THE EIGEN- AND SINGULAR VALUES OF THE SUM AND PRODUCT OF LINEAR OPERATORS

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Contents

Introduction

V.B. Lidskii [1] proved the following theorem, which in intuitive geometrical terms establishes a connection between the eigenvalues of the sum of two Hermitian matrices and the eigenvalues of the summands.

Let A and B be Hermitian matrices of order n, and K_1 (respectively K_2) the convex hull of the set of vectors of the form[1] $\{\lambda_j(B) + \lambda_{k_j}(A)\}_1^n$ *(respectively $\{\lambda_j(A) + \lambda_{k_j}(B)\}_1^n$), where $k_1, k_2, \ldots, k_n$ is an arbitrary permutation of the numbers $1, 2, \ldots, n$. Then the vector $\{\lambda_j(A + B)\}_1^n$ is in the intersection of K_1 and K_2.*

Also in [1] an analogous theorem was established for the eigenvalues of the product of two positive definite matrices.[2] This theorem was later carried over to the case of unitary matrices by A.A. Nudel'man and P.A. Shvartsman [3]. (For the formulation of these two theorems see § 2, 3.)

In his paper [4] Ky Fan established (Theorem 1) a property of the eigenvalues of an Hermitian matrix from which it follows immediately that

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[1] $\{\lambda_j(A)\}_1^n$ denotes the complete system of eigenvalues of A, numbered in decreasing order.

[2] As Lidskii mentions, the latter result had been obtained earlier by I.M. Gel'fand and M.A. Naimark in the course of their investigations into the theory of group representations. Proofs of both these theorems based on group-theoretical methods are given in the paper [2] by F.A. Berezin and I.M. Gel'fand.

$$\sum_{j=1}^{k} \lambda_j(A+B) \leqslant \sum_{j=1}^{k} \lambda_j(A) + \sum_{j=1}^{k} \lambda_j(B) \quad (k=1, 2, \ldots, n). \tag{0.1}$$

Shortly afterwards [5] he proved this same inequality for the singular values[1] of arbitrary matrices A and B, while Horn [6] established an analogous inequality for the singular values of the product of two matrices (to be precise, they established these results for arbitrary completely continuous operators in Hilbert space).

Apparently no connections between Lidskii's geometrical propositions and inequalities of the type (0.1) were discovered at the time.

In 1955 Wielandt [7], with the help of minimax properties of the sums of eigenvalues of Hermitian matrices which he had proved, established in generalization of (0.1) that for any system of indices $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$

$$\sum_{p=1}^{k} \lambda_{j_p}(A+B) \leqslant \sum_{p=1}^{k} \lambda_{j_p}(A) + \sum_{j=1}^{k} \lambda_j(B). \tag{0.2}$$

He discovered that by a result of Birkhoff [8] (see Remark 1.1 below) the inequalities (0.2) (together with those obtained from (0.2) by interchanging A and B) are equivalent to the above theorem of Lidskii.

One year later Amir-Moéz, developing Wielandt's method, obtained a number of new inequalities for the eigenvalues of the sum of two Hermitian matrices and the product of two positive definite matrices, and for the singular values of sums and products of matrices. In particular, he proved that for any systems of indices $1 \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n$ and $1 \leqslant j_1 \leqslant \ldots \leqslant j_k \leqslant n$ satisfying the conditions

$$k \leqslant n \quad \text{and} \quad i_p + j_p \leqslant n - k + p + 1 \quad (p=1, 2, \ldots, k),$$

the following inequality holds:

$$\sum_{p=1}^{k} \lambda_{m(l_p)}(A+B) \leqslant \sum_{p=1}^{k} \lambda_{m(i_p)}(A) + \sum_{p=1}^{k} \lambda_{m(j_p)}(B), \tag{0.3}$$

where

$$l_p = i_p + j_p - 1 \quad (p=1, 2, \ldots, k)$$

and

$$m(i_1) = i_1, \; m(i_p) = \max(i_p, m(i_{p-1}) + 1) \quad (p=2, 3, \ldots, k). \tag{0.4}$$

It is easy to verify that for $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$ and $j_p = 1$ ($p = 1, 2, \ldots, k$) the inequality (0.3) becomes (0.2). For $k = 1$ on the other hand (0.3) becomes the well-known inequality of H. Weyl [10]

$$\lambda_{i+j-1}(A+B) \leqslant \lambda_i(A) + \lambda_j(B) \qquad (i+j \leqslant n+1), \tag{0.5}$$

[1] The singular values of a matrix A are the non-negative square roots of the eigenvalues of the matrix A^*A: $s_j(A) = (\lambda_j(A^*A))^{1/2}$ $(j=1, 2, \ldots, n)$.

which was established as long ago as 1912 and seems to be the earliest known relation connecting the eigenvalues of A, B, and $A + B$.

Unfortunately the general inequality (0.3) hardly admits of an intuitive geometrical interpretation.

In [11] some additions to the above-mentioned results were obtained, and generalizations of some of them to the infinite-dimensional case.

Recently certain new connections between the eigenvalues of A, B, and $A + B$ have been established in [12] and [13].[1] In Horn's paper [12] a method is indicated for obtaining new relations between the numbers $\lambda_j(A)$, $\lambda_j(B)$, $\lambda_j(A + B)$ $(j = 1, \ldots, n)$ from known ones. By repeated application of this method one can, in particular, arrive at the inequality (0.2), starting from (0.1).

In the present paper the main results on connections between the eigen- and singular values of the sum (product) of linear operators and those of the summands (factors) are set forth, with complete proofs as far as possible. Attention is paid principally to results that admit of a geometrical formulation similar to that of Lidskii's theorem given above.

In § 2 a proof of Lidskii's theorem is given which uses ideas from Wielandt's paper [7], but not his minimax property. In the same section some propositions about the singular values of matrices, analogous to Lidskii's theorems, are proved.

As in Wielandt's paper, the main results are first established in the form of certain inequalities between the eigenvalues (or singular values). These are then put into geometrical form with the help of theorems on the convex hulls of certain sets of vectors, which are proved for the finite-dimensional case in § 1 and for the infinite-dimensional case in § 4.

The results of § 4 are based on the concept of a symmetric gauge function due to J. von Neumann and R. Schatten, and on certain propositions about such functions established in § 3.

In § 5 the results of § 2 are generalized to the case of completely continuous operators in Hilbert space.

The author is grateful to I.Ts. Gokhberg for valuable discussions of the questions considered here.

§1. Theorems on the convex hulls of the rearrangements of a vector (finite-dimensional case)

Let R^n be an n-dimensional Euclidean space and $\beta = \{\beta_j\}_1^n$ an arbitrary vector in R^n. Denote by $\Delta(\beta)$ the convex hull of the set of vectors obtained from β by all possible rearrangements of its coordinates, and by $\Gamma(\beta)$ the convex hull of the set of all vectors of the form

$$\{\varepsilon_k \beta_{j_k}\}_1^n, \tag{1.1}$$

where $\varepsilon_k = \pm 1$ $(k = 1, 2, \ldots, n)$ and $j_1, j_2, \ldots, n$ is an arbitrary permutation of the numbers $1, 2, \ldots, n$.

In the present section two auxiliary propositions about $\Delta(\beta)$ and $\Gamma(\beta)$

[1] We mention that among the inequalities obtained in [13] there are some that are new as compared with (0.3). We give one of them (for $n = 5$):
$\lambda_2(A+B)+\lambda_3(A+B)+\lambda_4(A+B) \leqslant \lambda_1(A)+\lambda_3(A)+\lambda_4(A)+\lambda_1(B)+\lambda_2(B)+\lambda_4(B)$.

will be established. Their proofs are based on the following simple proposition, which constitutes a finite-dimensional analogue of the well-known theorem of M.G. Krein and D P. Mil'man [14] on the extreme points of a convex compact set.

1°. *Every convex bounded closed set in R^n is the convex hull of the set of its extreme points.*[1]

For any vector $\alpha = \{\alpha_j\}_1^n$ $(\in R^n)$, denote by $\alpha^* = \{\alpha_j^*\}_1^n$ the vector obtained from α by rearranging its coordinates in non-increasing order.

Let α, $\beta \in R^n$. We shall write $\alpha \lll \beta$, if

$$\sum_{j=1}^{k} \alpha_j^* \leqslant \sum_{j=1}^{k} \beta_j^* \quad (k=1,\ 2,\ \ldots,n). \tag{1.2}$$

If $\alpha \lll \beta$ and in addition there is equality in the last of the relations (1.2), i.e.

$$\sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \beta_j,$$

then we shall write[2] $\alpha \prec \beta$.

THEOREM 1.1. *If $\beta \in R^n$, then $\alpha \prec \beta$ if and only if $\alpha \in \Delta(\beta)$.*

PROOF. It is easy to verify that the set $D(\beta)$ of all vectors α for which $\alpha \prec \beta$, is convex, bounded, and closed. Therefore, by Proposition 1°, it is sufficient to establish that a vector γ is an extreme point of $D(\beta)$ if and only if it can be obtained from β by rearranging its coordinates.

Without loss of generality we suppose that $\beta^* = \beta$. Since either both or neither of the vectors γ and γ^* are extreme points of $D(\beta)$, and since by 1° the set of extreme points of $D(\beta)$ is not empty, it is sufficient to prove that a vector $\gamma \in D(\beta)$ such that $\gamma^* = \gamma$ and $\gamma \neq \beta$ is not an extreme point of $D(\beta)$. We now prove this assertion.

Denote by k the least index for which $\gamma_k \neq \beta_k$, and by m the least index such that $m > k$ and

$$\sum_{j=1}^{m} \gamma_j = \sum_{j=1}^{m} \beta_j.$$

Obviously

$$\sum_{j=1}^{t} \gamma_j < \sum_{j=1}^{t} \beta_j \quad (t=k,\ \ldots,\ m-1).$$

Also, if $k > 1$, then $\gamma_k < \beta_k \leqslant \beta_{k-1} = \gamma_{k-1}$, and if $m < n$, then $\gamma_m > \beta_m \geqslant \beta_{m+1} \geqslant \gamma_{m+1}$.

Choose $\varepsilon > 0$ such that the following candidates are satisfied:

[1] A point of a convex set M is called an *extreme point* of M if it is not the midpoint of any interval with end-points in M.

[2] This notation is borrowed from [15] and [16].

$$\varepsilon < \sum_{j=1}^{t} \beta_j - \sum_{j=1}^{t} \gamma_j \quad (t = k, \ldots, m-1), \tag{1.3}$$

$$\varepsilon < \gamma_{k-1} - \gamma_k, \quad \text{if} \quad k > 1, \tag{1.4}$$

$$\varepsilon < \gamma_m - \gamma_{m+1}, \quad \text{if} \quad m < n, \tag{1.5}$$

$$\varepsilon < \frac{1}{2}(\gamma_k - \gamma_m), \quad \text{if} \quad \gamma_k > \gamma_m, \tag{1.6}$$

and consider the two vectors

$$\sigma = \{\gamma_1 \ \ldots, \gamma_{k-1}, \ \gamma_k + \varepsilon, \ \gamma_{k+1}, \ldots, \ \gamma_{m-1}, \ \gamma_m - \varepsilon, \ \gamma_{m+1}, \ldots, \ \gamma_n\},$$
$$\delta = \{\gamma_1, \ldots, \ \gamma_{k-1}, \ \gamma_k - \varepsilon, \ \gamma_{k+1}, \ldots, \ \gamma_{m-1}, \ \gamma_m + \varepsilon, \ \gamma_{m+1}, \ldots, \ \gamma_n\}.$$

From (1.4) and (1.5) it follows that $\sigma^* = \sigma$. Using this and (1.3) it is easy to verify that $\sigma \in D(\beta)$. If $\gamma_k = \gamma_m$, then δ is obtained from σ by transposing its k-th and m-th coordinates, and therefore belongs to $D(\beta)$ together with σ. If $\gamma_k > \gamma_m$, then using (1.6) it is not difficult to verify that $\delta \in D(\beta)$. Since $\gamma = (\sigma + \delta)/2$, $\sigma \neq \delta$, and σ, $\delta \in D(\beta)$, it follows that γ is not an extreme point of $D(\beta)$, and the theorem is proved.

REMARK 1.1. Theorem 1.1 seems to have been first established by Rado [17]; his proof was based on a theorem about the separation of convex sets by hyperplanes. However, as Horn [18] observed, it can also be obtained by combining earlier results of Hardy, Littlewood, and Pólya ([15], p.49) and Birkhoff [8]. These results are as follows. In [15] it is proved that the condition $\alpha \prec \beta$ is equivalent to $\alpha = M\beta$, where $M = \|m_{jk}\|_1^n$ is a *doubly stochastic matrix*, i.e., $m_{jk} \geqslant 0$ (j, $k = 1, 2, \ldots, n$) and

$$\sum_{k=1}^{n} m_{j_k} = \sum_{k=1}^{n} m_{kj} = 1 \quad (j = 1, \ 2, \ldots, n).$$

Birkhoff [8] established that a matrix M is doubly stochastic if and only if $M = \sum_{j=1}^{s} t_j P_j$, where $t_j \geqslant 0, \sum_{j=1}^{s} t_j = 1$, and the P_j ($j = 1, \ldots, s$) are permutation matrices, i.e., matrices in which in each row and in each column one element is equal to one and all the others to zero. Since the action of a permutation matrix on a vector is to permute its coordinates, Theorem 1.1 follows immediately from these two results.

