Drinfeld type realization of cyclotomic q-Schur algebras

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Introduction

$\mathscr{H}_{n,r}$: Ariki-Koike alg. $/_{\text{a field}}$ ass. to $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$.

$\mathscr{S}_{n,r}$: cyclotomic q-Schur algebra ass. to $\mathscr{H}_{n,r}$.

$\mathscr{S}_{n,r}$: a quasi-hereditary cover of $\mathscr{H}_{n,r}$.

Schur-Weyl duality ($r = 1$)

$$U_q(\mathfrak{gl}_m) \curvearrowright V^{\otimes n} \curvearrowleft \mathscr{H}_{n,1}$$

This duality holds for any field amd parameter.

$$\mathrm{Im}\,(U_q(\mathfrak{gl}_m) \to \mathrm{End}(V^{\otimes n})) \cong \mathrm{End}_{\mathscr{H}_{n,1}}(V^{\otimes n}) \cong \mathscr{S}_{n,1}.$$

Schur-Weyl duality ($r > 1$)

$$\begin{array}{ccc} U_q(\mathfrak{gl}_m) \curvearrowright & V^{\otimes n} & \curvearrowleft \mathscr{H}_{n,1} \\ \cup & & \cap \\ U_q(\mathfrak{g}) \curvearrowright & V^{\otimes n} & \curvearrowleft \mathscr{H}_{n,r} \end{array} \quad (\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \subset \mathfrak{gl}_m)$$

This duality holds only the case $\mathscr{H}_{n,r}$: semi-simple.

“ $V^{\otimes n}$ ” : too small !

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$\rightsquigarrow V^{\otimes n}_{\mu} \cong \mathrm{Ind}^{\mathscr{H}_{n,1}}_{\mathscr{H}(\mathfrak{S}_{\mu})} \mathbf{1}$ as $\mathscr{H}_{n,1}$-module (permutation module).

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$$\mathscr{S}_{n,r} := \mathrm{End}_{\mathscr{H}_{n,r}}\left(\bigoplus_{\mu} M^{\mu}\right)$$; cyclotomic q-Schur alg.

Today

- Introduce an algebra $\mathcal{U}$ ass. to Cartan data of $\mathfrak{gl}_m$ s.t. $\mathcal{U} \twoheadrightarrow \mathscr{S}_{n,r}$ (Drinfeld type presentation).
- Representation theory of $\mathcal{U}$ and $\mathscr{S}_{n,r}$.
  - highest weight modules.
  - Harish-Chandra Ind and Res ($\mathcal{U}^L \hookrightarrow \mathcal{U}^P \hookrightarrow \mathcal{U}$)
  - evaluation functor $\mathcal{O}^{\mathfrak{g}} \to \mathcal{U}^L$-mod $(\mathcal{O}^{\mathfrak{g}} \subset U_q(\mathfrak{g})\text{-mod})$.

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generators: $T_0, T_1, \dots, T_{n-1}$.

defining relations:

$$(T_0 - q^{c_1})(T_0 - q^{c_2}) \dots (T_0 - q^{c_r}) = 0 \qquad (c_1, \dots, c_r \in \mathbb{Z}),$$

$$(T_i - q)(T_i + q^{-1}) = 0 \qquad (1 \le i \le n-1),$$

$+$ braid relations

$L_i := T_{i-1} \dots T_1 T_0 T_1 \dots T_{i-1}$ $(1 \le i \le n)$: Jucys-Murphy elements.

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Borel subalgebras

$\mathbf{r = 1}$

e_i, f_i $(1 \le i \le m-1)$, $K_j^{\pm}$,$(1 \le j \le m)$: Chevalley gen. of $U_q(\mathfrak{gl}_m)$.
Recall $\rho : U_q(\mathfrak{gl}_m) \twoheadrightarrow \mathscr{S}_{n,1}$.

- $U_q^{\ge 0} := \langle e_i, K_j^{\pm} \mid 1 \le i \le m-1,\ 1 \le j \le m \rangle_{\text{alg.}} \subset U_q(\mathfrak{gl}_m)$
- $U_q^{\le 0} := \langle f_i, K_j^{\pm} \mid 1 \le i \le m-1,\ 1 \le j \le m \rangle_{\text{alg.}} \subset U_q(\mathfrak{gl}_m)$

$\rightsquigarrow \mathscr{S}_{n,1} = \rho(U_q^{\le 0}) \cdot \rho(U_q^{\ge 0})$

Theorem (Du-Rui)

$\exists\, \mathscr{S}_{n,r}^{\ge 0}$, $\exists\, \mathscr{S}_{n,r}^{\le 0} \subset_{\text{alg.}} \mathscr{S}_{n,r}$ s.t. $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}^{\le 0} \cdot \mathscr{S}_{n,r}^{\ge 0}$.

Moreover, $\mathscr{S}_{n,r}^{\le 0} \cong \rho(U_q^{\le 0})$, $\quad \mathscr{S}_{n,r}^{\ge 0} \cong \rho(U_q^{\ge 0})$

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Theorem (Du-Rui)

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Theorem (Du-Rui)

$^{\exists}\mathscr{S}_{n,r}^{\ge 0}, {}^{\exists}\mathscr{S}_{n,r}^{\le 0} \subset_{alg.} \mathscr{S}_{n,r}$ s.t. $\mathscr{S}_{n,r} = \mathscr{S}_{n,r}^{\le 0} \cdot \mathscr{S}_{n,r}^{\ge 0}$.

Moreover, $\mathscr{S}_{n,r}^{\le 0} \cong \rho(U_q^{\le 0})$, $\mathscr{S}_{n,r}^{\ge 0} \cong \rho(U_q^{\ge 0})$

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Cartan data

- $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ s.t. $m_k \geq n$. Put $m = m_1 + \cdots + m_r$.
- $\{1, 2, \ldots, m\} \overset{1:1}{\longleftrightarrow} \Gamma(\mathbf{m}) := \{(i,k) \mid 1 \leq i \leq m_k, 1 \leq k \leq r\}$

  $$\begin{matrix} \Psi & & \Psi \\ \sum_{l=1}^{k-1} m_l + i & \longleftrightarrow & (i,k) \end{matrix}$$

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  Denote $\lambda \in \Lambda_{n,r}(\mathbf{m})$ by $\lambda = (\lambda^{[1]}, \dots, \lambda^{[r]})$,
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generators of $\mathscr{S}_{n,r}$

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- $\Lambda_{n,r}(\mathbf{m}) := \left\{ \lambda = \sum_{(i,k)\in\Gamma(\mathbf{m})} \lambda_{(i,k)}\varepsilon_{(i,k)} \in P_{\geq 0} \,\middle|\, \sum_{(i,k)\in\Gamma(\mathbf{m})} \lambda_{(i,k)} = n \right\}$
  Denote $\lambda \in \Lambda_{n,r}(\mathbf{m})$ by $\lambda = (\lambda^{[1]}, \dots, \lambda^{[r]})$,
  where $\lambda^{[k]} = (\lambda_{(1,k)}, \lambda_{(2,k)}, \dots, \lambda_{(m_k,k)})$.

