.3) holds.

Now by applying (2.6.1), we see that $v_{Jw}^{dw}T_{w^{-1}}$ coincides with $v_J^d$. It follows that $v_J^d T_w F_\alpha T_{w^{-1}} = v_J^d$ by (2.12.2).

Next we consider the case where $b \neq d$. If $bx$ is not equal to $dw$ for any $x \leq w$, then $v_J^b T_w F_\alpha = 0$ by (2.12.1). Assume that there exists $x < w$ such that $bx = dw$. Then $\alpha(d) = \alpha(b)$. Clearly we have $l(w(b)) \leq l(x) < l(w(d))$. By (2.12.1), $v_J^b T_w F_\alpha$ is written as a linear combination of $v_{Jx}^{bx}$ for $x < w$. Hence $v_J^b T_w F_\alpha$ is contained in $v_J^b \mathcal{H}_n$. This proves the second equality, and the lemma follows. □

By making use of Lemma 2.12, we show the following.

**Proposition 2.13.** *The set $\widetilde{\mathcal{C}} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \text{r-Std}(\mu), \mu \in \widetilde{\mathcal{P}}_{n,r}\}$ spans the $R$-module $\mathcal{H}$.*

*Proof.* Let $\mu \in \widetilde{\mathcal{P}}_{n,r}$. By Lemma 2.5, and its counter parts for $M^\mu = m_\mu\mathcal{H}$, any elements in $m_\mu\mathcal{H}$ can be written as a linear combination of $m_\mu T_{d(\mathfrak{t})}$ with $\mathfrak{t} \in \text{r-Std}(\mu)$. Similarly $\mathcal{H}m_\mu = (m_\mu\mathcal{H})^*$ is spanned by the elements $(m_\mu T_{d(\mathfrak{s})})^* = T^*_{d(\mathfrak{s})}m_\mu$ with $\mathfrak{s} \in \text{r-Std}(\mu)$. Thus $\mathcal{H}m_\mu\mathcal{H}$ is spanned by elements $T^*_{d(\mathfrak{s})}m_\mu T_{d(\mathfrak{t})} = m_{\mathfrak{s}\mathfrak{t}} \in \widetilde{\mathcal{C}}$. It follows that the $R$-span of $\widetilde{\mathcal{C}}$ coincides with the set $\bigcup_\mu \mathcal{H}m_\mu\mathcal{H}$. In particular it is a two sided ideal of $\mathcal{H}$. Hence the proposition is proved if one shows that $1 \in \mathcal{H}$ is expressed as a linear combination of elements in $\widetilde{\mathcal{C}}$.

Let $\mathcal{L}$ be the $R$-span of all the elements $m_{\mathfrak{s}\mathfrak{t}}$, $\mathfrak{s}, \mathfrak{t} \in \text{Std}(\lambda)$, where $\lambda$ runs over the $r$-partitions of the form $\lambda = (1^{n_1}, 2^{n_2}, \dots, r^{n_r})$ with $\alpha(\lambda) = (n_1, \dots, n_r)$. We show that

(2.13.1) The $R$-space $v_0\mathcal{L}$ contains $v_0$.

Note that the proposition follows from (2.13.1), since it implies that $1 \in \mathcal{L}$ by (1.4.5). We shall show (2.13.1) by applying Lemma 2.12. First note that if $\lambda \in \mathcal{P}_{n,r}$

is as above, then $m_{\{\lambda\}} = 1$ and $\mathcal{D}_\lambda = \mathfrak{S}_n$. It follows that any element of the form $T_{w'}F_\alpha T_w$ $(w, w' \in \mathfrak{S}_n)$ with $\alpha = \alpha(\lambda)$ is expressed as $m_{\mathfrak{s}\mathfrak{t}}$ for some $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$. In particular, $\mathcal{L}$ is a right $\mathcal{H}_n$-module.

Now $v_0$ is written as $v_0 = \sum_{d\in[1,r]^n} v_J^d$ with $J = (n, n-1, \dots, 1)$. We shall show that $v_J^d \in v_0\mathcal{L}$ for any $d \in [1,r]^n$. Take $d \in [1,r]^n$, and $\alpha = \alpha(d)$. Then by Lemma 2.12 we have

$$v_0 T_w F_\alpha T_{w^{-1}} = v_J^d + \sum_b v_J^b h_b$$

with $h_b \in \mathcal{H}_n$, where $b$ runs over the elements in $[1,r]^n$ under the condition that $\alpha(b) = \alpha(d)$ and that $l(w(b)) < l(w(d))$. By induction on $l(w(d))$, we may assume that $v_J^v \in v_0\mathcal{L}$. Since $v_0\mathcal{L}$ is a right $\mathcal{H}_n$-module, then $v_J^b h_b \in v_0\mathcal{L}$ also. Since $v_0 T_w F_\alpha T_{w^{-1}} = v_0 m_{\mathfrak{s}\mathfrak{t}} \in v_0\mathcal{L}$, we see that $v_J^d \in v_0\mathcal{L}$. Thus all the $v_J^d$ are contained in $v_0\mathcal{L}$, and so $v_0 \in v_0\mathcal{L}$. (2.13.1) is now proved, and the proposition follows. $\square$

**2.14.** We define a dominance order $\lambda \trianglerighteq \mu$ on $\widetilde{\mathcal{P}}_{n,r}$ by $\{\lambda\} \trianglerighteq \{\mu\}$, where the latter is the usual dominance order for 1-compositions of $n$, i.e., for $\{\lambda\} = (\lambda_1, \dots, \lambda_m)$, $\{\mu\} = (\mu_1, \dots, \mu_m) \in \widetilde{\mathcal{P}}_{n,1}$, we require that $\sum_{i=1}^k \lambda_i \ge \sum_{i=1}^k \mu_i$ for $k = 1, \dots, m$. Let $\mu$ be an $r$-composition. As in the case of 1-composition, for $\mathfrak{t} \in \text{r-Std}(\mu)$ and for a positive integer $k \le n$, we define $\mathfrak{t}\downarrow k$ as the row standard tableau obtained from $\mathfrak{t}$ by deleting all of the entries containing $k+1, \dots, n$. For row standard tableaux $\mathfrak{t}$ and $\mathfrak{s}$ (of possibly different shape), we write $\mathfrak{t} \trianglerighteq \mathfrak{s}$ if $\mathrm{Shape}(\mathfrak{t}\downarrow k) \trianglerighteq \mathrm{Shape}(\mathfrak{s}\downarrow k)$ for $k = 1, \dots, n$. Then it is clear that $\mathfrak{t}^\mu \trianglerighteq \mathfrak{t}$ for any $\mathfrak{t} \in \text{r-Std}(\mu)$. We have the following lemma.

**Lemma 2.15.** *Let $\mu$ be an $r$-composition of $n$ and $\mathfrak{s}, \mathfrak{t} \in \text{r-Std}(\mu)$. Then $m_{\mathfrak{s}\mathfrak{t}}$ can be written as an $R$-linear combination of $m_{\mathfrak{u}\mathfrak{v}}$, where $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$ with $\mu \in \mathcal{P}_{n,r}$ such that $\mathfrak{u} \trianglerighteq \mathfrak{s}, \mathfrak{v} \trianglerighteq \mathfrak{t}$ (hence $\lambda \trianglerighteq \mu$).*

*Proof.* This lemma can be proved in a completely similar way as [DJM, Proposition 3.18]. However, this fact (the similarity to [DJM]) itself is important for later discussions, we give here an outline of the proof.

Let $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ be of type $\alpha = (n_1, \dots, n_r)$. We write $\mathfrak{s} = (\mathfrak{s}^{(1)}, \dots, \mathfrak{s}^{(r)})$, where $\mathfrak{s}^{(i)} \in \text{r-Std}(\mu^{(i)})$, and similarly for $\mathfrak{t} = (\mathfrak{t}^{(1)}, \dots, \mathfrak{t}^{(r)})$. Then there exist $w_1, w_2 \in \mathfrak{S}_n$, and $\mathfrak{p} = (\mathfrak{p}^{(1)}, \dots, \mathfrak{p}^{(r)}), \mathfrak{q} = (\mathfrak{q}^{(1)}, \dots, \mathfrak{q}^{(r)}) \in \text{r-Std}(\mu)$ satisfying the following; $\mathfrak{s} = \mathfrak{p}w_1, \mathfrak{t} = \mathfrak{q}w_2$, the entries of $\mathfrak{p}^{(i)}, \mathfrak{q}^{(i)}$ consist of numbers $a_i + 1, \dots, a_{i+1}$, where $\mathbf{a}(\mu) = (a_1, \dots, a_r)$. Those $w_1, w_2, \mathfrak{p}, \mathfrak{q}$ are unique under the condition that $w_1, w_2 \in \mathcal{D}_\alpha$. We define $d(\mathfrak{p}^{(i)}), d(\mathfrak{q}^{(q)}) \in \mathfrak{S}_{n_i}$ by

$$\mathfrak{t}^{\mu^{(i)}} d(\mathfrak{p}^{(i)}) = \mathfrak{p}^{(i)}, \qquad \mathfrak{t}^{\mu^{(i)}} d(\mathfrak{q}^{(i)}) = \mathfrak{q}^{(i)} \qquad (1 \le i \le r),$$

where $\mathfrak{t}^\mu = (\mathfrak{t}^{\mu^{(1)}}, \dots, \mathfrak{t}^{\mu^{(r)}})$. Then we have $d(\mathfrak{s}) = d(\mathfrak{p})w_1, d(\mathfrak{t}) = d(\mathfrak{q})w_2$ with

$$d(\mathfrak{p}) = d(\mathfrak{p}^{(1)}) \cdots d(\mathfrak{p}^{(r)}), \qquad d(\mathfrak{q}) = d(\mathfrak{q}^{(1)}) \cdots d(\mathfrak{q}^{(r)}).$$

Since $m_{\{\mu\}}$ can be written as $m_{\{\mu\}} = m_{\mu^{(1)}} \cdots m_{\mu^{(r)}}$, and $F_\alpha$ commutes with $d(\mathfrak{p}^{(i)}), d(\mathfrak{q}^{(i)})$ for any $i$, we can write

$$m_{\mathfrak{s}\mathfrak{t}} = T_{w_1}^*(x_1 x_2 \cdots x_r) F_\alpha T_{w_2}$$

with $x_i = T^*_{d(\mathfrak{p}^{(i)})} m_{\mu^{(i)}} T_{d(\mathfrak{q}^{(i)})}$.

Now by applying Murphy's result [Mu, 4.18] on $\mathcal{H}_{n_i}$ for $x_i$, we see that $x_i$ can be written as a linear combination of $y_i = T^*_{d(\mathfrak{u}^{(i)})} m_{\mu^{(i)}} T_{d(\mathfrak{v}^{(i)})}$, where $\mu^{(i)}$ is a partition of $n_i$ and $\mathfrak{u}^{(i)}, \mathfrak{v}^{(i)} \in \mathrm{Std}(\mu^{(i)})$ with $\mathfrak{u}^{(i)} \unrhd \mathfrak{p}^{(i)}, \mathfrak{v}^{(i)} \unrhd \mathfrak{q}^{(i)}$. It follows that $m_{\mathfrak{s}\mathfrak{t}}$ can be written as a linear combination of

$$T^*_{w_1}(y_1 y_2 \cdots y_r) F_\alpha T_{w_2}. \tag{2.15.1}$$

Set $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mathfrak{u} = (\mathfrak{u}^{(1)}, \dots, \mathfrak{u}^{(r)}), \mathfrak{v} = (\mathfrak{v}^{(1)}, \dots, \mathfrak{v}^{(r)})$. Then we see that $\lambda \in \mathcal{P}_{n,r}$ with type $\alpha$, and $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)$ with $\mathfrak{u} \unrhd \mathfrak{p}, \mathfrak{v} \unrhd \mathfrak{q}$. The formula in (2.15.1) coincides with

$$T^*_{w_1} T^*_{d(\mathfrak{u})} m_{\{\lambda\}} T_{d(\mathfrak{v})} F_\alpha T_{w_2} = T^*_{w_1} T^*_{d(\mathfrak{u})} m_\lambda T_{d(\mathfrak{v})} T_{w_2}.$$

If we put $\mathfrak{u}' = \mathfrak{u} w_1, \mathfrak{v}' = \mathfrak{v} w_2$, we have $\mathfrak{u}', \mathfrak{v}' \in \mathrm{Std}(\lambda)$. Thus the lemma follows. □

Summing up the above arguments, we have

**Proposition 2.16.** *Let $\mathcal{C} = \{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathcal{P}_{n,r}\}$. Then the set $\mathcal{C}$ forms a free basis of the $R$-module $\mathcal{H}$.*

*Proof.* By Proposition 2.13, together with Lemma 2.15, we see that the set $\mathcal{C}$ spans the $R$-module $\mathcal{H}$. The linearly independence follows from the fact that the cardinality of $\mathcal{C}$ coincides with $|W_{n,r}| = n! r^n$, which is equal to the rank of $\mathcal{H}$. The proposition follows. □

For $\mu \in \mathcal{P}_{n,r}$, let $\check{N}^\mu$ be the $R$-submodule of $\mathcal{H}$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$ for various $\mu \in \mathcal{P}_{n,r}$ such that $\mu \rhd \mu$. We have the following lemma.

**Lemma 2.17.** *Let $\mu \in \mathcal{P}_{n,r}$ and $\mathfrak{t} \in \mathrm{Std}(\mu)$. Then for $w \in \mathfrak{S}_n$, there exist $r_\mathfrak{v} \in R$ such that*

$$m_{\mathfrak{t}^\mu \mathfrak{t}} T_w \equiv \sum_{\mathfrak{v} \in \mathrm{Std}(\mu)} r_\mathfrak{v} m_{\mathfrak{t}^\mu \mathfrak{v}} \mod \check{N}^\mu. \tag{2.17.1}$$

*Proof.* By Lemma 2.6, $m_{\mathfrak{t}^\mu \mathfrak{t}} T_w$ can be written as a linear combination of $m_{\mathfrak{t}^\mu \mathfrak{q}}$ with $\mathfrak{q} \in \text{r-Std}(\mu)$. Then by Lemma 2.15, $m_{\mathfrak{t}^\mu \mathfrak{t}} T_w$ can be written as a linear combination of $m_{\mathfrak{u}\mathfrak{v}}$ for $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\mu)$ for various $r$-partitions $\mu$ such that $\mathfrak{u} \unrhd \mathfrak{t}^\mu, \mathfrak{v} \unrhd \mathfrak{t}$. We note that $\mu \unrhd \mu$ since $\mathfrak{u} \unrhd \mathfrak{t}^\mu$. To prove the lemma, it is enough to show that $\mu \rhd \mu$ if $\mathfrak{u} \neq \mathfrak{t}^\mu$. Now assume that $\mathfrak{u} \neq \mathfrak{t}^\mu$. Then $\mathfrak{u} \rhd \mathfrak{t}^\mu$. If $\mu = \mu$, we have $\mathfrak{t}^\mu \unrhd \mathfrak{u}$ by the remark in 2.14. This is a contradiction, and the lemma follows. □

**Theorem 2.18.** *Let $\mathcal{C}$ be the $R$-free basis of $\mathcal{H}$ given above. Then $\mathcal{C}$ gives rise to a cellular basis of $\mathcal{H}$ in the sense of* [GL]*, i.e., the following properties hold.*

(i) *The $R$-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \mapsto m_{\mathfrak{t}\mathfrak{s}}$ ($m_{\mathfrak{s}\mathfrak{t}} \in \mathcal{C}$) is an anti-linear automorphism on $\mathcal{H}$.*

(ii) *For a given $h \in \mathcal{H}$ and $\mathfrak{t} \in \mathrm{Std}(\mu)$, there exist $r_\mathfrak{v} \in R$ such that for all $\mathfrak{s} \in \mathrm{Std}(\mu)$, we have*

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \mathrm{Std}(\mu)} r_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \mod \check{N}^\mu. \tag{2.18.1}$$

*Proof.* (i) is clear since $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$ and $*$ is an anti-automorphism of the $R$-algebra $\mathcal{H}$. We show (ii). First we show the formula (2.18.1) in the case where $h = T_w \in \mathcal{H}_n$ by induction on the dominance order of $\mathcal{P}_{n,r}$. Assume that $\mu = (n; -; \dots; -)$. Then $\mathfrak{S}_\mu = \mathfrak{S}_n$ and $\mathfrak{t}^\mu$ is the unique standard tableau. Hence $m_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}^\mu\mathfrak{t}^\mu} = m_\mu$, and $m_\mu T_w = q^{l(w)} m_\mu$. So (2.18.1) holds. Now assume that (2.18.1) holds for $h = T_w$ for any $\lambda$ such that $\lambda \rhd \mu$. By noticing that $m_{\mathfrak{s}\mathfrak{v}} = T^*_{d(\mathfrak{s})} m_{\mathfrak{t}^\mu \mathfrak{v}}$, we multiply $T^*_{d(\mathfrak{s})}$ from the left on both sides of (2.17.1). If one shows that $T^*_{d(\mathfrak{s})} \check{N}^\mu \subset \check{N}^\mu$, we obtain (2.18.1) for $h = T_w$. Take $\lambda$ such that $\lambda \rhd \mu$, and define $N^\lambda$ as the $R$-submodule of $\mathcal{H}$ spanned by $m_{\mathfrak{u}\mathfrak{v}}$ for $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\nu)$ such that $\nu \trianglerighteq \lambda$. Note that, for $\mathfrak{s} \in \mathrm{Std}(\mu)$ and for $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}(\lambda)$, we have

$$T^*_{d(\mathfrak{s})} m_{\mathfrak{u}\mathfrak{v}} = (m_{\mathfrak{u}\mathfrak{v}} T_{d(\mathfrak{s})})^* \in N^\lambda$$

since induction works for $m_{\mathfrak{u}\mathfrak{v}} T_{d(\mathfrak{s})}$, and $N^\mu$ is stable by $*$. Clearly we have $N^\lambda \subseteq \check{N}^\mu$, and so $T^*_{d(\mathfrak{s})} m_{\mathfrak{u}\mathfrak{v}} \in \check{N}^\mu$. This proves that $T^*_{d(\mathfrak{s})} \check{N}^\mu \subseteq \check{N}^\mu$, and so (2.18.1) holds also for $\mu$.