THEOREM 1.2. *If $\beta \in R^n$, then for a vector $\alpha \in R^n$ the condition*

$$\{|\alpha_j|\}_1^n \prec\!\prec \{|\beta_j|\}_1^n \tag{1.7}$$

is satisfied if and only if $\alpha \in \Gamma(\beta)$.

PROOF. It is easy to verify that the set $G(\beta)$ of all vectors $\alpha \in R^n$ for which (1.7) is satisfied is convex, bounded, and closed. Therefore by Proposition 1° it is sufficient to prove that a vector γ is an extreme point of $G(\beta)$ if and only if it is of the form (1.1).

Without loss of generality we may suppose that $\beta_j \geqslant 0$ ($j = 1, 2, \ldots, n$) and $\beta^* = \beta$. Since either both or neither of γ and any vector of the form

$$\{\varepsilon_k \gamma_{j_k}\}_1^n,$$

where $\varepsilon_k = \pm 1$ and $\gamma_1, \ldots, \gamma_n$ is a permutation of $1, \ldots, n$, are extreme points of $G(\beta)$, and since by 1° the set of extreme points of $G(\beta)$ is not empty, it is sufficient to establish that a vector $\gamma \in G(\beta)$ satisfying the conditions $\gamma_j \geqslant 0$ $(j = 1, 2, \ldots, n)$, $\gamma^* = \gamma$, and $\gamma \neq \beta$ is not an extreme point of $G(\beta)$. We pass to the proof of this assertion.

Denote by k the least index for which $\gamma_k \neq \beta_k$. Obviously, if $k > 1$, then $\gamma_k < \beta_k \leqslant \beta_{k-1} = \gamma_{k-1}$. Consider the two possible cases:

$$1) \quad \sum_{j=1}^{t} \gamma_j < \sum_{j=1}^{t} \beta_j \quad (t = k, \ldots, n).$$

Choose $\varepsilon > 0$ such that

$$\varepsilon < \Big(\sum_{j=1}^{t} \beta_j - \sum_{j=1}^{t} \gamma_j\Big)/2 \quad (t = k, \ldots, n), \tag{1.8}$$

$$\varepsilon < \gamma_{k-1} - \gamma_k, \quad \text{если} \quad k > 1, \tag{1.9}$$

and put

$$\mu = \{\gamma_1, \ldots, \gamma_{k-1}, \gamma_k + \varepsilon, \gamma_{k+1}, \ldots, \gamma_n - \varepsilon\}$$
$$\tau = \{\gamma_1, \ldots, \gamma_{k-1}, \gamma_k - \varepsilon, \gamma_{k+1}, \ldots, \gamma_n + \varepsilon\}.$$

Using (1.8) and (1.9) it is not difficult to verify that $\mu, \tau \in G(\beta)$. Since $\gamma = (\mu + \tau)/2$ and $\mu \neq \tau$, γ is not an extreme point of $G(\beta)$.

2) For some index $m > k$

$$\sum_{j=1}^{m} \gamma_j = \sum_{j=1}^{m} \beta_j. \tag{1.10}$$

Let m be the least number $> k$ for which (1.10) holds.

Since $\{\gamma_1, \gamma_2, \ldots, \gamma_m\} \prec \{\beta_1, \beta_2, \ldots, \beta_m\}$ and these vectors are distinct, then, as was shown in the proof of Theorem 1.1, we can construct vectors $\sigma = \{\sigma_j\}_1^m$ and $\delta = \{\delta_j\}_1^m$ such that $\{\gamma_1, \gamma_2, \ldots, \gamma_m\} = (\sigma + \delta)/2$, $\sigma \neq \delta$, and

$$\sigma \prec \{\beta_1, \beta_2, \ldots, \beta_m\}, \quad \delta \prec \{\beta_1, \beta_2, \ldots, \beta_m\}. \tag{1.11}$$

If $m = n$, then since $\sigma, \delta \in G(\beta)$, we conclude that γ is not an extreme point of $G(\beta)$. If $m < n$, then $\gamma_m > \beta_m \geqslant \beta_{m+1} \geqslant \gamma_{m+1}$. Consequently if we choose the number ε in the proof of Theorem 1.1 sufficiently small, we can ensure that the vectors σ and δ also satisfy

$$\sigma_m^* > \gamma_{m+1}, \quad \delta_m^* > \gamma_{m+1}. \tag{1.12}$$

Consider the vectors

$$\tilde{\sigma} = \{\sigma_1, \ldots, \sigma_m, \gamma_{m+1}, \ldots, \gamma_n\} \text{ and } \tilde{\delta} = \{\delta_1, \ldots, \delta_m, \gamma_{m+1}, \ldots, \gamma_n\}.$$

From (1.11) and (1.12) it follows that $\tilde{\sigma}, \tilde{\delta} \in G(\beta)$. Since $\gamma = (\tilde{\sigma} + \tilde{\delta})/2$ and $\tilde{\sigma} \neq \tilde{\delta}$, γ is not an extreme point of $G(\beta)$, and the theorem is proved.

§2. The eigen- and singular values of the sum and product of matrices

In what follows we shall not distinguish between a square matrix of order n and the operator in n-dimensional unitary coordinate space $\mathfrak{H}^n$ determined by it. If A is an Hermitian matrix of order n, we shall denote by $\lambda(A) = \{\lambda_j(A)\}_1^n$ the complete system of eigenvalues of A, numbered in decreasing order and with multiplicities taken into account.

THEOREM 2.1 (Wielandt [7]). *If A and B are Hermitian matrices of order n and $C = A + B$, then $\lambda(C) - \lambda(B) \prec \lambda(A)$.*

PROOF. For $n = 1$ this is obvious; now suppose it true for $n - 1$.

Denote by φ_j (respectively ψ_j) the eigenvectors of the operator $B(C)$ in $\mathfrak{H}^n$ corresponding to the eigenvalues $\lambda_j(B)$ $(\lambda_j(C))$ $(j = 1, 2, \ldots, n)$. Denote by $\mathfrak{N}$ any $(n-1)$-dimensional subspace of $\mathfrak{H}^n$ containing the vectors φ_j $(j = m + 1, \ldots, n)$ and ψ_j $(j = 1, \ldots, m - 1)$, where m satisfies

$$\lambda_m(C) - \lambda_m(B) = \min_{1 \leqslant j \leqslant n} (\lambda_j(C) - \lambda_j(B)). \tag{2.1}$$

Let P be the orthogonal projector onto $\mathfrak{N}$ in $\mathfrak{H}^n$, and $\hat{A}$, $\hat{B}$, $\hat{C}$ the operators induced in $\mathfrak{N}$ by PA, PB, PC respectively.

As is known, the eigenvalues of A and $\hat{A}$ are connected by the inequalities

$$\lambda_{j+1}(A) \leqslant \lambda_j(\hat{A}) \leqslant \lambda_j(A) \quad (j = 1, 2, \ldots, n-1), \tag{2.2}$$

and similarly for B, $\hat{B}$ and C, $\hat{C}$.

Since $\mathfrak{N}$ contains the vectors φ_j $(j = m + 1, \ldots, n)$ and ψ_j $(j = 1, \ldots, m-1)$, the numbers $\lambda_j(B)$ $(j = m + 1, \ldots, n)$ are eigenvalues of $\hat{B}$, and the numbers $\lambda_j(C)$ $(j = 1, \ldots, m-1)$ eigenvalues of $\hat{C}$.

Using (2.2) for B, $\hat{B}$ and C, $\hat{C}$, we obtain

$$\lambda_j(\hat{B}) = \lambda_{j+1}(B) \quad (j = m, \ldots, n-1), \quad \lambda_j(\hat{C}) = \lambda_j(C) \quad (j = 1, \ldots. m-1).$$

From these inequalities and the relations (2.2) for B, $\hat{B}$ and C, $\hat{C}$ it follows that

$$\lambda_j(\hat{C}) - \lambda_j(\hat{B}) \geqslant \lambda_j(C) - \lambda_j(B) \quad (j = 1, \ldots, m-1), \tag{2.3}$$

$$\lambda_j(\hat{C}) - \lambda_j(\hat{B}) \geqslant \lambda_{j+1}(C) - \lambda_{j+1}(B) \quad (j = m, \ldots, n-1). \tag{2.4}$$

By the inductive hypothesis

$$\lambda(\hat{C}) - \lambda(\hat{B}) \prec \lambda(\hat{A}),$$

and consequently

$$\sum_{j=1}^{k} (\lambda_j(\hat{C}) - \lambda_j(\hat{B}))^* \leqslant \sum_{j=1}^{k} \lambda_j(\hat{A}) \quad (k = 1, 2, \ldots, n-1). \tag{2.5}$$

Since by (2.1)

$$(\lambda_n(C) - \lambda_n(B))^* = \lambda_m(C) - \lambda_m(B),$$

it follows from (2.2) - (2.5) that

$$\sum_{j=1}^{k}(\lambda_j(C)-\lambda_j(B))^*\leqslant\sum_{j=1}^{k}\lambda_j(A)\quad(k=1,\ 2,\ \ldots,\ n-1),$$

and since

$$\operatorname{Sp} C-\operatorname{Sp} B=\operatorname{Sp} A,$$

we have

$$\sum_{j=1}^{n}(\lambda_j(C)-\lambda_j(B))=\sum_{j=1}^{n}\lambda_j(A).$$

The theorem is now proved.

THEOREM 2.2 (Lidskii [1]). *Let A and B be Hermitian matrices of order n, K_1 the convex hull of the set of all vectors of the form*

$$\{\lambda_j(B)+\lambda_{k_j}(A)\}_1^n \tag{2.6}$$

and K_2 the convex hull of the set of all vectors of the form

$$\{\lambda_j(B)+\lambda_{k_j}(A)\}_1^n,$$

where $k_1, \ldots, k_n$ is an arbitrary permutation of $1, \ldots, n$. If $C = A + B$, then the vector $\lambda(C)$ is in the intersection of K_1 and K_2.

PROOF. Since, by Theorem 2.1, $\lambda(C) - \lambda(B) \prec \lambda(A)$, therefore by Theorem 1.1 the vector $\lambda(C) - \lambda(B)$ is in the convex hull $\Delta(\lambda(A))$ of the set of all vectors obtained by permuting the coordinates of $\lambda(A)$. Consequently $\lambda(C) \in K_1$, and since the matrices A and B may be interchanged, $\lambda(C) \in K_2$.

COROLLARY 2.1 (Lidskii [1]). *If either*

$$\lambda_1(A)-\lambda_n(A)<\lambda_k(B)-\lambda_{k+1}(B)\quad(k=1,\ 2,\ \ldots,\ n-1) \tag{2.7}$$

or

$$\lambda_1(B)-\lambda_n(B)<\lambda_k(A)-\lambda_{k+1}(A)\quad(k=1,\ 2,\ \ldots,\ n-1),$$

then C has no repeated eigenvalues.

In fact, if for example (2.7) holds, then all points of the form (2.6), and therefore also their convex hull K_1, lie in the half-space of R^n defined by the inequalities $x_k - x_{k+1} > 0$ $(k = 1, 2, \ldots, n-1)$.

2 Let A be a complex matrix of order n. We shall denote by $s(A) = \{s_j(A)\}_1^n$ the complete system, arranged in decreasing order, of *singular numbers* of A, i.e., square roots of the eigenvalues of A^*A, where A^* is the Hermitian-conjugate matrix of A.

THEOREM 2.3 (L. Mirsky [16]). *If A and B are matrices of order n and $C = A + B$, then*

$$\{|s_j(C)-s_j(B)|\}_1^n \prec\!\!\prec s(A) \tag{2.8}$$

PROOF. Denote by $\tilde{A}$ the following Hermitian matrix of order $2n$:

$$\tilde{A}=\begin{pmatrix}0 & A\\ A^* & 0\end{pmatrix}.$$

It is easy to verify that the eigenvalues of A are the numbers $\pm\, s_j(A)$ $(j = 1, 2, \ldots, n)$.[1]

[1] This remark is due to Wielandt.

Applying Theorem 2.1 to $\tilde{A}$ and to the similarly defined matrices $\tilde{B}$ and $\tilde{C}$, we obtain

$$\{s_1(C)-s_1(B), \ldots, s_n(C)-s_n(B), s_n(B)-s_n(C), \ldots, s_1(B)-s_1(C)\} \prec$$
$$\prec \{s_1(A), \ldots, s_n(A), -s_n(A), \ldots, -s_1(A)\}.$$

From this (2.8) follows immediately.