Recall $\mathscr{S}_{n,r} := \mathscr{S}_{n,r}^{\leq 0} \cdot \mathscr{S}_{n,r}^{\geq 0}$ and $\mathscr{S}_{n,r}^{\leq 0} \cong \rho(U_q^{\leq 0})$, $\mathscr{S}_{n,r}^{\geq 0} \cong \rho(U_q^{\geq 0})$

$$\rho : U_q(\mathfrak{gl}_m) \twoheadrightarrow \mathscr{S}_{n,1} = \rho(U_q^{\leq 0}) \cdot \rho(U_q^{\geq 0}).$$

Define $\mathcal{X}^{\pm}_{(i,k),0}, \mathcal{K}^{\pm}_{(j,l)} \in \mathscr{S}_{n,r}$ $((i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma(\mathbf{m}))$ by

- $\mathcal{X}^{+}_{(i,k),0}$: image of $e_{(i,k)}$ in $\mathscr{S}_{n,r}^{\geq 0} \cong \rho(U_q^{\geq 0})$.
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Symmetric polynomials $\Phi_t^{\pm}$

Define $\Phi_t^{\pm}(X_1, \dots, X_k) \in \mathbb{Z}[q, q^{-1}][X_1, \dots, X_k]$ ($t \geq 1$) by

- $\Phi_1^{\pm}(X_1, \dots, X_k) := X_1 + X_2 + \cdots + X_k$.
- $\Phi_{t+1}^{\pm}(X_1, \dots, X_k)$

$$:= X_1^{t+1} + \sum_{s=2}^{k} \Big(\Phi_t^{\pm}(X_1, \dots, X_s) X_s - q^{\mp 2} \Phi_t^{\pm}(X_1, \dots, X_{s-1}) X_s \Big)$$

$$\Big(\mathbb{Z}[q, q^{-1}][X_1, \dots, X_s] \hookrightarrow \mathbb{Z}[q, q^{-1}][X_1, \dots, X_k], X_i \mapsto X_i \Big)$$

Lemma

$$\Phi_t^{\pm}(X_1, \dots, X_k) \in \mathbb{Z}[q, q^{-1}][X_1, \dots, X_k]^{\mathfrak{S}_k}.$$

Remark

$\Phi_t^{\pm}(X_1, \dots, X_{k-1}, 0) = \Phi_t^{\pm}(X_1, \dots, X_{k-1})$

$\rightsquigarrow$ ${}^{\exists}\Phi_t^{\pm}(X_1, X_2, \dots)$: symmetric function

s.t. $\Phi_t^{\pm}(X_1, \dots, X_k, 0, \dots) = \Phi_t^{\pm}(X_1, \dots, X_k)$

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Recall $\mathscr{S}_{n,r} = \mathrm{End}_{\mathscr{H}_{n,r}}\Big(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} m_\mu \cdot \mathscr{H}_{n,r}\Big)$.

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- $\mathcal{H}^{+}_{(j,l),t}(m_\mu) := \dfrac{1}{q^{-t+1}(q-q^{-1})^{t-1}} m_\mu \Phi^{+}_t(L_N, L_{N-1}, \dots, L_{N-\mu^{(l)}_j+1})$,
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generators of $\mathscr{S}_{n,r}$

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Definition

$\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{>0}$.

$\mathcal{U}$: associative algebra over $\mathbb{Q}(q)$ defined by

generators: $\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{K}^{\pm}_{(j,l)}, \mathcal{H}^{\pm}_{(j,l),t}, C_k$
$((i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r)$

defining relations:

C_k : central elements,

$$\mathcal{K}^+_{(j,l)}\mathcal{K}^-_{(j,l)} = \mathcal{K}^-_{(j,l)}\mathcal{K}^+_{(j,l)} = 1, \quad \mathcal{H}^{\pm}_{(i,k),0} = 1,$$

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$$(\varepsilon, \varepsilon' \in \{+,-\})$$

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Algebra $\mathcal{U}$

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$$[\mathcal{H}^+_{(j,l),s+1}, \mathcal{X}^{\pm}_{(i,k),t}] = q^{\pm a_{(i,k)(j,l)}}\mathcal{H}^+_{(j,l),s}\mathcal{X}^{\pm}_{(i,k),t+1} - q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^{\pm}_{(i,k),t+1}\mathcal{H}^+_{(j,l),s}$$

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Definition

$\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{>0}$.

$\mathcal{U}$: associative algebra over $\mathbb{Q}(q)$ defined by

generators: $\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{K}^{\pm}_{(j,l)}, \mathcal{H}^{\pm}_{(j,l),t}, C_k$
$((i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma(\mathbf{m}), t \geq 0, 1 \leq k \leq r)$

defining relations:

C_k : central elements,

$$\mathcal{K}^{+}_{(j,l)}\mathcal{K}^{-}_{(j,l)} = \mathcal{K}^{-}_{(j,l)}\mathcal{K}^{+}_{(j,l)} = 1, \quad \mathcal{H}^{\pm}_{(i,k),0} = 1,$$

$$[\mathcal{K}^{\varepsilon}_{(i,k)}, \mathcal{K}^{\varepsilon'}_{(j,l)}] = [\mathcal{K}^{\varepsilon}_{(i,k)}, \mathcal{H}^{\varepsilon'}_{(j,l),s}] = [\mathcal{H}^{\varepsilon}_{(i,k),t}, \mathcal{H}^{\varepsilon'}_{(j,l),s}] = 0$$

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defining relations (cont.):

$$\mathcal{X}^{\varepsilon}_{(i,k),t}\mathcal{X}^{\varepsilon}_{(j,l),s} - \mathcal{X}^{\varepsilon}_{(j,l),s}\mathcal{X}^{\varepsilon}_{(i,k),t} = 0 \quad \text{if } (j,l) \neq (i,k), (i \pm 1, k),$$

$$\mathcal{X}^{\varepsilon}_{(i\pm1,k),0}(\mathcal{X}^{\varepsilon}_{(i,k),0})^2 - (q+q^{-1})\mathcal{X}^{\varepsilon}_{(i,k),0}\mathcal{X}^{\varepsilon}_{(i\pm1,0),0}\mathcal{X}^{\varepsilon}_{(i,k),0} + (\mathcal{X}^{\varepsilon}_{(i,k),0})^2\mathcal{X}^{\varepsilon}_{(i\pm1,k),0} = 0$$

$$[\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{-}_{(j,l),s}] = \delta_{(i,k)(j,l)} \begin{cases} \dfrac{\mathcal{J}^{+}_{(i,k),s+t} - \mathcal{J}^{-}_{(i,k),s+t}}{q-q^{-1}} & \text{if } i \neq m_k, \\ -C_{k+1}\dfrac{\mathcal{J}^{+}_{(m_k,k),s+t} - \mathcal{J}^{-}_{(m_k,k),s+t}}{q-q^{-1}} + (\mathcal{J}^{+}_{(m_k,k),s+t+1} - \mathcal{J}^{-}_{(m_k,k),s+t+1}) & \text{if } i = m_k, \end{cases}$$

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Theorem (W)

Assume that $m_k \geq n$ for all $k = 1, \dots, r-1$.