Now by Lemma 2.6 (ii), $\xi_j$ acts on $m_{\mathfrak{s}\mathfrak{t}}$ as a scalar multiplication by $Q_{\varepsilon(j)}$. Since any element of $\mathcal{H}$ is written as a linear combination of $\xi_1^{c_1} \cdots \xi_n^{c_n} T_w$, (2.18.1) holds for any $h \in \mathcal{H}$. This proves (ii). □

**2.19.** For $\lambda \in \mathcal{P}_{n,r}$, $\check{N}^\lambda$ is a two sided ideal of $\mathcal{H}$ by (2.18.1). Let $\overline{m}_\lambda$ be the image of $m_\lambda$ under the algebra homomorphism $\mathcal{H} \to \mathcal{H}/\check{N}^\lambda$. We denote by $S^\lambda$ the right $\mathcal{H}$-submodule of $\mathcal{H}/\check{N}^\lambda$ generated by $\overline{m}_\lambda$, which is called the Specht module associated to $\lambda$. By Theorem 2.18, $S^\lambda$ is a free $R$-module with basis $\{\overline{m}_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$. By a general theory of cellular basis [GL], one can define an $\mathcal{H}$-module $D^\lambda$ of $S^\lambda$ for each $\lambda \in \mathcal{P}_{n,r}$ as the quotient of $S^\lambda$ by the radical of certain bilinear form on $S^\lambda$. If $R$ is a filed, the set $\{D^\lambda \neq 0 \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a complete set of non-isomorphic irreducible $\mathcal{H}$-modules.

## 3. Hu's action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$

**3.1.** In [H], J. Hu constructed a representation of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$, independently from [$SS$]. He claims in [H, Remark 3.5] that his $\widetilde{\mathcal{H}}$-module structure on $V^{\otimes n}$ is different from ours. In fact, he gave a counter example in the case where $n = r = 2, m_1 = m_2 = 1$, by comparing the result of [ATY]. In turn, our representation coincides with the one in [ATY] in the special case where $m_1 = m_2 = \cdots = m_r = 1$. Nevertheless, in this section we shall show that Hu's representation essentially coincides with ours, i.e., we show that a modified Hu's representation actually coincides with ours. (The same result was proved by F. Stoll independently, and is announced in [HS] without proof.)

In this section, we consider the general setting as in the beginning of 1.4, without any restriction on $\mathbf{m}$, and only assuming that $q, Q_1, \dots, Q_r$ are in $R$, with $q$ invertible. In order to explain Hu's construction, we review some results in [DJM] concerning a cellular basis of $\widetilde{\mathcal{H}}$. We define $L_i = T_i T_{i-1} \cdots T_2 T_1 T_2 \cdots T_{i-1} T_i$ for $i = 1, 2, \dots, n$. For an $r$-tuple $\mathbf{a} = (a_1, \dots a_r)$ of integers such that $0 = a_1 \le a_2 \le \cdots \le a_r \le n$, let

$u^+_{\mathbf{a}} = u_{\mathbf{a},1}u_{\mathbf{a},2}\cdots u_{\mathbf{a},r}$, where

$$u_{\mathbf{a},k} = \prod_{i=1}^{a_k}(L_i - Q_k)$$

for $1 \le k \le r$. For $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we define an element $m^\natural_\mu \in \mathcal{H}$ by

$$m^\natural_\mu = u^+_{\mathbf{a}} m_{\{\mu\}},$$

where $m_{\{\mu\}}$ is as in 2.2, and $\mathbf{a} = \mathbf{a}(\mu)$ is as in 2.1.

Now, for $\mathfrak{s}, \mathfrak{t} \in$ r-Std$(\mu)$ with $\mu \in \widetilde{\mathcal{P}}_{n,r}$, put

$$m^\natural_{\mathfrak{st}} = T^*_{d(\mathfrak{s})} m^\natural_\mu T_{d(\mathfrak{t})}. \tag{3.1.1}$$

Then it is known by [DJM, 3.26] that

(3.1.2) The set $\mathcal{C}^\natural = \{m^\natural_{\mathfrak{st}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda), \lambda \in \mathcal{P}_{n,r}\}$ forms a cellular basis of $\widetilde{\mathcal{H}}$.

**Remark 3.2.** There are some discrepancies in the notation in [DJM], [H] and ours. The algebra $\widetilde{\mathcal{H}}$ in [DJM] is obtained from ours by putting $T_0 = T_1$ and $T_i = qT_{i+1}$ $(1 \le i \le n-1)$ and by replacing $q$ by $q^{1/2}$. While the algebra in [H] is obtained from ours by putting $T_i = T_{i+1}$ for $(0 \le i \le n-1)$ and by replacing $q$ by $q^{-1}$. Thus, $L_i$ in 3.1 corresponds to $L_i$, and our $m_{\{\mu\}}$ (resp. $m^\natural_\mu$) corresponds to $x_\mu$ (resp. $m_\mu$) in [DJM] under the above identification. Moreover, $m_{\{\mu\}}$ corresponds to $y_\mu$ in [H]. Thus our $m^\natural_\mu$ also corresponds to $m_{\underline{\mu}}$ in [H].

Also we note that the element $m^\natural_{\mathfrak{st}}$ in (3.1.1) corresponds to $q^{-l(d(\mathfrak{s}))-l(d(\mathfrak{t}))}m_{\mathfrak{st}}$ (and replacing $q$ by $q^{1/2}$) in [DJM].

**3.3.** We define a bijection $\phi : \widetilde{\mathcal{P}}_{n,r} \to \widetilde{\mathcal{P}}_{n,r}$ by $\phi(\mu) = \nu$ for $\mu \in \widetilde{\mathcal{P}}_{n,r}$, where $\nu = (\nu^{(1)}, \dots, \nu^{(r)})$ is given by

$$\nu^{(k)} = (\mu^{(r-k+1)}_{m_k}, \mu^{(r-k+1)}_{m_{k-1}}, \dots, \mu^{(r-k+1)}_1) \in \mathbb{Z}^{m_k}_{\ge 0}$$

if $\mu^{(k)} = (\mu^{(k)}_1, \mu^{(k)}_2, \dots, \mu^{(k)}_{m_k}) \in \mathbb{Z}^{m_k}_{\ge 0}$. Hence $\nu$ is of type $(n_r, \dots, n_1)$. We often write $\mu' = \phi(\mu)$. Don't confuse $\mu'$ with the dual partition of $\mu$.

We consider the action of $U_q$ on $V^{\otimes n}$. The weights of $U_q$ are given by $r$-compositions of $n$. For $\mu \in \widetilde{\mathcal{P}}_{n,r}$, we denote by $V^{\otimes}_\mu$ the weight subspace of $V^{\otimes n}$ corresponding to the weight $\mu$. Then $V^{\otimes n}_\mu$ has a basis vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$ such that $\sharp\{a \mid v_{i_a} = v^{(k)}_j\} = \mu^{(k)}_j$. In particular, we have

$$V^{\otimes n} = \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} V^{\otimes n}_\mu \tag{3.3.1}$$

Clearly the weight space $V^{\otimes n}_\mu$ is stable by $\mathcal{H}_n$. Put

$$v^\natural_\mu = \underbrace{v^{(r)}_{m_r} \otimes \cdots \otimes v^{(r)}_{m_r}}_{\mu^{(r)}_{m_r} \text{ factors}} \otimes \underbrace{v^{(r)}_{m_r-1} \otimes \cdots \otimes v^{(r)}_{m_r-1}}_{\mu^{(r)}_{m_r-1} \text{ factors}} \otimes \cdots \otimes \underbrace{v^{(1)}_1 \otimes \cdots \otimes v^{(1)}_1}_{\mu^{(1)}_1 \text{ factors}}. \tag{3.3.2}$$

Then it is easy to see that $V_\mu^{\otimes n}$ is generated by $v_\mu^\natural$ as a right $\mathcal{H}_n$-module, and is isomorphic to $\mathrm{Ind}_{\mathcal{H}_{\{\mu'\}}}^{\mathcal{H}_n} \theta_q$ with $\mu' = \phi(\mu)$. Hence we have

$$V_\mu^{\otimes n} = v_\mu^\natural \mathcal{H}_n \simeq v_{\mu'}\mathcal{H}_n \simeq m_{\{\mu'\}}\mathcal{H}_n \tag{3.3.3}$$

as $\mathcal{H}_n$-modules (cf. Lemma 2.5). The isomorphism $m_{\{\mu'\}}\mathcal{H}_n \to V_\mu^{\otimes n}$ is given by $m_{\{\mu'\}}h \mapsto v_\mu^\natural h, (h \in \mathcal{H}_n)$.

We define a partial order on the set $\mathbb{Z}_{\geq 0}^r$ by $\mathbf{a} \geq \mathbf{a}'$ for $\mathbf{a} = (a_1, \dots, a_r), \mathbf{a}' = (a_1', \dots, a_r') \in \mathbb{Z}_{\geq 0}^r$ if $a_i \geq a_i'$ for any $i$. We write $\mathbf{a} > \mathbf{a}'$ if $\mathbf{a} \geq \mathbf{a}'$ and $\mathbf{a} \neq \mathbf{a}'$. It is clear that

(3.3.4) If $\lambda \trianglerighteq \mu$, then $\mathbf{a}(\lambda) \geq \mathbf{a}(\mu)$ for $\lambda, \mu \in \widetilde{\mathcal{P}}_{n,r}$.

For an $r$-composition $\mu$ of $n$, we define a right ideal of $\widetilde{\mathcal{H}}$ by $M_\natural^\mu = m_\mu^\natural \widetilde{\mathcal{H}}$. Then we define $\widehat{N}^{\mathbf{a}(\mu)}$ as an $R$-submodule of $\widetilde{\mathcal{H}}$ generated by $m_{\mathfrak{s}\mathfrak{t}}^\natural$ such that $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\nu)$ with $\mathbf{a}(\nu) > \mathbf{a}(\mu)$. By Lemma 3.2 in [H], $\widehat{N}^{\mathbf{a}(\mu)}$ is a two sided ideal of $\widetilde{\mathcal{H}}$, and so if we put $\widehat{M}_\natural^\mu = M_\natural^\mu \cap \widehat{N}^{\mathbf{a}(\mu)}$, $M_\natural^\mu / \widehat{M}_\natural^\mu$ has a natural structure of a right $\widetilde{\mathcal{H}}$-module.

Hu showed the following.

**Theorem 3.4** ([H, Theorem 3.3])**.** *Let* $\mathbf{a} = \mathbf{a}(\mu)$. *The $R$-linear map*

$$m_{\{\mu\}}\mathcal{H}_n \to M_\natural^\mu / \widehat{M}_\natural^\mu, \qquad m_{\{\mu\}}h \mapsto u_\mathbf{a}^+ m_{\{\mu\}}h + \widehat{M}_\natural^\mu$$

*gives an isomorphism of right $\mathcal{H}_n$-modules.* (*Note that* $u_\mathbf{a}^+ m_{\{\mu\}} = m_\mu^\natural$).

**3.5.** By using the isomorphism in the above theorem, together with (3.3.3), one can define an action of $\widetilde{\mathcal{H}}$ on $V_\mu^{\otimes n} \simeq M_\natural^{\mu'} / \widehat{M}_\natural^{\mu'}$. Thus by (3.3.1), $V^{\otimes n}$ is endowed with an $\mathcal{H}$-module structure. He showed that the thus obtained action of $\widetilde{\mathcal{H}}$ commutes with the action of $U_q$ and that the Schur-Weyl reciprocity holds for $U_q$-$\widetilde{\mathcal{H}}$-bimodule $V^{\otimes n}$ ([H, Theorem 4.3]). We call this action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$ as Hu's action (however see Remark 3.7 below).

We now show the following theorem.

**Theorem 3.6.** *Hu's action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$ coincides with the action given in 1.4.*

*Proof.* In the following proof, we understand that the action of $\widetilde{\mathcal{H}}$ always means our action, while Hu's action is denoted as Hu's action. It is clear from the definition (1.4.4) that $V_\mu^{\otimes n}$ is stable by $\widetilde{\mathcal{H}}$ for any $\mu \in \widetilde{\mathcal{P}}_{n,r}$. Hence we have only to show the following statement.

(3.6.1) For each $\mu \in \widetilde{\mathcal{P}}_{n,r}$, Hu's action of $T_1$ on $V_\mu^{\otimes n}$ coincides with our action.

We show (3.6.1). Put $\mu' = \phi(\mu)$ as before. Let $\alpha = (n_1, \dots, n_r)$ be the type of $\mu$ and $\alpha' = (n_r, \dots, n_1)$, the type of $\mu'$. We write $\mathbf{a} = \mathbf{a}(\mu') = (a_1, \dots, a_r)$. First we describe the Hu's action of $T_1$. Now $V_\mu^{\otimes n}$ has a basis $\{v_\mu^\natural T_d \mid d \in \mathcal{D}_{\mu'}\}$. We fix $d$ as above. Then $d$ can be written as $d = yc$, where $y$ (resp, $c$) is a distinguished right coset representative of $\mathfrak{S}_{\mu'}$ in $\mathfrak{S}_{\alpha'}$ (resp. of $\mathfrak{S}_{\alpha'}$ in $\mathfrak{S}_n$). Assume that the letter 1 is in the $k$-th row of the row standard tableau $t^{\alpha'}c$. Then $c$ can be further decomposed as $c = wc'$, where $w = s_{b_k+1}s_{b_k} \cdots s_3 s_2$ and $c' \in \mathfrak{S}_{n-1}$ stabilizing 1, such that $l(c) = l(w) + l(c')$.

By using the arguments in the proof of [H, Theorem 3.3], (see also [DJM, 3.4]), we see that Hu's action of $T_1$ is given as

$$v_\mu^\natural T_d T_1 = Q_k v_\mu^\natural T_y T_{b_k+1}^{-1} \cdots T_2^{-1} T_{c'}. \tag{3.6.2}$$

Next we compute our action of $T_1$ on the vector $v_\mu^\natural T_d$ following the definition in (1.4.4). Since $T_{c'}$ is a product of $T_3, \dots, T_n$, we see that $T_1$ commutes with $T_{c'}$. Hence

$$v_\mu^\natural T_d T_1 = v_\mu^\natural T_y T_w T_1 T_{c'}. \tag{3.6.3}$$

We can express $v_\mu^\natural = v_I^p$ with

$$p = (p_1, \dots, p_n) = (\underbrace{r, \dots, r}_{n_r \text{times}}, \underbrace{r-1, \dots, r-1}_{n_{r-1} \text{times}}, \dots, \underbrace{1, \dots, 1}_{n_1 \text{times}})$$

for some $I$. Since $y \in \mathfrak{S}_{\alpha'}$, $v_\lambda^\natural T_y$ is a linear combination of basis vectors of the form $v_J^p$ ($p$ is unchanged) with various $J$. We simply write one of them as $v_J$ and compute $v_J T_w$. It is easy to see that $v_J T_w = v_J w = v'_J$, where $v'_J$ is obtained from $v_J$ by permuting the factors according to the cyclic permutation $1 \to 2 \to \cdots \to b_k + 1 \to 1$. We apply $T_1$ on $v'_J$. Since the first factor of $v'_J$ is in $V_k$, we have $v'_J \xi_1 = Q_k v'_J$. Then we apply $S_2 \cdots S_n$ on it. The actions of $S_2, \dots S_{b_k+1}$ are just given by permutations $s_2, \dots, s_{b_k+1}$ since in each step, the corresponding factors are in different $V_j$. Hence we see that

$$v_J T_w \xi_1 S_2 \cdots S_{b_k+1} = Q_k v_J. \tag{3.6.4}$$

In the next step, the successive actions of $S_j$ for $j = b_k + 2, \dots, b_{k+1}$ are given by $T_j$ since the corresponding factors are in the same subspace $V_k$. Here note that $Q_k v_J T_{b_k+2} \cdots T_{b_{k+1}}$ is expressed as a linear combination of basis vectors such that its $b_{k+1} - 1$ factor belongs to $V_k$, and the factors in the latter part from there belong to $V_l$ with $l < k$. This implies that the subsequent actions of $S_j$ for $j = b_{k+1}, \dots, n$ on these vectors coincide with $s_j$, and which also coincide with $T_j$. It follows from this, together with (3.6.4) that

$$v_J T_w \xi_1 S_2 \cdots S_n = Q_k v_J T_{b_k+2} \cdots T_n$$

and we have

$$v_J T_w T_1 = v_J T_w \xi_1 S_2 \cdots S_n T_n^{-1} \cdots T_2^{-1} = Q_k v_J T_{b_k+1}^{-1} \cdots T_2^{-1}.$$

This shows that our action of $T_1$ on $v_\lambda^\natural T_d$ is also given by the right hand side of (3.6.2). Hence (3.6.1) holds, and the theorem is proved. □

**Remark 3.7.** The action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$ given in 3.5 is not exactly the same as Hu's original one. He uses the right action of $\widetilde{\mathcal{H}}$ on $V^{\otimes n}$, induced from the left action by changing the generators $T_i$ by $T_i^{-1}$, i.e., changing the parameters $q, Q_i$ by $q^{-1}, Q_i^{-1}$. He also uses a different type of cellular basis of $\widetilde{\mathcal{H}}$, a modified one from the basis of

[DJM], which does not coincide with ours even if the parameters are changed to their inverses.

