# Spinor Representations and Symmetric Functions

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## Representations of $\mathbf{GL}_n$, $\mathbf{Sp}_{2n}$ and $\mathbf{O}_N$

| Group | Partitions | Irreducible characters |
|---|---|---|
| $\mathbf{GL}_n$ | $l(\lambda) \leq n$ | $S_\lambda = \det\left(H_{\lambda_i - i + j}\right)$ |
| $\mathbf{Sp}_{2n}$ | $l(\lambda) \leq n$ | $S_{\langle\lambda\rangle} = \frac{1}{2}\det\left(H_{\lambda_i - i + j} + H_{\lambda_i - i - j + 2}\right)$ |
| $\mathbf{O}_N$ | ${}^t\lambda_1 + {}^t\lambda_2 \leq N$ | $S_{[\lambda]} = \det\left(H_{\lambda_i - i + j} - H_{\lambda_i - i - j}\right)$ |

where

$$H_k = \text{character of the } k\text{th symmetric power } S^k(V)$$
$$\text{of the vector representation } V \text{ of } G$$

## Universal characters (Littlewood, King, Koike–Terada)

Let $\Lambda$ be the ring of symmetric functions. For any partition $\lambda$, we define $s_\lambda$, $s_{\langle\lambda\rangle}$, $s_{[\lambda]}$ by putting

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)$$

(Schur function),

$$s_{\langle\lambda\rangle} = \frac{1}{2}\det\left(h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 2}\right)$$

(symplectic universal character),

$$s_{[\lambda]} = \det\left(h_{\lambda_i - i + j} - h_{\lambda_i - i - j}\right)$$

(orthogonal universal character),

where $h_k$ is the $k$th complete symmetric function.

## Specializations

For each classical group $G = \mathbf{GL}_n$, $\mathbf{Sp}_{2n}$, $\mathbf{O}_N$, let $\mathrm{Rep}(G)$ denote the representation ring of $G$ and define $\pi_G : \Lambda \to \mathrm{Rep}(G)$ by

$$\pi_G(h_k) = H_k \quad (k \geq 0).$$

Then we have

$$\begin{aligned} \pi_{\mathbf{GL}_n}(s_\lambda) &= S_\lambda && \text{if } l(\lambda) \leq n, \\ \pi_{\mathbf{Sp}_{2n}}(s_{\langle\lambda\rangle}) &= S_{\langle\lambda\rangle} && \text{if } l(\lambda) \leq n, \\ \pi_{\mathbf{O}_N}(s_{[\lambda]}) &= S_{[\lambda]} && \text{if } {}^t\lambda_1 + {}^t\lambda_2 \leq N. \end{aligned}$$

and $\pi_{\mathbf{GL}_n}(s_\lambda) = 0$ if $l(\lambda) > n$. There are algorithms for expressing the images $\pi_{\mathbf{Sp}_{2n}}(s_{\langle\lambda\rangle})$ ($l(\lambda) > n$) and $\pi_{\mathbf{O}_N}(s_{[\lambda]})$ (${}^t\lambda_1 + {}^t\lambda_2 > N$) in terms of irreducible characters.

Hence the computation of the irreducible decomposition of tensor products and restrictions can be reduced to the manipulation of symmetric functions. For example, we can show the stability of tensor product multiplicities and restriction multiplicities.

**Goal :** To give such a framework for spinor representations of $\mathbf{Pin}_N$.

## Orthogonal Universal Characters and Representations of $\mathrm{O}_N$

(Littlewood, King, Koike–Terada)