There exists a surjective homomorphism

$$\mathcal{U} \twoheadrightarrow \mathscr{S}_{n,r}$$

s.t. $C_k \mapsto q^{c_k}, \mathcal{X}^{\pm}_{(i,k),t} \mapsto \mathcal{X}^{\pm}_{(i,k),t}, \mathcal{K}^{\pm}_{(j,l)} \mapsto \mathcal{K}^{\pm}_{(j,l)}, \mathcal{H}^{\pm}_{(j,l),t} \mapsto \mathcal{H}^{\pm}_{(j,l),t}$.

Proposition

1. *There exists a surjective homomorphism*

$$\mathcal{U} \twoheadrightarrow U_q(\mathfrak{gl}_m)$$

s.t. $C_k \mapsto -1, \mathcal{X}^{+}_{(i,k),0} \mapsto e_{(i,k)}, \mathcal{X}^{-}_{(i,k),0} \mapsto f_{(i,k)}, \mathcal{K}^{\pm}_{(j,l)} \mapsto K^{\pm}_{(j,l)}$,
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$$U_q(\mathfrak{g}) \hookrightarrow \mathcal{U}$$

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Highest weight modules of $\mathcal{U}$

Define subalgebras $\mathcal{U}^0$ and $\mathcal{U}^{\pm}$ of $\mathcal{U}$ by

- $\mathcal{U}^0 := \langle \mathcal{K}^{\pm}_{(j,l)},\ \mathcal{H}^{\pm}_{(j,l),t},\ C_k \mid (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r \rangle_{\text{alg.}}$.
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Lemma

$$\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+ \qquad (\mathcal{U} \cong \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+\ ???)$$

For $\lambda \in P_{\geq 0}$, $\varphi = (\varphi^{\pm}_{(i,k),t})_{(i,k)\in\Gamma(\mathbf{m}),t\geq 1}$ $(\varphi^{\pm}_{(i,k),t} \in \mathbb{Q}(q))$ and $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{Z}^r$,

$\mathcal{U}$-module V is a highest weight module (of type 1) with h.w. $(\lambda, \varphi, \mathbf{c})$ if $^\exists v_0 \in V$ s.t.

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Highest weight modules of $\mathcal{U}$

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For $(\lambda, \varphi, \mathbf{c})$, define the Verma module $V(\lambda, \varphi, \mathbf{c})$ in usual way.

- h.w. module with h.w. $(\lambda, \varphi, \mathbf{c})$ is a quotient of $V(\lambda, \varphi, \mathbf{c})$.
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Problem

When is $L(\lambda, \varphi, \mathbf{c})$ finite dimensional?.

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$L(\lambda, \varphi, \mathbf{c})$: finite dim.
$\rightsquigarrow \lambda$: r-partition. i.e. $\lambda_{(1,k)} \geq \lambda_{(2,k)} \geq \cdots \geq \lambda_{(m_k,k)} \geq 0$ for $k = 1, \ldots, r$.

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Proposition

1. *simple object of $\mathcal{O}^{\mathbf{c}}$ is a h.w. module.*
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Category $\mathcal{O}^{\mathbf{c}}$

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Example

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- $\mathcal{X}^{-}_{(j,l),t} \cdot v_{(i,k)} = \begin{cases} \dfrac{q^{tc_1}}{(q-q^{-1})^t} v_{(i+1,k)} & \text{if } (j,l) = (i,k) \\ 0 & \text{otherwise} \end{cases}$
- $\mathcal{X}^{+}_{(j,l),t} \cdot v_{(i,k)}$

$$= \begin{cases} \dfrac{q^{tc_1}}{(q-q^{-1})^t} v_{(i-1,k)} & \text{if } i \neq 1 \text{ and } (j,l) = (i-1,k) \\ \dfrac{q^{tc_1}}{(q-q^{-1})^t} (q^{c_1} - q^{c_k}) v_{(m_{k-1},k-1)} & \text{if } i = 1 \text{ and } (j,l) = (m_{k-1}, k-1) \\ 0 & \text{otherwise} \end{cases}$$

Example

$V := \bigoplus_{(i,k)\in\Gamma(\mathbf{m})} \mathbb{Q}(q)v_{(i,k)}$ with the following action:

- $\mathcal{K}^{+}_{(j,l)} \cdot v_{(i,k)} = \begin{cases} qv_{(i,k)} & \text{if } (j,l) = (i,k), \\ 0 & \text{othewise.} \end{cases}$
- $\mathcal{H}^{\pm}_{(j,l).t} \cdot v_{(i,k)} = \begin{cases} q^{tc_1} & \text{if } (j,l) = (i,k), \\ 0 & \text{otherwise.} \end{cases}$
- $\mathcal{X}^{-}_{(j,l),t} \cdot v_{(i,k)} = \begin{cases} \dfrac{q^{tc_1}}{(q-q^{-1})^t} v_{(i+1,k)} & \text{if } (j,l) = (i,k) \\ 0 & \text{otherwise} \end{cases}$
- $\mathcal{X}^{+}_{(j,l),t} \cdot v_{(i,k)}$

$$= \begin{cases} \dfrac{q^{tc_1}}{(q-q^{-1})^t} v_{(i-1,k)} & \text{if } i \neq 1 \text{ and } (j,l) = (i-1,k) \\ \dfrac{q^{tc_1}}{(q-q^{-1})^t}(q^{c_1} - q^{c_k}) v_{(m_{k-1},k-1)} & \text{if } i = 1 \text{ and } (j,l) = (m_{k-1}, k-1) \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{U}$-module V :