We now return to the setting in Section 2. As a corollary to Theorem 3.6, combined with Theorem 3.4, we have the following result.

**Corollary 3.8.** *Assume that (1.2.1) holds for $\mathcal{H}$. We regard $M^\mu$ as an $\widetilde{\mathcal{H}}$ module via the map $\rho_0 : \widetilde{\mathcal{H}} \to \mathcal{H}$. Then for each $\mu \in \widetilde{\mathcal{P}}_{n,r}$, there exists an exact sequence of $\widetilde{\mathcal{H}}$-modules*

$$0 \longrightarrow \widehat{M}^\mu_\natural \longrightarrow M^\mu_\natural \xrightarrow{\ f\ } M^\mu \longrightarrow 0,$$

*where the map $f$ is given by $m^\natural_\mu h \mapsto m_\mu h$ for $h \in \widetilde{\mathcal{H}}$.*

*Proof.* It is easy to see that $V_{\mu'}^{\otimes n} = v^\natural_{\mu'}\mathcal{H} \simeq \operatorname{Ind}_{\mathcal{H}_\mu}^{\mathcal{H}} \theta^\mu_q$ as $\mathcal{H}$-modules. Hence by Lemma 2.5, $M^\mu$ is isomorphic to $V_{\mu'}^{\otimes n}$, via $m_\mu h \to v^\natural_{\mu'} h$ ($h \in \mathcal{H}$), and is isomorphic to $M^\mu_\natural/\widehat{M}^\mu_\natural$ as $\widetilde{\mathcal{H}}$-modules by Theorem 3.6. This proves the corollary. □

## 4. Specht filtrations for $M^\mu_\natural$ and $M^\mu$

**4.1.** In [DJM], they constructed a semistandard basis for $M^\mu_\natural$ by using their cellular basis, and defined a filtration whose successive quotients are Specht modules. In this section, we construct a similar basis of $M^\mu$ and its filtration by using our cellular basis, and compare those two filtrations.

First we recall some notation and results on semistandard bases in [DJM, 4]. For an $r$-composition $\lambda$ of $n$, a $\lambda$-Tableau $S = (S^{(1)}, \dots, S^{(r)})$ is a map $S : [\lambda] \to \{1, \dots, n\} \times \{1, \dots, r\}$. As in the case of usual $\lambda$-tableau, we regard $S$ as the diagram $[\lambda]$, together with an ordered pair $(i, k)\,(1 \le i \le n, 1 \le k \le r)$ attached to each node. Suppose that $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \widetilde{\mathcal{P}}_{n,r}$. A $\lambda$-Tableau $S$ is said to be of type $\mu$ if the number of $(i, k)$ in the entry of $S$ is equal to $\mu^{(k)}_i$. Assume that $\mathfrak{s}$ is a standard $\lambda$-tableau. We define a $\lambda$-Tableau $\mu(\mathfrak{s})$ by replacing each entry $m$ in $\mathfrak{s}$ by $(i, k)$ if $m$ is in the $i$-th row of the $k$-th component of $\mathfrak{t}^\mu$. Clearly $\mu(\mathfrak{s})$ is a $\lambda$-Tableau of type $\mu$.

We define a total order on the set of pairs $(i, k)$ as above, by $(i_1, k_1) < (i_2, k_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $i_1 < i_2$. Suppose that $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \widetilde{\mathcal{P}}_{n,r}$. Let $S = (S^{(1)}, \dots, S^{(r)})$ be a $\lambda$-Tableau of type $\mu$. $S$ is said to be semistandard if each component $S^{(k)}$ is semistandard with respect to the entries $\{(i, l)\}$ (i.e., non-decreasing in rows, strictly increasing in columns), and each entry of $S^{(k)}$ consist of $(i, l)$ with $l \ge k$. We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard $\lambda$-Tableaux of type $\mu$.

Assume that $S \in \mathcal{T}_0(\lambda, \mu)$. Then there exists a unique $\mathfrak{s}_1 = \operatorname{first}(S) \in \operatorname{Std}(\lambda)$ satisfying the property that $\mu(\mathfrak{s}_1) = S$ and that $\mathfrak{s}_1 \trianglerighteq \mathfrak{s}$ for any $\mathfrak{s} \in \operatorname{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$.

By [DJM, 4.5], the following property of semistandard Tableaux is known. Recall that the order of $r$-tuples $\mathbf{a} \in \mathbb{Z}^r_{\ge 0}$ is given as in 3.3.

(4.1.1) Assume that $\mathcal{T}_0(\lambda, \mu)$ is non-empty. Then we have $\mathbf{a}(\lambda) \ge \mathbf{a}(\mu)$.

**4.2.** In the special case where $r = 1$, the above definition coincides with the usual definition of the semistandard $\lambda$-Tableaux of type $\mu$ for $\lambda \in \mathcal{P}_{n,1}, \mu \in \widetilde{\mathcal{P}}_{n,1}$. Following [M1, Chap.4], we recall the basic properties of semistandard tableaux. Let $\lambda, \mu$ be

1-compositions of $n$. A Tableau $S$ of type $\mu$ is said be row semistandard if the entries in each row of $S$ are non-decreasing. Moreover, in the case where $\lambda$ is a 1-partition, $S$ is said to be semistandard if $S$ is row semistandard and the entries in each column are strictly increasing. Assume that $\lambda, \mu \in \widetilde{\mathcal{P}}_{n,1}$ and put $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$. Then $\mathcal{D}_{\lambda\mu}$ is the set of minimal length elements in the double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu$. The set of row semistandard $\lambda$-tableaux of type $\mu$ is in one to one correspondence with the set $\mathcal{D}_{\lambda\mu}$, by associating $\lambda$-tableau $\mu(\mathfrak{t}^\lambda d)$ to $d \in \mathcal{D}_{\lambda\mu}$.

Let $d \in \mathcal{D}_{\lambda\mu}$ and put $S = \mu(\mathfrak{t}^\lambda d)$, $T = \lambda(\mathfrak{t}^\mu d^{-1})$. Then $S$ and $T$ are both row semistandard, and we have ([M1, 4.6])

$$\sum_{\substack{y \in \mathcal{D}_\mu \\ \lambda(\mathfrak{t}^\mu y) = T}} q^{l(y)} T_y^* m_{\{\mu\}} = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} q^{l(w)} T_w = \sum_{\substack{x \in \mathcal{D}_\lambda \\ \mu(\mathfrak{t}^\lambda x) = S}} q^{l(x)} m_{\{\lambda\}} T_x. \tag{4.2.1}$$

Note that if $\lambda$ is a partition and $S$ is semistandard, then the condition $\mu(\mathfrak{t}^\lambda x) = S$ is equivalent to the condition that $\mathfrak{t}^\lambda x$ is a standard tableau. Moreover, in this setting, $d$ is given by $\operatorname{first}(S) = \mathfrak{t}^\lambda d$.

**4.3.** Assume that $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \widetilde{\mathcal{P}}_{n,r}$. We define a subset $\mathcal{T}_0^+(\lambda, \mu)$ of $\mathcal{T}_0(\lambda, \mu)$ by

$$\mathcal{T}_0^+(\lambda, \mu) = \{S \in \mathcal{T}_0(\lambda, \mu) \mid \mathbf{a}(\lambda) = \mathbf{a}(\mu)\}.$$

Note that the condition $\mathbf{a}(\lambda) = \mathbf{a}(\mu)$ is also expressed as $\alpha(\lambda) = \alpha(\mu)$. Take $S \in \mathcal{T}_0(\lambda, \mu)$. Then one can check that $S \in \mathcal{T}_0^+(\lambda, \mu)$ if and only if each entry of $S^{(k)}$ is of the form $(i, k)$ for some $i$. Hence in this case $S^{(k)}$ can be identified with a semistandard $\lambda^{(k)}$-tableau of type $\mu^{(k)}$ as in 4.2, and we have a bijection

$$\mathcal{T}_0^+(\lambda, \mu) \simeq \mathcal{T}_0(\lambda^{(1)}, \mu^{(1)}) \times \cdots \times \mathcal{T}_0(\lambda^{(r)}, \mu^{(r)}) \tag{4.3.1}$$

via $S \leftrightarrow (S^{(1)}, \dots, S^{(r)})$. Moreover, if $\mathfrak{s} \in \operatorname{Std}(\lambda)$ is such that $\mu(\mathfrak{s}) = S$ with $S \in \mathcal{T}_0^+(\lambda, \mu)$, then the entries of $i$-th component of $\mathfrak{s}$ consist of numbers $a_i + 1, \dots, a_{i+1}$ for $\mathbf{a}(\lambda) = (a_1, \dots, a_r)$. In particular, $d(\mathfrak{s}) \in \mathfrak{S}_\alpha$ for $\alpha = \alpha(\lambda)$.

Take $S \in \mathcal{T}_0^+(\lambda, \mu)$ and let $\mathfrak{s}_1 = \operatorname{first}(S)$. Let $\alpha = \alpha(\lambda) = \alpha(\mu)$. Then $d = d(\mathfrak{s}_1)$ is given as $d = (d_1, \dots, d_r) \in \mathfrak{S}_\alpha$, with $d_k$ a distinguished double coset representative in $\mathfrak{S}_{\lambda^{(k)}} \backslash \mathfrak{S}_{n_k} / \mathfrak{S}_{\mu^{(k)}}$. Then by combining this with (4.2.1), together with the remark after that, we have

$$\sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s}))} m_{\{\lambda\}} T_{d(\mathfrak{s})} = \sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} q^{l(w)} T_w = h T_d m_{\{\mu\}}, \tag{4.3.2}$$

where $h = \sum T_v$, the sum running over certain elements $v \in \mathfrak{S}_\lambda$.

Let us take $S \in \mathcal{T}_0(\lambda, \mu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$ for $\lambda \in \mathcal{P}_{n,r}$ and $\mu \in \widetilde{\mathcal{P}}_{n,r}$. We define $m^\natural_{S\mathfrak{t}} \in \widetilde{\mathcal{H}}$ by

$$m^\natural_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \operatorname{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m^\natural_{\mathfrak{s}\mathfrak{t}}. \tag{4.3.3}$$

Then we have

**Theorem 4.4** ([DJM, Theorem 4.14]). *$M_\natural^\mu$ is a free $R$-module with basis*

$$\{m^\natural_{S\mathfrak{t}} \mid S \in \mathcal{T}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_{n,r}\}.$$

As a corollary, we have

**Corollary 4.5.** (i) *$\widehat{M}_\natural^\mu$ is a free $R$-module with basis*

$$\{m^\natural_{S\mathfrak{t}} \mid S \in \mathcal{T}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda), \mathbf{a}(\lambda) > \mathbf{a}(\mu)\}.$$

(ii) *Let $f : M_\natural^\mu \to M^\mu$ be the map given in Corollary 3.8. Then the set*

$$\{f(m^\natural_{S\mathfrak{t}}) \mid S \in \mathcal{T}_0^+(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_{n,r}\}$$

*gives a basis of $M^\mu$.*

*Proof.* (i) is clear from the definition. It is known by [DJM, Corollary 3.24] that for $\mu \in \widetilde{\mathcal{P}}_{n,r}$, any element of $M_\natural^\mu$ can be written as a linear combination of $m^\natural_{\mathfrak{s}\mathfrak{t}}$, where $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ such that $\lambda \trianglerighteq \mu$. Hence by (3.3.4), we have $\mathbf{a}(\lambda) \geq \mathbf{a}(\mu)$ for such $m^\natural_{\mathfrak{s}\mathfrak{t}}$. Now (ii) follows from this and (i). □

**4.6.** We now construct a basis of $M^\mu$ related to the basis $m_{\mathfrak{s}\mathfrak{t}}$ as an analogy of $m^\natural_{S\mathfrak{t}}$ in $M^\mu_\natural$. Let us define, for $S \in \mathcal{T}_0^+(\lambda,\mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$,

$$m_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}}. \tag{4.6.1}$$

We have

**Lemma 4.7.** *Let $S \in \mathcal{T}_0^+(\lambda,\mu)$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$. Then $m_{S\mathfrak{t}} \in M^\mu$.*

*Proof.* The proof is similar to Lemma 4.10 in [DJM] and simpler than it. By (4.3.2) we have

$$m_{S\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = S}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} T^*_{d(\mathfrak{s})} m_{\{\lambda\}} F_\alpha T_{d(\mathfrak{t})} = q^{l(d(\mathfrak{t}))} m_{\{\mu\}} T_d^* h^* F_\alpha T_{d(\mathfrak{t})},$$

where $\alpha = \alpha(\lambda)$. Since $h \in \mathcal{H}(\mathfrak{S}_\lambda) \subset \mathcal{H}_\lambda$, we have $h^* F_\alpha = F_\alpha h^*$. Furthermore, since $S \in \mathcal{T}_0^+(\lambda,\mu)$, $\mathfrak{s}_1$ satisfies the property that the entries of $k$th component of $\mathfrak{s}_1$ consist of the same numbers as the entries of $k$th component of $\mathfrak{t}^\lambda$ for each $k$. It follows that $d \in \mathfrak{S}_\alpha$, and $F_\alpha$ commutes with $T_d \in \mathcal{H}(\mathfrak{S}_\alpha)$ by Lemma 2.9. If we note that $\alpha = \alpha(\mu)$, we see that

$$m_{\{\mu\}} T_d^* h^* F_\alpha = m_{\{\mu\}} F_\alpha T_d^* h^* \in m_\mu \mathcal{H} = M^\mu,$$

and $m_{S\mathfrak{t}} \in M^\mu$ as asserted. □

**Proposition 4.8.** *The set $\{m_{S\mathfrak{t}} \mid S \in \mathcal{T}_0^+(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathcal{P}_{n,r}\}$ gives rise to an $R$-basis of $M^\mu$.*

*Proof.* The basis elements $m_{\mathfrak{s}\mathfrak{t}}$ contained in the expression of $m_{S\mathfrak{t}}$ are disjoint for different $m_{S\mathfrak{t}}$. It follows that $m_{S\mathfrak{t}}$ are linearly independent. By Corollary 4.5 (ii), $M^\mu$ is a free $R$-module, and its rank is equal to the number of $\{m_{S\mathfrak{t}}\}$ given in the theorem. Hence, any element in $M^\mu$ can be written as a linear combination of various $m_{S\mathfrak{t}}$ with coefficients in the quotient field of $R$. But since the set $\{q^{l(d(\mathfrak{s}))+l(d(\mathfrak{t}))}m_{\mathfrak{s}\mathfrak{t}}\}$ is an $R$-basis of $\mathcal{H}$, these coefficients are actually in $R$. This proves the proposition. $\square$

We can now state

**Theorem 4.9.** *Let $\mu \in \widetilde{\mathcal{P}}_{n,r}$, and $f : M^\mu_\natural \to M^\mu$ be as in Corollary 3.8. Let $\lambda \in \mathcal{P}_{n,r}$ and consider $m_{S\mathfrak{t}}$ for $S \in \mathcal{T}_0(\lambda,\mu), \mathfrak{t} \in \mathrm{Std}(\lambda)$. Then we have*

$$f(m^\natural_{S\mathfrak{t}}) = \begin{cases} m_{S\mathfrak{t}} & \text{if } S \in \mathcal{T}_0^+(\lambda,\mu), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The second equality is clear from Corollary 4.5 (i). We show the first one. We know that $\{m_\mu T_{d(\mathfrak{t})} \mid \mathfrak{t} \in \text{r-Std}(\mu)\}$ is a basis of $M^\mu$ by Lemma 2.5 and 2.7. Since $m_\mu T_{d(\mathfrak{t})} = m_{\mathfrak{t}^\mu\mathfrak{t}}$, $m_\mu T_{d(\mathfrak{t})}$ can be written as a linear combination of various $m_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u},\mathfrak{v} \in \mathrm{Std}(\lambda)$ for $\lambda \trianglerighteq \mu$, by the procedure described in Lemma 2.15. This procedure also determines to write down $m_\mu T_{d(\mathfrak{t})}$ as a linear combination of the basis $m_{S\mathfrak{t}'}$ of $M^\mu$. On the other hand, by the definition of $f$, the element $m^\natural_\mu T_{d(\mathfrak{t})}$ is mapped by $f$ to $m_\mu T_{d(\mathfrak{t})}$. Since $m^\natural_\mu T_{d(\mathfrak{t})} = m^\natural_{\mathfrak{t}^\mu\mathfrak{t}}$, $m^\natural_\mu T_{d(\mathfrak{t})}$ can be written as a linear combination of $m^\natural_{\mathfrak{u}\mathfrak{v}}$ with $\mathfrak{u},\mathfrak{v} \in \mathrm{Std}(\lambda)$ for $\lambda \trianglerighteq \mu$, by the procedure given in [DJM, Prop. 3.18]. However, as is seen in the proof of Lemma 2.15, these two procedures are completely parallel. Hence the coefficient of $m_{\mathfrak{u}\mathfrak{v}}$ in $m_\mu T_{d(\mathfrak{t})}$ is the same as that of $m^\natural_{\mathfrak{u}\mathfrak{v}}$ in $m^\natural_\mu T_{d(\mathfrak{t})}$. The same is true for the coefficients of $m_{S\mathfrak{t}'}$ and of $m^\natural_{S\mathfrak{t}'}$. It follows that the basis $\{f(m^\natural_{S\mathfrak{t}})\}$ and the basis $\{m_{S\mathfrak{t}}\}$ have the same transition matrix for the basis $\{m_\mu T_{d(\mathfrak{t})}\}$ of $M^\mu$. This proves that $f(m^\natural_{S\mathfrak{t}}) = m_{S\mathfrak{t}}$ as asserted. $\square$

**4.10.** The Specht module $S^\lambda_\natural$ for $\lambda \in \mathcal{P}_{n,r}$ is defined in [DJM] by using the cellular basis $\mathcal{C}^\natural$ of $\widetilde{\mathcal{H}}$, which is the submodule of $\widetilde{\mathcal{H}}/\check{N}^\lambda_\natural$ generated by the image $\overline{m}^\natural_\lambda$ of $m^\natural_\lambda$ under the map $\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}/\check{N}^\lambda_\natural$, where $\check{N}^\lambda_\natural$ is the $R$-submodule of $\widetilde{\mathcal{H}}$ spanned by $m^\natural_{\mathfrak{s}\mathfrak{t}}$ with $\mathfrak{s},\mathfrak{t} \in \mathrm{Std}(\nu)$ such that $\nu \triangleright \lambda$. Hence $S^\lambda_\natural$ is a free $R$-module with basis $\{\overline{m}^\natural_\lambda T_{d(\mathfrak{t})} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$.