## Representations of $\mathbf{O}_N$

The irreducible representations of $\mathbf{O}_N$ are parametrized by partitions $\lambda$ such that ${}^t\lambda_1 + {}^t\lambda_2 \leq N$. We call such a partition an $N$-orthogonal partition. For an $N$-orthogonal partition $\lambda$, the corresponding irreducible character $S_{[\lambda]}$ is given by

$$S_{[\lambda]} = \det\left(H_{\lambda_i - i + j} - H_{\lambda_i - i - j}\right)_{1 \leq i,j \leq r}$$

$$= \det\begin{pmatrix} H_{\lambda_1} - H_{\lambda_1 - 2} & H_{\lambda_1+1} - H_{\lambda_1 - 3} & H_{\lambda_1+2} - H_{\lambda_1-4} & \cdots \\ H_{\lambda_2-1} - H_{\lambda_2-3} & H_{\lambda_2} - H_{\lambda_2-4} & H_{\lambda_2+1} - H_{\lambda_2-5} & \cdots \\ H_{\lambda_3-2} - H_{\lambda_3-4} & H_{\lambda_3-1} - H_{\lambda_3-5} & H_{\lambda_3} - H_{\lambda_3-6} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $r \geq l(\lambda)$ and $H_k$ is the character of the $k$-th symmetric tensor $S^k(\mathbb{C}^N)$ of the vector representation of $\mathbf{O}_N$.

## Orthogonal universal characters

For any partition $\lambda$, we define a symmetric function $s_{[\lambda]}$ (called an orthogonal universal character) by

$$s_{[\lambda]} = \det\left(h_{\lambda_i - i + j} - h_{\lambda_i - i - j}\right)_{1 \le i,j \le l(\lambda)}.$$

Let $\mathrm{Rep}(\mathbf{O}_N)$ be the representation ring of $\mathbf{O}_N$, and $\pi_N = \pi_{\mathbf{O}_N} : \Lambda \to \mathrm{Rep}(\mathbf{O}_N)$ be the ring homomorphism defined by

$$\pi_N(h_k) = H_k \quad (k \ge 0).$$

Then we have

$$\pi_N(s_{[\lambda]}) = S_{[\lambda]} \quad \text{if } {}^t\lambda_1 + {}^t\lambda_2 \le N.$$

## Properties of orthogonal universal characters

- Cauchy–type identity :

$$\sum_{\lambda} s_{[\lambda]}(X) s_{\lambda}(U) = \frac{\prod_{i \le j}(1 - u_i u_j)}{\prod_{i,j}(1 - x_i u_j)}.$$

- Schur function expansion :

$$s_{[\lambda]} = \sum_{\mu} \left( \sum_{\kappa=(\alpha+1|\alpha)} (-1)^{|\kappa|/2} \, \mathrm{LR}^{\lambda}_{\mu,\kappa} \right) s_{\mu}.$$

where $\mathrm{LR}^{\lambda}_{\mu,\nu}$ is the Littlewood–Richardson coefficient.

- Dual Jacobi–Trudi type identity :

$$s_{[\lambda]} = \frac{1}{2} \det \left( e_{{}^t\lambda_i - i + j} + e_{{}^t\lambda_i - i - j + 2} \right),$$

where $e_k$ is the $k$th elementary symmetric function.

## Specialization

By the dual Jacobi–Trudi type identity, we have

$$\pi_N(s_{[\lambda]}) = \frac{1}{2}\det\Big(E_{{}^t\lambda_i - i + j} + E_{{}^t\lambda_i - i - j + 2}\Big),$$

where

$$E_k = \text{character of the } k\text{th exterior power } \textstyle\bigwedge^k(\mathbb{C}^N).$$

This can be rewritten as

$$\pi_N(s_{[\lambda]}) = \det{}^t\Big(\overrightarrow{E}_{\alpha_1}, \overrightarrow{E}_{\alpha_2}, \cdots, \overrightarrow{E}_{\alpha_r}\Big),$$

where

$$\alpha = ({}^t\lambda_1, {}^t\lambda_2 - 1, \cdots, {}^t\lambda_r - (r-1)), \quad r = l({}^t\lambda),$$

and $\overrightarrow{E}_k$ is the row vector given by

$$\overrightarrow{E}_k = (E_k, E_{k+1} + E_{k-1}, E_{k+2} + E_{k-2}, \cdots, E_{k+(r-1)} + E_{k-(r-1)}).$$