THEOREM 2.4. *Let A and B be matrices of order n, and L_1 (L_2) the convex hull of the set of all vectors of the form $\{s_j(B) + \varepsilon_j s_{k_j}(A)\}_1^n$ (respectively $\{s_j(A) + \varepsilon_j s_{k_j}(B)\}_1^n$), where $\varepsilon_j = \pm 1$ and $k_1, \ldots, k_n$ is an arbitrary permutation of $1, \ldots, n$. If $C = A + B$, then the vector $s(C)$ is in the intersection of L_1 and L_2.*

PROOF. From Theorems 2.3 and 1.2 it follows that $s(C) - s(B) \in \Gamma(s(A))$. Therefore $s(C) \in L_1$, and since A and B can be interchanged, $s(C) \in L_2$.

COROLLARY 2.2. *If either*

$$s_1(A)+s_2(A) < s_k(B)-s_{k+1}(B) \qquad (k=1, 2, \ldots, n-1) \tag{2.9}$$

or

$$s_1(B)+s_2(B) < s_k(A)-s_{k+1}(A) \qquad (k=1, 2, \ldots, n-1), \tag{2.10}$$

then all the numbers $s_j(C)$ $(j = 1, \ldots, n)$ are distinct.

3. We now formulate the theorems on the eigenvalues of the products of positive definite and unitary matrices established in [1] and [3], and we prove an analogous theorem on the singular values of the products of matrices.

THEOREM 2.5 (Lidskii [1]).[1] *Let A and B be positive definite matrices of order n, and M_1 (M_2) the convex hull of the set of all vectors of the form $\{\ln \lambda_j(B) + \ln \lambda_{k_j}(A)\}_1^n$ (respectively $\{\ln \lambda_j(A) + \ln \lambda_{k_j}(B)\}_1^n$ where $k_1, \ldots, k_n$ is an arbitrary permutation of $1, \ldots, n$. If $C = AB$, then the vector*[2] *$\{\ln \lambda_j(C)\}_1^n$ is in the intersection of M_1 and M_2.*

If U is a unitary matrix of order n, then we denote by $\omega_k(U)$ $(k = 1, \ldots, n)$ the numbers defined by the conditions

1) $\{\exp(i\omega_k(U))\}_1^n$ is the complete system of eigenvalues of U;
2) $2\pi > \omega_1(U) \geqslant \ldots \geqslant \omega_k(U) \geqslant 0$.

THEOREM 2.6 (Nudel'man and Shvartsman [3]). *Let U and V be unitary matrices of order n for which*

$$\omega_1(U)+\omega_1(V)-\omega_n(U)-\omega_n(V) < 2\pi.$$

If N_1 (N_2) is the convex hull of the set of all vectors of the form $\{\omega_j(U) + \omega_{k_j}(V)\}_1^n$ (respectively $\{\omega_j(V) + \omega_{k_j}(U)\}_1^n$), where $k_1, \ldots, k_n$ is an arbitrary permutation of $1, \ldots, n$, and if $W = UV$, then the vector $\{\omega_j(W)\}_1^n$ is in the intersection of N_1 and N_2.

THEOREM 2.7. *Let A and B be non-degenerate matrices of order n, and F_1 (F_2) the convex hull of the set of all vectors of the form*

[1] A result equivalent to Theorem 2.5 was obtained later by Amir-Moéz ([9], Theorem 3.12) in the form of certain inequalities between the numbers $\lambda_j(A)$, $\lambda_j(B)$ and $\lambda_j(C)$ but apparently he did not notice the equivalence of the theorems.

[2] Although C is positive definite only when $AB = BA$, the eigenvalues of C are always positive. We number them in decreasing order.

$\{\ln s_j(B) + \ln s_{k_j}(A)\}_1^n$ *(respectively* $\{\ln s_j(A) + \ln s_{k_j}(B)\}_1^n$*), where* $k_1, \dots, k_n$ *is an arbitrary permutation of* $1, \dots, n$. *If* $C = AB$, *then the vector* $\{\ln s_j(C)\}_1^n$ *is in the intersection of* F_1 *and* F_2.

PROOF. As Amir-Moéz has proved ([9], Theorem 3.10), for any set of indices $j_1 < j_2 < \dots < j_k$ $(\leqslant n)$

$$\prod_{t=1}^{k} s_{j_t}(C) \leqslant \prod_{t=1}^{k} s_t(A) \prod_{t=1}^{k} s_{j_t}(B). \tag{2.11}$$

Consequently

$$\sum_{j=1}^{k} (\ln s_j(C) - \ln s_j(B))^* \leqslant \sum_{j=1}^{k} \ln s_j(A) \quad (k = 1, 2, \dots, n). \tag{2.12}$$

Since $\det C = \det A \det B$ and $\det A = \prod_{j=1}^{n} s_j(A)$, there is equality in the last of the relations (2.12). Thus, $\{\ln s_j(C) - \ln s_j(B)\}_1^n \prec \{\ln s_j(A)\}_1^n$. By Theorem 1.1 this means that

$$\{\ln s_j(C) - \ln s_j(B)\}_1^n \in \Delta(\{\ln s_j(A)\}_1^n),$$

and therefore $\{\ln s_j(C)\}_1^n \in F_1$. Since $C^* = B^*A^*$ and in the passage to the conjugate matrix the singular values do not change, $\{\ln s_j(C)\}_1^n \in F_2$, and the theorem is proved.

COROLLARY 2.3. *If either*

$$\frac{s_1(A)}{s_n(A)} < \frac{s_k(B)}{s_{k+1}(B)} \qquad (k = 1, 2, \dots, n-1)$$

or

$$\frac{s_1(B)}{s_n(B)} < \frac{s_k(A)}{s_{k+1}(A)} \qquad (k = 1, 2, \dots, n-1),$$

then all the numbers $s_j(C)$ $(j = 1, \dots, n)$ *are distinct.*

4. We shall now briefly discuss the problem in what cases the theorems given in 1. – 3. are exact.

First we take Theorem 2.2. Let $\alpha = \{\alpha_j\}_1^n$, $\beta = \{\beta_j\}_1^n$ be vectors of R^n with $\alpha_j \geqslant \alpha_{j+1}$, $\beta_j \geqslant \beta_{j+1}$ $(j = 1, \dots, n-1)$. Denote by $K(\alpha, \beta)$ the convex hull of the set of all vectors of the form $\{\alpha_j + \beta_{k_j}\}_1^n$, where $k_1, \dots, k_n$ is an arbitrary permutation of $1, \dots, n$, and put $\tilde{K}(\alpha, \beta) = K(\alpha, \beta) \cap K(\beta, \alpha)$. Denote by $E(\alpha, \beta)$ the set of all vectors $\gamma \in R^n$ of the form $\gamma = \lambda(A + B)$, where

$$A = A^*, \quad B = B^*, \quad \lambda(A) = \alpha, \quad \lambda(B) = \beta. \tag{2.13}$$

Theorem 2.2 signifies that always $E(\alpha, \beta) \subseteq \tilde{K}(\alpha, \beta)$. It is easy to see that in the case $n = 2$ always $E(\alpha, \beta) = \tilde{K}(\alpha, \beta)$ (this is also true in the trivial case $n = 1$). In fact, if for definiteness $\alpha_1 - \alpha_2 \geqslant \beta_1 - \beta_2$, then $\tilde{K}(\alpha, \beta)$ is the interval with the end-points $\{\alpha_1 + \beta_2, \alpha_2 + \beta_1\}$ and $\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\}$. We observe that the set $E(\alpha, \beta)$ may be treated as the set of all vectors $\gamma \in R^n$ of the form $\gamma = \lambda(A + U^*BU)$, where A and B are fixed matrices satisfying (2.13) and U ranges over the group of unitary matrices. Since this group is connected, so is $E(\alpha, \beta)$. Since

$E(\alpha,\beta) \subseteq \tilde{K}(\alpha,\beta)$ and the ends of the interval $\tilde{K}(\alpha,\beta)$ are in $E(\alpha,\beta)$, therefore $E(\alpha,\beta) = \tilde{K}(\alpha,\beta)$.

However, the equality $E(\alpha,\beta) = \tilde{K}(\alpha,\beta)$ is not true in general for any $n > 2$. Further, for any $n > 2$ there exist vectors α, $\beta \in R^n$ such that $E(\alpha,\beta) \neq \hat{K}(\alpha,\beta)$, where $\hat{K}(\alpha,\beta)$ denotes the intersection of $\tilde{K}(\alpha,\beta)$ with the set L of all points $\gamma = \{\gamma_j\}_1^n \in R^n$ for which $\gamma_1 \geqslant \gamma_2 \geqslant \dots \geqslant \gamma_n$ ($E(\alpha,\beta)$ is by definition contained in L). We give an example taken from [9]. Let $n \geqslant 3$ and $\alpha = \{4, 2, 2, 0, \dots, 0\}$, $\beta = \{4, 1, 1, 0, \dots, 0\}$. It is obvious that $\gamma = \{6, 4, 4, 0, \dots, 0\} \in \hat{K}(\alpha,\beta)$, but $\gamma \notin E(\alpha,\beta)$. In fact, if $\gamma \in E(\alpha,\beta)$, then by (0.5) we should have

$$\gamma_3 \leqslant \alpha_2 + \beta_2,$$

which is false.

An important case in which $E(\alpha,\beta) = \tilde{K}(\alpha,\beta)$ was pointed out in [1] (Theorem 3).

THEOREM 2.8 (Lidskii [1]). *If $\alpha_1 - \alpha_n \leqslant \beta_k - \beta_{k+1}$ for $k = 1, \dots, n-1$, or $\beta_1 - \beta_n \leqslant \alpha_k - \alpha_{k+1}$ for $k = 1, \dots, n-1$, then $E(\alpha,\beta) = \tilde{K}(\alpha,\beta) (= \hat{K}(\alpha,\beta))$.*

The problem of describing $E(\alpha,\beta)$ completely in the general case appears to be very difficult. All the necessary conditions so far known (see [9], [12], [13]) for a vector γ to belong to $E(\alpha,\beta)$, apart from the obvious condition

$$\sum_{j=1}^{n} \gamma_j = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_j, \tag{2.14}$$

reduce to inequalities of the form

$$\sum_{p=1}^{k} \gamma_{l_p} \leqslant \sum_{p=1}^{k} \alpha_{i_p} + \sum_{p=1}^{k} \beta_{j_p}, \tag{2.15}$$

where $1 \leqslant l_1 < \dots < l_k \leqslant n$, $1 \leqslant i_1 < \dots < i_k \leqslant n$, $1 \leqslant j_1 < \dots < j_k \leqslant n$. For $n \leqslant 8$ Horn [12] obtained a more complete description of the set $E(\alpha,\beta)$, by proving that it is characterized by (2.14) and a finite number of inequalities of the form (2.15). He conjectured that this also holds for any n. However, so far it has not even been proved that $E(\alpha,\beta)$ is convex.

Concerning Theorem 2.4 we mention only that, as is easily seen, its conditions are exact for $n = 1$, and that this is false, in general, for any $n > 1$, as is shown by the example $\alpha = \{3, 2, 0, \dots, 0\}$, $\beta = \{4, 1, 0, \dots, 0\}$, and $\gamma = \{1, 0, \dots, 0\}$. One might conjecture by analogy with Theorem 2.8 that if the inequalities (2.9) or (2.10) are satisfied (with possible equalities), then the conditions of Theorem 2.4 are exact.

Lidskii [1] established that the conditions of Theorem 2.5 are exact if

$$\frac{\lambda_1(A)}{\lambda_n(A)} \leqslant \frac{\lambda_k(B)}{\lambda_{k+1}(B)} \quad (k = 1, 2, \dots, n-1)$$

or

$$\frac{\lambda_1(B)}{\lambda_n(B)} \leqslant \frac{\lambda_k(A)}{\lambda_{k+1}(A)} \quad (k = 1, 2, \dots, n-1).$$

One naturally expects analogous assertions to be valid for Theorems 2.6 and 2.7.

3. Two theorems on symmetric gauge functions

An essential part will be played in the sequel by the concept of a symmetric gauge function, which was introduced by von Neumann and Schatten (see [19], p.89 or [20], p.68).

Denote by K the vector space of all finite sequences $\alpha = \{\alpha_j\}_1^\infty$ of real numbers (i.e., sequences in which only a finite number of coordinates are different from zero). A function $\Phi(\alpha)$, defined on K is called a *symmetric gauge function* if it has the following properties:

a) $\Phi(\alpha) > 0$ $(\alpha \in K,\ \alpha \neq 0)$;

b) for any real number λ

$$\Phi(\lambda\alpha) = |\lambda|\Phi(\alpha) \qquad (\alpha \in K);$$

c) $\Phi(\alpha+\beta) \leqslant \Phi(\alpha)+\Phi(\beta)$ $(\alpha,\ \beta \in K)$;

d) if $\alpha = \{\alpha_j\}_1^\infty \in K$ and $\alpha' = \{\varepsilon_j \alpha_{j_k}\}_1^\infty$, where $\{k_j\}_1^\infty$ is an arbitrary permutation of the set of natural numbers and $\varepsilon_j = \pm 1$, then

$$\Phi(\alpha') = \Phi(\alpha);$$

e) $\Phi(\{1, 0, 0, \ldots\}) = 1$.

