Modular Springer Correspondence for classical groups

Karine Sorlin

Université de Picardie Jules Verne

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W Weyl group of G.
ℓ a prime number distinct from p,

$\mathbb{K}$ sufficiently large finite extension of $\mathbb{Q}_\ell$,

Springer Correspondence in characteristic 0 (1976)

$$\mathrm{Irr}\mathbb{K}W \hookrightarrow \mathfrak{P}_{\mathbb{K}}$$

- $\mathrm{Irr}\mathbb{K}W$: set of representatives of isomorphism classes of simple $\mathbb{K}W$-modules.
- $\mathfrak{P}_{\mathbb{K}}$: set of pairs (x, ρ) up to G-conjugacy, where x is a nilpotent element of $\mathrm{Lie}(G)$ and $\rho \in \mathrm{Irr}\mathbb{K}A(x)$. Where $A(x) = C_G(x)/C_G(x)^0$.

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ℓ a prime number distinct from p,
$(\mathbb{K}, \mathbb{O}, \mathbb{F})$ ℓ-modular system:
$\mathbb{K}$ sufficiently large finite extension of $\mathbb{Q}_\ell$,
$\mathbb{O}$ valuation ring, $\mathbb{F}$ residue field.

Springer Correspondence in characteristic ℓ (Juteau, 2007)

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  Where $A(x) = C_G(x)/C_G(x)^0$.

The Springer Correspondence in characteristic 0 was:

- explicitely determined in the case of classical groups by Shoji (1979).
- generalized by Lusztig to include all pairs (x, ρ) (1984).
- The Springer correspondence was used by Shoji in an algorithm which computes Green functions of a finite reductive group G^F, where G is a reductive group over $\overline{\mathbb{F}}_p$ endowed with a $\mathbb{F}_q$-rational structure ($q = p^n$) given by a Frobenius endomorphism F.

- Subject of this talk: common work with Daniel Juteau (Université de Caen) and Cédric Lecouvey (Université de Tours).
- Our purpose was to determine explicitly the modular Springer correspondence for classical groups.
- Strategy: we used the explicit description of the Springer Correspondence in characteristic 0 and unitriangularity properties of the decomposition matrices (both for the Weyl group and perverse sheaves).

Geometric construction of the Springer Correspondence

Simple perverse sheaves on the nilpotent cone

Let $\mathcal{N} \subset \mathfrak{g} = \mathrm{Lie}(G)$ be the nilpotent cone.

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$(\mathbb{K}, \mathbb{O}, \mathbb{F})$ an ℓ-modular system as before, $\mathbb{E} = \mathbb{K}$ or $\mathbb{F}$.
We consider the abelian category $\mathrm{Perv}_G(\mathcal{N}, \mathbb{E})$ of G-equivariant $\mathbb{E}$-perverse sheaves on $\mathcal{N}$.
We recall the notation:

$$\mathfrak{P}_{\mathbb{E}} = \{(x, \rho) \text{ up to } G\text{-conjugacy } | x \in \mathcal{N}, \rho \in \mathrm{Irr}\, \mathbb{E}A(x)\}$$

where $A(x) = C_G(x)/C_G(x)^0$.

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where $A(x) = C_G(x)/C_G(x)^0$.
These pairs parametrize the simple objects in $\mathrm{Perv}_G(\mathcal{N}, \mathbb{E})$:

$$\begin{array}{ccc} \mathfrak{P}_{\mathbb{E}} & \simeq & \mathrm{Irr}\, \mathrm{Perv}_G(\mathcal{N}, \mathbb{E}) \\ (x, \rho) & \mapsto & \mathbf{IC}_{\mathbb{E}}(x, \rho) \end{array}$$

Lusztig's construction (1981)

Let $\mathcal{B}$ be the flag variety.
Let $\tilde{\mathfrak{g}} = \{(x, B) \in \mathfrak{g} \times \mathcal{B} | x \in Lie(B)\}$
$\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ projection onto the first factor

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We have a diagram with cartesian squares:

$$\begin{array}{ccccc} \tilde{\mathfrak{g}}_{rs} & \overset{\tilde{j}_{rs}}{\hookrightarrow} & \tilde{\mathfrak{g}} & \overset{i_{\tilde{\mathcal{N}}}}{\hookleftarrow} & \tilde{\mathcal{N}} \\ \downarrow \pi_{rs} & & \downarrow \pi & & \downarrow \pi_{\mathcal{N}} \\ \mathfrak{g}_{rs} & \underset{j_{rs}}{\hookrightarrow} & \mathfrak{g} & \underset{i_{\mathcal{N}}}{\hookleftarrow} & \mathcal{N} \end{array}$$

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where $\mathfrak{g}_{rs}$ is the open dense subset of regular semi-simple elements of $\mathfrak{g}$.
One can define an action of the Weyl group W on $K = \pi_* \mathbb{E}_{\tilde{\mathfrak{g}}}$.
And $K|_{\mathcal{N}}[\dim(\mathcal{N})] \in \mathrm{Perv}_G(\mathcal{N}, \mathbb{E})$.