Let $\mu \in \widetilde{\mathcal{P}}_{n,r}$. We define a total order $S_1, S_2, \dots, S_l$ on the set of semistandard Tableaux of type $\mu$ satisfying the following property; Assume that $S_i \in \mathcal{T}_0(\lambda_i,\mu)$ and $S_j \in \mathcal{T}_0(\lambda_j,\mu)$. Then $i > j$ if $\mathbf{a}(\lambda_i) > \mathbf{a}(\lambda_j)$ or $\lambda_i \triangleright \lambda_j$. Hence the first $k$ terms of $S_i$ are contained in $\mathcal{T}_0^+(\lambda,\mu)$ for some $\lambda$. Let us define $M^\natural_i$ as an $R$-submodule of $M^\mu_\natural$ with basis $\{m^\natural_{S_j\mathfrak{t}} \mid j \ge i, \mathfrak{t} \in \mathrm{Std}(\lambda_j)\}$. Similarly, we define $M_i$ $(1 \le i \le k)$ as an $R$-submodule of $M^\mu$ with basis $\{m_{S_j\mathfrak{t}} \mid j \ge i, \mathfrak{t} \in \mathrm{Std}(\lambda_j)\}$. Then by the property of cellular bases, $M^\natural_i$ is a right ideal of $\widetilde{\mathcal{H}}$, and $M_i$ is a right ideal of $\mathcal{H}$. Thus, we have filtrations

$$M^\mu_\natural = M^\natural_1 \supset M^\natural_2 \supset \cdots \supset M^\natural_k \supset \cdots \supset M^\natural_l \supset M^\natural_{l+1} = 0,$$
$$M^\mu = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0,$$

which are called the Specht filtrations for $M^\mu_\natural, M^\mu$, respectively.

As a corollary to Theorem 4.9, we have the following.

**Corollary 4.11.** *Let $f : M^\mu_\natural \to M^\mu$ be as in Corollary 3.8. Then*

(i) *The map $f$ is compatible with the above filtrations, i.e., $f(M^\natural_i) = M_i$ for $1 \le i \le k$, and $f(M^\natural_i) = 0$ otherwise.*
(ii) *$M^\natural_i/M^\natural_{i+1}$ is isomorphic to $S^{\lambda_i}_\natural$ as $\widetilde{\mathcal{H}}$-modules for $1 \le i \le l$.*
(iii) *$M_i/M_{i+1}$ is isomorphic to $S^{\lambda_i}$ as $\mathcal{H}$-modules for $1 \le i \le k$.*
(iv) *$S^{\lambda_i}_\natural$ is isomorphic to $S^{\lambda_i}$ as $\widetilde{\mathcal{H}}$-modules for $1 \le i \le k$.*

*Proof.* (i) is immediate from Theorem 4.9. (ii) is nothing but Corollary 4.15 in [DJM]. A similar argument as in [DJM] works also for (iii). Now (iv) follows from (i) $\sim$ (iii) since $f^{-1}(M_i) = M^\natural_i$ for $1 \le i \le k$, and $f$ is $\widetilde{\mathcal{H}}$-equivariant. □

**4.12.** For each $\lambda \in \mathcal{P}_{n,r}$, the quotient module $D^\lambda_\natural$ of $S^\lambda_\natural$ is defined as in 2.19. When $R$ is a field, the set $\{D^\lambda_\natural \ne 0 \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a complete set of non-isomorphic irreducible $\widetilde{\mathcal{H}}$-modules. We denote by $d^\natural_{\lambda\mu} = [S^\lambda_\natural : D^\mu_\natural]$ the composition multiplicity of $D^\mu_\natural$ in $S^\lambda_\natural$. Similarly we define $d_{\lambda\mu} = [S^\lambda : D^\mu]$ for $\mathcal{H}$-modules $S^\lambda$ and $D^\mu$. By a general theory of cellular algebras, it is known, in the case where $R$ is a field, that $d_{\mu\mu} = 1$, and $d_{\lambda\mu} = 0$ unless $\lambda \trianglerighteq \mu$, where $\lambda, \mu \in \mathcal{P}_{n,r}$ such that $D^\mu \ne 0$, and similar facts hold also for $\widetilde{\mathcal{H}}$. Note that the $\mathcal{H}$-modules $S^\lambda, D^\mu$ can be regarded as $\widetilde{\mathcal{H}}$-modules via the homomorphism $\rho_0 : \widetilde{\mathcal{H}} \to \mathcal{H}$.

Next result gives an estimate of the decomposition numbers for the Ariki-Koike algebra $\widetilde{\mathcal{H}}$ in terms of the decomposition numbers for the modified Ariki-Koike algebra $\mathcal{H}$.

**Theorem 4.13.** *Assume that $R$ is a field.*

(i) *For $\widetilde{\mathcal{H}}$-modules $D^\mu$ and $D^\nu_\natural$, we have $[D^\mu : D^\nu_\natural] = 0$ unless $\nu \trianglerighteq \mu$.*
(ii) *If $D^\mu_\natural \ne 0$, then $D^\mu \ne 0$. Moreover in this case, we have $[D^\mu : D^\mu_\natural] = 1$.*
(iii) *For any $\lambda, \mu \in \mathcal{P}_{n,r}$ such that $D^\mu_\natural \ne 0$, we have $[S^\lambda : D^\mu] \le [S^\lambda_\natural : D^\mu_\natural]$.*
(iv) *Assume that $\mathcal{H} \simeq \widetilde{\mathcal{H}}$. Then $D^\mu \simeq D^\mu_\natural$ for any $\mu$, and the parametrization of irreducible modules in terms of the cellular basis $\mathcal{C}$ and of $\mathcal{C}^\natural$ coincides with each other.*

*Proof.* For a Specht module $S^\lambda$, we consider a composition series whose successive quotients are of the form $D^\mu$. This gives a filtration of $S^\lambda$ as $\widetilde{\mathcal{H}}$-modules, and by an isomorphism $S^\lambda \simeq S^\lambda_\natural$ it gives a filtration of $S^\lambda_\natural$ whose successive quotients are isomorphic to $D^\mu$ as $\widetilde{\mathcal{H}}$-modules. Now (i) follows from this by induction on the dominance order, by making use of the property of decomposition numbers $d_{\lambda\mu}$ and $d^\natural_{\lambda\mu}$.

We show (ii). Assume that $D^\mu_\natural \ne 0$. If $D^\mu = 0$, the Specht module $S^\mu$ has a composition series whose quotients consist of $D^\nu$ such that $\nu \triangleright \mu$. Then by (i), $S^\mu_\natural$ does not contain $D^\mu_\natural$, which contradicts $d^\natural_{\mu\mu} = 1$. Hence we have $D^\mu \ne 0$. The equality $[D^\mu : D^\mu_\natural] = 1$ follows from (i) noticing that $d_{\mu\mu} = 1, d^\natural_{\mu\mu} = 1$.

Now assume that $D^\mu_\natural \ne 0$. Then by (ii), $D^\mu \ne 0$, and the $\widetilde{\mathcal{H}}$-module $D^\mu$ contains the $\widetilde{\mathcal{H}}$-module $D^\mu_\natural$ with multiplicity one, as a composition factor. This proves (iii).

Finally (iv) follows easily from (ii). □

## 5. Cyclotomic $q$-Schur algebras

**5.1.** Let $\mu,\nu \in \widetilde{\mathcal{P}}_{n,r}$ be such that $\alpha(\mu) = \alpha(\nu) = \alpha$. Put $\alpha = (n_1, \dots, n_r)$. Let us take $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha$. We have $d = (d_1, \dots, d_r)$ with $d_k \in \mathcal{D}_{\mu^{(k)}\nu^{(k)}}$ with respect to $\mathfrak{S}_{n_k}$. Then, by a similar argument as in the proof of Lemma 4.7, one can define a map $\varphi^d_{\mu\nu} : M^\nu \to M^\mu$ by

$$\varphi^d_{\mu\nu}(m_\nu h) = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} F_\alpha T_w h$$

for all $h \in \mathcal{H}$. In fact, by (4.2.1), (see also (4.3.2)), we have

$$\sum_{\substack{y \in \mathcal{D}_\nu \cap \mathfrak{S}_\alpha \\ \mu(\mathfrak{t}^\nu y) = T}} q^{l(y)} F_\alpha T^*_y m_{\{\nu\}} = \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} F_\alpha T_w = \sum_{\substack{x \in \mathcal{D}_\mu \cap \mathfrak{S}_\alpha \\ \nu(\mathfrak{t}^\mu x) = S}} q^{l(x)} m_\mu T_x, \tag{5.1.1}$$

where $S = \mu(\mathfrak{t}^\nu d)$, $T = \nu(\mathfrak{t}^\mu d^{-1})$ are row semistandard Tableaux. Since $y \in \mathfrak{S}_\alpha$, we have $F_\alpha T^*_y m_{\{\nu\}} = T^*_y F_\alpha m_{\{\nu\}} = T^*_y m_\nu$, and $\varphi^d_{\mu\nu}$ is well-defined.

As an analogy to the case of $\mathcal{H}_n$ ([M1, Theorem 4.7]), we have the following result.

**Proposition 5.2.** *Let $\mu,\nu \in \widetilde{\mathcal{P}}_{n,r}$. Then*

(i) *Assume that $\alpha(\mu) \neq \alpha(\nu)$. Then* $\operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu) = 0$.

(ii) *Assume that $\alpha(\mu) = \alpha(\nu)$. Then* $\operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ *is a free $R$-module with basis* $\{\varphi^d_{\mu\nu} \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$.

*Proof.* Suppose that $\varphi \in \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$. Then for $h \in \mathcal{H}$, we have $\varphi(m_\nu h) = \varphi(m_\nu)h$, and $\varphi$ is determined completely by $\varphi(m_\nu)$. Since $\varphi(m_\nu) \in M^\mu$ one can write, by Lemma 2.5, that $\varphi(m_\nu) = \sum_{x \in \mathcal{D}_\mu} c_x m_\mu T_x$, with $c_x \in R$. Then by Lemma 2.6 (iii), we have

$$\varphi(m_\nu \xi_k) = \varphi(Q_{\varepsilon(k)} m_\nu) = \sum_{x \in \mathcal{D}_\mu} Q_{\varepsilon(k)} c_x m_\mu T_x \tag{5.2.1}$$

for each $k$, where $\varepsilon(k)$ is determined by the condition $v_\alpha = v^{(\varepsilon(1))}_{j_1} \otimes \cdots \otimes v^{(\varepsilon(n))}_{j_n}$, with $\alpha = \alpha(\nu)$. Now assume that $c_y \neq 0$ for some $y \in \mathcal{D}_\mu$. Then we have

$$(m_\mu T_y)\xi_k = Q_{\varepsilon'(k)} m_\mu T_y, \tag{5.2.2}$$

where $\varepsilon'(k)$ is determined by the condition that $v_{\alpha'} T_y = v^{(\varepsilon'(1))}_{j_1} \otimes \cdots \otimes v^{(\varepsilon'(n))}_{j_n}$, with $\alpha' = \alpha(\mu)$. By comparing (5.2.1) and (5.2.2), we have $\varepsilon(k) = \varepsilon'(k)$ for $k = 1, \dots, r$. This implies that $\alpha = \alpha'$ since $T_y$ preserves the weight spaces. So (i) is proved.

Now assume that $\alpha(\mu) = \alpha(\nu)$. Since $T_y$ acts as a permutation of factors on $v_\alpha$, the condition $\varepsilon(k) = \varepsilon'(k)$ implies that $y \in \mathfrak{S}_\alpha$. Let $d$ be the unique minimal length element in $\mathfrak{S}_\mu y \mathfrak{S}_\nu$. Then $d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha$, and a similar argument as in the proof of [M1, Theorem 4.7] implies that $c_d \neq 0$. Put $\psi = \varphi - c_d \varphi^d_{\mu\nu}$. Then $\psi \in \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$, and $\psi(m_\nu)$ can be written as $\psi(m_\nu) = \sum_{x \in \mathcal{D}_\mu} a_x m_\mu T_x$, where $a_x = c_x$ if $\mathfrak{S}_\mu x \mathfrak{S}_\nu \neq \mathfrak{S}_\mu d \mathfrak{S}_\nu$, and $a_x = 0$ for $x \in \mathfrak{S}_\mu d\mathfrak{S}_\nu$ by the argument there. Thus by induction, $\varphi$ can be written as a linear combination of $\varphi^d_{\mu\nu}$.

We have only to show that $\{\varphi^d_{\mu\nu} \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$ is linearly independent. But $\{\varphi^d_{\mu\nu}(m_\nu)\}$ is a linearly independent subset of $M^\mu$ since a set $\{F_\alpha T_w\}$ is linearly independent by (1.2.7). $\square$

We write $M^{\nu *} = (M^\nu)^* = m_\nu \mathcal{H}$. As a corollary to Proposition 5.2, we have the following.

**Corollary 5.3.** *Let $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}$.*

(i) $\operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$ *and $M^{\nu *} \cap M^\mu$ are canonically isomorphic as $R$-modules. The isomorphism* $\operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu) \to M^{\nu *} \cap M^\mu$ *is given by $\varphi \mapsto \varphi(m_\nu)$.*

(ii) *Assume that $\alpha = \alpha(\mu) = \alpha(\nu)$. Then we have*

$$M^{\nu *} \cap M^\mu = F_\alpha\big(\mathcal{H}_\alpha m_{\{\nu\}} \cap m_{\{\mu\}}\mathcal{H}_\alpha\big),$$

*where $m_{\{\nu\}}$ and $m_{\{\mu\}}$ are regarded as elements in $\mathcal{H}_\alpha = \mathcal{H}(\mathfrak{S}_\alpha)$ in a natural way.*

*Proof.* The assertion (i) is proved in a similar way as [M1, Corollary 4.8], thanks to Proposition 5.2. We show (ii). By (i) together with Proposition 5.2, we see that

$$\{F_\alpha \sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} T_w \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$$

forms an $R$-basis of $M^{\nu *} \cap M^\mu$. But it is already known from the results in the case of $\mathcal{H}_\alpha$ (cf [M1, 4.6, 4.8]) that $\{\sum_{w \in \mathfrak{S}_\mu d \mathfrak{S}_\nu} q^{l(w)} T_w \mid d \in \mathcal{D}_{\mu\nu} \cap \mathfrak{S}_\alpha\}$ forms a basis of $\mathcal{H}_\alpha m_{\{\nu\}} \cap m_{\{\mu\}}\mathcal{H}_\alpha$. Thus the assertion (ii) holds. $\square$

**5.4.** We define a cyclotomic $q$-Schur algebra $\mathcal{S}(\mathbf{m}, n)$ by $\mathcal{S}(\mathbf{m}, n) = \operatorname{End}_{\mathcal{H}} V^{\otimes n}$. By (3.3.1), we have $V^{\otimes n} = \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} V^{\otimes n}_\mu \simeq \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^{\mu'}$. Thus one can write

$$\mathcal{S}(\mathbf{m}, n) = \operatorname{End}_{\mathcal{H}}\big(\bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^\mu\big) \simeq \bigoplus_{\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}} \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu).$$

For an $r$-tuples $\alpha \in \widetilde{\mathcal{P}}_{n,1}$, let $M^\alpha = \bigoplus_{\mu; \alpha(\mu) = \alpha} M^\mu$. Then by Proposition 5.2 (i), we have

$$\mathcal{S}(\mathbf{m}, n) \simeq \bigoplus_{\alpha \in \widetilde{\mathcal{P}}_{n,1}} \operatorname{End}_{\mathcal{H}} M^\alpha \tag{5.4.1}$$

as $R$-algebras.

Recall that $U_q$ is the quantum algebra over $R$. Since $U_q$ acts on $V^{\otimes n}$ commuting with $\mathcal{H}$, we have an algebra homomorphism $\tau : U_q \to \operatorname{End}_{\mathcal{H}} V^{\otimes n} = \mathcal{S}(\mathbf{m}, n)$. The following result is in analogy to the result of Ariki [A, Theorem 3.2], where he proved a similar fact for the cyclotomic $q$-Schur algebra $\operatorname{End}_{\widetilde{\mathcal{H}}} V^{\otimes n}$ (associated to the Ariki-Koike algebra $\widetilde{\mathcal{H}}$) under the separation condition

(5.4.2) $q^{2k}Q_i - Q_j \in R$ are inverstible in $R$ for $|k| < n, i \neq j$,

but without any restriction on $\mathbf{m} = (m_1, \dots, m_r)$. Note that the condition (5.4.2) is stronger than our condition (1.2.1).