$$
\begin{array}{ccccccc}
\mathbb{Q}(q)v_{(1,1)} & \mathbb{Q}(q)v_{(1,2)} & & \mathbb{Q}(q)v_{(1,k)} & & \mathbb{Q}(q)v_{(1,r)} \\
X^-_{(1,1),t} \downarrow \uparrow X^+_{(1,1),t} & X^-_{(1,2),t} \downarrow \uparrow X^+_{(1,2),t} & & X^-_{(1,k),t} \downarrow \uparrow X^+_{(1,k),t} & & X^-_{(1,r),t} \downarrow \uparrow X^+_{(1,r),t} \\
\mathbb{Q}(q)v_{(2,1)} & \mathbb{Q}(q)v_{(2,2)} & & \mathbb{Q}(q)v_{(2,k)} & & \mathbb{Q}(q)v_{(2,r)} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(i-1,1)} & \mathbb{Q}(q)v_{(i-1,2)} & & \mathbb{Q}(q)v_{(i-1,k)} & & \mathbb{Q}(q)v_{(i-1,r)} \\
X^-_{(i-1,1),t} \downarrow \uparrow X^+_{(i-1,1),t} & X^-_{(i-1,2),t} \downarrow \uparrow X^+_{(i-1,2),t} & & X^-_{(i-1,k),t} \downarrow \uparrow X^+_{(i-1,k),t} & & X^-_{(i-1,r),t} \downarrow \uparrow X^+_{(i-1,r),t} \\
\mathbb{Q}(q)v_{(i,1)} & \mathbb{Q}(q)v_{(i,2)} & \dots & \mathbb{Q}(q)v_{(i,k)} & \dots & \mathbb{Q}(q)v_{(i,r)} \\
X^-_{(i,1),t} \downarrow \uparrow X^+_{(i,1),t} & X^-_{(i,2),t} \downarrow \uparrow X^+_{(i,2),t} & & X^-_{(i,k),t} \downarrow \uparrow X^+_{(i,k),t} & & X^-_{(i,r),t} \downarrow \uparrow X^+_{(i,r),t} \\
\mathbb{Q}(q)v_{(i+1,1)} & \mathbb{Q}(q)v_{(i+1,2)} & & \mathbb{Q}(q)v_{(i+1,k)} & & \mathbb{Q}(q)v_{(i+1,r)} \\
X^-_{(i+1,1),t} \downarrow \uparrow X^+_{(i+1,1),t} & X^-_{(i+1,2),t} \downarrow \uparrow X^+_{(i+1,2),t} & & X^-_{(i+1,k),t} \downarrow \uparrow X^+_{(i+1,k),t} & & X^-_{(i+1,r),t} \downarrow \uparrow X^+_{(i+1,r),t} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(m_1-1,1)} & \mathbb{Q}(q)v_{(m_2-1,2)} & & \mathbb{Q}(q)v_{(m_k-1,k)} & & \mathbb{Q}(q)v_{(m_r-1,r)} \\
X^-_{(m_1-1,1),t} \downarrow \uparrow X^+_{(m_1-1,1),t} & X^-_{(m_2-1,2),t} \downarrow \uparrow X^+_{(m_2-1,2),t} & & X^-_{(m_k-1,k),t} \downarrow \uparrow X^+_{(m_k-1,k),t} & & X^-_{(m_r-1,r),t} \downarrow \uparrow X^+_{(m_r-1,r)} \\
\mathbb{Q}(q)v_{(m_1,1)} & \mathbb{Q}(q)v_{(m_2,2)} & & \mathbb{Q}(q)v_{(m_k,k)} & & \mathbb{Q}(q)v_{(m_r,r)} \\
X^-_{(m_1,1),t} \downarrow \uparrow X^+_{(m_1,1),t} & X^-_{(m_2,2),t} \downarrow \uparrow X^+_{(m_2,2),t} & & X^-_{(m_k,k),t} \downarrow \uparrow X^+_{(m_k,k),t} & &
\end{array}
$$

$\mathrm{Res}^{\mathcal{U}}_{U_q(\mathfrak{g})} V$ (red : omit, only $t = 0$) :

$$
\begin{array}{ccccccc}
\mathbb{Q}(q)v_{(1,1)} & \mathbb{Q}(q)v_{(1,2)} & & \mathbb{Q}(q)v_{(1,k)} & & \mathbb{Q}(q)v_{(1,r)} \\
x^-_{(1,1),t} \downarrow \uparrow x^+_{(1,1),t} & x^-_{(1,2),t} \downarrow \uparrow x^+_{(1,2),t} & & x^-_{(1,k),t} \downarrow \uparrow x^+_{(1,k),t} & & x^-_{(1,r),t} \downarrow \uparrow x^+_{(1,r),t} \\
\mathbb{Q}(q)v_{(2,1)} & \mathbb{Q}(q)v_{(2,2)} & & \mathbb{Q}(q)v_{(2,k)} & & \mathbb{Q}(q)v_{(2,r)} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(i-1,1)} & \mathbb{Q}(q)v_{(i-1,2)} & & \mathbb{Q}(q)v_{(i-1,k)} & & \mathbb{Q}(q)v_{(i-1,r)} \\
x^-_{(i-1,1),t} \downarrow \uparrow x^+_{(i-1,1),t} & x^-_{(i-1,2),t} \downarrow \uparrow x^+_{(i-1,2),t} & & x^-_{(i-1,k),t} \downarrow \uparrow x^+_{(i-1,k),t} & & x^-_{(i-1,r),t} \downarrow \uparrow x^+_{(i-1,r),t} \\
\mathbb{Q}(q)v_{(i,1)} & \mathbb{Q}(q)v_{(i,2)} & \cdots & \mathbb{Q}(q)v_{(i,k)} & \cdots & \mathbb{Q}(q)v_{(i,r)} \\
x^-_{(i,1),t} \downarrow \uparrow x^+_{(i,1),t} & x^-_{(i,2),t} \downarrow \uparrow x^+_{(i,2),t} & & x^-_{(i,k),t} \downarrow \uparrow x^+_{(i,k),t} & & x^-_{(i,r),t} \downarrow \uparrow x^+_{(i,r),t} \\
\mathbb{Q}(q)v_{(i+1,1)} & \mathbb{Q}(q)v_{(i+1,2)} & & \mathbb{Q}(q)v_{(i+1,k)} & & \mathbb{Q}(q)v_{(i+1,r)} \\
x^-_{(i+1,1),t} \downarrow \uparrow x^+_{(i+1,1),t} & x^-_{(i+1,2),t} \downarrow \uparrow x^+_{(i+1,2),t} & & x^-_{(i+1,k),t} \downarrow \uparrow x^+_{(i+1,k),t} & & x^-_{(i+1,r),t} \downarrow \uparrow x^+_{(i+1,r),t} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(m_1-1,1)} & \mathbb{Q}(q)v_{(m_2-1,2)} & & \mathbb{Q}(q)v_{(m_k-1,k)} & & \mathbb{Q}(q)v_{(m_r-1,r)} \\
x^-_{(m_1-1,1),t} \downarrow \uparrow x^+_{(m_1-1,1),t} & x^-_{(m_2-1,2),t} \downarrow \uparrow x^+_{(m_2-1,2),t} & & x^-_{(m_k-1,k),t} \downarrow \uparrow x^+_{(m_k-1,k),t} & & x^-_{(m_r-1,r),t} \downarrow \uparrow x^+_{(m_r-1,r)} \\
\mathbb{Q}(q)v_{(m_1,1)} & \mathbb{Q}(q)v_{(m_2,2)} & & \mathbb{Q}(q)v_{(m_k,k)} & & \mathbb{Q}(q)v_{(m_r,r)} \\
\color{red}{x^-_{(m_1,1),t} \downarrow \uparrow x^+_{(m_1,1),t}} & \color{red}{x^-_{(m_2,2),t} \downarrow \uparrow x^+_{(m_2,2),t}} & & \color{red}{x^-_{(m_k,k),t} \downarrow \uparrow x^+_{(m_k,k),t}} & &
\end{array}
$$

Weyl modules and simple modules of $\mathscr{S}_{n,r}$

Assume that $m_k \geq n$ for all $k = 1, \ldots, r-1$.
Then we have $\mathcal{U} \twoheadrightarrow \mathscr{S}_{n,r}$.