In characteristic 0

Borho-MacPherson Theorem (1981)

1. $K[\dim(\mathcal{N})]|_{\mathcal{N}}$ is a semi-simple object in $\mathrm{Perv}_G(\mathcal{N}, \mathbb{K})$ and
$$K[\dim(\mathcal{N})]|_{\mathcal{N}} \simeq \bigoplus_{(x,\rho)\in\mathcal{P}_{\mathbb{K}}} V_{(x,\rho)} \otimes \mathbf{IC}(x,\rho)$$
2. For any $(x,\rho) \in \mathfrak{P}_{\mathbb{K}}$, we get $V_{(x,\rho)} \in \mathrm{Irr}\,\mathbb{K}W$ and we get an injective map
$$\mathrm{Irr}\,\mathbb{K}W \hookrightarrow \mathcal{P}_{\mathbb{K}}$$
which is the Springer Correspondance over $\mathbb{K}$.

Proof based on the Beilinson-Bernstein-Deligne decomposition theorem of perverse sheaves.

A method one can still use in characteristic ℓ

Fourier-Deligne transform is an autoequivalence $\mathcal{F}$ of the category $\mathrm{Perv}_G(\mathfrak{g}, \mathbb{E})$ such that

$$\mathcal{F}(K[\dim(\mathcal{N})]|_{\mathcal{N}}) \simeq K[\dim(\mathfrak{g})]$$

Theorem ($\mathbb{E} = \mathbb{K}$ Brylinski (1986), $\mathbb{E} = \mathbb{F}$ Juteau (2007))

Using a Fourier-Deligne transform, on can define an injective map $\Psi_{\mathbb{E}} : \mathrm{Irr}\, \mathbb{E}W \hookrightarrow \mathfrak{P}_{\mathbb{E}}$.

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- The two versions of the Springer correspondence in char. 0 are related by tensoring with the sign character.

$$E \in \mathrm{Irr}\, \mathbb{K}W \mapsto E \otimes_{\mathbb{K}} Sgn \in \mathrm{Irr}\, \mathbb{K}W$$

Example: $G = GL_n(\overline{\mathbb{F}}_p)$

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- Here, W is the symmetric group $\mathfrak{S}_n$: The simple modules of $\mathbb{K}\mathfrak{S}_n$ are the Specht modules S^λ, for $\lambda \vdash n$.

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Springer correspondence in char. 0 for $GL_n(\overline{\mathbb{F}}_p)$

$\Psi_{\mathbb{K}}$ is a bijection and maps $S^\lambda \in \operatorname{Irr} \mathbb{K}\mathfrak{S}_n$ to $\mathcal{O}_{\lambda^*} \in \mathcal{P}_{\mathbb{K}}$, where λ^* is the transpose partition of λ.

How to use the known results in characteristic 0 to solve the case of characteristic ℓ?

Decomposition matrix for the Weyl group W

As for any finite group, we can define for the Weyl group W an ℓ-modular decomposition matrix

$$D^W := (d^W_{E,F})_{E \in \operatorname{Irr} \mathbb{K}W,\ F \in \operatorname{Irr} \mathbb{F}W}$$

where $d^W_{E,F}$ is the composition multiplicity of the simple $\mathbb{F}W$-module F in $\mathbb{F} \otimes_{\mathbb{O}} E_{\mathbb{O}}$, where $E_{\mathbb{O}}$ is some integral form of E. This is independent of the choice of $E_{\mathbb{O}}$.