**Theorem 5.5.** *Let $\mathcal{S}(\mathbf{m}, n)$ be the cyclotomic $q$-Schur algebra associated to the modified Ariki-Koike algebra $\mathcal{H}$. Then we have*

(i) *Let $\mathcal{S}(m_i, n_i)$ be the $q$-Schur algebra assocaited to the Iwahori-Hecke algebra $\mathcal{H}_{n_i}$ as in [M1, Chap. 4]. Then there exists an isomorphism of $R$-algebras*

$$\mathcal{S}(\mathbf{m}, n) \simeq \bigoplus_{\substack{(n_1,\dots,n_r) \\ n=n_1+\cdots+n_r}} \mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r). \tag{5.5.1}$$

(ii) *The map $\tau : U_q \to \mathcal{S}(\mathbf{m}, n)$ is surjective.*

*Proof.* First we show (i). For each $\mu \in \widetilde{\mathcal{P}}_{n,r}$ we denote by $\overline{M}^{\mu}$ the right ideal $m_{\{\mu\}}\mathcal{H}_\alpha$ of $\mathcal{H}_\alpha$. For $\alpha \in \widetilde{\mathcal{P}}_{n,1}$, let $\overline{M}^{\alpha} = \bigoplus_{\mu;\, \alpha(\mu)=\alpha} \overline{M}^{\mu}$. It is easy to see that

$$\operatorname{End}_{\mathcal{H}_\alpha} \overline{M}^{\alpha} \simeq \mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r)$$

as $R$-algebras, with $\alpha = (n_1, \dots, n_r)$. Thus, in view of (5.4.1), it is enough to show that

$$\operatorname{End}_{\mathcal{H}} M^{\alpha} \simeq \operatorname{End}_{\mathcal{H}_\alpha} \overline{M}^{\alpha} \tag{5.5.2}$$

as $R$-algebras. We show (5.5.2). By Corollary 5.3 (ii), we have an isomorphism of $R$-modules $\overline{M}^{\nu *} \cap \overline{M}^{\mu} \to M^{\nu *} \cap M^{\mu}$ via $h \mapsto F_\alpha h$. By applying Corollary 5.3 (i) for both sides, one can define an $R$-linear map $g : \operatorname{End}_{\mathcal{H}_\alpha} \overline{M}^{\alpha} \to \operatorname{End}_{\mathcal{H}} M^{\alpha}$, which is clearly bijective. It remains to show that the map $g$ is a homomorphism of $R$-algebras.

Let $\tau, \nu, \mu \in \widetilde{\mathcal{P}}_{n,r}$ be such that $\alpha(\tau) = \alpha(\nu) = \alpha(\mu)$. Take $x \in M^{\tau *} \cap M^{\nu}$, $y \in M^{\nu *} \cap M^{\mu}$, and let $\varphi_x \in \operatorname{Hom}_{\mathcal{H}}(M^{\tau}, M^{\nu})$, $\varphi_y \in \operatorname{Hom}_{\mathcal{H}}(M^{\nu}, M^{\mu})$ be the elements corresponding to $x, y$ under the isomorphism in Corollary 5.3 (i), respectively. By using Corollary 5.3 (ii), one can write $x = m_\nu h = F_\alpha m_{\{\nu\}} h$ for some $h \in \mathcal{H}_\alpha$. By definition, we have $\varphi_x(m_\tau) = x$, $\varphi_y(m_\nu) = y$. Hence $\varphi_y\varphi_x(m_\tau) = \varphi_y(m_\nu h) = \varphi_y(m_\nu)h = yh$. Thus we see that $yh \in M^{\tau *} \cap M^{\mu}$ corresponds to $\varphi_y\varphi_x \in \operatorname{Hom}_{\mathcal{H}}(M^{\tau}, M^{\mu})$. If we write $x = F_\alpha x', y = F_\alpha y'$ with $x' \in \overline{M}^{\tau *} \cap \overline{M}^{\nu}, y' \in \overline{M}^{\nu *} \cap \overline{M}^{\mu}$, a completely similar rule for the composition holds between $\varphi_{x'} \in \operatorname{Hom}_{\mathcal{H}_\alpha}(\overline{M}^{\tau}, \overline{M}^{\nu})$ and $\varphi_{y'} \in \operatorname{Hom}_{\mathcal{H}_\alpha}(\overline{M}^{\nu}, \overline{M}^{\mu})$, i.e., $y'h$ corresponds to $\varphi_{y'}\varphi_{x'} \in \operatorname{Hom}_{\mathcal{H}_\alpha}(\overline{M}^{\tau}, \overline{M}^{\mu})$. It follows that $g : \operatorname{End}_{\mathcal{H}_\alpha} \overline{M}^{\alpha} \to \operatorname{End}_{\mathcal{H}} M^{\alpha}$ is a homomorphism. Hence (5.5.2) holds, and (i) follows.

Next we show (ii). Take $\alpha = (n_1, \dots, n_r) \in \widetilde{\mathcal{P}}_{n,1}$ and put $\alpha' = (n_r, \dots, n_1)$. Put $V_\alpha^{\otimes n} = \bigoplus_{\mu;\alpha(\mu)=\alpha} V_\mu^{\otimes n}$. Then $V^{\otimes n} = \bigoplus_\alpha V_\alpha^{\otimes n}$, and $V_\alpha^{\otimes n}$ is isomorphic to $M^{\alpha'}$ as $\mathcal{H}$-modules. Moreover, $V_\alpha^{\otimes n}$ can be expressed as

$$V_\alpha^{\otimes n} = \bigoplus_{w \in \mathfrak{S}_{\alpha'} \backslash \mathfrak{S}_n} \big(V_r^{\otimes n_r} \otimes V_{r-1}^{\otimes n_{r-1}} \otimes \cdots \otimes V_1^{\otimes n_1}\big) w.$$

Here $V_r^{\otimes n_r}\otimes\cdots\otimes V_1^{\otimes n_1}$ is isomorphic to $\overline{M}^{\alpha'}$ as $\mathcal{H}_{\alpha'}$-modules. We note that, if we choose $w\in\mathcal{D}_{\alpha'}$,

$$(V_r^{\otimes n_r}\otimes\cdots\otimes V_1^{\otimes n_1})w=(V_r^{\otimes n_r}\otimes\cdots\otimes V_1^{\otimes n_1})T_w.$$

Hence the map $\varphi_r\otimes\cdots\otimes\varphi_1\in\operatorname{End}_{\mathcal{H}_{\alpha'}}(V_r^{\otimes n_r}\otimes\cdots\otimes V_1^{\otimes n_1})$ with $\varphi_i\in\operatorname{End}_{\mathcal{H}_{n_i}}V_i^{\otimes n_i}$ determines the map $\varphi\in\operatorname{End}V_\alpha^{\otimes n}$. It follows from (5.5.2) that the correspondence $\varphi_r\otimes\cdots\otimes\varphi_1\mapsto\varphi$ gives an $R$-isomorphism between $(\operatorname{End}_{\mathcal{H}_{n_r}}V_r^{\otimes n_r})\otimes\cdots\otimes(\operatorname{End}_{\mathcal{H}_{n_1}}V_1^{\otimes n_1})$ and $\operatorname{End}_{\mathcal{H}}V_\alpha^{\otimes n}$. On the other hand, it is known by J. Du [D, Theorem 3.4, Corollary 3.5] that the natural map

$$U_{q,R}(\mathfrak{gl}_m)\to\operatorname{End}_{\mathcal{H}_n}V^{\otimes n}$$

(i.e., the map $\tau$ in the case where $r=1$) is surjective. It follows that the map

$$\tau_1:U_q(\mathfrak{gl}_{m_r})\otimes\cdots\otimes U_q(\mathfrak{gl}_{m_1})\to(\operatorname{End}_{\mathcal{H}_{n_r}}V_r^{\otimes n_r})\otimes\cdots\otimes(\operatorname{End}_{\mathcal{H}_{n_1}}V_1^{\otimes n_1})$$

is surjective. But the map $\tau_1$ is compatible with $\tau:U_q\to\operatorname{End}_{\mathcal{H}}V^{\otimes n}$ under the identifications

$$(\operatorname{End}_{\mathcal{H}_{n_r}}V_r^{\otimes n_r})\otimes\cdots\otimes(\operatorname{End}_{\mathcal{H}_{n_1}}V_1^{\otimes n_1})\simeq\operatorname{End}_{\mathcal{H}}V_\alpha^{\otimes n}\hookrightarrow\operatorname{End}_{\mathcal{H}}V^{\otimes n}.$$

and $U_q\simeq U_{q,R}(\mathfrak{gl}_{m_r})\otimes\cdots\otimes U_{q,R}(\mathfrak{gl}_{m_1})$. Since $\operatorname{End}_{\mathcal{H}}V^{\otimes n}\simeq\bigoplus_\alpha\operatorname{End}_{\mathcal{H}}V_\alpha^{\otimes n}$ by (5.4.1), this proves the surjectivity of $\tau$. □

**5.6** Let $\mu,\nu\in\widetilde{\mathcal{P}}_{n,r}$ and $\lambda\in\mathcal{P}_{n,r}$. We assume that $\alpha(\mu)=\alpha(\nu)=\alpha(\lambda)$. For $S\in\mathcal{T}_0^+(\lambda,\mu)$ and $T\in\mathcal{T}_0^+(\lambda,\nu)$, put

$$m_{ST}=\sum_{\mathfrak{s},\mathfrak{t}}m_{\mathfrak{s}\mathfrak{t}},\tag{5.6.1}$$

where the sum is taken over all $\mathfrak{s},\mathfrak{t}\in\operatorname{Std}(\lambda)$ such that $\mu(\mathfrak{s})=S$ and $\nu(\mathfrak{t})=T$. Since

$$m_{ST}=\sum_{\substack{\mathfrak{s}\in\operatorname{Std}(\lambda)\\ \mu(\mathfrak{s})=S}}(m_{T\mathfrak{s}})^*=\sum_{\substack{\mathfrak{t}\in\operatorname{Std}(\lambda)\\ \nu(\mathfrak{t})=T}}m_{S\mathfrak{t}},$$

we see that $m_{ST}\in M^\mu\cap M^{\nu*}$ by Lemma 4.7. The following result can be proved in a similar way as in [DJM, Prop. 6.3], [M1, Cor. 4.11], by using Proposition 4.8.

**Proposition 5.7.** *Let $\mu,\nu\in\widetilde{\mathcal{P}}_{n,r}$ be such that $\alpha(\mu)=\alpha(\nu)$. Then the set*

$$\{m_{ST}\mid S\in\mathcal{T}_0^+(\lambda,\mu),T\in\mathcal{T}_0^+(\lambda,\nu)\textit{ for some }\lambda\in\mathcal{P}_{n,r}\}$$

*is a basis of $M^{\nu*}\cap M^\mu$.*

**5.8.** Let $S\in\mathcal{T}_0^+(\lambda,\mu)$ and $T\in\mathcal{T}_0^+(\lambda,\nu)$. In view of Proposition 5.7, one can define $\varphi_{ST}\in\operatorname{Hom}_{\mathcal{H}}(M^\nu,M^\mu)$ by

$$\varphi_{ST}(m_\nu h)=m_{ST}h.$$

We extend $\varphi_{ST}$ to an element of $\mathcal{S}(\mathbf{m},n)$ by defining $\varphi_{ST}$ is zero on $M^\kappa$ for $\nu \neq \kappa \in \widetilde{\mathcal{P}}_{n,r}$. For any partition $\lambda \in \mathcal{P}_{n,r}$, let $\mathcal{T}_0^+(\lambda) = \bigcup_{\mu\in\widetilde{\mathcal{P}}_{n,r}} \mathcal{T}_0^+(\lambda,\mu)$. We denote by $\check{\mathcal{S}}(\mathbf{m},n)^\lambda$ the $R$-submodule of $\mathcal{S}(\mathbf{m},n)$ spanned by $\varphi_{ST}$ such that $S,T \in \mathcal{T}_0^+(\alpha)$ with $\alpha \rhd \lambda$. Similarly, we define $\mathcal{S}(\mathbf{m},n)^\lambda$ as the $R$-submodule spanned by $\varphi_{ST}$ with $\alpha \unrhd \lambda$. We have the following theorem.

**Theorem 5.9.** *The cyclotomic q-Schur algebra $\mathcal{S}(\mathbf{m},n)$ has a free basis*

$$\mathcal{C}(\mathbf{m},n) = \{\varphi_{ST} \mid \lambda \in \mathcal{P}_{n,r}, S,T \in \mathcal{T}_0^+(\lambda)\}.$$

*satisfying the following properties.*

(i) *The $R$-linear map $* : \mathcal{S}(\mathbf{m},n) \to \mathcal{S}(\mathbf{m},n)$ determined by $\varphi_{ST}^* = \varphi_{TS}$, for all $S,T \in \mathcal{T}_0^+(\lambda)$ and all $\lambda \in \mathcal{P}_{n,r}$, is an anti-automorphism of $\mathcal{S}(\mathbf{m},n)$.*

(ii) *Let $T \in \mathcal{T}_0^+(\lambda)$. Then for all $\varphi \in \mathcal{S}(\mathbf{m},n)$, and every $V \in \mathcal{T}_0^+(\lambda)$, there exists $r_V = r_{V,T,\varphi} \in R$ such that for any $S \in \mathcal{T}_0^+(\lambda)$,*

$$\varphi_{ST}\varphi \equiv \sum_{V\in\mathcal{T}_0^+(\lambda)} r_V\varphi_{SV} \mod \check{\mathcal{S}}(\mathbf{m},n)^\lambda. \tag{5.9.1}$$

*In particular, $\mathcal{C}(\mathbf{m},n)$ is a cellular basis of $\mathcal{S}(\mathbf{m},n)$.*

*Proof.* The proof is similar to [M1, Theorem 4.13] and [DJM, Theorem 6.6]. By Proposition 5.7 and Corollary 5.3, $\mathcal{C}(\mathbf{m},n)$ gives an $R$-basis of $\mathcal{S}(\mathbf{m},n)$. Take $S \in \mathcal{T}_0^+(\lambda,\mu), T \in \mathcal{T}_0^+(\lambda,\nu)$. Then $\varphi_{ST}^*(m_\mu) = m_{TS} = m_{ST}^* = \varphi_{ST}(m_\nu)^*$, and by the $R$-linearity, we have $\varphi^*(m_\mu) = \varphi(m_\nu)^*$ for any $\varphi \in \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$. Then the assertion (i) follows from the standard functorial argument (see the proof of [M1, Theorem 4.13]).

Next we show (ii). Take $S \in \mathcal{T}_0^+(\lambda,\mu), T \in \mathcal{T}_0^+(\lambda,\nu)$. We may assume that $\varphi \in \operatorname{Hom}_{\mathcal{H}}(M^\tau, M^\nu)$ for some $\tau \in \widetilde{\mathcal{P}}_{n,r}$ such that $\alpha(\lambda) = \alpha(\tau)$. We have $\varphi(m_\tau) = m_\nu h$ for some $h \in \mathcal{H}$. Then $\varphi_{ST}\varphi(m_\tau) = m_{ST}h$. By Corollary 5.3, we see that $m_{ST}h \in M^{\tau*} \cap M^\mu$. Hence by Proposition 5.7, one can write $m_{ST}h = \sum_{U,V} r_{UV}m_{UV}$ with $r_{UV} \in R$, where $U,V$ are elements in $\mathcal{T}_0^+(\alpha)$ for some $\alpha \in \mathcal{P}_{n,r}$. But in view of Theorem 2.19 (ii), $m_{ST}h$ can be rewritten as

$$m_{ST}h = \sum_{V\in\mathcal{T}_0^+(\lambda,\tau)} r_V m_{SV} + \sum_{\substack{\alpha\in\mathcal{P}_{n,r}\\ \alpha\rhd\lambda}} \sum_{\substack{U'\in\mathcal{T}_0^+(\alpha,\mu)\\ V'\in\mathcal{T}_0^+(\alpha,\tau)}} r_{U'V'}m_{U'V'}$$

with $r_V, r_{U'V'} \in R$. This implies the formula (5.9.1) evaluated at $m_\tau$, and so (5.9.1) follows. $\square$

**5.10.** For an $r$-partition $\lambda \in \mathcal{P}_{n,r}$, let $T^\lambda = \lambda(\mathfrak{t}^\lambda)$. Then $T^\lambda \in \mathcal{T}_0^+(\lambda,\lambda)$, and in fact $T^\lambda$ is the unique semistandard $\lambda$-Tableau of type $\lambda$. Moreover $\mathfrak{t} = \mathfrak{t}^\lambda$ is the unique element in $\operatorname{Std}(\lambda)$ such that $\lambda(\mathfrak{t}) = T^\lambda$. Thus, $m_{T^\lambda T^\lambda} = m_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda} = m_\lambda$, and $\varphi_\lambda = \varphi_{T^\lambda T^\lambda}$ is the identity map on $M^\lambda$. We define the Weyl module $W^\lambda$ as the right $\mathcal{S}(\mathbf{m},n)$-submodule of $\mathcal{S}(\mathbf{m},n)/\check{\mathcal{S}}(\mathbf{m},n)^\lambda$ spanned by the image of $\varphi_\lambda$. For each $S \in \mathcal{T}_0^+(\lambda,\mu)$, we denote by $\varphi_T$ the image of $\varphi_{T^\lambda T}$ in $\mathcal{S}(\mathbf{m}, n)/\check{\mathcal{S}}(\mathbf{m}, n)^\lambda$. Then by Theorem 5.9, we see that

(5.10.1) The Weyl module $W^\lambda$ is $R$-free with basis $\{\varphi_T \mid T \in \mathcal{T}_0^+(\lambda)\}$.