$\mathscr{S}_{n,r}$: quasi-hereditary algebra.

$\Lambda^+_{n,r}(\mathbf{m}) := \{\lambda \in \Lambda_{n,r}(\mathbf{m}) \mid \lambda_{(1,k)} \geq \lambda_{(2,k)} \geq \cdots \geq \lambda_{(m_k,k)}$ for all $k = 1, \ldots, r\}$

- $\{W(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\}$: a set of standard (Weyl) modules of $\mathscr{S}_{n,r}$.
- $\{L(\lambda) := \operatorname{Top} W(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\} = \{$ simple $\mathscr{S}_{n,r}$-modules$\}/_{\text{iso.}}$.

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Theorem (W)

Under $U_q(\mathfrak{g}) \hookrightarrow \mathcal{U}$,

$$W(\lambda) \cong \bigoplus_{\mu \in \Lambda^+_{n,r}(\mathbf{m})} \Big(W(\mu^{[1]}) \otimes W(\mu^{[2]}) \otimes \cdots \otimes W(\mu^{[r]})\Big)^{\oplus \beta_{\lambda\mu}} \quad \text{as } U_q(\mathfrak{g})\text{-modules},$$

$\beta_{\lambda\mu}$ is computed by a generalization of Littlewood-Richardson rule.

Assume that $m_k \geq n$ for all $k = 1, \dots, r-1$.
Then we have $\mathcal{U} \twoheadrightarrow \mathscr{S}_{n,r}$.

$\mathscr{S}_{n,r}$: quasi-hereditary algebra.

$\Lambda^+_{n,r}(\mathbf{m}) := \{\lambda \in \Lambda_{n,r}(\mathbf{m}) \mid \lambda_{(1,k)} \geq \lambda_{(2,k)} \geq \cdots \geq \lambda_{(m_k,k)}$ for all $k = 1, \dots, r\}$

- $\{W(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\}$: a set of standard (Weyl) modules of $\mathscr{S}_{n,r}$.
- $\{L(\lambda) := \operatorname{Top} W(\lambda) \mid \lambda \in \Lambda^+_{n,r}(\mathbf{m})\} = \{$ simple $\mathscr{S}_{n,r}$-modules$\}/_{\text{iso.}}$.

Proposition

As a $\mathcal{U}$-module,
$W(\lambda)$ (resp. $L(\lambda)$) is a h.w. module with h.w. $(\lambda, \varphi, \mathbf{c})$, where

$$\varphi^{\pm}_{(i,k),t} = \frac{\pm 1}{q^{\pm(1-t)}(q-q^{-1})} \Phi^{\pm}_t (q^{c_k+2(1-i)}, q^{c_k+2(2-i)}, \dots, q^{c_k+2(\lambda_{(i,k)}-i)})$$

Recall the generators of $\mathcal{U}$:

$$\mathbb{X} = \left\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,l)},\ \mathcal{H}^{\pm}_{(j,l),t},\ C_k \,\middle|\, (i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r \right\}.$$

Put

- $\mathcal{U}^P$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{-}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$
- $\mathcal{U}^L$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{\pm}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$

Lemma

$$\begin{array}{ccccc} \mathcal{U}^L & \hookrightarrow & \mathcal{U}^P & \hookrightarrow & \mathcal{U} \\ & \overset{id}{\searrow} & \downarrow g & & \\ & & \mathcal{U}^L \cong \mathcal{U}^{[1]} \otimes \mathcal{U}^{[2]} \otimes \cdots \otimes \mathcal{U}^{[r]} & & \end{array}$$

where $\mathcal{U}^{[k]}$ is an ass. algebra generated by

$$\left\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,k)},\ \mathcal{H}^{\pm}_{(j,k),t},\ C_k \,\middle|\, 1 \leq i \leq m_k - 1,\ 1 \leq j \leq m_k,\ t \geq 0 \right\}$$

with the same defining relations of $\mathcal{U}$.

Recall the generators of $\mathcal{U}$:

$$\mathbb{X} = \left\{\mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,l)},\ \mathcal{H}^{\pm}_{(j,l),t},\ C_k \,\middle|\, (i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r\right\}.$$

Put

- $\mathcal{U}^P$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{-}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$
- $\mathcal{U}^L$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{\pm}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$

Lemma

$$\mathcal{U}^L \hookrightarrow \mathcal{U}^P \hookrightarrow \mathcal{U}$$

$$\mathcal{U}^P \xrightarrow{g} \mathcal{U}^L \cong \mathcal{U}^{[1]} \otimes \mathcal{U}^{[2]} \otimes \cdots \otimes \mathcal{U}^{[r]}, \qquad \mathcal{U}^L \xrightarrow{id} \mathcal{U}^L$$

where $\mathcal{U}^{[k]}$ is an ass. algebra generated by

$$\left\{\mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,k)},\ \mathcal{H}^{\pm}_{(j,k),t},\ C_k \,\middle|\, 1 \leq i \leq m_k - 1,\ 1 \leq j \leq m_k,\ t \geq 0\right\}$$

with the same defining relations of $\mathcal{U}$.

Recall the generators of $\mathcal{U}$:

$$\mathbb{X} = \Big\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,l)},\ \mathcal{H}^{\pm}_{(j,l),t},\ C_k \ \Big|\ (i,k) \in \varGamma'(\mathbf{m}),\ (j,l) \in \varGamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r \Big\}.$$

Put

- $\mathcal{U}^P$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{-}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$
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Lemma

$$\begin{array}{ccccc} \mathcal{U}^L & \hookrightarrow & \mathcal{U}^P & \hookrightarrow & \mathcal{U} \\ & \overset{id}{\searrow} & \downarrow g & & \\ & & \mathcal{U}^L \cong \mathcal{U}^{[1]} \otimes \mathcal{U}^{[2]} \otimes \cdots \otimes \mathcal{U}^{[r]} & & \end{array}$$

where $\mathcal{U}^{[k]}$ is an ass. algebra generated by

$$\Big\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,k)},\ \mathcal{H}^{\pm}_{(j,k),t},\ C_k \ \Big|\ 1 \leq i \leq m_k - 1,\ 1 \leq j \leq m_k,\ t \geq 0 \Big\}$$

with the same defining relations of $\mathcal{U}$.