It is not difficult to see that for any symmetric gauge function $\Phi(\alpha)$

$$\max_j |\alpha_j| \leqslant \Phi(\alpha) \leqslant \sum_j |\alpha_j| \qquad (\alpha = \{\alpha_j\} \in K). \tag{3.1}$$

If $\alpha = \{\alpha_j\}_1^\infty$ is an arbitrary sequence of real numbers, then we put

$$\alpha^{(n)} = \{\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots\}, \quad \alpha_{(n)} = \{\alpha_{n+1}, \alpha_{n+2}, \ldots\} \qquad (n = 1, 2, \ldots).$$

With each symmetric gauge function $\Phi(\alpha)$ two (in general) distinct Banach spaces of sequences are associated (see [19], p.91-92).

The first, which we shall denote by l_Φ, consists of all sequences of real numbers $\alpha = \{\alpha_j\}_1^\infty$ for which

$$\sup_n \Phi(\alpha^{(n)}) < \infty;$$

the norm in l_Φ is defined by the equation

$$|\alpha|_\Phi = \lim_{n\to\infty} \Phi(\alpha^{(n)})\ (= \Phi(\alpha)) \qquad (\alpha \in l_\Phi).$$

The second, denoted by $l_\Phi^{(0)}$, is the subspace of l_Φ, obtained by forming the closure with respect to the norm $|\cdot|_\Phi$ of the set K of all finite sequences. A vector $\alpha \in l_\Phi$ belongs to $l_\Phi^{(0)}$ if and only if

$$\lim_{n\to\infty} |\alpha_{(n)}|_\Phi \left(= \lim_{n\to\infty} \Phi(\alpha_{(n)})\right) = 0. \tag{3.2}$$

We give a few examples of symmetric gauge functions and the associated spaces l_Φ and $l_\Phi^{(0)}$.

The simplest example is the function $\Phi_p(\alpha)$, where

$$\Phi_p(\alpha) = \left(\sum_j |\alpha_j|^p\right)^{1/p} \ (1 \leqslant p < \infty),\ \Phi_\infty(\alpha) = \max_j |\alpha_j| \ (\alpha \in K).$$

If $p < \infty$, the spaces l_{Φ_0} and $l_{\Phi_p}^{(0)}$ coincide and are the same as l_p.

For $p=\infty$, $l_{\Phi_\infty}=m$, while $l^{(0)}_{\Phi_\infty}=c_0$. For simplicity we shall denote the norm $|\alpha|_{\Phi_1}=\sum_j|\alpha_j|$ by $|\alpha|_1$.

We take the following example from [21]. Let $\pi=\{\pi_j\}_1^\infty$ be a non-increasing sequence of non-negative numbers tending to zero. Associate with it the following two symmetric gauge functions

$$\Phi_\pi(\alpha)=\sum_{j=1}^{\infty}\pi_j\alpha_j^* \text{ and } \Phi_\Pi(\alpha)=\sup_n\Big(\sum_{j=1}^{n}\alpha_j^*\Big/\sum_{j=1}^{n}\pi_j\Big)\qquad(\alpha\in K),$$

where $\alpha^*=\{\alpha_j^*\}_1^\infty$ denotes the sequence obtained by arranging the numbers $|\alpha_1|$, $|\alpha_2|$, ... in decreasing order.

It turns out that always $l^{(0)}_{\Phi_\pi}=l_{\Phi_\pi}$, while $l^{(0)}_{\Phi_\Pi}=l_{\Phi_\Pi}$ if and only if

$$\sum_{j=1}^{\infty}\pi_j<\infty.$$

Later we shall also need the definition of certain relations between sequences of numbers analogous to those for the finite-dimensional case considered in § 1.

For sequences $\alpha=\{\alpha_j\}_1^\infty$ and $\beta=\{\beta_j\}_1^\infty$ of real numbers we shall write $\alpha\prec\!\!\prec\beta$, if

$$\sup\sum_{m=1}^{k}\alpha_{j_m}\leqslant\sup\sum_{m=1}^{k}\beta_{j_m}\qquad(k=1,2,\dots),$$

the upper bound being extended over all systems of distinct natural numbers $j_1,\dots,j_k$.

If the sequences α and β are in l_1, $\alpha\prec\!\!\prec\beta$, $-\alpha\prec\!\!\prec-\beta$ and

$$\sum_{j=1}^{\infty}\alpha_j=\sum_{j=1}^{\infty}\beta_j,$$

then we shall write $\alpha\prec\beta$.

Later we shall need a result of Ky Fan ([5], Theorem 4). We give a formulation convenient for our purposes, and a simple proof using Theorem 1.2.

THEOREM 3.1. *Let Φ be an arbitrary symmetric gauge function and $\alpha=\{\alpha_j\}_1^\infty$, $\beta=\{\beta_j\}_1^\infty$ sequences of real numbers. If $\beta\in l_\Phi$, $|\beta_j|\geqslant|\beta_{j+1}|$ $(j=1,2,\dots)$, and $\{|\alpha_j|\}_1^\infty\prec\!\!\prec\{|\beta_j|\}_1^\infty$, then $\alpha\in l_\Phi$ and $\Phi(\alpha)\leqslant\Phi(\beta)$.*

PROOF. It is obvious that for any natural number n

$$\{|\alpha_1|,|\alpha_2|,\dots,|\alpha_n|,0,0,\dots\}\prec\!\!\prec\{|\beta_1|,|\beta_2|,\dots,|\beta_n|,0,0,\dots\},$$

and consequently, by Theorem 1.2,

$$\alpha^{(n)}=\sum_{j=1}^{s}\lambda_jQ_j\beta^{(n)},$$

where $\lambda_j\geqslant0$, $\Sigma\lambda_j\geqslant1$, and the $Q_j\beta^{(n)}$ are vectors obtained from $\beta^{(n)}$ by rearranging its coordinates and multiplying them by ±1. Using properties b) – d) we obtain

$$\Phi(\alpha^{(n)})\leqslant\sum_j\lambda_j\Phi(Q_j\beta^{(n)})=\Phi(\beta^{(n)}).$$

Therefore
$$\Phi(\alpha)=\lim_{n\to\infty}\Phi(\alpha^{(n)})\leqslant\lim_{n\to\infty}\Phi(\beta^{(n)})=\Phi(\beta).$$

The theorem is proved.

In order to establish an analogous result for the space $l_\Phi^{(0)}$, we shall need the following

LEMMA 3.1. *If* α, $\beta \in c_0$ *are non-increasing sequences of non-negative numbers, then the condition* $\alpha \prec\!\!\prec \beta$ *is satisfied if and only if there exist numbers* m_{jk} $(j,\ k = 1,\ 2,\ \ldots)$ *such that*

$$\alpha_j=\sum_{k=1}^{\infty} m_{jk}\beta_k \qquad (j=1,\ 2,\ \ldots) \tag{3.3}$$

and

$$m_{jk}\geqslant 0\ (j,\ k=1,\ 2,\ \ldots),\quad \sum_{k=1}^{\infty} m_{jk}\leqslant 1,\quad \sum_{k=1}^{\infty} m_{kj}\leqslant 1\ \ (j=1,\ 2,\ \ldots). \tag{3.4}$$

PROOF. To prove the sufficiency of these conditions we consider the numbers

$$s_k=\sum_{j=1}^{n} m_{jk}\quad (k=1,\ 2,\ \ldots),$$

where n is any natural number. Obviously

$$\sum_{j=1}^{n}\alpha_j\leqslant\sum_{k=1}^{n-1}s_k\beta_k+\Big(n-\sum_{k=1}^{n-1}s_k\Big)\beta_n\leqslant\sum_{k=1}^{n-1}(\beta_k-\beta_n)+n\beta_n=\sum_{k=1}^{n}\beta_k,$$

i.e., $\alpha \prec\!\!\prec \beta$.

Conversely, let $\alpha \prec\!\!\prec \beta$. If $\alpha_1 = 0$, then it is sufficient to set $m_{jk} = 0$ $(j,\ k = 1,\ 2,\ \ldots)$. If $\alpha_1 > 0$, then there exists an index n such that $\beta_n \geqslant \alpha_1 \geqslant \beta_{n+1}$. Obviously $\alpha_1 = t\beta_n + (1-t)\beta_{n+1}$, where $0 \leqslant t \leqslant 1$.

If $\alpha_2 = 0$, we put $m_{jk} = 0$ $(j = 2,\ 3,\ \ldots;\ k = 1,\ 2,\ \ldots)$. If $\alpha_2 > 0$, then we consider the sequences $\{\alpha_j^{(1)}\}$ and $\{\beta_j^{(1)}\}$ defined as follows:

$$\beta_j^{(1)}=\beta_j\quad (j<n),\ \ \beta_n^{(1)}=\beta_n+\beta_{n+1}-\alpha_1,\ \ \beta_j^{(1)}=\beta_{j+1}\quad (j>n),$$
$$\alpha_j^{(1)}=\alpha_{j+1}\quad (j=1,\ 2,\ \ldots).$$

It is easy to verify that $\{\alpha_j^{(1)}\}\prec\!\!\prec\{\beta_j^{(1)}\}$, and that the numbers $\beta_j^{(1)}$ $(j = 1,\ 2,\ \ldots)$ are averages[1] of the numbers β_j $(j = 1,\ 2,\ \ldots)$. Applying the above reasoning to the sequences $\{\alpha_j^{(1)}\}$ and $\{\beta_j^{(1)}\}$, we observe that the number $\alpha_1^{(1)} = \alpha_2$ is an average of some pair of numbers $\beta_k^{(1)}$, $\beta_{k+1}^{(1)}$, and therefore an average of the numbers $\{\beta_j\}$. Repeating this reasoning n times and remembering that averages of averages are again averages, we obtain that if $\alpha_n > 0$, then $\alpha_n = \sum_k m_{nk}\beta_k$, where $\sum_k m_{nk} = 1$.

If $\alpha_n = 0$, then we put $m_{jk} = 0$ $(j = n,\ n+1,\ \ldots;\ k = 1,\ 2,\ \ldots)$.

From the method of construction of the sequences $\{\beta_j^{(k)}\}$ it is not difficult to deduce that

[1] A number a is said to be an average of the numbers $\{b_j\}$ if $a = \sum_j t_j b_j$, where $t_j \geqslant 0$ and $\sum_j t_j = 1$.

$$\sum_{j=1}^{\infty} m_{jk} \leqslant 1 \quad (k = 1, 2, \ldots).$$

The lemma is proved.

THEOREM 3.2. *Let Φ be an arbitrary symmetric gauge function and $\alpha = \{\alpha_j\}_1^\infty$, $\beta = \{\beta_j\}_1^\infty$ sequences of real numbers. If $\beta \in l_\Phi^{(0)}$ and $\{|\alpha_j|\}_1^\infty \prec\!\!\prec \{|\beta_j|\}_1^\infty$, then $\alpha \in l_\Phi^{(0)}$.*

PROOF. It follows from (3.1) that $l_\Phi^{(0)} \subseteq c_0$, and therefore $\beta \in c_0$. We first show that also $\alpha \in c_0$. In fact, if this were not so, then there would exist a subsequence $\{\alpha_{n_j}\}$ of α such that $|\alpha_{n_j}| > \delta > 0$ $(j = 1, 2, \ldots)$. Consequently, for some subsequence $\{\beta_{n_j}\}$ of β we should have

$$n^{-1} \sum_{j=1}^{n} |\beta_{k_j}| > \delta \quad (n = 1, 2, \ldots),$$

which is impossible, because $\lim \beta_j = 0$.

Since α, $\beta \in c_0$, by property c) we may suppose without loss of generality that these sequences consist of non-negative numbers and are decreasing. Therefore, by Lemma 3.1 there exist numbers m_{jk} $(j, k = 1, 2, \ldots)$ such that (3.3) and (3.4) are satisfied.

To prove the theorem it is sufficient to establish that (3.2) is satisfied. Let ε be an arbitrary positive number. Since $\beta \in l_\Phi^{(0)}$, we can find a natural number r such that

$$\Phi(\beta_{(r)}) < \varepsilon. \tag{3.5}$$

Choose a natural number n such that

$$\sum_{j=n+1}^{\infty} m_{jk} < \varepsilon r^{-1} \quad (k = 1, 2, \ldots, r). \tag{3.6}$$

Obviously

$$\Phi(\alpha_{(n)}) \leqslant \Phi\Big(\Big\{\sum_{k=1}^{r} m_{jk}\beta_k\Big\}_{j=n+1}^{\infty}\Big) + \Phi(\gamma), \tag{3.7}$$

where

$$\gamma = \{\gamma_j\}_1^\infty, \; \gamma_j = \sum_{k=r+1}^{\infty} m_{n+j,\,k}\beta_k \quad (j = 1, 2, \ldots). \tag{3.8}$$

From (3.8) it follows by Lemma 3.1 that $\gamma \prec\!\!\prec \beta_{(r)}$, and therefore by Theorem 3.1

$$\Phi(\gamma) \leqslant \Phi(\beta_{(r)}). \tag{3.9}$$

On the other hand it follows from (3.1) and (3.6) that

$$\Phi\Big(\Big\{\sum_{k=1}^{r} m_{jk}\beta_k\Big\}_{j=n+1}^{\infty}\Big) \leqslant \sum_{j=n+1}^{\infty} \sum_{k=1}^{r} m_{jk}\beta_k < \varepsilon\beta_1. \tag{3.10}$$

Together (3.7), (3.9), (3.5), and (3.10) imply that

$$\Phi(\alpha_{(n)}) < \varepsilon\beta_1 + \varepsilon.$$

Thus,

$$\lim_{n\to\infty}\Phi(\alpha_{(n)})=0.$$

The theorem is proved.