The Weyl module $W^\lambda$ posses an associative symmetric bilinear form, defined by the equation

$$\varphi_{T^\lambda S}\varphi_{TT^\lambda} \equiv \langle \varphi_S, \varphi_T\rangle\, \varphi_\lambda \mod \check{\mathcal{S}}(\mathbf{m}, n)^\lambda.$$

for all $S, T \in \mathcal{T}_0^+(\lambda)$. Note that $\langle \varphi_S, \varphi_T\rangle = 0$ unless $S$ and $T$ are semistandard Tableau of the same type. Let $L^\lambda = W^\lambda/\operatorname{rad} W^\lambda$, where $\operatorname{rad} W^\lambda = \{x \in W^\lambda \mid \langle x, y\rangle = 0 \text{ for all } y \in W^\lambda\}$.

As in the case of $q$-Schur algebras ([M1, Theorem 4.15]), we have

**Proposition 5.11.** *Suppose that $R$ is a field.*

(i) *For an $r$-partition $\lambda \in \mathcal{P}_{n,r}$, $L^\lambda$ is an absolutely irreducible $\mathcal{S}(\mathbf{m}, n)$-module.*
(ii) *$\{L^\lambda \mid \lambda \in \mathcal{P}_{n,r}\}$ is a complete set of non-isomorphic irreducible $\mathcal{S}(\mathbf{m}, n)$-modules.*

*Proof.* We have

$$\varphi_{T^\lambda T^\lambda}\varphi_{T^\lambda T^\lambda} \equiv \langle \varphi_{T^\lambda}, \varphi_{T^\lambda}\rangle\, \varphi_\lambda \mod \check{\mathcal{S}}(\mathbf{m}, n)^\lambda.$$

But since $\varphi_{T^\lambda T^\lambda}\varphi_{T^\lambda T^\lambda} = \varphi_\lambda$ is the identity map on $M^\lambda$, we see that $\langle \varphi_{T^\lambda}, \varphi_{T^\lambda}\rangle = 1$, and so $L^\lambda$ is non-zero. Then the assertions (i) and (ii) follow from the general properties of cellular algebras [GL]. □

**5.12.** We give here a more explicit description of the Weyl module $W^\lambda$. We consider $\operatorname{Hom}_{\mathcal{H}}(\bigoplus_\mu M^\mu, S^\lambda) = \operatorname{Hom}_{\mathcal{H}}(V^{\otimes n}, S^\lambda)$, which has a structure of right $\mathcal{S}(\mathbf{m}, n)$-module by the composition of maps. For each $\mathfrak{t} \in \operatorname{Std}(\lambda)$, let $m_\mathfrak{t} \in S^\lambda$ be the image of $m_{\mathfrak{t}^\lambda\mathfrak{t}}$ under the map $\mathcal{H} \to \mathcal{H}/\check{N}^\lambda$. Then $\{m_\mathfrak{t}\} = \{\overline{m}_\lambda T_{d(\mathfrak{t})}\}$ gives a basis of $S^\lambda$ by 2.19. For $T \in \mathcal{T}_0^+(\lambda, \mu)$, put $m_T = \sum_\mathfrak{t} m_\mathfrak{t} \in S^\lambda$, where the sum is taken over all $\mathfrak{t}$ such that $\mu(\mathfrak{t}) = T$. Since $m_T$ is the image of $m_{T^\lambda T}$, one obtains a well-defined map $\varphi_T \in \operatorname{Hom}_{\mathcal{H}}(M^\mu, S^\lambda)$ by $\varphi_T(m_\mu) = m_T$, which we regard as an element of $\operatorname{Hom}_{\mathcal{H}}(\bigoplus M^\mu, S^\lambda)$ by extending by 0 outside. In a similar way as in [M1, 4.14], we see that

(5.12.1) $W^\lambda$ is isomorphic to the $\mathcal{S}(\mathbf{m}, n)$-submodule of $\operatorname{Hom}_{\mathcal{H}}(\bigoplus_\mu M^\mu, S^\lambda)$ with basis $\{\varphi_T \mid T \in \mathcal{S}_0^+(\lambda)\}$.

**5.13.** We shall describe the structures of $W^\lambda$ and $L^\mu$ in connection with the decomposition of $\mathcal{S}(\mathbf{m}, n)$ in (5.5.1). Note that in the case where $r = 1$, the notation for $\mathcal{S}(\mathbf{m}, n)$ coincides with the standard notation for $q$-Schur algebras discussed as in [M1, Chap.4]. So, we use freely such a notation. For $\lambda, \mu \in \mathcal{P}_{n,r}$, we denote by $[W^\lambda : L^\mu]$ the composition multiplicity of $L^\mu$ in $W^\lambda$. Similarly, $[W^{\lambda^{(i)}} : L^{\mu^{(i)}}]$ $(1 \le i \le r)$ is defined as the composition multiplicity of $L^{\mu^{(i)}}$ in $W^{\lambda^{(i)}}$ for $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ and $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$. Then we have the following.

**Proposition 5.14.** *Suppose that $R$ is a field. Let $\lambda, \mu \in \mathcal{P}_{n,r}$. Then under the isomorphism (5.5.1), we have the following.*

(i) $W^\lambda \simeq W^{\lambda^{(1)}} \otimes \cdots \otimes W^{\lambda^{(r)}}$.

(ii) $L^\mu \simeq L^{\mu^{(1)}} \otimes \cdots \otimes L^{\mu^{(r)}}$.

(iii) $[W^\lambda : L^\mu] = \begin{cases} \prod_{i=1}^r [W^{\lambda^{(i)}} : L^{\mu^{(i)}}] & \text{if } \alpha(\lambda) = \alpha(\mu), \\ 0 & \text{otherwise.} \end{cases}$

*(In (i) and (ii), the right hand sides are regarded as $\mathcal{S}(\mathbf{m}, n)$-modules by the projection from $\mathcal{S}(\mathbf{m}, n)$ to $\mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r)$ via (5.5.1).)*

*Proof.* We shall see how the cellular basis of $\mathcal{S}(\mathbf{m}, n)$ is mapped by the isomorphism (5.5.1). Let $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}$ be such that $\alpha(\mu) = \alpha(\nu) = (n_1, \dots, n_r)$. Take $S \in \mathcal{T}_0^+(\lambda, \mu)$ and $T \in \mathcal{T}_0^+(\lambda, \nu)$ with $S = (S^{(1)}, \dots, S^{(r)}), T = (T^{(1)}, \dots, T^{(r)})$. Then under the bijection in (4.3.1), $S^{(i)} \in \mathcal{T}_0(\lambda^{(i)}, \mu^{(i)})$ and $T^{(i)} \in \mathcal{T}_0(\lambda^{(i)}, \nu^{(i)})$ are semistandard $\lambda^{(i)}$-Tableaux of type $\mu^{(i)}$ or $\nu^{(i)}$ in the case of type $A$. We define $\overline{m}_{ST} \in \overline{M}^{\nu *} \cap \overline{M}^\mu$, and $\overline{\varphi}_{ST} \in \mathrm{Hom}_{\mathcal{H}_\alpha}(\overline{M}^\nu, \overline{M}^\mu)$ by

$$\begin{aligned} \overline{m}_{ST} &= m_{S^{1)}T^{(1)}} \otimes \cdots \otimes m_{S^{r)}T^{(r)}}, \\ \overline{\varphi}_{ST} &= \varphi_{S^{(1)}T^{(1)}} \otimes \cdots \otimes \varphi_{S^{(r)}T^{(r)}}, \end{aligned}$$

where $m_{S^{(i)}T^{(i)}}$ (resp. $\varphi_{S^{(i)}T^{(i)}}$) are corresponding elements in $\mathcal{H}_{n_i} m_{\nu^{(i)}} \cap m_{\mu^{(i)}} \mathcal{H}_{n_i}$ (resp. $\mathrm{Hom}_{\mathcal{H}_{\nu_i}}(m_{\nu^{(i)}} \mathcal{H}_{n_i}, m_{\mu^{(i)}} \mathcal{H}_{n_i})$. Then under the $R$-isomorphism $M^{\nu *} \cap M^\mu \simeq \overline{M}^{\nu *} \cap \overline{M}^\mu$, the basis elements $m_{ST}$ are mapped to $\overline{m}_{ST}$. (Note that any element $m_{\mathfrak{s}\mathfrak{t}}$ occurring in $m_{ST}$ satisfies the property that $d(\mathfrak{s}), d(\mathfrak{t}) \in \mathfrak{S}_\alpha$ by 4.3). It follows that $\varphi_{ST}$ are mapped to $\overline{\varphi}_{ST}$ under the $R$-isomorphism $\mathrm{Hom}_{\mathcal{H}}(M^\nu, M^\mu) \simeq \mathrm{Hom}_{\mathcal{H}_\alpha}(\overline{M}^\mu, \overline{M}^\mu)$. This shows that, under the isomorphism (5.5.1), the basis element $\varphi_{ST}$ of $\mathcal{S}(\mathbf{m}, n)$ is mapped to the basis element $\overline{\varphi}_{ST}$ in $\mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r)$. Note that for $\lambda, \kappa \in \mathcal{P}_{n,r}$ such that $\alpha(\lambda) = \alpha(\kappa)$, the condition $\kappa \rhd \lambda$ is equivalent to the condition $\kappa^{(i)} \rhd \lambda^{(i)}$ for $i = 1, \dots, r$, where $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ an d $\kappa = (\kappa^{(1)}, \dots, \kappa^{(r)})$. This implies that, via (5.5.1), $\check{\mathcal{S}}(\mathbf{m}, n)^\lambda$ is mapped to $\check{\mathcal{S}}(m_1, n_1)^{\lambda^{(1)}} \otimes \cdots \otimes \check{\mathcal{S}}(m_r, n_r)^{\lambda^{(r)}}$. Since $\varphi_\lambda$ is mapped to $\overline{\varphi}_\lambda = \varphi_{\lambda^{(1)}} \otimes \cdots \otimes \varphi_{\lambda^{(r)}}$, and $\varphi_T$ is mapped to $\overline{\varphi}_T = \varphi_{T(1)} \otimes \cdots \otimes \varphi_{T^{(r)}}$, we see that the Weyl module $W^\lambda$ is mapped onto $W^{\lambda^{(1)}} \otimes \cdots \otimes W^{\lambda^{(r)}}$. This proves (i).

Next we show (ii) and (iii). Let $L_1^\mu = L^{\mu^{(1)}} \otimes \cdots \otimes L^{\mu^{(r)}}$ be the irreducible module of $\mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r)$. Then under the isomorphism (5.5.1), $L_1^\mu$ gives an irreducible $\mathcal{S}(\mathbf{m}, n)$-module, and all the irreducible modules of $\mathcal{S}(\mathbf{m}, n)$ are obtained in this way. We have only to show that $L^\mu \simeq L_1^\mu$. Now the decomposition matrix for $L^\mu$ with respect to $W^\lambda$ is unitriangular with respect to the dominance order of $\mathcal{P}_{n,r}$ by [GL]. On the other hand, by (i), and by the results in the case of type $A$, the decomposition matrix for $L_1^\mu$ is also unitriangular. Hence by induction on the dominance order, we can conclude that $L^\mu \simeq L_1^\mu$. This argument also shows that the decomposition number $[W^\lambda : L^\mu]$ is equal to zero unless $\alpha(\lambda) = \alpha(\mu)$, and in that case it coincides with the corresponding decomposition number in $\mathcal{S}(m_1, n_1) \otimes \cdots \otimes \mathcal{S}(m_r, n_r)$. Hence (ii) and (iii) holds. □

## 6. A comparison of cyclotomic $q$-Schur algebras

**6.1.** We shall compare the cyclotomic $q$-Schur algebras $\mathcal{S}(\mathbf{m}, n)$ with the original cyclotomic $q$-Schur algebras defined by Dipper-James-Mathas [DJM], associated to $\widetilde{\mathcal{H}}$.

Recall that the cyclotomic $q$-Schur algebra $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ defined in [DJM] is of the form

$$\widetilde{\mathcal{S}}(\mathbf{m}, n) = \operatorname{End}_{\widetilde{\mathcal{H}}}\Big( \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M_\natural^\mu \Big) = \bigoplus_{\mu,\nu \in \widetilde{\mathcal{P}}_{n,r}} \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M_\natural^\nu, M_\natural^\mu).$$

(Note that the algebra $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ given above is the cyclotomic $q$-Schur algebra $\mathcal{S}(\varLambda)$ with $\varLambda = \widetilde{\mathcal{P}}_{n,r}$, under the notation of [DJM].) We review some properties of $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ following [DJM, 6]. For each $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}$ and $\lambda \in \mathcal{P}_{n,r}$, take $S \in \mathcal{T}_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)$. Let

$$m_{ST}^\natural = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}^\natural$$

where the sum is taken over all $\mathfrak{s}, \mathfrak{t} \in \operatorname{Std}(\lambda)$ such that $\mu(\mathfrak{s}) = S$, $\nu(\mathfrak{t}) = T$. Then

(6.1.1) The set $\{m_{ST}^\natural \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)\}$ gives a basis of $M_\natural^{\nu *} \cap M_\natural^\mu$.

Let $\lambda, \mu, \nu$ be as above. For each $S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)$, we define $\varphi_{ST}^\natural \in \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M_\natural^\nu, M_\natural^\mu)$ by

$$\varphi_{ST}^\natural(m_\nu^\natural h) = m_{ST}^\natural h.$$

for any $h \in \widetilde{\mathcal{H}}$. Then $\varphi_{ST}^\natural$ is well-defined. We extend $\varphi_{TS}^\natural$ to an element in $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ by defining $\varphi_{ST}^\natural = 0$ on $M_\natural^\kappa$ for $\nu \ne \kappa \in \widetilde{\mathcal{P}}_{n,r}$. Let $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \widetilde{\mathcal{P}}_{n,r}} \mathcal{T}_0(\lambda, \mu)$. By [DJM, Theorem 6.6] we have

(6.1.2) $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ is a cellular algebra with cellular basis

$$\mathcal{C}^\natural(\mathbf{m}, n) = \{\varphi_{ST}^\natural \mid \lambda \in \mathcal{P}_{n,r}, S, T \in \mathcal{T}_0(\lambda)\}.$$

For each $\lambda \in \mathcal{P}_{n,r}$, let $\widetilde{\mathcal{S}}^\vee(\mathbf{m}, n)^\lambda$ be the $R$-span of $\varphi_{ST}^\natural$ such that $S, T \in \mathcal{T}_0(\alpha)$ with $\alpha \rhd \lambda$, which is a two-sided ideal of $\widetilde{\mathcal{S}}(\mathbf{m}, n)$. We define the Weyl module $W_\natural^\lambda$ by the right $\widetilde{\mathcal{S}}(\mathbf{m}, n)$-submodule of $\widetilde{\mathcal{S}}(\mathbf{m}, n)/\widetilde{\mathcal{S}}^\vee(\mathbf{m}, n)^\lambda$ generated by the image $\varphi_\lambda^\natural$ of $\varphi_{T^\lambda T^\lambda}^\natural \in \widetilde{\mathcal{S}}(\mathbf{m}, n)$. For each $T \in \mathcal{T}_0(\lambda)$, let $\varphi_T^\natural$ be the image of $\varphi_{T^\lambda T}^\natural$ in $W_\natural^\lambda$. Then

(6.1.3) The Weyl module $W_\natural^\lambda$ is $R$-free with basis $\{\varphi_T^\natural \mid T \in \mathcal{T}_0(\lambda)\}$.

A bilinear form $\langle\, , \rangle_\natural$ on $W_\natural^\lambda$ is defined in a similar way as in 5.10, and the $\widetilde{\mathcal{S}}(\mathbf{m}, n)$-module $L_\natural^\lambda$ is defined as the quotient of $W_\natural^\lambda$ by $\operatorname{rad} W_\natural^\lambda$, where $\operatorname{rad} W_\natural^\lambda$ is the radical of the bilinear form $\langle\, , \rangle_\natural$. Then in the case where $R$ is a field, $\{L_\natural^\lambda \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a complete set of non-isomorphic irreducible $\widetilde{\mathcal{S}}(\mathbf{m}, n)$-modules.

**6.2.** Let $\widehat{M_\natural} = \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} \widehat{M}_\natural^\mu$, where $\widehat{M}_\natural^\mu$ is as in 3.3. If $\varphi \in \widetilde{\mathcal{S}}(\mathbf{m}, n)$ satisfies the property that $\varphi(\widehat{M_\natural}) \subset \widehat{M_\natural}$, then $\varphi$ induces a map $\overline{\varphi} \in \operatorname{End} M$, where $M = \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^\mu \simeq M_\natural/\widehat{M_\natural}$. Here $\overline{\varphi}$ is $\widetilde{\mathcal{H}}$-equivariant, but not necessarily $\mathcal{H}$-equivariant. We define an $R$-subalgebra $\mathcal{S}^0(\mathbf{m}, n)$ of $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ by

$$\mathcal{S}^0(\mathbf{m}, n) = \{\varphi \in \widetilde{\mathcal{S}}(\mathbf{m}, n) \mid \varphi(\widehat{M_\natural}) \subset \widehat{M_\natural}, \overline{\varphi} \in \operatorname{End}_{\mathcal{H}} M\}.$$

Clearly, we have

$$\mathcal{S}^0(\mathbf{m}, n) = \bigoplus_{\mu,\nu \in \widetilde{\mathcal{P}}_{n,r}} \operatorname{Hom}^0_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural)$$

where

$$\begin{aligned}\operatorname{Hom}^0_{\widetilde{\mathcal{H}}}&(M^\nu_\natural, M^\mu_\natural) = \\ &\{\varphi \in \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural) \mid \varphi(\widehat{M}^\nu_\natural) \subset \widehat{M}^\mu_\natural, \overline{\varphi} \in \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)\}.\end{aligned}$$

We have the following lemma.