Decomposition matrix for perverse sheaves (Juteau, 2007)

- For $\mathbb{E} \in \{\mathbb{K}, \mathbb{F}\}$,
  $\mathrm{Perv}_G(\mathcal{N}, \mathbb{E})$: category of G-equivariant $\mathbb{E}$-perverse sheaves on $\mathcal{N}$.
  Simple objects: $\mathbf{IC}(x, \rho)$ where $(x, \rho) \in \mathfrak{P}_{\mathbb{E}}$
- One can define a decomposition matrix for G-equivariant perverse sheaves on $\mathcal{N}$:

$$D^{\mathcal{N}} := \left(d^{\mathcal{N}}_{(x,\rho),(y,\sigma)}\right)_{(x,\rho)\in\mathfrak{P}_{\mathbb{K}},\ (y,\sigma)\in\mathfrak{P}_{\mathbb{F}}}$$

Where $d^{\mathcal{N}}_{(x,\rho),(y,\sigma)}$ is the composition multiplicity of $\mathbf{IC}(y, \sigma)$ in $\mathbb{F} \otimes^{L}_{\mathbb{O}} \mathbf{IC}(x, \rho_{\mathbb{O}})$ and $\rho_{\mathbb{O}}$ is some integral form of ρ.

D^W can be seen as a submatrix of $D^{\mathcal{N}}$

Theorem (Juteau, 2007)

For $E \in \operatorname{Irr} \mathbb{K}W$ and $F \in \operatorname{Irr} \mathbb{F}W$, we have

$$d^W_{E,F} = d^{\mathcal{N}}_{\Psi_{\mathbb{K}}(E),\Psi_{\mathbb{F}}(F)}.$$

Where $\Psi_{\mathbb{E}} : \operatorname{Irr} \mathbb{E}W \to \mathfrak{P}_{\mathbb{E}}$ is the Springer correspondence over $\mathbb{E}$.

Till the end of this talk

We will suppose that $G = GL_n(\mathbb{K})$ or G is a classical group and that $\ell \neq 2$.
Then, ℓ does not divide $|A(x)|$, hence we can identify $\operatorname{Irr} \mathbb{F}A(x)$ with $\operatorname{Irr} \mathbb{K}A(x)$ and $\mathfrak{P}_{\mathbb{K}}$ with $\mathfrak{P}_{\mathbb{F}}$.

Unitriangularity of the decomposition matrix of perverse sheaves

Definition: partial order on the nilpotent orbits

$\mathcal{O} \leq \mathcal{O}' \Leftrightarrow \mathcal{O} \subset \overline{\mathcal{O}'}$

$D^{\mathcal{N}}$ has the following unitriangularity property:

Proposition (Juteau, 2007)

$$d^{\mathcal{N}}_{(x,\rho),(y,\sigma)} = \begin{cases} 0 & \text{unless } \mathcal{O}_y \leq \mathcal{O}_x, \\ \delta_{\rho,\sigma} & \text{if } y = x. \end{cases}$$

Where $\mathcal{O}_x$ (resp. $\mathcal{O}_y$) is the orbit of x (resp. y).

Example: $GL_4(\overline{\mathbb{F}}_p)$, $\ell = 3$, $p \neq 3$

	$\overline{\chi_4}$	$\overline{\chi_{31}}$	$\overline{\chi_{1^4}}$	$\overline{\chi_{21^2}}$
χ_4	1	0	0	0
χ_{31}	0	1	0	0
χ_{2^2}	1	0	1	0
χ_{21^2}	0	0	0	1
χ_{1^4}	0	0	1	0

Decomposition matrix of $\mathfrak{S}_4$

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Decomposition matrix $D^{\mathcal{N}}$

		1^4	21^2	2^2	31	4
χ_4	1^4	1	0	0	0	0
χ_{31}	21^2		1	0	0	0
χ_{2^2}	2^2			1	0	0
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$$
\begin{array}{c} \\ \chi_4 \\ \chi_{31} \\ \chi_{2^2} \\ \chi_{21^2} \\ \chi_{1^4} \end{array}
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\left(\begin{array}{cccc}
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\end{array}\right) \end{array}
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Decomposition matrix of $\mathfrak{S}_4$
$\{\chi_4, \chi_{31}, \chi_{2^2}, \chi_{21^2}\}$
is called a basic set for $\mathfrak{S}_4$

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Decomposition matrix $D^{\mathcal{N}}$

Basic data, Springer basic data

Basic set datum

Definition

A basic set datum for W is a pair $\mathfrak{B} = (\leq, \beta)$, consisting of a partial order $\leq$ on $\operatorname{Irr} \mathbb{K}W$, and an injection $\beta : \operatorname{Irr} \mathbb{F}W \hookrightarrow \operatorname{Irr} \mathbb{K}W$ such that:

$$d^W_{\beta(F),F} = 1 \text{ for all } F \in \operatorname{Irr} \mathbb{F}W,$$

$$d^W_{E,F} \neq 0 \Rightarrow E \leq \beta(F) \text{ for } E \in \operatorname{Irr} \mathbb{K}W,\ F \in \operatorname{Irr} \mathbb{F}W.$$

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Proposition

Let $(\leq_1, \beta_1)$ and $(\leq_2, \beta_2)$ be two basic set data for W.
If $\leq_2$ is a finer order than $\leq_1$, then, $\beta_1 = \beta_2$.