§ 4. Theorems on the convex hulls of the rearrangements of a vector (infinite-dimensional case)

Let Φ be an arbitrary symmetric gauge function and β some vector of the space l_Φ. Denote by $\Gamma_\Phi(\beta)$ (respectively $\Delta_\Phi(\beta)$) the convex closed hull with respect to the norm $|\cdot|_\Phi$ of the set of all vectors of the form $\{\varepsilon_k\beta_{j_k}\}_1^\infty$, where $\varepsilon_k = \pm 1$ and $j_1, j_2, \ldots$ is an arbitrary permutation of the set of natural numbers (respectively of the set of all vectors obtained by permuting the coordinates of β).

It is obvious that $\Delta_\Phi(\beta)\subset\Gamma_\Phi(\beta)$ and that if $\beta\in l_\Phi^{(0)}$, then $\Gamma_\Phi(\beta)\subset l_\Phi^{(0)}$.

THEOREM 4.1. *Let $\Phi(\xi)$ be an arbitrary symmetric gauge function such that the norm $|\xi|_\Phi$ is not equivalent to the norm $|\xi|_1=\sum_j|\xi_j|$. If $\beta\in l_\Phi^{(0)}$, then for a real vector $\alpha=\{\alpha_j\}_1^\infty$*

$$\alpha\prec\!\!\prec\beta \text{ and } -\alpha\prec\!\!\prec-\beta \tag{4.1}$$

if and only if $\alpha\in\Delta_\Phi(\beta)$.

PROOF. It is not difficult to verify that by (3.1) the functionals

$$p_k(\alpha)=\sup_{j_1<\ldots<j_k}\sum_{t=1}^{k}\alpha_{j_t}\quad(\alpha\in l_\Phi^{(0)};\ k=1,\ 2,\ \ldots)$$

are convex and continuous in $l_\Phi^{(0)}$. Consequently, for every vector $\alpha\in\Delta_\Phi(\beta)$ the conditions (4.1) are satisfied.

Conversely, suppose that (4.1) holds for a vector α. It obviously follows that

$$\{|\alpha_j|\}_1^\infty\prec\!\!\prec\{|\beta_j|\}_1^\infty,$$

and therefore, by Theorem 3.2, $\alpha\in l_\Phi^{(0)}$. Consequently by (3.2) for any $\varepsilon>0$ we can choose an index n such that

$$|\alpha_{(n)}|_\Phi<\frac{\varepsilon}{3}.$$

Denote by $r\geqslant 0$ (respectively $l\geqslant 0$) the number of positive (negative) coordinates among the α_j ($j = 1, \ldots, n$), and select an index m such that

$$|\beta_{(m)}|_\Phi<\frac{\varepsilon}{3},$$

$$\max_{j_1<\ldots<j_p\leqslant n}\sum_{t=1}^{p}\alpha_{j_t}\leqslant\max_{\substack{i_1<\ldots<i_q\leqslant m\\ q\leqslant p}}\sum_{t=1}^{q}\beta_{i_t}\quad(p=1,\ 2,\ \ldots,\ r),\tag{4.2}$$

$$\min_{j_1<\ldots<j_p\leqslant n}\sum_{t=1}^{p}\alpha_{j_t}\geqslant\min_{\substack{i_1<\ldots<i_q\leqslant m\\ q\leqslant p}}\sum_{t=1}^{q}\beta_{i_t}\quad(p=1,\ 2,\ \ldots,\ l).\tag{4.3}$$

Let $\varrho=\sum_{j=1}^{m}\beta_j-\sum_{j=1}^{n}\alpha_j$. Since the norms $|\cdot|_\Phi$ and $|\cdot|_1$ are not equivalent, there exists a finite vector $\sigma=\{\sigma_1, \sigma_2, \dots, \sigma_k, 0, 0, \dots\}$ such that the coordinates σ_i $(i = 1, \dots, k)$ have the same sign as ρ and

$$|\sigma_i|<\min_{\substack{j\leqslant n\\ \alpha_j\neq 0}}|\alpha_j|,\quad |\sigma_i|<\min_{\substack{j\leqslant m\\ \beta_j\neq 0}}|\beta_j|\quad (i=1, 2, \dots, k),$$

$$\sum_{i=1}^{k}\sigma_i=\varrho,\quad |\sigma|_\Phi<\frac{\varepsilon}{3}.$$

Consider the vectors

$\beta^{(m)}=\{\beta_1, \beta_2, \dots, \beta_m, 0, 0, \dots\}$ and $\widetilde{\alpha}=\{\alpha_1, \alpha_2, \dots, \alpha_n, \sigma_1, \sigma_2, \dots, \sigma_k, 0, 0, \dots\}$.

It is not difficult to verify that $\widetilde{\alpha}\prec\beta^{(m)}$. Since the vectors $\tilde{\alpha}$ and $\beta^{(m)}$ are finite, by Theorem 1.1

$$\widetilde{\alpha}=\sum_{j=1}^{s}\lambda_j P_j\beta^{(m)},$$

where $\lambda_j\geqslant 0$, $\sum_j\lambda_j = 1$, and the P_j $(j = 1, \dots, s)$ are linear transformations effecting certain permutations of the coordinates of a vector.

Obviously

$$|\alpha-\widetilde{\alpha}|_\Phi\leqslant|\alpha_{(n)}|_\Phi+|\sigma|_\Phi<\frac{2\varepsilon}{3},\quad |\beta-\beta^{(m)}|_\Phi=|\beta_{(m)}|_\Phi<\frac{\varepsilon}{3},$$

and consequently

$$|\alpha-\sum_{j=1}^{s}\lambda_jP_j\beta|_\Phi\leqslant|\alpha-\widetilde{\alpha}|_\Phi+|\sum_{j=1}^{s}\lambda_jP_j\beta-\sum_{j=1}^{s}\lambda_jP_j\beta^{(m)}|_\Phi=$$

$$=|\alpha-\widetilde{\alpha}|_\Phi+|\sum_{j=1}^{s}\lambda_jP_j(\beta-\beta^{(m)})|_\Phi\leqslant|\alpha-\widetilde{\alpha}|_\Phi+|\beta_{(m)}|_\Phi<\varepsilon.$$

The theorem is proved.

THEOREM 4.2. *If $\beta\in l_1$, then for a real vector $\alpha=\{\alpha_j\}_1^\infty$ the condition $\alpha\prec\beta$ is satisfied if and only if $\alpha\in\Delta_{\Phi_1}(\beta)$, where $\alpha\in\Delta_{\Phi_1}(\beta)$*

PROOF. It follows from (3.1) that $\Delta_{\Phi_1}(\beta)\subseteq\Delta_\Phi(\beta)$ for any symmetric gauge function Φ, and therefore by Theorem 4.1 the conditions (4.1) hold for every vector $\alpha\in\Delta_{\Phi_1}(\beta)$. Observing that the functional

$$f(\xi)=\sum_j\xi_j\quad(\xi\in l_1)$$

is linear and continuous, we obtain that for every vector $\alpha\in\Delta_{\Phi_1}(\beta)$

$$\sum_j\alpha_j=\sum_j\beta_j$$

and consequently $\alpha\prec\beta$.

Conversely, let $\alpha\prec\beta$. Obviously $\alpha\in l_1$. For an arbitrary $\varepsilon>0$ choose an index n such that $|\alpha_{(n)}|_1<\frac{\varepsilon}{4}$.

Denote by $r \geqslant 0$ ($l \geqslant 0$) the number of positive (negative) coordinates among the α_j ($j = 1, \ldots, n$), and select an index m such that (4.2) and (4.3) and the inequality

$$|\beta_{(m)}|_1 < \frac{\varepsilon}{4}.$$

are satisfied. Let

$$\varrho = \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{n} \alpha_j.$$

Obviously

$$|\varrho| \leqslant \left|\sum_{j=n+1}^{\infty} \alpha_j\right| + \left|\sum_{j=m+1}^{\infty} \beta_j\right| \leqslant |\alpha_{(n)}|_1 + |\beta_{(m)}|_1 < \frac{\varepsilon}{2}.$$

Consider the vectors

$$\alpha^{(n)} = \{\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots\} \text{ and } \widetilde{\beta} = \{\beta_1, \beta_2, \ldots, \beta_m, \varrho, 0, 0, \ldots\}.$$

It is not difficult to verify that $\alpha^{(n)} \prec \widetilde{\beta}$, and therefore by Theorem 1.1

$$\alpha^{(n)} = \sum_{j=1}^{s} \lambda_j P_j \widetilde{\beta},$$

where $\lambda_j \geqslant 0$, $\sum_j \lambda_j = 1$, and the P_j ($j = 1, \ldots, s$) are linear transformations effecting certain permutations of the coordinates of a vector. Obviously

$$|\alpha - \alpha^{(n)}|_1 = |\alpha_{(n)}|_1 < \frac{\varepsilon}{4}, \quad |\beta - \widetilde{\beta}|_1 \leqslant |\varrho| + |\beta_{(m)}|_1 < \frac{3\varepsilon}{4},$$

and consequently

$$\left|\alpha - \sum_{j=1}^{s} \lambda_j P_j \beta\right|_1 \leqslant |\alpha - \alpha^{(n)}|_1 + \left|\sum_{j=1}^{s} \lambda_j P_j (\beta - \widetilde{\beta})\right|_1 < \varepsilon.$$

The theorem is now proved.

REMARK 4.1. It follows from Theorems 4.1 and 4.2 that if Φ and Ψ are symmetric gauge functions for which $|\xi|_\Phi$ and $|\xi|_\Psi$ are not equivalent to $|\xi|_1$, and if $\beta \in l_\Phi^{(0)}$, $\beta \in l_\Psi^{(0)}$, then $\Delta_\Phi(\beta) = \Delta_\Psi(\beta)$. If in addition $\beta \in l_1$ and $\beta \neq 0$, then $\Delta_\Phi(\beta) \neq \Delta_{\Phi_1}(\beta)$.

THEOREM 4.3. *Let $\Phi(\xi)$ be an arbitrary symmetric gauge function. If $\beta \in l_\Phi^{(0)}$, then for a real vector $\alpha = \{\alpha_j\}_1^\infty$ we have*

$$\{|\alpha_j|\}_1^\infty \prec\prec \{|\beta_j|\}_1^\infty \tag{4.4}$$

if and only if $\alpha \in \Gamma_\Phi(\beta)$.

PROOF. Since the functionals

$$q_k(\alpha) = \sup_{j_1 < \cdots < j_k} \sum_{t=1}^{k} |\alpha_{j_t}| \quad (\alpha \in l_\Phi^{(0)};\ k = 1, 2, \ldots)$$

are convex and continuous in $l_\Phi^{(0)}$, (4.4) is satisfied for every vector $\alpha \in \Gamma_\Phi(\beta)$

Conversely, let (4.4) be satisfied for a vector α. By Theorem 3.2

$\alpha \in l_\Phi^{(0)}$, and therefore for any number $\varepsilon > 0$ there is an index n such that

$$|\alpha_{(n)}|_\Phi < \frac{\varepsilon}{2}.$$

Choose an index m ($\geqslant n$) such that

$$|\beta_{(m)}|_\Phi < \frac{\varepsilon}{2}$$

and

$$\max_{j_1 < \dots < j_p \leqslant n} \sum_{t=1}^{p} |\alpha_{j_t}| \leqslant \max_{i_1 < \dots < i_p \leqslant m} \sum_{t=1}^{p} |\beta_{i_t}| \quad (p = 1, 2, \dots, n).$$

Since

$$\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_n|, 0, 0, \dots\} \prec\!\prec \{|\beta_1|, |\beta_2|, \dots, |\beta_m|, 0, 0, \dots\},$$

by Theorem 1.2

$$\alpha^{(n)} = \sum_{j=1}^{s} \lambda_j Q_j \beta^{(m)},$$

where $\lambda_j \geqslant 0$, $\Sigma \lambda_j = 1$, and the Q_j are linear transformations of the following form: $Q_j = \{\varepsilon_k \gamma_{n_k}\}_1^\infty$, where $\varepsilon_k = \pm 1$ and $n_1, n_2, \dots$ is a permutation of the set of natural numbers. Obviously

$$|\alpha - \alpha^{(n)}|_\Phi = |\alpha_{(n)}|_\Phi < \frac{\varepsilon}{2}, \quad |\beta - \beta^{(m)}|_\Phi = |\beta_{(m)}|_\Phi < \frac{\varepsilon}{2},$$

and consequently

$$|\alpha - \sum_{j=1}^{s} \lambda_j Q_j \beta|_\Phi \leqslant |\alpha - \alpha^{(n)}|_\Phi + |\sum_{j=1}^{s} \lambda_j Q_j (\beta - \beta^{(m)})|_\Phi < \varepsilon$$

The theorem is proved.