Harish-Chandra Ind and Res

Recall the generators of $\mathcal{U}$:

$$\mathbb{X} = \left\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,l)},\ \mathcal{H}^{\pm}_{(j,l),t},\ C_k \;\middle|\; (i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0,\ 1 \leq k \leq r \right\}.$$

Put

- $\mathcal{U}^P$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{-}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$
- $\mathcal{U}^L$: subalg. of $\mathcal{U}$ gen. by $\mathbb{X} \setminus \{\mathcal{X}^{\pm}_{(m_k,k),t} \mid 1 \leq k \leq r-1,\ t \geq 0\}$

Lemma

$$\begin{array}{ccccc} \mathcal{U}^L & \hookrightarrow & \mathcal{U}^P & \hookrightarrow & \mathcal{U} \\ & \overset{id}{\searrow} & \downarrow g & & \\ & & \mathcal{U}^L \cong \mathcal{U}^{[1]} \otimes \mathcal{U}^{[2]} \otimes \cdots \otimes \mathcal{U}^{[r]} & & \end{array}$$

where $\mathcal{U}^{[k]}$ is an ass. algebra generated by

$$\left\{ \mathcal{X}^{\pm}_{(i,k),t},\ \mathcal{K}^{\pm}_{(j,k)},\ \mathcal{H}^{\pm}_{(j,k),t},\ C_k \;\middle|\; 1 \leq i \leq m_k - 1,\ 1 \leq j \leq m_k,\ t \geq 0 \right\}$$

with the same defining relations of $\mathcal{U}$.

$\mathcal{U}$-module V :

$$
\begin{array}{cccccc}
\mathbb{Q}(q)v_{(1,1)} & \mathbb{Q}(q)v_{(1,2)} & & \mathbb{Q}(q)v_{(1,k)} & & \mathbb{Q}(q)v_{(1,r)} \\
x^-_{(1,1),t} \downarrow \uparrow x^+_{(1,1),t} & x^-_{(1,2),t} \downarrow \uparrow x^+_{(1,2),t} & & x^-_{(1,k),t} \downarrow \uparrow x^+_{(1,k),t} & & x^-_{(1,r),t} \downarrow \uparrow x^+_{(1,r),t} \\
\mathbb{Q}(q)v_{(2,1)} & \mathbb{Q}(q)v_{(2,2)} & & \mathbb{Q}(q)v_{(2,k)} & & \mathbb{Q}(q)v_{(2,r)} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(i-1,1)} & \mathbb{Q}(q)v_{(i-1,2)} & & \mathbb{Q}(q)v_{(i-1,k)} & & \mathbb{Q}(q)v_{(i-1,r)} \\
x^-_{(i-1,1),t} \downarrow \uparrow x^+_{(i-1,1),t} & x^-_{(i-1,2),t} \downarrow \uparrow x^+_{(i-1,2),t} & & x^-_{(i-1,k),t} \downarrow \uparrow x^+_{(i-1,k),t} & & x^-_{(i-1,r),t} \downarrow \uparrow x^+_{(i-1,r),t} \\
\mathbb{Q}(q)v_{(i,1)} & \mathbb{Q}(q)v_{(i,2)} & \cdots & \mathbb{Q}(q)v_{(i,k)} & \cdots & \mathbb{Q}(q)v_{(i,r)} \\
x^-_{(i,1),t} \downarrow \uparrow x^+_{(i,1),t} & x^-_{(i,2),t} \downarrow \uparrow x^+_{(i,2),t} & & x^-_{(i,k),t} \downarrow \uparrow x^+_{(i,k),t} & & x^-_{(i,r),t} \downarrow \uparrow x^+_{(i,r),t} \\
\mathbb{Q}(q)v_{(i+1,1)} & \mathbb{Q}(q)v_{(i+1,2)} & & \mathbb{Q}(q)v_{(i+1,k)} & & \mathbb{Q}(q)v_{(i+1,r)} \\
x^-_{(i+1,1),t} \downarrow \uparrow x^+_{(i+1,1),t} & x^-_{(i+1,2),t} \downarrow \uparrow x^+_{(i+1,2),t} & & x^-_{(i+1,k),t} \downarrow \uparrow x^+_{(i+1,k),t} & & x^-_{(i+1,r),t} \downarrow \uparrow x^+_{(i+1,r),t} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(m_1-1,1)} & \mathbb{Q}(q)v_{(m_2-1,2)} & & \mathbb{Q}(q)v_{(m_k-1,k)} & & \mathbb{Q}(q)v_{(m_r-1,r)} \\
x^-_{(m_1-1,1),t} \downarrow \uparrow x^+_{(m_1-1,1),t} & x^-_{(m_2-1,2),t} \downarrow \uparrow x^+_{(m_2-1,2),t} & & x^-_{(m_k-1,k),t} \downarrow \uparrow x^+_{(m_k-1,k),t} & & x^-_{(m_r-1,r),t} \downarrow \uparrow x^+_{(m_r-1,r)} \\
\mathbb{Q}(q)v_{(m_1,1)} & \mathbb{Q}(q)v_{(m_2,2)} & & \mathbb{Q}(q)v_{(m_k,k)} & & \mathbb{Q}(q)v_{(m_r,r)} \\
x^-_{(m_1,1),t} \downarrow \uparrow x^+_{(m_1,1),t} & x^-_{(m_2,2),t} \downarrow \uparrow x^+_{(m_2,2),t} & & x^-_{(m_k,k),t} \downarrow \uparrow x^+_{(m_k,k),t} & &
\end{array}
$$