Springer basic set datum

Definition: Springer order on $\operatorname{Irr} \mathbb{K}W$

For $i \in \{1, 2\}$, let $E_i \in \operatorname{Irr} \mathbb{K}W$, and let us write $\Psi_{\mathbb{K}}(E_i) = (x_i, \rho_i)$.
Then, $E_1 \leq^{\mathcal{N}} E_2 \iff (E_1 = E_2 \text{ or } \mathcal{O}_{x_2} < \mathcal{O}_{x_1})$

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Proposition

Let $F \in \operatorname{Irr} \mathbb{F}W$, and let us write $\Psi_{\mathbb{F}}(F) = (x, \sigma)$. Then there exists a unique $E \in \operatorname{Irr} \mathbb{K}W$ such that $\Psi_{\mathbb{K}}(E) = (x, \sigma)$.

Definition

We define the map $\beta^{\mathcal{N}} : \operatorname{Irr} \mathbb{F}W \to \operatorname{Irr} \mathbb{K}W$ by the condition $\beta^{\mathcal{N}}(F) = E \Leftrightarrow \Psi_{\mathbb{F}}(F) = \Psi_{\mathbb{E}}(E)$.

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$(\leq^{\mathcal{N}}, \beta^{\mathcal{N}})$ is a basic set datum for W, we will call it the **Springer basic set datum** for W.

The case of $GL_n(\overline{\mathbb{F}}_p)$

The decomposition matrix of $\mathfrak{S}_n$ (James, 1976)

$\operatorname{Irr}\mathbb{K}\mathfrak{S}_n = \{S^\lambda ; \lambda \vdash n\}$ (Specht modules)
S^λ is defined over $\mathbb{Z}$ and is endowed with a scalar product which is also defined over $\mathbb{Z}$, and thus one can reduce them to get a module for $\mathbb{F}\mathfrak{S}_n$, still denoted by S^λ, endowed with a symmetric bilinear form f, which no longer needs to be non-degenerate.

Then $S^\lambda / \operatorname{Ker}(f) = \begin{cases} D^\lambda \in \operatorname{Irr}\mathbb{F}\mathfrak{S}_n \text{ if } \lambda \text{ is } \ell\text{-regular} \\ 0 \text{ otherelse} \end{cases}$

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James basic set datum $(\leq^{DJ}, \beta^{DJ})$

- $S^\lambda \leq^{DJ} S^\mu \Leftrightarrow \lambda \leq \mu$ (dominance order)
- $\beta^{DJ} : \operatorname{Irr} \mathbb{F}\mathfrak{S}_n \to \operatorname{Irr} \mathbb{K}\mathfrak{S}_n$ is defined by $\beta^{DJ}(D^\lambda) = S^\lambda$

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- The Springer and James basic set data involve the same order relation, hence they coincide:
  $S^\lambda = \beta^{\mathcal{N}}(D^\lambda)$ is the unique $E \in \operatorname{Irr} \mathbb{K}\mathfrak{S}_n$ such that $\Psi_{\mathbb{F}}(D^\lambda) = \Psi_{\mathbb{K}}(E)$.

- Nilpotent orbits of GL_n: $\{\mathcal{O}_\lambda, \lambda \vdash n\}$.
  The orbit closure order is given by the dominance order on partitions.
- $\Psi_{\mathbb{K}}$ maps $S^\lambda \in \operatorname{Irr} \mathbb{K}\mathfrak{S}_n$ to the orbit $\mathcal{O}_{\lambda^*}$
- Springer order on $\operatorname{Irr} \mathbb{K}\mathfrak{S}_n$:

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Modular Springer correspondence for GL_n

$$\Psi_{\mathbb{F}} : D^\mu \in \operatorname{Irr} \mathbb{F}\mathfrak{S}_n \mapsto \mathcal{O}_{\mu^*}$$