**Lemma 6.3.** *Suppose that $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}$. Then*

$$\operatorname{Hom}^0_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural) = \begin{cases} \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural) & \text{if } \alpha(\mu) = \alpha(\nu), \\ \{\varphi \in \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural) \mid \varphi(m^\natural_\nu) \in \widehat{M}^\mu_\natural\} & \text{if } \alpha(\mu) \neq \alpha(\nu). \end{cases}$$

*Proof.* First assume that $\alpha(\mu) = \alpha(\nu)$. Take $\varphi \in \operatorname{Hom}_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural)$. We show that $\varphi$ maps $\widehat{M}^\nu_\natural$ into $\widehat{M}^\mu_\natural$. By [DJM, 5.16], there exists $h_\varphi \in \mathcal{H}$ such that $\varphi(m^\natural_\nu h) = h_\varphi m^\natural_\nu h$ for any $h \in \mathcal{H}$. Note that any element in $\widehat{M}^\nu_\natural$ is a linear combination of $m^\natural_{S\mathfrak{t}}$ with $S \in \mathcal{T}_0(\lambda, \nu)$ and $\mathfrak{t} \in \operatorname{Std}(\lambda)$ such that $\mathbf{a}(\lambda) > \mathbf{a}(\nu)$. Assume that $m^\natural_\nu h = m^\natural_{S\mathfrak{t}}$ for such $m^\natural_{S\mathfrak{t}}$. Then by the property of cellular basis, $\varphi(m^\natural_\nu h) = h_\varphi m^\natural_{S\mathfrak{t}}$ is a linear combination of $m^\natural_{\mathfrak{s}'\mathfrak{t}'}$, where $\mathfrak{s}', \mathfrak{t}' \in \operatorname{Std}(\lambda')$ with $\mathbf{a}(\lambda') \geq \mathbf{a}(\lambda) > \mathbf{a}(\nu)$. Since $\mathbf{a}(\nu) = \mathbf{a}(\mu)$ by our assumption, we see that $\varphi(m^\nu_\natural h) \in \widehat{M}^\mu_\natural$. Hence $\varphi$ induces an $\widetilde{\mathcal{H}}$-equivariant map $\overline{\varphi} : M^\nu \to M^\mu$. In particular $\overline{\varphi}$ is $\mathcal{H}_n$-equivariant, and $\overline{\varphi}$ is determined uniquely by $\overline{\varphi}(m_\nu)$. Thus, in order to see $\overline{\varphi}$ is $\mathcal{H}$-equivariant, we have only to show that $\overline{\varphi}(m_\nu) \in M^{\nu*} \cap M^\mu$. But by our construction, $\overline{\varphi}(m_\nu)$ coincides with $f(\varphi(m^\natural_\nu))$ under the map $f : M^\mu_\natural \to M^\mu$. Our assertion then follows from the following fact

(6.3.1) Let $f : M^\mu_\natural \to M^\mu$ be the natural map as before. Then we have $f(m^\natural_{ST}) = m_{ST}$ for any $S \in \mathcal{T}_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)$. In particular, $f(M^{\nu*}_\natural \cap M^\mu_\natural) = M^{\nu*} \cap M^\mu$.

In fact, by (6.1.1), $M^{\nu*}_\natural \cap M^\mu_\natural$ has a basis $\{m^\natural_{ST} \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu))\}$. By Proposition 5.7, $M^{\nu*} \cap M^\mu$ has a basis $\{m_{ST} \mid S \in \mathcal{T}^+_0(\lambda, \mu), T \in \mathcal{T}^+_0(\lambda, \nu)\}$. Moreover by Theorem 4.9, $f$ maps $M^{\nu*}_\natural \cap M^\mu_\natural$ onto $M^{\nu*} \cap M^\mu$. This shows (6.3.1), and (i) follows.

Next assume that $\alpha(\mu) \neq \alpha(\nu)$, and take $\varphi \in \operatorname{Hom}^0_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural)$. Then $\varphi$ induces a map $\overline{\varphi} \in \operatorname{Hom}_{\mathcal{H}}(M^\nu, M^\mu)$, which is a zero map by Proposition 5.2 (i). It follows that $\varphi(M^\nu_\natural) \subseteq \widehat{M}^\mu_\natural$, which is equivalent to $\varphi(m^\natural_\nu) \in \widehat{M}^\mu_\natural$. This proves the second assertion. □

The following result gives a relationship between Dipper-James-Mathas' cyclotomic $q$-Schur algebra $\widetilde{\mathcal{S}}(\mathbf{m}, n)$ and our algebra $\mathcal{S}(\mathbf{m}, n)$ by means of $\mathcal{S}^0(\mathbf{m}, n)$.

**Proposition 6.4.** (i) *$\mathcal{S}^0(\mathbf{m}, n)$ is free as an $R$-module with basis*

$$\begin{aligned}\mathcal{C}^0(\mathbf{m}, n) = \bigcup_{\mu,\nu \in \widetilde{\mathcal{P}}_{n,r}, \lambda \in \mathcal{P}_{n,r}} &\{\varphi^\natural_{ST} \in \mathcal{C}^\natural(\mathbf{m}, n) \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu), \\ &\mathbf{a}(\lambda) > \mathbf{a}(\mu) \text{ if } \alpha(\mu) \neq \alpha(\nu)\}.\end{aligned}$$

(ii) *There exists a surjective algebra homomorphism* $\widehat{f} : \mathcal{S}^0(\mathbf{m}, n) \to \mathcal{S}(\mathbf{m}, n)$ *satisfying the following. For* $\varphi^\natural_{ST} \in \mathcal{C}^0(\mathbf{m}, n)$ *with* $S, T \in \mathcal{T}_0(\lambda)$,

$$\widehat{f}(\varphi^\natural_{ST}) = \begin{cases} \varphi_{ST} & \text{if } S, T \in \mathcal{T}_0^+(\lambda), \\ 0 & \text{otherwise.} \end{cases} \tag{6.4.1}$$

*Proof.* By (6.1.1), $M^{\nu *}_\natural \cap \widehat{M}^\mu_\natural$ is $R$-free with basis $\{m^\natural_{ST}\}$ such that $S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)$ for some $\lambda \in \mathcal{P}_{n,r}$ with $\mathbf{a}(\lambda) > \mathbf{a}(\mu)$. Then (i) follows from Lemma 6.5 and the fact that $\operatorname{Hom}_{\widetilde{\mathcal{H}}}(M^\nu_\natural, M^\mu_\natural) \simeq M^{\nu *}_\natural \cap M^\mu_\natural$ by $\varphi \mapsto \varphi(m^\natural_\nu)$ (cf. [DJM, 5.2]).

Next we show (ii). Take $\varphi \in \widehat{\mathcal{S}}(\mathbf{m}, n)$. By definition, it induces an endomorphism $\overline{\varphi} \in \operatorname{End}_{\mathcal{H}} M = \mathcal{S}(\mathbf{m}, n)$. The map $\varphi \mapsto \overline{\varphi}$ gives an algebra homomorphism $\widehat{f} : \widehat{\mathcal{S}}(\mathbf{m}, n) \to \mathcal{S}(\mathbf{m}, n)$. The equation (6.4.1) is immediate from (6.3.1). The surjectivity of $\widehat{f}$ then follows from (6.4.1). $\square$

**Remark 6.5.** The basis $\mathcal{C}^0(\mathbf{m}, n)$ satisfies the following property. Let $\mu, \nu \in \widetilde{\mathcal{P}}_{n,r}, \lambda \in \mathcal{P}_{n,r}$ and $\varphi \in \mathcal{S}^0(\mathbf{m}, n)$. Then for any $\kappa \in \widetilde{\mathcal{P}}_{n,r}$ and $T' \in \mathcal{T}_0(\lambda, \kappa)$, there exists $r_{T'} \in R$ such that for all $S \in \mathcal{T}_0(\lambda, \mu)$, we have

$$\varphi^\natural_{ST}\varphi \equiv \sum_{\kappa \in \widetilde{\mathcal{P}}_{n,r}} \sum_{T' \in \mathcal{T}_0(\lambda, \kappa)} r_{T'} \varphi^\natural_{ST'} \mod \check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda \tag{6.5.1}$$

where $\varphi^\natural_{ST}, \varphi^\natural_{ST'} \in \mathcal{C}^0(\mathbf{m}, n)$ and $\check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda = \mathcal{S}^0(\mathbf{m}, n) \cap \widetilde{\mathcal{S}}^\vee(\mathbf{m}, n)^\lambda$. However, $\mathcal{C}^0(\mathbf{m}, n)$ is not a cellular basis since $* : \varphi^\natural_{ST} \mapsto \varphi^\natural_{TS}$ is not defined.

**6.6.** For $\lambda \in \mathcal{P}_{n,r}$, $\varphi^\natural_\lambda = \varphi^\natural_{T^\lambda T^\lambda}$ is an element in $\mathcal{S}^0(\mathbf{m}, n)$. Hence one can define a Weyl module $Z^\lambda$ of $\mathcal{S}^0(\mathbf{m}, n)$ as the right $\mathcal{S}^0(\mathbf{m}, n)$-submodule of $\mathcal{S}^0(\mathbf{m}, n)/\check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda$ spanned by the image of $\varphi^\natural_\lambda$. We denote by $\varphi^0_T$ the image of $\varphi^\natural_{T^\lambda T}$ in $\mathcal{S}^0(\mathbf{m}, n)/\check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda$. It follows by (6.5.1) that

(6.6.1) The Weyl module $Z^\lambda$ is $R$-free with basis $\{\varphi^0_T \mid T \in \mathcal{T}_0^+(\lambda)\}$.

The following lemma describes the relationship of various Weyl modules.

**Lemma 6.7.** *Let* $W^\lambda_\natural, Z^\lambda, W^\lambda$ *be the Weyl modules of* $\widetilde{\mathcal{S}}(\mathbf{m}, n), \mathcal{S}^0(\mathbf{m}, n), \mathcal{S}(\mathbf{m}, n)$, *respectively.*

(i) *There exists an injective* $R$*-linear map* $f_1 : Z^\lambda \to W^\lambda_\natural$, *compatible with the action of* $\mathcal{S}^0(\mathbf{m}, n)$ *such that* $f_1(\varphi^0_T) = \varphi^\natural_T$ *for* $T \in \mathcal{T}_0^+(\lambda)$.

(ii) *There exists an* $\mathcal{S}^0(\mathbf{m}, n)$*-module isomorphism* $f_2 : Z^\lambda \to W^\lambda$ *such that* $f_2(\varphi^0_T) = \varphi_T$ *for any* $T \in \mathcal{T}_0^+(\lambda)$, *where* $W^\lambda$ *is regarded as an* $\mathcal{S}^0(\mathbf{m}, n)$*-module via the map* $\widehat{f}$ *in Proposition 6.4 (ii).*

*Proof.* The inclusion $\mathcal{S}^0(\mathbf{m}, n) \hookrightarrow \widetilde{\mathcal{S}}(\mathbf{m}, n)$ induces an injective $\mathcal{S}^0(\mathbf{m}, n)$-equivariant map $\mathcal{S}^0(\mathbf{m}, n)/\check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda \to \widetilde{\mathcal{S}}(\mathbf{m}, n)/\widetilde{\mathcal{S}}^\vee(\mathbf{m}, n)^\lambda$, which maps $\varphi^0_T$ to $\varphi^\natural_T$ for any $T \in \mathcal{T}_0(\lambda)$. Hence (i) holds.

The surjection $\widehat{f} : \mathcal{S}^0(\mathbf{m}, n) \to \mathcal{S}(\mathbf{m}, n)$ induces a map $\mathcal{S}^0(\mathbf{m}, n)/\check{\mathcal{S}}^0(\mathbf{m}, n)^\lambda \to \mathcal{S}(\mathbf{m}, n)/\check{\mathcal{S}}^\lambda(\mathbf{m}, n)$, which is comaptible with the action of $\mathcal{S}^0(\mathbf{m}, n)$. This map sends the basis $\{\varphi^0_T\}$ of $Z^\lambda$ to the basis $\{\varphi_T\}$ of $W^\lambda$. Thus (ii) holds. □

Concerning the irreducible modules, we have the following.

**Lemma 6.8.** *Suppose that $R$ is a field. Let $L^\mu_\natural$ (resp. $L^\mu$) be the irreducible $\widetilde{\mathcal{S}}(\mathbf{m}, n)$-module (resp. $\mathcal{S}(\mathbf{m}, n)$-module). We regard them as $\mathcal{S}^0(\mathbf{m}, n)$-modules, via the restriction and via the map $\widehat{f}$. Then $\mathcal{S}^0(\mathbf{m}, n)$-module $L^\mu_\natural$ contains $L^\mu$ as an irreducible submodule.*

*Proof.* Let $\langle\, ,\, \rangle_\natural$ (resp. $\langle\, ,\, \rangle$) be the bilinear form on $W^\lambda_\natural$ (resp. on $W^\lambda$ ). By identifying $Z^\lambda$ with $W^\lambda$ by the map $f_2$, we show that $f_1 : Z^\lambda \to W^\lambda_\natural$ maps $\operatorname{rad} W^\lambda$ into $\operatorname{rad} W^\lambda_\natural$. In fact, it is easy to see that $\langle\varphi^\natural_S, \varphi^\natural_T\rangle_\natural = \langle\varphi_S, \varphi_T\rangle$ for any $S, T \in \mathcal{T}^+_0(\lambda)$. On the other hand, if $S \in \mathcal{T}^+_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)\backslash\mathcal{T}^+_0(\lambda, \nu)$, then $\alpha(\mu) \ne \alpha(\nu)$, and so $\mu \ne \nu$. It follows that $\varphi^\natural_{T^\lambda S}\varphi^\natural_{TT^\lambda} = 0$, and $\langle\varphi^\natural_S, \varphi^\natural_T\rangle_\natural = 0$. Hence $f_1(\operatorname{rad} W^\lambda) \subset \operatorname{rad} W^\lambda_\natural$. Now $f_1$ induces an $\mathcal{S}^0(\mathbf{m}, n)$-equivariant map $L^\lambda \to L^\lambda_\natural$, which is clearly non-trivial. Since $L^\mu$ is an irreducible $\mathcal{S}^0(\mathbf{m}, n)$-module, $L^\mu_\natural$ contains an irreducible module isomorphic to $L^\mu$. □

## 7. Schur functors

**7.1.** Recall the $R$-submodule $\mathcal{U}$ of $V^{\otimes n}$ defined in 1.4. Under the identification $V^{\otimes n} \simeq \bigoplus_{\mu\in\widetilde{\mathcal{P}}_{n,r}} M^\mu$ in 5.4, $\mathcal{U}$ can be identified with $\bigoplus_{\omega\in\Omega} M^\omega$, where $\Omega$ is the set of $r$-compositions $\omega \in \widetilde{\mathcal{P}}_{n,r}$ of the form $\omega = (\omega^{(1)}, \dots, \omega^{(r)})$ such that $\omega^{(k)}_j \in \{0, 1\}$ and that $\sum_{1\le k\le r} \omega^{(k)}_j = 1$ for $j = 1, \dots, n$. $\mathcal{U} = v_0\mathcal{H}$ is a regular $\mathcal{H}$-module, and $\operatorname{Hom}_\mathcal{H}(\mathcal{U}, \mathcal{U})$ is isomorphic to $\mathcal{H}$ as $R$-algebras via $\varphi_h \leftrightarrow h$, where $\varphi_h$ is a map on $\mathcal{U}$ determined by $\varphi_h(v_0) = v_0h$. Put $\varphi_\Omega = \sum_{\omega\in\Omega} \varphi_{T^\omega T^\omega}$. Then $\varphi_\Omega$ is the identity map on $\mathcal{U}$ and is zero on the complement of $\mathcal{U}$. For each $\omega \in \Omega$, we consider $v^\natural_\omega \in V^{\otimes n}$ (see (3.3.2)). Then there exists a unique $d(\omega) \in \mathcal{D}_{\alpha(\omega)}$ such that $v^\natural_\omega T_{d(\omega)} = v^c_J$ with $J = (n, n-1, \dots, 1)$ and some $c \in [1, r]^n$. It follows that $v_0$ can be written as

$$v_0 = \sum_{\omega\in\Omega} v^\natural_\omega T_{d(\omega)}, \tag{7.1.1}$$

and so $v_0$ corresponds to an element $\sum_{\omega\in\Omega} m_\omega T_{d(\omega)} \in \bigoplus_{\omega\in\Omega} M^\omega$.

Now $\varphi_\Omega$ is an idempotent in $\mathcal{S}(\mathbf{m}, n)$ and $\varphi_\Omega\mathcal{S}(\mathbf{m}, n)\varphi_\Omega = \operatorname{Hom}_\mathcal{H}(\mathcal{U}, \mathcal{U}) \simeq \mathcal{H}$. Hence one can define a functor, which we call a Schur functor, from the category of right $\mathcal{S}(\mathbf{m}, n)$-modules to the category of right $\mathcal{H}$-modules, by associating $M\varphi_\Omega$ to any $\mathcal{S}(\mathbf{m}, n)$-module $M$.

The following result is an analogy of the case of Iwahori-Hecke algebras $\mathcal{H}_n$ ([M1, 4.18]), and of $\widetilde{\mathcal{H}}$ ([JM]).

**Proposition 7.2.** *For each $\lambda \in \mathcal{P}_{n,r}$, there exist isomorphisms of right $\mathcal{H}$-modules.*

(i) $W^\lambda\varphi_\Omega \simeq S^\lambda$,

(ii) $L^\lambda\varphi_\Omega \simeq D^\lambda$.