REMARK 4.2. It follows from Theorem 4.3 that if Φ and Ψ are any two symmetric gauge functions and $\beta \in l_\Phi^{(0)}$, $\beta \in l_\Psi^{(0)}$, then $\Gamma_\Phi(\beta) = \Gamma_\Psi(\beta)$.

§ 5. The eigen- and singular values of the sum and product of completely continuous operators

1. Let $\mathfrak{H}$ be a separable Hilbert space, $\mathfrak{R}$ the normed ring of all linear bounded operators on $\mathfrak{H}$, and $\mathfrak{S}_\infty$ the ideal of all completely continuous operators of $\mathfrak{R}$.

If $A \in \mathfrak{S}_\infty$, then $s(A) = \{s_j(A)\}_1^\infty$ denotes the sequence of *singular values* of A, i.e., the sequence of eigenvalues of the operator $(A^*A)^{1/2}$, numbered in decreasing order with multiplicities taken into account.

We shall need later the following two well-known properties of the singular values:

1°. *If* $A \in \mathfrak{S}_\infty$ *and* $B \in \mathfrak{R}$, *then*

$$s_j(AB) \leqslant |B| s_j(A), \quad s_j(BA) \leqslant |B| s_j(A) \quad (j = 1, 2, \dots), \tag{5.1}$$

where $|B|$ *is the norm of the operator* B (*i.e.* $|B| = \sup_{\varphi \in \mathfrak{H}} |B\varphi|/|\varphi|$).

2°. *If* $A \in \mathfrak{S}_\infty$, $\mathfrak{N}$ *is the invariant sub-space of* A, *and* A *is the* *operator induced in* $\mathfrak{N}$ *by* A, *then*

$$s_j(\hat{A}) \leqslant s_j(A) \quad (j = 1,\ 2,\ \ldots,\ \dim \mathfrak{N}). \tag{5.2}$$

Let $\mathfrak{K}$ be the set of all finite-dimensional operators of $\mathfrak{R}$. Following [20] (p.54) we shall call a real function $\|A\|$ $(A \in \mathfrak{K})$ defined on $\mathfrak{K}$ a *unitarily invariant norm* if the following conditions are satisfied:

a) $\|A\| > 0$ if $A \neq 0$, $A \in \mathfrak{K}$;

b) $\|\lambda A\| = |\lambda|\ \|A\|$ for any number λ and any $A \in \mathfrak{K}$;

c) $\|A + B\| \leqslant \|A\| + \|B\|\ (A,\ B \in \mathfrak{K})$;

d)[1] $\|UAV\| = \|A\|$ for any $A \in \mathfrak{K}$ and any unitary operators U, V on $\mathfrak{H}$;

e) if K is a one-dimensional operator, $K = (\cdot,\ \varphi)\,\psi$, then

$$\|K\| = |K| = |\varphi|\,|\psi|.$$

The following result is due to von Neumann and Schatten (see [20], p.69):

If $\Phi(\alpha)$ *is an arbitrary symmetric gauge function, then the equation*

$$\|A\| = \Phi(s(A)) \quad (A \in \mathfrak{K}) \tag{5.3}$$

defines a unitarily invariant norm on $\mathfrak{K}$. *Conversely, for any unitarily invariant norm* $\|A\|$ $(A \in \mathfrak{K})$ *there exists a symmetric gauge function* $\Phi(\alpha)$ *such that* (5.3) *holds.*

We shall denote by $\|A\|_\Phi$ $(A \in \mathfrak{K})$ the invariant norm generated by the symmetric gauge function $\Phi(\alpha)$ in accordance with (5.3).

Let Φ be a symmetric gauge function, and $\mathfrak{S}_\Phi$ the set of all operators $A \in \mathfrak{S}_\infty$ for which $s(A) \in l_\Phi$. If the norm $\|A\|_\Phi = |s(A)|_\Phi = \Phi(s(A))$ is introduced into $\mathfrak{S}_\Phi$, then it becomes a Banach space. We denote by $\mathfrak{S}_\Phi^{(0)}$ the subspace of $\mathfrak{S}_\Phi$ obtained by taking the closure of $\mathfrak{K}$.

It is obvious that always $\mathfrak{S}_\Phi^{(0)} \subseteq \mathfrak{S}_\Phi \subseteq \mathfrak{S}_\infty$, while $\mathfrak{S}_\infty$ itself is $\mathfrak{S}_{\Phi_\infty} = \mathfrak{S}_{\Phi_\infty}^{(0)}$. It is not difficult to see that an operator $A (\in \mathfrak{S}_\infty)$ is in $\mathfrak{S}_\Phi^{(0)}$ if and only if the vector $s(A) \in l_\Phi^{(0)}$.

An important rôle is played in the theory of non-self-adjoint operators by the spaces $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\Phi^{(0)}$ for the case when $\Phi = \Phi_p$, $\Phi = \Phi_\Pi$ and $\Phi = \Phi_\pi$ (see § 2). The spaces $\mathfrak{S}_{\Phi_p} = \mathfrak{S}_{\Phi_p}^{(0)}$ $(1 \leqslant p < \infty)$, usually denoted by $\mathfrak{S}_p$, were first considered by von Neumann and Schatten, and the spaces $\mathfrak{S}_{\Phi_\Pi}^{(0)}$, $\mathfrak{S}_{\Phi_\Pi}$, and $\mathfrak{S}_{\Phi_\pi}^{(0)} = \mathfrak{S}_{\Phi_\pi}$ by I.Ts. Gokhberg and M.G. Krein [21].

The space $\mathfrak{S}_1$ is the set of all *kernel* operators [22]. We shall denote the norm $\|A\|_{\Phi_1}$ in $\mathfrak{S}_1$ by $\|A\|_1$.

As is known, $A \in \mathfrak{R}$ is a kernel operator if and only if for any orthonormal basis $\{\varphi_j\}_1^\infty$ of $\mathfrak{H}$ the following series converges:

$$\sum_{j=1}^{\infty} (A\varphi_j,\ \varphi_j). \tag{5.4}$$

If $A \in \mathfrak{S}_1$, then the sum of this series does not depend on the choice of basis $\{\varphi_j\}_1^\infty$; it is called the *trace* of A and denoted by Sp A. We mention that

[1] Condition d) in this definition can be replaced by the following condition (see [20], p.71):

d) $\|AKB\| \leqslant |A|\,|B|\,\|K\|$ $(K \in \mathfrak{K};\ A,\ B \in \mathfrak{R})$.

$$|\operatorname{Sp} A| \leqslant \|A\|_1 \quad (A \in \mathfrak{S}_1). \tag{5.5}$$

2. If A is a self-conjugate operator of $\mathfrak{S}_\infty$, we shall denote by $\lambda(A) = \{\lambda_j(A)\}'^{\infty}_{-\infty}$ the complete system of eigenvalues of A, numbered (taking multiplicities into account) by indices running from $-\infty$ to ∞ with omission of the index $j = 0$ and with [1]

$$\lambda_j(A) \geqslant 0, \quad \lambda_{-j}(A) \leqslant 0, \quad \lambda_j(A) \geqslant \lambda_{j+1}(A), \quad \lambda_{-j}(A) \leqslant \lambda_{-j-1}(A) \quad (j=1,\, 2,\, \ldots)^{1}).$$

In this paragraph we shall denote by $l_{\Phi}^{(\varphi)}$ spaces that differ from those introduced in § 3 only in that they consist of sequences $\alpha = \{\alpha_j\}'^{\infty}_{-\infty}$, where the prime signifies omission of the index $j = 0$.

THEOREM 5.1. *If A and B are self-adjoint completely continuous operators and $C = A + B$, then*

$$\lambda(C) - \lambda(B) \prec\!\prec \lambda(A) \text{ and } \lambda(B) - \lambda(C) \prec\!\prec -\lambda(A). \tag{5.6}$$

PROOF. Since $-C = -A - B$, it is sufficient to prove the first half of (5.6), in other words, that for any finite system of indices $-i_1 < -i_2 < \ldots < -i_k < 0 < j_l < \ldots < j_2 < j_1$

$$\sum_{t=1}^{k} (\lambda_{-i_t}(C) - \lambda_{-i_t}(B)) + \sum_{t=1}^{l} (\lambda_{j_t}(C) - \lambda_{j_t}(B)) \leqslant \sum_{j=1}^{k+l} \lambda_j(A). \tag{5.7}$$

Without loss of generality we may suppose that to the numbers $\lambda_{-i}(B)$ ($i = 1, \ldots, i_k$) there corresponds an orthonormal system of eigenvectors φ_i ($i = 1, \ldots, i_k$) of B, and to the numbers $\lambda_j(C)$ ($j = 1, \ldots, j_l$) an orthonormal system of eigenvectors ψ_j ($j = 1, \ldots, j_l$) of C. For, if among the values $\lambda_{-i}(B)$ ($i = 1, \ldots, i_k$) and $\lambda_j(C)$ ($j = 1, \ldots, j_l$) there are p that are equal to zero and to which no eigenvectors correspond, then instead of $\mathfrak{H}$ we consider a space $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathfrak{M}$, where $\dim \mathfrak{M} = p$, and instead of A, B, C operators $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ that coincide with A, B, C on $\mathfrak{H}$ and vanish on $\mathfrak{M}$. It is obvious that $\lambda(A) = \lambda(\tilde{A})$, $\lambda(B) = \lambda(\tilde{B})$, $\lambda(C) = \lambda(\tilde{C})$, and that $\tilde{B}$ and $\tilde{C}$ have the required properties.

Let $\mathfrak{N}$ be a subspace of $\mathfrak{H}$ of dimension $n = i_k + j_l$ containing the vectors φ_{-i} ($i = 1, \ldots, i_k$) and ψ_j ($j = 1, \ldots, j_l$), P the orthogonal projector onto $\mathfrak{N}$ in $\mathfrak{H}$, and $\hat{A}$, $\hat{B}$, $\hat{C}$ the operators induced on $\mathfrak{N}$ by PA, PB, PC, respectively. As is known, the eigenvalues of A and $\hat{A}$ are connected by the inequalities

$$\lambda_{j-n-1}(A) \leqslant \lambda_j(\hat{A}) \leqslant \lambda_j(A) \quad (j=1,\, 2,\, \ldots,\, n), \tag{5.8}$$

and similarly for B, $\hat{B}$ and C, $\hat{C}$. Since the vectors φ_{-i} ($i = 1, \ldots, i_k$) and ψ_j ($j = 1, \ldots, j_l$) are in $\mathfrak{N}$, the $\lambda_{-i}(B)$ ($i = 1, \ldots, i_k$) are

[1] The term "complete" does not have altogether the usual meaning here, because if A has infinitely many positive and infinitely many negative eigenvalues, then the sequence $\{\lambda_j(A)\}'^{\infty}_{-\infty}$ will consist of these alone, and therefore contain no eigenvalues equal to zero, even if A possesses some; but if A has only a finite number n ($\geqslant 0$) of positive (negative) eigenvalues, then we put

$$\lambda_{n+j}(A) = 0 \quad (j=1,\, 2,\, \ldots) \quad (\lambda_{-n-j}(A) = 0 \quad (j=1,\, 2,\, \ldots))$$

independently of whether or not 0 is an eigenvalue of A.

eigenvalues of $\hat{B}$ and the $\lambda_j(C)$ ($j = 1, \ldots, j_l$) of $\hat{C}$. Using (5.8) for B, $\hat{B}$ and C, $\hat{C}$, we obtain

$$\lambda_{n+1-i}(\hat{B}) = \lambda_{-i}(B) \quad (i = 1, 2, \ldots, i_k)$$
$$\lambda_j(\hat{C}) = \lambda_j(C) \quad (j = 1, 2, \ldots, j_l).$$

From these equations and (5.8) for B, $\hat{B}$ and C, $\hat{C}$ it follows that

$$\lambda_{n+1-i}(\hat{C}) - \lambda_{n+1-i}(\hat{B}) \geqslant \lambda_{-i}(C) - \lambda_{-i}(B) \quad (i = 1, 2, \ldots, i_h), \tag{5.9}$$
$$\lambda_j(\hat{C}) - \lambda_j(\hat{B}) \geqslant \lambda_j(C) - \lambda_j(B) \quad (j = 1, 2, \ldots, j_l). \tag{5.10}$$

Applying Theorem 2.1 to $\hat{A}$, $\hat{B}$, and $\hat{C}$, we can write, in particular,

$$\sum_{t=1}^{k} (\lambda_{n+1-i_t}(\hat{C}) - \lambda_{n+1-i_t}(\hat{B})) + \sum_{t=1}^{l} (\lambda_{j_t}(\hat{C}) - \lambda_{j_t}(\hat{B})) \leqslant \sum_{j=1}^{k+l} \lambda_j(\hat{A}).$$

From this inequality and (5.8) - (5.10) the inequality (5.7) now follows, and the theorem is proved.