$\mathrm{Res}^{\mathcal{U}}_{\mathcal{U}^P} V$ (red : omit) :

$$
\begin{array}{cccccc}
\mathbb{Q}(q)v_{(1,1)} & \mathbb{Q}(q)v_{(1,2)} & & \mathbb{Q}(q)v_{(1,k)} & & \mathbb{Q}(q)v_{(1,r)} \\
x^-_{(1,1),t} \downarrow \uparrow x^+_{(1,1),t} & x^-_{(1,2),t} \downarrow \uparrow x^+_{(1,2),t} & & x^-_{(1,k),t} \downarrow \uparrow x^+_{(1,k),t} & & x^-_{(1,r),t} \downarrow \uparrow x^+_{(1,r),t} \\
\mathbb{Q}(q)v_{(2,1)} & \mathbb{Q}(q)v_{(2,2)} & & \mathbb{Q}(q)v_{(2,k)} & & \mathbb{Q}(q)v_{(2,r)} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(i-1,1)} & \mathbb{Q}(q)v_{(i-1,2)} & & \mathbb{Q}(q)v_{(i-1,k)} & & \mathbb{Q}(q)v_{(i-1,r)} \\
x^-_{(i-1,1),t} \downarrow \uparrow x^+_{(i-1,1),t} & x^-_{(i-1,2),t} \downarrow \uparrow x^+_{(i-1,2),t} & & x^-_{(i-1,k),t} \downarrow \uparrow x^+_{(i-1,k),t} & & x^-_{(i-1,r),t} \downarrow \uparrow x^+_{(i-1,r),t} \\
\mathbb{Q}(q)v_{(i,1)} & \mathbb{Q}(q)v_{(i,2)} & \cdots & \mathbb{Q}(q)v_{(i,k)} & \cdots & \mathbb{Q}(q)v_{(i,r)} \\
x^-_{(i,1),t} \downarrow \uparrow x^+_{(i,1),t} & x^-_{(i,2),t} \downarrow \uparrow x^+_{(i,2),t} & & x^-_{(i,k),t} \downarrow \uparrow x^+_{(i,k),t} & & x^-_{(i,r),t} \downarrow \uparrow x^+_{(i,r),t} \\
\mathbb{Q}(q)v_{(i+1,1)} & \mathbb{Q}(q)v_{(i+1,2)} & & \mathbb{Q}(q)v_{(i+1,k)} & & \mathbb{Q}(q)v_{(i+1,r)} \\
x^-_{(i+1,1),t} \downarrow \uparrow x^+_{(i+1,1),t} & x^-_{(i+1,2),t} \downarrow \uparrow x^+_{(i+1,2),t} & & x^-_{(i+1,k),t} \downarrow \uparrow x^+_{(i+1,k),t} & & x^-_{(i+1,r),t} \downarrow \uparrow x^+_{(i+1,r),t} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(m_1-1,1)} & \mathbb{Q}(q)v_{(m_2-1,2)} & & \mathbb{Q}(q)v_{(m_k-1,k)} & & \mathbb{Q}(q)v_{(m_r-1,r)} \\
x^-_{(m_1-1,1),t} \downarrow \uparrow x^+_{(m_1-1,1),t} & x^-_{(m_2-1,2),t} \downarrow \uparrow x^+_{(m_2-1,2),t} & & x^-_{(m_k-1,k),t} \downarrow \uparrow x^+_{(m_k-1,k),t} & & x^-_{(m_r-1,r),t} \downarrow \uparrow x^+_{(m_r-1,r)} \\
\mathbb{Q}(q)v_{(m_1,1)} & \mathbb{Q}(q)v_{(m_2,2)} & & \mathbb{Q}(q)v_{(m_k,k)} & & \mathbb{Q}(q)v_{(m_r,r)} \\
{\color{red} x^-_{(m_1,1),t} \downarrow} \uparrow x^+_{(m_1,1),t} & {\color{red} x^-_{(m_2,2),t} \downarrow} \uparrow x^+_{(m_2,2),t} & & {\color{red} x^-_{(m_k,k),t} \downarrow} \uparrow x^+_{(m_k,k),t} & &
\end{array}
$$

$\mathrm{Res}^{\mathcal{U}}_{\mathcal{U}^L} V$ (red : omit) :

$$
\begin{array}{ccccccc}
\mathbb{Q}(q)v_{(1,1)} & \mathbb{Q}(q)v_{(1,2)} & & \mathbb{Q}(q)v_{(1,k)} & & \mathbb{Q}(q)v_{(1,r)} \\
x^-_{(1,1),t} \downarrow \uparrow x^+_{(1,1),t} & x^-_{(1,2),t} \downarrow \uparrow x^+_{(1,2),t} & & x^-_{(1,k),t} \downarrow \uparrow x^+_{(1,k),t} & & x^-_{(1,r),t} \downarrow \uparrow x^+_{(1,r),t} \\
\mathbb{Q}(q)v_{(2,1)} & \mathbb{Q}(q)v_{(2,2)} & & \mathbb{Q}(q)v_{(2,k)} & & \mathbb{Q}(q)v_{(2,r)} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(i-1,1)} & \mathbb{Q}(q)v_{(i-1,2)} & & \mathbb{Q}(q)v_{(i-1,k)} & & \mathbb{Q}(q)v_{(i-1,r)} \\
x^-_{(i-1,1),t} \downarrow \uparrow x^+_{(i-1,1),t} & x^-_{(i-1,2),t} \downarrow \uparrow x^+_{(i-1,2),t} & & x^-_{(i-1,k),t} \downarrow \uparrow x^+_{(i-1,k),t} & & x^-_{(i-1,r),t} \downarrow \uparrow x^+_{(i-1,r),t} \\
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x^-_{(i,1),t} \downarrow \uparrow x^+_{(i,1),t} & x^-_{(i,2),t} \downarrow \uparrow x^+_{(i,2),t} & & x^-_{(i,k),t} \downarrow \uparrow x^+_{(i,k),t} & & x^-_{(i,r),t} \downarrow \uparrow x^+_{(i,r),t} \\
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x^-_{(i+1,1),t} \downarrow \uparrow x^+_{(i+1,1),t} & x^-_{(i+1,2),t} \downarrow \uparrow x^+_{(i+1,2),t} & & x^-_{(i+1,k),t} \downarrow \uparrow x^+_{(i+1,k),t} & & x^-_{(i+1,r),t} \downarrow \uparrow x^+_{(i+1,r),t} \\
\vdots & \vdots & & \vdots & & \vdots \\
\mathbb{Q}(q)v_{(m_1-1,1)} & \mathbb{Q}(q)v_{(m_2-1,2)} & & \mathbb{Q}(q)v_{(m_k-1,k)} & & \mathbb{Q}(q)v_{(m_r-1,r)} \\
x^-_{(m_1-1,1),t} \downarrow \uparrow x^+_{(m_1-1,1),t} & x^-_{(m_2-1,2),t} \downarrow \uparrow x^+_{(m_2-1,2),t} & & x^-_{(m_k-1,k),t} \downarrow \uparrow x^+_{(m_k-1,k),t} & & x^-_{(m_r-1,r),t} \downarrow \uparrow x^+_{(m_r-1,r)} \\
\mathbb{Q}(q)v_{(m_1,1)} & \mathbb{Q}(q)v_{(m_2,2)} & & \mathbb{Q}(q)v_{(m_k,k)} & & \mathbb{Q}(q)v_{(m_r,r)} \\
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\end{array}
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Harish-Chandra Ind and Res