By using the relations

$$\overrightarrow{E}_k = 0 \quad \text{for } k \geq N + r, \qquad E_N \overrightarrow{E}_k = \overrightarrow{E}_{N-k},$$

we can express

$$\pi_N(s_{[\lambda]}) = \det {}^t\!\left(\overrightarrow{E}_{\alpha_1}, \overrightarrow{E}_{\alpha_2}, \cdots, \overrightarrow{E}_{\alpha_r}\right) \quad ({}^t\!\lambda_1 + {}^t\!\lambda_2 > N)$$

in terms of irreducible characters $S_{[\mu]}$ of $\mathbf{O}_N$.

(1) If $\alpha_i \geq N + r$ for some $i$, then we have

$$\pi_N(s_{[\lambda]}) = 0.$$

(2) If $\alpha_i + \alpha_j = N$ for some $i$ and $j$, then we have

$$\pi_N(s_{[\lambda]}) = 0.$$

(3) Otherwise we can find a permutation $\sigma \in \mathfrak{S}_r$ and an $N$-orthogonal partition $\mu$ such that

$$\pi_N(s_{[\lambda]}) = \mathrm{sgn}(\sigma) S_{[\mu]}.$$

## Tensor product

In the ring $\Lambda$ of symmetric functions, we can show that

$$s_{[\mu]}s_{[\nu]} = \sum_{\lambda} \left( \sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi}\, \mathrm{LR}^{\nu}_{\tau,\eta}\, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) s_{[\lambda]},$$

where $\lambda$ and $\tau$, $\xi$, $\eta$ run over all partitions.

If $\mu$ and $\nu$ are $N$-orthogonal partitions, then we have

$$S_{[\mu]}S_{[\nu]} = \sum_{\lambda} \left( \sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi}\, \mathrm{LR}^{\nu}_{\tau,\eta}\, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) \pi_N(s_{[\lambda]}).$$

Together with the algorithm computing $\pi_N(s_{[\lambda]})$, we obtain the actual decomposition of $S_{[\mu]}S_{[\nu]}$ in the representation ring $\mathrm{Rep}(\mathbf{O}_N)$.

## Stability of tensor product decomposition

Note that

$$\mathrm{LR}^{\alpha}_{\beta,\gamma} = 0$$

unless

$${}^t\beta_1 + {}^t\beta_2 + {}^t\gamma_1 + {}^t\gamma_2 \geq {}^t\alpha_1 + {}^t\alpha_2 \geq \max({}^t\beta_1 + {}^t\beta_2, {}^t\gamma_1 + {}^t\gamma_2).$$

Hence we see that, if ${}^t\mu_1 + {}^t\mu_2 + {}^t\nu_1 + {}^t\nu_2 \leq N$, then we have

$$S_{[\mu]} S_{[\nu]} = \sum_{\lambda} \left( \sum_{\tau,\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\nu}_{\tau,\eta} \, \mathrm{LR}^{\lambda}_{\xi,\eta} \right) S_{[\lambda]},$$

where $\lambda$ runs over all $N$-orthogonal partitions, i.e., the decomposition rule of tensor products is stable in $N$.

## Restriction

Let $\Delta : \Lambda \to \Lambda \otimes \Lambda$ be the ring homomorphism defined by

$$\Delta(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}.$$

Note that

$$h_k(X \cup Y) = \sum_{i=0}^{k} h_i(X) h_{k-i}(Y).$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \Lambda & \stackrel{\Delta}{\longrightarrow} & \Lambda \otimes \Lambda \\ \pi_{M+N} \Big\downarrow & & \Big\downarrow \pi_M \otimes \pi_N \\ \mathrm{Rep}(\mathbf{O}_{M+N}) & \underset{\mathrm{Res}}{\longrightarrow} & \mathrm{Rep}(\mathbf{O}_M) \otimes \mathrm{Rep}(\mathbf{O}_N) \end{array}$$