Later we shall need the following

LEMMA 5.1. *If $H \in \mathfrak{S}_\infty$ and $T \in \mathfrak{S}_1$ are self-adjoint operators, then*

$$\sum_{j=-\infty}^{\infty}{}' (\lambda_j(H+T) - \lambda_j(H)) = \operatorname{Sp} T, \tag{5.11}$$

the series on the left being absolutely convergent [1].

PROOF. Since by Theorem 5.1

$$\lambda(H+T) - \lambda(H) \prec\!\prec \lambda(T) \text{ and } \lambda(H) - \lambda(H+T) \prec\!\prec -\lambda(T),$$

the series on the left of (5.11) converges absolutely, with

$$\sum_{j=-\infty}^{\infty} |\lambda_j(H+T) - \lambda_j(H)| \leqslant \sum_{j=-\infty}^{\infty} |\lambda_j(T)| = \|T\|_1. \tag{5.12}$$

Let ε be an arbitrary positive number and let

$$K = \sum_{j=1}^{n} (\cdot, \varphi_j)\varphi_j \quad ((\varphi_j, \varphi_k) = 0, \ j \neq k)$$

be a self-adjoint finite-dimensional operator such that

$$\|T - K\|_1 < \frac{\varepsilon}{4}. \tag{5.13}$$

Further, let $\{\psi_k\}_1^\infty$ be an orthonormal basis of $\mathfrak{H}$ composed of eigenvectors of H, and let

$$P_m = \sum_{k=1}^{m} (\cdot, \psi_k)\psi_k \quad (m = 1, 2, \ldots).$$

Choose m so large that

$$|P_m\varphi_j - \varphi_j| < \varepsilon(16n^2|K|^{1/2})^{-1} \quad (j = 1, 2, \ldots, n),$$

and put

[1] As M.G. Krein has pointed out to the author, Lemma 5.1 can be derived from his trace formula.

$$\widetilde{K} = \sum_{j=1}^{n} (\cdot,\ P_m\varphi_j) P_m\varphi_j.$$

For any $x \in \mathfrak{H}$ and $j = 1, \ldots, n$

$$|(x,\ P_m\varphi_j) P_m\varphi_j - (x,\ \varphi_j)\varphi_j| \leqslant |(x,\ P_m\varphi_j)(P_m\varphi_j - \varphi_j)| + $$
$$+ |(x,\ P_m\varphi_j - \varphi_j)\varphi_j| \leqslant |P_m\varphi_j - \varphi_j|\,(|P_m\varphi_j| +$$
$$+ |\varphi_j|)\,|x| \leqslant 2\,|K|^{1/2}\,|P_m\varphi_j - \varphi_j|\,|x|.$$

Consequently

$$|\widetilde{K} - K| \leqslant \sum_{j=1}^{n} 2\,|K|^{1/2}\,|P_m\varphi_j - \varphi_j| < \varepsilon\,(8n)^{-1}.$$

Since $\dim(\widetilde{K} - K) \leqslant 2n$,

$$\|\widetilde{K} - K\|_1 \leqslant 2n\,|\widetilde{K} - K| < \frac{\varepsilon}{4}. \tag{5.14}$$

From (5.13) and (5.14),

$$\|T - \widetilde{K}\|_1 < \frac{\varepsilon}{2}. \tag{5.15}$$

Denote by $\mathfrak{N}$ the subspace spanned by the vectors ψ_j $(j = 1, \ldots, m)$, and by $\mathfrak{N}_1$ its orthogonal complement in $\mathfrak{H}$. Obviously $\mathfrak{N}$ and $\mathfrak{N}_1$ are invariant for H and $\widetilde{K}$, and $\widetilde{K}\mathfrak{N}_1 = 0$. Since $\mathfrak{N}$ is finite-dimensional, it follows that the complete systems of eigenvalues of H and $H + \widetilde{K}$ differ by only a finite number of elements. Therefore we can find natural numbers k and l, integers p and q $(p + k > 1,\ q - l \leqslant -1)$, and a finite-dimensional subspace $\mathfrak{M} \supseteq \mathfrak{N}$ such that

1) $\lambda_j(H + \widetilde{K}) = \lambda_{j+p}(H)\quad (j \geqslant k),\quad \lambda_{-i}(H + \widetilde{K}) = \lambda_{q-i}(H)\quad (i \geqslant l)$;

2) $H\mathfrak{M} \subseteq \mathfrak{M},\quad \widetilde{K}\mathfrak{M} \subseteq \mathfrak{M}$;

3) the spectra of H and $H + \widetilde{K}$ in $\mathfrak{M}$ coincide, respectively, except possibly for zero values, with the systems

$$\{\lambda_j(H)\}_{q-l+1}^{p+k-1} \text{ and } \{\lambda_j(H + \widetilde{K})\}_{-l+1}^{k-1}.$$

Obviously

$$\sum_{j=-\infty}^{\infty}{}' (\lambda_j(H + \widetilde{K}) - \lambda_j(H)) = \sum_{j=-l+1}^{k-1} \lambda_j(H + \widetilde{K}) - \sum_{j=q-l+1}^{p+k-1} \lambda_j(H).$$

The right hand side of this equation coincides with the trace of the part of the operator $\widetilde{K}$ in the subspace $\mathfrak{M}$, but since $\widetilde{K}(\mathfrak{H} \ominus \mathfrak{M}) = 0$, it is equal to $\operatorname{Sp}\widetilde{K}$. Thus,

$$\sum_{j=-\infty}^{\infty}{}' (\lambda_j(H + \widetilde{K}) - \lambda_j(H)) = \operatorname{Sp}\widetilde{K}. \tag{5.16}$$

Applying (5.12) to $H + T$ and $H + \widetilde{K}$, we obtain

$$\sum_{j=-\infty}^{\infty}{}' |\lambda_j(H + T) - \lambda_j(H + \widetilde{K})| \leqslant \|T - \widetilde{K}\|_1. \tag{5.17}$$

We note also that by (5.5)

$$|\operatorname{Sp} T - \operatorname{Sp} \widetilde{K}| \leqslant \|T - \widetilde{K}\|_1. \tag{5.18}$$

From (5.16) - (5.18) and (5.13) it follows that

$$\Big|\sum_{j=-\infty}^{\infty}{}' (\lambda_j(H+T) - \lambda_j(H)) - \operatorname{Sp} T\Big| < \varepsilon.$$

Since ε is arbitrary, (5.11) now follows, and the lemma is proved.

THEOREM 5.2. *If* $A \in \mathfrak{S}_1$ *and* $B \in \mathfrak{S}_\infty$ *are self-adjoint operators and* $C = A + B$, *then*

$$\lambda(C) - \lambda(B) \prec \lambda(A).$$

For by Theorem (5.1) the conditions (5.6) are satisfied, while by Lemma 5.1

$$\sum_{j=-\infty}^{\infty}{}' (\lambda_j(C) - \lambda_j(B)) = \sum_{j=-\infty}^{\infty}{}' \lambda_j(A).$$

THEOREM 5.3. *Let* Φ *be an arbitrary symmetric gauge function,* $A \in \mathfrak{S}_\Phi^{(0)}$ *and* $B \in \mathfrak{S}_\infty$ *self-adjoint operators, and* $C = A + B$. *Then the vector* $\lambda(C) - \lambda(B)$ *is contained in* $\Delta_\Phi(\lambda(A))$, *the convex closed hull, with respect to the norm* $|\cdot|_\Phi$ *of the set of all vectors obtained by permuting the coordinates of* $\lambda(A)$.

If the norm $|\alpha|_\Phi$, is not equivalent to $|\alpha|_1 = \Sigma|\alpha_j|$, then Theorem 5.3 follows from Theorems 5.1 and 4.1, and if $|\alpha|_\Phi$ is equivalent to $|\alpha|_1$, it follows from Theorems 5.2 and 4.2.

COROLLARY 5.1. *Let* Φ *be an arbitrary symmetric gauge function,* A *and* B *two self-adjoint operators of* $\mathfrak{S}_\Phi^{(0)}$, *and* K_1 (K_2) *the convex closed hull, with respect to the norm* $l_\Phi^{(0)}$, *of the set of all vectors of the form* $\{\lambda_j(B) + \lambda_{k_j}(A)\}_{-\infty}^{\infty}$ *(respectively* $\{\lambda_j(A) + \lambda_{k_j}(B)\}_{-\infty}^{\prime\infty}$*), where* $\{k_j\}_{-\infty}^{\prime\infty}$ *is an arbitrary permutation of the sequence* $\{j\}_{-\infty}^{\prime\infty}$. *If* $C = A + B$, *then* $\lambda(C) \in K_1 \cap K_2$.

3. THEOREM 5.4. *If* A *and* B *are completely continuous operators and* $C = A + B$, *then*

$$\{|s_j(C) - s_j(B)|\}_1^\infty \prec\!\!\prec s(A). \tag{5.19}$$

PROOF. Consider the self-adjoint operator, acting on the space $\mathfrak{H} \times \mathfrak{H}$,

$$\widetilde{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$$

and similarly defined operators $\widetilde{B}$ and $\widetilde{C}$.

Applying Theorem 5.1 to $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, we obtain

$$\lambda(\widetilde{C}) - \lambda(\widetilde{B}) \prec\!\!\prec \lambda(\widetilde{A}), \quad \lambda(\widetilde{B}) - \lambda(\widetilde{C}) \prec\!\!\prec -\lambda(\widetilde{A}). \tag{5.20}$$

It is easy to verify that

$$\lambda_j(\widetilde{A}) = s_j(A), \ \lambda_{-j}(A) = -s_j(A) \qquad (j = 1, 2, \ldots),$$

and similarly for B, $\widetilde{B}$ and C, $\widetilde{C}$. Therefore (5.19) follows from (5.20).

COROLLARY 5.2. *If $A, B \in \mathfrak{S}_\infty$ and $C = A + B$, then for any finite system of indices $j_1 < j_2 < \ldots < j_k$*

$$\sum_{t=1}^{k} s_{j_t}(C) \leqslant \sum_{t=1}^{k} s_t(A) + \sum_{t=1}^{k} s_{j_t}(B).$$

For the case $j_t = t$ ($t = 1, \ldots, k$) this was established by Ky Fan ([5], Theorem 5).

From Theorems 1.2 and 5.4 follows

COROLLARY 5.3. *If Φ is an arbitrary symmetric gauge function, $A, B \in \mathfrak{S}_\infty$, and $A - B \in \mathfrak{S}_\Phi$, then*

$$\| A - B \|_\Phi \geqslant \Phi(s(A) - s(B)).$$

For the finite-dimensional case this result was obtained by Mirsky [16].

THEOREM 5.5. *Let Φ be an arbitrary symmetric gauge function, $A \in \mathfrak{S}_\Phi^{(0)}$, $B \in \mathfrak{S}_\infty$, and $C = A + B$. Then*

$$s(C) - s(B) \in \Gamma_\Phi(s(A)).$$

This follows from Theorems 5.4 and 4.3.

4. THEOREM 5.6. *If $A, B \in \mathfrak{S}_\infty$ and $C = AB$, then for any finite system of indices $j_1 < j_2 < \ldots < j_k$*

$$\prod_{t=1}^{k} s_{j_t}(C) \leqslant \prod_{t=1}^{k} s_t(A) \prod_{t=1}^{k} s_{j_t}(B). \tag{5.21}$$

PROOF. Represent C in polar form: $C = UH$, where U is a partially isometric and H a non-negative operator. Obviously $H = U^*AB$ and $\lambda_j(H) = s_j(C)$ ($j = 1, 2, \ldots$).

Let $\mathfrak{N}$ be the subspace of $\mathfrak{H}$ spanned by the eigenvectors of H corresponding to the eigenvalues $\lambda_j(H)$ ($j = 1, \ldots, j_k$), and P the orthogonal projector of $\mathfrak{H}$ onto $\mathfrak{N}$. Choose a unitary operator V such that $VB\mathfrak{N} \subseteq \mathfrak{N}$, and put $A_1 = PU^*AV^*$, $B_1 = VB$. Denote by $\hat{H}$, $\hat{A}_1$, $\hat{B}_1$ the operators induced on $\mathfrak{N}$ by H, A_1, B_1, respectively. Obviously $\hat{H} = \hat{A}_1 \hat{B}_1$. Applying to $\hat{H}$, $\hat{A}_1$, and $\hat{B}_1$ the finite-dimensional analogue of a theorem proved earlier (see inequality (2.9)), we obtain

$$\prod_{t=1}^{k} s_{j_t}(\hat{H}) \leqslant \prod_{t=1}^{k} s_t(\hat{A}_1) \prod_{t=1}^{k} s_{j_t}(\hat{B}_1). \tag{5.22}$$

By (5.1) and (5.2)

$$s_j(\hat{A}_1) \leqslant s_j(A), \quad s_j(\hat{B}_1) \leqslant s_j(B) \quad (j = 1, 2, \ldots, j_k). \tag{5.23}$$

From (5.22), (5.23), and the obvious equations

$$s_j(\hat{H}) = \lambda_j(\hat{H}) = \lambda_j(H) = s_j(C) \quad (j = 1, 2, \ldots, j_k)$$

(5.21) follows, and the theorem is proved.

We mention that for the case $j_t = t$ ($t = 1, \ldots, k$) Theorem 5.6 was established by Horn [6].