The case of groups of type B or C

Decomposition matrix for the Weyl group W_n of type B_n (Dipper-James, 1990)

$\operatorname{Irr} \mathbb{K} W_n = \{\mathbf{S}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathfrak{Bip}_n\}$, where $\mathfrak{Bip}_n$ is the set of bipart. of n.
$\operatorname{Irr} \mathbb{F} W_n = \{\mathbf{D}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathfrak{Bip}_n^{(\ell)}\}$, where $\mathfrak{Bip}_n^{(\ell)}$ is the set of $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \in \mathfrak{Bip}_n$ s.t. $\lambda^{(1)}$ and $\lambda^{(2)}$ are ℓ-regular.

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Dipper-James order on bipartitions

$$\boldsymbol{\lambda} \leq^{DJ} \boldsymbol{\mu} \Leftrightarrow \begin{cases} |\lambda^{(i)}| = |\mu^{(i)}|, \\ \lambda^{(i)} \leq \mu^{(i)} \end{cases} \quad \text{for } i \in \{1, 2\}$$

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Dipper-James basic set datum $(\leq^{DJ}, \beta^{DJ})$

- Order on $\operatorname{Irr} \mathbb{K}W_n$ induced by $\leq^{DJ}$,
- $\beta^{DJ} : \operatorname{Irr} \mathbb{F}W_n \to \operatorname{Irr} \mathbb{K}W_n$ is defined by $\beta^{DJ}(\mathbf{D}^{\boldsymbol{\lambda}}) := \mathbf{S}^{\boldsymbol{\lambda}}$.

Springer correspondence in characteristic 0

Let G be a connected reductive group of type $X_n \in \{B_n, C_n\}$. $\mathfrak{P}_{\mathbb{K}}$ is then parametrized by a set $\mathbf{Symb}(X_n)$ of "symbols" which are some pairs (α, β) of finite increasing sequences of positive integers satisfying some specific conditions.

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Combinatorial description of the Springer correspondence over $\mathbb{K}$ (Shoji, Lusztig)

$$\Lambda : \mathfrak{Bip}_n \hookrightarrow \mathbf{Symb}(X_n)$$

Example: type C_3

$$((\lambda_1^{(1)} \leq \lambda_2^{(1)} \leq \lambda_3^{(1)} \leq \lambda_4^{(1)}), (\lambda_1^{(2)} \leq \lambda_2^{(2)} \leq \lambda_3^{(2)})) \in \mathfrak{Bip}_3$$

$$\mapsto \begin{pmatrix} \lambda_1^{(1)} & & \lambda_2^{(1)}+2 & & \lambda_3^{(1)}+4 & & \lambda_4^{(1)}+6 \\ & \lambda_1^{(2)}+1 & & \lambda_2^{(2)}+3 & & \lambda_3^{(2)}+5 & \end{pmatrix} \in \mathbf{Symb}(C_3)$$

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To get $\Psi_{\mathbb{K}}$, we first need to send $(\lambda^{(1)}, \lambda^{(2)})$ to $(\lambda^{(2)*}, \lambda^{(1)*})$

Springer order on $\operatorname{Irr} \mathbb{K}W$

Using the Jordan canonical form, one can parametrize nilpotent orbits of a group of type $X_n \in \{B_n, C_n\}$ by some set $\mathcal{P}(X_n)$ of partitions.
Hence we get a combinatorial process which sends a bipartition of $\mathfrak{Bip}_n$ to a partition of $\mathcal{P}(X_n)$.

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Hence we get a combinatorial process which sends a bipartition of $\mathfrak{Bip}_n$ to a partition of $\mathcal{P}(X_n)$.

The orbit closure order on nilpotent orbits is still given by the dominance order on partitions.
By making use of the above combinatorial process, we can define on $\mathfrak{Bip}_n$ the Springer order.
We would like to compare Dipper-James order and Springer order.