For the orthogonal universal characters, we have

$$s_{[\lambda]}(X \cup Y) = \sum_{\mu,\nu} \left( \sum_{\kappa} \mathrm{LR}^{\lambda}_{\mu,\nu,\kappa} \right) s_{[\mu]}(X) s_{[\nu]}(Y),$$

where $\mu$, $\nu$ run over all partitions and $\kappa$ runs over all partitions such that all parts are even. And $\mathrm{LR}^{\lambda}_{\mu,\nu,\kappa}$ denotes the coefficients of $s_\lambda$ in the product $s_\mu s_\nu s_\kappa$. Hence we have, for $(M+N)$-orthogonal partition $\lambda$,

$$\mathrm{Res}^{\mathbf{O}_{M+N}}_{\mathbf{O}_M \times \mathbf{O}_N} S_{[\lambda]} = \sum_{\mu,\nu} \left( \sum_{\kappa} \mathrm{LR}^{\lambda}_{\mu,\nu,\kappa} \right) \pi_M(s_{[\mu]}) \pi_N(s_{[\nu]}).$$

If $l(\lambda) \leq \min(M, N)$, then we have

$$\mathrm{Res}^{\mathbf{O}_{M+N}}_{\mathbf{O}_M \times \mathbf{O}_N} S_{[\lambda]} = \sum_{\mu,\nu} \left( \sum_{\kappa} \mathrm{LR}^{\lambda}_{\mu,\nu,\kappa} \right) S_{[\mu]} \otimes S_{[\nu]}.$$

# Spinor Universal Characters
# and
# Spinor Representations of $\mathrm{Pin}_N$

## Representations of $\mathbf{Pin}_N$

Let $\mathbf{Pin}_N$ be the pin group :

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{Pin}_N \xrightarrow{\pi} \mathbf{O}_N \longrightarrow 1.$$

So any representation of $\mathbf{O}_N$ can be viewed as a representation of $\mathbf{Pin}_N$. We use the same symbol to represent the character of $\mathbf{Pin}_N$ which is obtained by lifting the character of $\mathbf{O}_N$.

$$\begin{aligned} S_{[\lambda]} &= \text{lift of the irreducible character of } \mathbf{O}_N \\ &\qquad\qquad (\text{for an } N\text{-orthogonal partition } \lambda), \\ H_k &= \text{lift of the character of } \mathbf{O}_N\text{-module } S^k(\mathbb{C}^N), \\ E_k &= \text{lift of the character of } \mathbf{O}_N\text{-module } \textstyle\bigwedge^k(\mathbb{C}^N). \end{aligned}$$

Note that $E_N$ is a one-dimensional character and

$$\mathbf{Spin}_N = \operatorname{Ker} E_N.$$

We say that an irreducible representation of $\mathbf{Pin}_N$ is

- a tensor representation if it factors through $\mathbf{O}_N$,
- a spinor representation otherwise.

We put

$$\begin{aligned} \mathrm{Rep}(\mathbf{Pin}_N) &= \text{the representation ring of } \mathbf{Pin}_N, \\ \mathrm{Rep}^+(\mathbf{Pin}_N) &= \text{span of the tensor irreducible characters}, \\ \mathrm{Rep}^-(\mathbf{Pin}_N) &= \text{span of the spinor irreducible characters}. \end{aligned}$$

Then we have

$$\mathrm{Rep}(\mathbf{Pin}_N) = \mathrm{Rep}^+(\mathbf{Pin}_N) \oplus \mathrm{Rep}^-(\mathbf{Pin}_N),$$

and

$$\mathrm{Rep}^+(\mathbf{Pin}_N) \cong \mathrm{Rep}(\mathbf{O}_N).$$

## Spin representation

Let $\Delta_N$ be the character of the spin representation of $\mathbf{Pin}_N$, whose dimension is $2^{\lfloor N/2 \rfloor}$.