Define

- $$\begin{array}{ccccc} R^{\mathcal{U}}_{\mathcal{U}^L} : \mathcal{U}^L\text{-mod} & \to & \mathcal{U}^P\text{-mod} & \to & \mathcal{U}\text{-mod}. \\ \Uplus & & \Uplus & & \Uplus \\ N & \longmapsto & N \text{ (via } g) & \mapsto & \mathcal{U} \otimes_{\mathcal{U}^P} N. \end{array}$$

- $$\begin{array}{ccccc} {}^*R^{\mathcal{U}}_{\mathcal{U}^L} : \mathcal{U}\text{-mod} & \to & \mathcal{U}^P\text{-mod} & \to & \mathcal{U}^L\text{-mod}. \\ \Uplus & & \Uplus & & \Uplus \\ M & \longmapsto & \mathrm{Res}^{\mathcal{U}}_{\mathcal{U}^P} M & \mapsto & \{m \in M \mid \mathrm{Ker}\, g \cdot m = 0\}. \end{array}$$

$\rightsquigarrow$ $R^{\mathcal{U}}_{\mathcal{U}^L}$ is left adjoint to ${}^*R^{\mathcal{U}}_{\mathcal{U}^L}$,

i.e. $\mathrm{Hom}_{\mathcal{U}}(R^{\mathcal{U}}_{\mathcal{U}^L}(N), M) \cong \mathrm{Hom}_{\mathcal{U}^L}(N, {}^*R^{\mathcal{U}}_{\mathcal{U}^L}(M))$.

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Recall $\mathcal{U}^0 \hookrightarrow \mathcal{U}^L \hookrightarrow \mathcal{U}^P \hookrightarrow \mathcal{U}$.

Proposition

- $N_{(\lambda,\varphi,\mathbf{c})}$: *h.w.* $\mathcal{U}^L$*-module with h.w.* $(\lambda, \varphi, \mathbf{c})$
  $\rightsquigarrow R^{\mathcal{U}}_{\mathcal{U}^L} N_{(\lambda,\varphi,\mathbf{c})}$: *h.w.* $\mathcal{U}$*-module with h.w.* $(\lambda, \varphi, \mathbf{c})$.
  In particular,
  $$\mathrm{Top}\, R^{\mathcal{U}}_{\mathcal{U}^L} N_{(\lambda,\varphi,\mathbf{c})} \cong L(\lambda, \varphi, \mathbf{c}).$$
- *Assume that* $L(\lambda, \varphi, \mathbf{c})$ *is fin. dim.*
  $\rightsquigarrow {}^*R^{\mathcal{U}}_{\mathcal{U}^L} L(\lambda, \varphi, \mathbf{c})$: *simple h.w.* $\mathcal{U}^L$*-module with h.w.* $(\lambda, \varphi, \mathbf{c})$.

Harish-Chandra Ind and Res

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Harish-Chandra Ind and Res

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Recall $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \subset \mathfrak{gl}_m$.

$\mathcal{O}^{\mathfrak{g}}$: category of finite dim. $U_q(\mathfrak{g})$-modules
s.t. $M \in \mathcal{O}^{\mathfrak{g}}$ has the weight space decom.:

$$M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda, \text{ where } M_\lambda = \{m \in M \mid K_{(i,k)} \cdot m = q^{\lambda_{(i,k)}} m\}.$$

We have

$$\mathcal{O}^{\mathfrak{g}} = \bigoplus_{(n_1,\dots,n_r) \in \mathbb{Z}^r_{\geq 0}} \mathscr{S}_{n_1,1} \otimes \mathscr{S}_{n_2,1} \otimes \cdots \otimes \mathscr{S}_{n_r,1} \text{-mod}$$

For each $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{Z}^r$ and $(n_1, \dots, n_r) \in \mathbb{Z}^r_{\geq 0}$, we have

$$\mathcal{U}^L \cong \mathcal{U}^{[1]} \otimes \cdots \otimes \mathcal{U}^{[r]} \twoheadrightarrow \mathscr{S}_{n_1,1} \otimes \cdots \otimes \mathscr{S}_{n_r,1}$$

(Note the relation $(T_0 - q^{c_k}) = 0$ in $\mathscr{H}_{n_k,1}$)
Through this surjection, we have

$$\mathrm{ev}_{\mathbf{c}} : \mathcal{O}^{\mathfrak{g}} \to \mathcal{U}^L\text{-mod}$$

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Theorem (W)

Assume that $m_k \geq n$ for all $k = 1, \dots, r-1$.
For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, let

- $L(\lambda)$ *: simple $\mathscr{S}_{n,r}$-module with h.w. $(\lambda, \varphi, \mathbf{c})$.*
- $L(\lambda^{[1]}) \otimes \cdots \otimes L(\lambda^{[r]}) \in \mathcal{O}^{\mathfrak{g}}$ *: simple $U_q(\mathfrak{g})$-module with h.w. λ.*

We have

1. $L(\lambda) \cong \mathrm{Top}\left(R^{\mathcal{U}}_{\mathcal{U}^L} \circ \mathrm{ev}_{\mathbf{c}}(L(\lambda^{[1]}) \otimes \cdots \otimes L(\lambda^{[r]}))\right)$ *as $\mathcal{U}$-modules.*
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2. $L(\lambda^{[1]}) \otimes \cdots \otimes L(\lambda^{[r]}) \cong \mathrm{Res}^{\mathcal{U}^L}_{U_q(\mathfrak{g})} \circ {}^*R^{\mathcal{U}}_{\mathcal{U}^L}(L(\lambda))$ *as $U_q(\mathfrak{g})$-modules.*

$$\mathscr{O}^{\mathfrak{g}} \xrightarrow{\mathrm{ev}_{\mathbf{c}}} \mathcal{U}^L\text{-mod} \xrightarrow{R^{\mathcal{U}}_{\mathcal{U}^L}} \mathcal{U}\text{-mod}.$$

$$\mathcal{U}\text{-mod} \xrightarrow{{}^*R^{\mathcal{U}}_{\mathcal{U}^L}} \mathcal{U}^L\text{-mod} \xrightarrow{\mathrm{Res}^{\mathcal{U}^L}_{U_q(\mathfrak{g})}} U_q(\mathfrak{g})\text{-mod}$$

Theorem (W)

Assume that $m_k \geq n$ for all $k = 1, \dots, r-1$.

For $\lambda \in \Lambda^+_{n,r}(\mathbf{m})$, let

- $L(\lambda)$ *: simple $\mathscr{S}_{n,r}$-module with h.w. $(\lambda, \varphi, \mathbf{c})$.*
- $L(\lambda^{[1]}) \otimes \cdots \otimes L(\lambda^{[r]}) \in \mathscr{O}^{\mathfrak{g}}$ *: simple $U_q(\mathfrak{g})$-module with h.w. λ.*

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- Related combinatorics.
- Connection with (affine) Lie algebra, W-algebra, ...
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