If A and B ($\in \mathfrak{S}_\infty$) are non-negative operators, then $C = AB$ is non-negative only in case $AB = BA$. However, the eigenvalues of $C = A^{1/2} A^{1/2} B^{1/2} B^{1/2}$ also coincide in the general case with those of $(A^{1/2} B^{1/2} B^{1/2}) A^{1/2} = A^{1/2} B^{1/2} (A^{1/2} B^{1/2})^*$; consequently they are non-negative. Since the eigenvalues of $A^{1/2} B^{1/2} (A^{1/2} B^{1/2})^*$ are the squares of the singular values of $A^{1/2} B^{1/2}$, by applying Theorem 5.6 to the product of $A^{1/2}$ and $B^{1/2}$ we obtain the following proposition (in the sequence $\{\lambda_j(C)\}_1^\infty$ the eigenvalues of C are numbered in order of decreasing modulus and with multiplicities taken into account).

THEOREM 5.7.[1] *If A and B are non-negative operators of $\mathfrak{S}_\infty$, and $C = AB$, then for any finite system of indices $j_1 < j_2 < \ldots < j_k$*

$$\prod_{t=1}^{k} \lambda_{j_t}(C) \leqslant \prod_{t=1}^{k} \lambda_t(A) \prod_{t=1}^{k} \lambda_{j_t}(B).$$

5. If H is a self-adjoint operator of $\mathfrak{S}_\infty$, we put

$$\lambda_j(I+H) = 1 + \lambda_j(H) \quad (j = \pm 1, \pm 2, \ldots).$$

We introduce the notion of singular values for operators of the form $U + T$, where U is unitary and T completely continuous, as follows:

$$s_j(U+T) = (\lambda_j((U^*+T^*)(U+T)))^{1/2} = (1 + \lambda_j(U^*T + T^*U + T^*T))^{1/2} \quad (j = \pm 1, \pm 2, \ldots).$$

It is easy to verify that these singular values have the following properties:

1°. *If U and U_1 are unitary operators and $T \in \mathfrak{S}_\infty$, then*

$$s_j(U_1(U+T)) = s_j((U+T)U_1) = s_j(U+T) \; (j = \pm 1, \pm 2, \ldots). \tag{5.24}$$

2°. *If $\mathfrak{N}$ is an n-dimensional invariant subspace of $B = U + T$, and $\hat{B}$ is the operator induced by B in $\mathfrak{N}$, then*

$$s_{j-n-1}(B) \leqslant s_j(\hat{B}) \leqslant s_j(B) \quad (j = 1, 2, \ldots, n). \tag{5.25}$$

THEOREM 5.8. *Suppose that Φ is an arbitrary symmetric gauge function, $T_1 \in \mathfrak{S}_\Phi^{(0)}$, $T_2 \in \mathfrak{S}_\infty$, that the operators $A = I + T_1$ and $B = I + T_2$ vanish only at the origin, and that $C = AB$. Then*

$$\{\ln s_j(C) - \ln s_j(B)\}_{-\infty}^{\prime\infty} \in \Delta_\Phi(\{\ln s_j(A)\}_{-\infty}^{\prime\infty}).$$

PROOF. Since

$$\ln s_j(I+T) = \frac{1}{2} \lambda_j(T + T^* + T^*T) + o(\lambda_j(T + T^* + T^*T)),$$

for $T \in \mathfrak{S}_\Phi^{(0)}$ we have

$$\{\ln s_j(I+T)\}_{-\infty}^{\prime\infty} \in l_\Phi^{(0)}.$$

Therefore, by Theorems 4.1 and 4.2, it is sufficient to establish that

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1 The finite-dimensional case of Theorem 5.7, together with this method of deriving it from a theorem on the singular values of the product of two operators, is indicated by Amir-Moéz ([9], Theorem 3.12).

$$\{\ln s_j(C)-\ln s_j(B)\}_{-\infty}^{'\infty} \prec\prec \{\ln s_j(A)\}_{-\infty}^{'\infty} \tag{5.26}$$

$$\{\ln s_j(B)-\ln s_j(C)\}_{-\infty}^{'\infty} \prec\prec \{-\ln s_j(A)\}_{-\infty}^{'\infty}, \tag{5.27}$$

and if $T_1 \in \mathfrak{S}_1$, that

$$\{\ln s_j(C)-\ln s_j(B)\}_{-\infty}^{'\infty} \prec \{\ln s_j(A)\}_{-\infty}^{'\infty}. \tag{5.28}$$

Since $B = A^{-1}C$ and $s_j(A^{-1}) = s_j^{-1}(A)$ $(j = \pm 1, \pm 2, \ldots)$, (5.27) follows from (5.26). To prove (5.26) it is sufficient to establish that for any finite system of indices

$$-i_1 < -i_2 < \ldots < -i_k < 0 < j_l < j_{l-1} < \ldots < j_1$$

we have

$$\prod_{t=1}^{k} \frac{s_{-i_t}(C)}{s_{-i_t}(B)} \prod_{t=1}^{l} \frac{s_{j_t}(C)}{s_{j_t}(B)} \leqslant \prod_{j=1}^{k+l} s_j(A). \tag{5.29}$$

Represent C and B in polar form: $C = UH$, $B = U_1H_1$, where U, U_1 are unitary and H, H_1 positive. It is easy to verify that $H - I$ and $H_1 - I$ are completely continuous. Obviously

$$\lambda_j(H) = s_j(C),\ \lambda_j(H_1) = s_j(B) \qquad (j = \pm 1, \pm 2, \ldots). \tag{5.30}$$

Without loss of generality we may suppose that to the values $\lambda_j(H)$ $(j = 1, \ldots, j_l)$ there corresponds an orthonormal system of eigenvectors $\psi_j (j = 1, \ldots, j_l)$ of H, and to the values $\lambda_{-i}(H_1)$ $(i = 1, \ldots, i_k)$ an orthonormal system of eigenvectors $\varphi_{-i} (i = 1, \ldots, i_k)$ of H_1 (this is established in exactly the same way as the analogous assertion in the proof of Theorem 5.1). Let $\mathfrak{M}$ be the linear hull of the vectors $\psi_j (j = 1, \ldots, j_l)$, $\mathfrak{M}_1$ that of the vectors φ_{-i} $(i = 1, \ldots, i_k)$, and $\mathfrak{N}$ a subspace of $\mathfrak{H}$ of dimension $n = i_k + j_l$ containing $\mathfrak{M}$ and $\mathfrak{M}_1$. Denote by U_2 and U_3 unitary operators satisfying the conditions

$$U_2H_1\mathfrak{N} = \mathfrak{N},\quad U_2\varphi = \varphi \quad (\varphi \in \mathfrak{M}_1),$$
$$U_3H\mathfrak{N} = \mathfrak{N},\ U_3\psi = \psi \quad (\psi \in \mathfrak{M}),$$

and by $\hat{H}$, $\hat{A}$, $\hat{H}_1$ the operators induced in $\mathfrak{N}$ by U_3H, $U_3U^*AU_1U_2^*$, U_2H_1, respectively. Since $U_3H = U_3U^*AU_1U_2^*U_2H_1$, therefore $\hat{H} = \hat{A}\hat{H}_1$. Applying (2.11) to $\hat{H}$, $\hat{A}$, and $\hat{H}_1$, we obtain

$$\prod_{t=1}^{k} \frac{s_{n+1-i_t}(\hat{H})}{s_{n+1-i_t}(\hat{H}_1)} \prod_{t=1}^{l} \frac{s_{j_t}(\hat{H})}{s_{j_t}(\hat{H}_1)} \leqslant \prod_{j=1}^{k+l} s_j(\hat{A}). \tag{5.31}$$

By (5.24) and (5.25)

$$s_j(\hat{A}) \leqslant s_j(U_3U^*AU_1U_2^*) = s_j(A) \ (j = 1, 2, \ldots, k+l), \tag{5.32}$$

$$s_{n+1-i}(\hat{H}) \geqslant s_{-i}(U_3H) = s_{-i}(H) \ (i = 1, 2, \ldots, i_k), \tag{5.33}$$

$$s_j(\hat{H}_1) \leqslant s_j(U_2H_1) = s_j(H_1) \ (j = 1, 2, \ldots, j_l). \tag{5.34}$$

From (5.25), by the choice of the subspace $\mathfrak{N}$ and the operators U_2, U_3, it is not difficult to deduce that

$$s_j(\hat{H}) \geqslant \lambda_j(H)\,(j=1,2,\ldots,j_l),\ s_{n+1-i}(\hat{H}_1) \leqslant \lambda_{-i}(H_1)\,(i=1,2,\ldots,i_k). \tag{5.35}$$

Inequality (5.29) follows from (5.30 - (5.35).

Thus, to conclude the proof it remains to show that if $T_1 \in \mathfrak{S}_1$, then (5.28) holds, i.e.

$$\prod_{j=-\infty}^{\infty}{}' \frac{s_j(C)}{s_j(B)} = \prod_{j=-\infty}^{\infty}{}' s_j(A).$$

The proof of this relation is basically analogous to that of Lemma 5.1, and therefore we omit it.

Just as Theorem 5.7 is derived from Theorem 5.6, so we can easily obtain from Theorem 5.8 the following proposition, the infinite-dimensional analogue of Theorem 2.5.

THEOREM 5.9. *Let Φ be an arbitrary symmetric gauge function, and let the operators $A = I + H_1$ and $B = I + H_2$ be positive definite, where $H_1 \in \mathfrak{S}_\Phi^{(0)}$ and $H_2 \in \mathfrak{S}_\infty$. If $C = AB$, then*[1]

$$\{\ln \lambda_j(C) - \ln \lambda_j(B)\}_{-\infty}^{'\infty} \in \Delta_\Phi(\{\ln \lambda_j(A)\}_{-\infty}^{'\infty}).$$

It is obvious that corollaries similar to Corollary 5.1 also hold for Theorems 5.5, 5.8, and 5.9.

6. With the aid of the methods indicated above, the inequalities for the eigen- and singular values of matrices established in [9], [12], [13] can also be carried over to the infinite-dimensional case. We give as an example the analogue of one of the theorems of Amir-Moéz ([9], Theorem 3.6).

THEOREM 5.10. *If $A, B \in \mathfrak{S}_\infty$ and $C = A + B$, then for any system of indices*

$$i_1 \leqslant i_2 \leqslant \ldots \leqslant i_k \text{ and } j_1 \leqslant j_2 \leqslant \ldots \leqslant j_k$$

we have

$$\sum_{p=1}^{k} s_{m(l_p)}(C) \leqslant \sum_{p=1}^{k} s_{m(i_p)}(A) + \sum_{p=1}^{k} s_{m(j_p)}(B), \tag{5.36}$$

where $l_p = i_p + j_p - 1$ ($p = 1, \ldots, i_k$), and $m(i_p)$ is defined by (0.4).

PROOF. We represent C in polar form: $C = UH$, and denote by $\mathfrak{N}$ the subspace of $\mathfrak{H}$ spanned by the eigenvectors of H corresponding to the eigenvalues $\lambda_j(H) = s_j(C)$ ($j = 1, \ldots, m(l_k)$). If P is the orthogonal projector of $\mathfrak{H}$ onto $\mathfrak{N}$, we put $A_1 = PU^*A$ and $B_1 = PU^*B$. We denote by $\hat{H}$, $\hat{A}_1$, $\hat{B}_1$ respectively. Since $\hat{H} = \hat{A}_1\hat{B}_1$, by [9] (Theorem 3.6) the inequality (5.36) is valid for $\hat{H}$, $\hat{A}_1$, and $\hat{B}_1$, and since

$$s_j(\hat{H}) = \lambda_j(\hat{H}) = s_j(C), \quad s_j(\hat{A}_1) \leqslant s_j(\hat{A}),\ s_j(\hat{B}_1) \leqslant s_j(B)\,(j=1,2,\ldots,m(l_k)),$$

(5.36) is also valid for C, A, and B. The theorem is proved.

We mention in conclusion that by analyzing the proofs of Theorems 5.6

[1] It is easy to see that the eigenvalues of C are positive and tend to 1. We number those greater than unity in decreasing order by the indices 1 to ∞, and those less than unity in increasing order by the indices -1 to $-\infty$.

and 5.10 one can easily establish the validity of the following general assertion.

THEOREM 5.11. *If for arbitrary matrices A and B of order n we have*

$$\sum_{p=1}^{k} s_{l_p}(A+B) \leqslant \sum_{p=1}^{k} [s_{i_p}(A)+s_{j_p}(B)]$$

(respectively $\prod_{p=1}^{k} s_{l_p}(AB) \leqslant \prod_{p=1}^{k} s_{i_p}(A) s_{j_p}(B)$*), where the indices* i_p, j_p, l_p ($\leq n$) *are fixed, then this inequality also remains valid for arbitrary operators* $A, B \in \mathfrak{S}_\infty$.

A similar assertion also holds, for example, for an inequality of the form (0.3) (compare [12], Theorem 4), but the formulation of complete analogues of such inequalities for self-adjoint completely continuous operators is complicated somewhat by the method of enumerating their eigenvalues.

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