*Furthermore, for $\mu \in \mathcal{P}_{n,r}$ such that $D^\mu \neq 0$, we have $[W^\lambda : L^\mu] = [S^\lambda : D^\mu]$.*

*Proof.* Let $T \in \mathcal{T}_0^+(\lambda, \nu)$ and $\varphi_T \in \mathrm{Hom}(\bigoplus_\mu M^\mu, S^\lambda)$ be the element in the semistandard basis of $W^\lambda$ (cf. (5.12.1)). Then we see that $\varphi_T \varphi_\Omega = 0$ unless $\nu = \omega$ for some $\omega \in \Omega$. If $\nu = \omega$, we have $\varphi_T \varphi_\Omega = \varphi_T$. Thus $W^\lambda \varphi_\Omega$ is a free $R$-module with basis

$$\{\varphi_T \mid T \in \bigcup_{\omega \in \Omega} \mathcal{T}_0^+(\lambda, \omega)\}. \tag{7.2.1}$$

Here any $T \in \mathcal{T}_0^+(\lambda, \omega)$ can be seen as a standard tableau $\mathfrak{t} \in \mathrm{Std}(\lambda)$ in an obvious way. Then $\mathcal{T}_0^+(\lambda, \omega)$ is identified with the set of $\mathfrak{t} \in \mathrm{Std}(\lambda)$ such that the entries in $\mathfrak{t}^{(i)}$ consists of $\{j \mid \omega_j^{(i)} \neq 0\}$. It follows that the set $\bigcup_{\omega \in \Omega} \mathcal{T}_0^+(\lambda, \omega)$ may be identified with $\mathrm{Std}(\lambda)$. For $T \in \mathcal{T}_0^+(\lambda, \omega)$, there exists a unique $\mathfrak{t}_1 \in \mathrm{Std}(\lambda)$ such that $\lambda(\mathfrak{t}_1) = T$. Moreover it is easy to see that $\mathfrak{t} = \mathfrak{t}_1 d(\omega)$. We note that $\mathfrak{t}_1$ is a standard tableau of the type the entries of $\mathfrak{t}_1^{(i)}$ consist of integers in the set $\{p \mid a_i < p \leq a_{i+1}\}$ for $\mathbf{a}(\omega) = (a_1, \dots, a_r)$. We claim that

(7.2.2) For $T = \mathfrak{t} \in \mathcal{T}_0^+(\lambda, \omega)$, we have $\varphi_T(v_0) = m_\mathfrak{t} \in S^\lambda$.

In fact, by (7.1.1) and the remark after that, we have

$$\varphi_T(v_0) = \varphi_T(m_\omega T_{d(\omega)}) = \varphi_T(m_\omega) T_{d(\omega)} = m_T T_{d(\omega)}.$$

By definition, $m_T = m_{\mathfrak{t}_1} = m_{\mathfrak{t}^\lambda \mathfrak{t}_1} + \check{\mathcal{S}}(\mathbf{m}, n)^\lambda \in S^\lambda$. Since

$$m_{\mathfrak{t}^\lambda \mathfrak{t}_1} T_{d(\omega)} = m_\lambda T_{d(\mathfrak{t}_1)} T_{d(\omega)} = m_\lambda T_{d(\mathfrak{t})} = m_{\mathfrak{t}^\lambda \mathfrak{t}},$$

we obtain (7.2.2).

Now, by the $\mathcal{H}$-equivariant map $W^\lambda \varphi_\Omega \to S^\lambda$, $\varphi \mapsto \varphi(v_0)$, the basis of $W^\lambda \varphi_\Omega$ in (7.2.1) is mapped to the basis $\{m_\mathfrak{t} \mid \mathfrak{t} \in \mathrm{Std}(\lambda)\}$ of $S^\lambda$. This proves (i).

We show (ii). Let $S, T \in \mathcal{T}_0^+(\lambda, \omega)$, and let $\alpha = \alpha(\lambda) = \alpha(\omega)$. First we note that

$$m_{TT^\lambda} = F_\alpha m_{TT^\lambda} = m_\omega m_{TT^\lambda}. \tag{7.2.3}$$

In fact, $m_{TT^\lambda} = m_{\mathfrak{t}_1 \mathfrak{t}^\lambda} = T^*_{d(\mathfrak{t}_1)} m_\lambda$. Now, $m_\lambda \in \mathcal{H}$ corresponds to $v^\natural_{\lambda'} \in V^{\otimes n}_{\lambda'}$ under the isomorphism $m_\lambda \mathcal{H} = M^\lambda \simeq V^{\otimes n}_{\lambda'}$. Let $\alpha = \alpha(\lambda) = (n_1, \dots, n_r)$. Then $v^\natural_{\lambda'}$ is expressed as $v^\natural_{\lambda'} = v^c_I$ for some $I$ with

$$c = (\underbrace{r, \dots, r}_{n_1-\text{times}}, \dots, \underbrace{1, \dots, 1}_{n_r-\text{times}})$$

by (3.3.2). This implies that $m_\lambda F_\alpha = m_\lambda$ by Lemma 2.11. Since $d(\mathfrak{t}_1) \in \mathfrak{S}_\alpha$, $F_\alpha$ commutes with $m_\lambda$ and $T^*_{d(\mathfrak{t}_1)}$. This shows the first equality. Since $m_{\{\omega\}} = 1$, we have $m_\omega = F_\alpha m_{\{\omega\}} = F_\alpha$. The second equality follows from this.

It follows from (7.2.3) that

$$\varphi_{T^\lambda S} \varphi_{TT^\lambda}(m_\lambda) = \varphi_{T^\lambda S}(m_{TT^\lambda}) = \varphi_{T^\lambda S}(m_\omega) m_{TT^\lambda} = m_{T^\lambda S} m_{TT^\lambda}. \tag{7.2.4}$$

We now consider the defining equation

$$\langle \varphi_S, \varphi_T \rangle \, \varphi_{T^\lambda T^\lambda} \equiv \varphi_{T^\lambda S} \varphi_{T T^\lambda} \mod \check{\mathcal{S}}(\mathbf{m}, n)^\lambda.$$

By evaluating both sides by $m_\lambda$, together with (7.2.4), we have

$$\langle \varphi_S, \varphi_T \rangle \, m_\lambda \equiv m_{T^\lambda S} m_{T T^\lambda} \mod \check{N}^\lambda.$$

This implies that $\langle \varphi_S, \varphi_T \rangle = \langle m_{\mathfrak{s}}, m_{\mathfrak{t}} \rangle$ for $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ corresponding to $S, T$ respectively. (ii) follows from this.

The last statement is completely similar to [M1, 4.18]. We consider a composition series $W^\lambda = W^1 > \cdots > W^k > 0$. Then $S^\lambda \simeq W^1\varphi_\Omega \geq \cdots \geq W^k\varphi_\Omega \geq 0$, and this implies that $W^i\varphi_\Omega / W^{i+1}\varphi_\Omega \simeq L^{\mu_i}\varphi_\Omega \simeq D^{\mu_i}$ for some $\mu_i \in \mathcal{P}_{n,r}$. Hence if $D^\mu \neq 0$, the number of $D^\mu$ occurring in the series of $S^\lambda$ is the same as the number of $L^\mu$ occurring in the series of $W^\lambda$. $\square$

Recall that a 1-partition $\nu = (\nu_1, \dots, \nu_m) \in \mathcal{P}_{n,1}$ is called $e$-restricted if $\nu_i - \nu_{i+1} < e$ for each $i$, where $e$ is the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$. It is known in the case of Iwahori-Hecke algebra $\mathcal{H}_n$, $D^\nu \neq 0$ if and only if $\nu$ is $e$-restricted. The following result is a consequence of Corollary 4.13, Proposition 5.13 and Proposition 7.2.

**Corollary 7.3.** *Suppose that $R$ is a filed. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)}), \mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \mathcal{P}_{n,r}$. Suppose that $D^\mu_\natural \neq 0$ and that $\mu^{(i)}$ is $e$-restricted for $i = 1, \dots, r$.*

(i) *Suppose that $\alpha(\lambda) = \alpha(\mu) = (n_1, \dots, n_r)$. Then we have*

$$[S^\lambda_\natural : D^\mu_\natural] \geq \prod_{i=1}^{r} [S^{\lambda^{(i)}} : D^{\mu^{(i)}}],$$

*where $[S^{\lambda^{(i)}} : D^{\mu^{(i)}}]$ is the composition multiplicity in the Iwahori-Hecke algebra $\mathcal{H}_{n_i}$.*

(ii) *Suppose that $\mathcal{H} \simeq \widetilde{\mathcal{H}}$. Then $[S^\lambda_\natural : D^\mu_\natural] = 0$ unless $\alpha(\lambda) = \alpha(\mu)$. If $\alpha(\lambda) = \alpha(\mu)$, then the equality holds in (i).*

**Remark 7.4.** As the remark in the next section (7.3.2) shows, $\mathcal{H} \simeq \widetilde{\mathcal{H}}$ if the parameters satisfy the separation condition (5.4.2).

By applying the Schur functor on the isomorphism in Theorem 5.5, we have the following result, which was obtained by [HS] in the case where $R$ is a field, in a different method.

**Proposition 7.5.** *For each $r$-tuple $\alpha = (n_1, \dots, n_r)$ such that $n_1 + \cdots n_r = n$, put $n_\alpha = n!/(n_1! \cdots n_r!)$. Then we have an isomorphism of $R$-algebras*

$$\mathcal{H} \simeq \bigoplus_{\substack{\alpha = (n_1, \dots, n_r) \\ n_1 + \cdots + n_r = n}} M_{n_\alpha}(\mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_r}),$$

*where $M_n(A)$ denotes the matrix algebra of degree $n$ over the algebra $A$.*

*Proof.* We have
$$\mathcal{H} \simeq \varphi_\Omega \mathcal{S}(\mathbf{m}, n)\varphi_\Omega \simeq \bigoplus_\alpha \operatorname{End}_{\mathcal{H}}(M^\alpha \cap \mathcal{U}).$$
But by (5.5.2),
$$\operatorname{End}_{\mathcal{H}}(M^\alpha \cap \mathcal{U}) \simeq \bigoplus_{\substack{\omega,\omega' \in \Omega \\ \alpha(\omega)=\alpha(\omega')=\alpha}} \operatorname{Hom}_{\mathcal{H}_\alpha}(\overline{M}^\omega, \overline{M}^{\omega'}).$$
Here $\overline{M}^\omega$ are isomorphic each other as $\mathcal{H}_\alpha$-modules for any $\omega \in \Omega$ such that $\alpha(\omega) = \alpha$, and we have
$$\operatorname{End}_{\mathcal{H}_\alpha} \overline{M}^\omega \simeq \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_r}.$$
The proposition follows from this by noticing that $\sharp\{\omega \in \Omega \mid \alpha(\omega) = \alpha\} = n_\alpha$. $\square$

**Remark 7.6.** Proposition 7.5 implies that $D^\mu \neq 0$ if and only if $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ with all $\mu^{(i)}$ $e$-restricted. By combining this with Theorem 4.13 (ii), we recover the result in [DJMu, Theorem 6.9]; If $D^\mu_\natural \neq 0$, then all the partitions $\mu^{(i)}$ are $e$-restricted.

## 8. Schur-Weyl reciprocity

**8.1.** The tensor space $V^{\otimes n}$ has a structure of $U_q$-$\mathcal{H}$-bimodule (the left action of $U_q$ and the right action of $\mathcal{H}$), which is the same as the $\mathcal{S}(\mathbf{m}, n)$-$\mathcal{H}$-bimodule structure on $V^{\otimes n}$ by Theorem 5.5 (ii). It is known in [SS] that the Schur-Weyl reciprocity (i.e., the double centralizer property) between $U_q$ and $\mathcal{H}$ holds in the case where $R = K_0$. In this section, we shall extend this to the case of $R$.

**Theorem 8.2.** *Let $\rho : \mathcal{H} \to \operatorname{End} V^{\otimes n}$, $\tau : U_q \to \operatorname{End} V^{\otimes n}$ be the representations of $\mathcal{H}$, $U_q$ on $V^{\otimes n}$, respectively. Then we have*
$$\tau(U_q) = \operatorname{End}_{\mathcal{H}} V^{\otimes n}, \qquad \rho(\mathcal{H}) = \operatorname{End}_{U_q} V^{\otimes n}.$$

*Proof.* The first equality is equivalent to Theorem 5.5, (ii). We show the second equality. Instead of discussing with $U_q$, we will work on $\mathcal{S}(\mathbf{m}, n)$. It is enough to show that

$$\rho(\mathcal{H}) = \operatorname{End}_{\mathcal{S}(\mathbf{m},n)} V^{\otimes n}. \tag{8.2.1}$$

The proof is similar to [M1, Theorem 4.19]. For each $\mu \in \widetilde{\mathcal{P}}_{n,r}$, let $\varphi_\mu = \varphi_{T^\mu T^\mu}$ be the identity map on $M^\mu$. As before, we regard $\varphi_\mu$ as an element of $\mathcal{S}(\mathbf{m}, n)$. Then $\sum_\mu \varphi_\mu$ is the identity map of $\mathcal{S}(\mathbf{m}, n)$. First we note that $\varphi_\mu \mathcal{S}(\mathbf{m}, n)\varphi_\Omega \simeq M^\mu$ as $\mathcal{H}$-modules. Since $\varphi_\mu \mathcal{S}(\mathbf{m}, n)\varphi_\Omega = \bigoplus_{\omega\in\Omega} \operatorname{Hom}_{\mathcal{H}}(M^\omega, M^\mu)$, we see that $\varphi_\mu \mathcal{S}(\mathbf{m}, n)\varphi_\Omega$ is a free $R$-module with basis $\{\varphi_{ST}\}$, where $S \in \mathcal{T}_0^+(\lambda, \mu), T \in \mathcal{T}_0^+(\lambda, \omega)$ for some $\lambda \in \mathcal{P}_{n,r}$ and $\omega \in \Omega$. As in the proof of (8.2.2) (under the notation there), we have
$$\varphi_{ST}(v_0) = \varphi_{ST}(m_\omega T_{d(\omega)}) = m_{ST} T_{d(\omega)} = m_{S\mathfrak{t}},$$

where $\mathfrak{t} \in \operatorname{Std}(\lambda)$ is the underlying standard tableau of $T$. Since $\{m_{S\mathfrak{t}}\}$, where $S \in \mathcal{T}_0^+(\lambda, \mu), \mathfrak{t} \in \operatorname{Std}(\lambda)$ is a basis of $M^\mu$, we see that $\varphi \mapsto \varphi(v_0)$ gives an $\mathcal{H}$-module isomorphism between $\varphi_\mu \mathcal{S}(\mathbf{m}, n)\varphi_\Omega$ and $M^\mu$.

It follows that

$$V^{\otimes n} \simeq \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^\mu \simeq \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} \operatorname{Hom}_{\mathcal{H}}(\mathcal{U}, M^\mu) \simeq \bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} \varphi_\mu \mathcal{S}(\mathbf{m}, n)\varphi_\Omega.$$

Since $\sum_\mu \varphi_\mu$ is the identity map on $\bigoplus_{\mu \in \widetilde{\mathcal{P}}_{n,r}} M^\mu$, we see that $\bigoplus_\mu M^\mu \simeq \mathcal{S}(\mathbf{m}, n)\varphi_\Omega$ as left $\mathcal{S}(\mathbf{m}, n)$-modules. This implies that

$$\begin{aligned}\operatorname{End}_{\mathcal{S}(\mathbf{m},n)} V^{\otimes n} &\simeq \operatorname{End}_{\mathcal{S}(\mathbf{m},n)}(\mathcal{S}(\mathbf{m}, n)\varphi_\Omega) \\ &\simeq \varphi_\Omega \mathcal{S}(\mathbf{m}, n)\varphi_\Omega \\ &\simeq \mathcal{H}.\end{aligned}$$

This isomorphism is explicitly given by the map $\varphi \mapsto h_\varphi$ for $\varphi \in \operatorname{End}_{\mathcal{S}(\mathbf{m},n)} V^{\otimes n}$, where $h_\varphi \in \mathcal{H}$ is determined by the condition that $\varphi(v_0) = v_0 h_\varphi$. Now $\rho(\mathcal{H})$ is contained in $\operatorname{End}_{\mathcal{S}(\mathbf{m},n)} V^{\otimes n}$, and one can check that $h_\varphi = h$ for any $\varphi = \rho(h)$. This shows that $\rho(\mathcal{H}) = \operatorname{End}_{\mathcal{S}(\mathbf{m},n)} V^{\otimes n}$. Hence (8.2.1) holds. □

**Remarks 8.3.** (i) In [HS, Theorem 3.3], Hu and Stoll extended the action of $\mathcal{H}$ on the space $V^{\otimes n}$ to the general case where no restrictions on $m_i$ are assumed, and proved the second equality of Theorem 8.2 for this general case, in a different method, under the assumption that $R$ is a field.

(ii) They also proved in [HS, Theorem 1.10] a similar formula for the Ariki-Koike algebra $\widetilde{\mathcal{H}}$, subject to the condition that $R$ is a field with the separation condition (5.4.2), i.e.,

$$\widetilde{\rho}(\widetilde{\mathcal{H}}) = \operatorname{End}_{U_q} V^{\otimes n}. \tag{8.3.1}$$

This implies the following result.

(8.3.2) Suppose that $R$ is a field satisfying the separation condition. Then $\widetilde{\mathcal{H}} \simeq \mathcal{H}$.

In fact, we consider the case where $m_i \geq n$ for any $i$. By Theorem 8.2 and (8.3.1), we have $\rho(\mathcal{H}) = \widetilde{\rho}(\widetilde{\mathcal{H}})$. Thus by Lemma 1.5, we have $\mathcal{H} \simeq \widetilde{\mathcal{H}}$.

It is likely that the separation condition will be the necessary and sufficient condition for the isomorphism $\mathcal{H} \simeq \widetilde{\mathcal{H}}$.

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