Group of type C_3

$\mathfrak{Bip}_3$	**Symb**(C_3)	$\mathfrak{P}_{\mathbb{K}}$	$\mathfrak{Bip}_3$	**Symb**(C_3)	$\mathfrak{P}_{\mathbb{K}}$
$(3,-)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 9 \\ & 1 & & 3 & & 5 & \end{pmatrix}$		$(1,2)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 7 \\ & 1 & & 3 & & 7 & \end{pmatrix}$	
$(12,-)$	$\begin{pmatrix} 0 & & 2 & & 5 & & 8 \\ & 1 & & 3 & & 5 & \end{pmatrix}$		$(1,1^2)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 7 \\ & 1 & & 4 & & 6 & \end{pmatrix}$	
$(1^3,-)$	$\begin{pmatrix} 0 & & 3 & & 5 & & 7 \\ & 1 & & 3 & & 5 & \end{pmatrix}$		$(-,3)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 3 & & 8 & \end{pmatrix}$	
$(2,1)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 8 \\ & 1 & & 3 & & 6 & \end{pmatrix}$		$(-,12)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 4 & & 7 & \end{pmatrix}$	
$(1^2,1)$	$\begin{pmatrix} 0 & & 2 & & 5 & & 7 \\ & 1 & & 3 & & 6 & \end{pmatrix}$		$(-,1^3)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 2 & & 4 & & 6 & \end{pmatrix}$	

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$\mathfrak{Bip}_3$	$\mathbf{Symb}(C_3)$	$\mathfrak{P}_{\mathbb{K}}$	$\mathfrak{Bip}_3$	$\mathbf{Symb}(C_3)$	$\mathfrak{P}_{\mathbb{K}}$
$(3,-)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 9 \\ & 1 & & 3 & & 5 & \end{pmatrix}$	$(6,\mathbf{1})$	$(1,2)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 7 \\ & 1 & & 3 & & 7 & \end{pmatrix}$	$(3^2,\mathbf{1})$
$(12,-)$	$\begin{pmatrix} 0 & & 2 & & 5 & & 8 \\ & 1 & & 3 & & 5 & \end{pmatrix}$	$(1^2 4,\mathbf{1})$	$(1,1^2)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 7 \\ & 1 & & 4 & & 6 & \end{pmatrix}$	$(1^2 2^2,\mathbf{1})$
$(1^3,-)$	$\begin{pmatrix} 0 & & 3 & & 5 & & 7 \\ & 1 & & 3 & & 5 & \end{pmatrix}$	$(1^4 2,\mathbf{1})$	$(-,3)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 3 & & 8 & \end{pmatrix}$	
$(2,1)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 8 \\ & 1 & & 3 & & 6 & \end{pmatrix}$	$(24,\mathbf{1})$	$(-,12)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 4 & & 7 & \end{pmatrix}$	
$(1^2,1)$	$\begin{pmatrix} 0 & & 2 & & 5 & & 7 \\ & 1 & & 3 & & 6 & \end{pmatrix}$	$(2^3,\mathbf{1})$	$(-,1^3)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 2 & & 4 & & 6 & \end{pmatrix}$	$(1^6,\mathbf{1})$

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$(3,-)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 9 \\ & 1 & & 3 & & 5 & \end{pmatrix}$	$(6,\mathbf{1})$	$(1,2)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 7 \\ & 1 & & 3 & & 7 & \end{pmatrix}$	$(3^2,\mathbf{1})$
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$(2,1)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 8 \\ & 1 & & 3 & & 6 & \end{pmatrix}$	$(24,\mathbf{1})$	$(-,12)$	$\begin{pmatrix} 0 & & 2 & & 4 & & 6 \\ & 1 & & 4 & & 7 & \end{pmatrix}$	$(1^2 2^2,\varepsilon)$
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Springer correspondence in characteristic ℓ

The ordinary Springer correspondence can be described by a combinatorial process which sends a bipartition of $\mathfrak{Bip}_n$ to a partition of $\mathcal{P}(X_n)$.
Moreover, if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathfrak{Bip}_n$ are sent respectively to λ and μ by this process, then

$$\boldsymbol{\lambda} \leq^{DJ} \boldsymbol{\mu} \Rightarrow \lambda \leq \mu$$

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$$\boldsymbol{\lambda} \leq^{DJ} \boldsymbol{\mu} \Rightarrow \lambda \leq \mu$$

Hence, Dipper-James order for $\operatorname{Irr} \mathbb{K} W_n$ is coarser that Springer order.
Once again, Dipper-James and Springer basic sets coincide.

Theorem

The modular Springer correspondence for a group G of type B or C maps the simple $\mathbb{F}W$-module $D^{\boldsymbol{\lambda}}$ ($\boldsymbol{\lambda} \in \mathfrak{Bip}_n^{(\ell)}$) to the image of the simple $\mathbb{K}W$-module $S^{\boldsymbol{\lambda}}$ under the ordinary Springer correspondence.