If $N$ is odd, then

$$E_N \cdot \Delta_N \neq \Delta_N,$$

and

$$\Delta_N|_{\mathbf{Spin}_N} = \text{irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right).$$

If $N$ is even, then

$$E_N \cdot \Delta_N = \Delta_N,$$

and

$$\Delta_N|_{\mathbf{Spin}_N} = \text{irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \frac{1}{2}\right)$$
$$+ \text{ irred. character with h. w. } \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}\right).$$

## Irreducible spinor characters

**Theorem 1** For a partition $\lambda$ of length $\leq N/2$, we define a class function $S_{[\lambda+1/2]}$ on $\mathbf{Pin}_N$ by

$$S_{[\lambda+1/2]} = \Delta_N \cdot \det \left( H_{\lambda_i - i + j} - E_N H_{\lambda_i - i - j + 1} \right)_{1 \leq i,j \leq l(\lambda)}.$$

Then $S_{[\lambda+1/2]}$ is an irreducible character of $\mathbf{Pin}_N$. Moreover

(1) If $N$ is odd, then $\mathrm{Rep}^-(\mathbf{Pin}_N)$ has a basis

$$S_{[\lambda+1/2]}, \quad E_N \cdot S_{[\lambda+1/2]} \quad (l(\lambda) \leq N/2).$$

(2) If $N$ is even, then $\mathrm{Rep}^-(\mathbf{Pin}_N)$ has a basis

$$S_{[\lambda+1/2]} \quad (l(\lambda) \leq N/2).$$

**Idea of Proof of Theorem 1** : It is enough to show that

- $S_{[\lambda+1/2]}$ is a virtual character, i.e., an integral linear combination of characters,
- If $\langle \quad , \quad \rangle$ is the canonical symmetric bilinear form on the space of class functions of $\mathbf{Pin}_N$, then

$$\langle S_{[\lambda+1/2]}, S_{[\lambda+1/2]} \rangle = 1,$$

- The value of $S_{[\lambda+1/2]}$ at the identity element of $\mathbf{Pin}_N$ is positive.

## Spinor universal characters

We work in the ring $\widetilde{\Lambda}$ of symmetric functions with coefficients in the ring $\mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1)$ :

$$\widetilde{\Lambda} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[\varepsilon]/(\varepsilon^2 - 1).$$

For any partition $\lambda$, we define a symmetric function $s'_{[\lambda]}$ (called a spinor universal character) by putting

$$s'_{[\lambda]} = \det \left( h_{\lambda_i - i + j} - \varepsilon h_{\lambda_i - i - j + 1} \right)_{1 \le i,j \le l(\lambda)}.$$

Let $\widetilde{\pi}_N : \widetilde{\Lambda} \to \mathrm{Rep}(\mathbf{Pin}_N)$ be the ring homomorphism given by

$$\widetilde{\pi}_N(h_k) = H_k \quad (k \ge 0) \quad \text{and} \quad \widetilde{\pi}_N(\varepsilon) = E_N.$$

Then we have, for a partition $\lambda$ of length $\le N/2$,

$$S_{[\lambda+1/2]} = \Delta_N \cdot \widetilde{\pi}_N(s'_{[\lambda]}).$$

## Properties of spinor universal characters $s'_{[\lambda]}$

- Cauchy–type identity :

$$\sum_{\lambda} s'_{[\lambda]}(X) s_{\lambda}(U) = \frac{\prod_i (1-\varepsilon u_i) \prod_{i<j}(1-u_i u_j)}{\prod_{i,j}(1-x_i u_j)}.$$

- Schur function expansion :

$$s'_{[\lambda]} = \sum_{\mu} \left( \sum_{\nu = {}^t\nu} (-1)^{(|\nu|+l(\nu))/2} \varepsilon^{|\nu|} \, \mathrm{LR}^{\lambda}_{\mu,\nu} \right) s_{\mu},$$

  where the inner summation is taken over all self-conjugate partitions $\nu$.

- $\{s'_{[\lambda]}, \varepsilon s'_{[\lambda]}\}_{\lambda}$ form a $\mathbb{Z}$-basis of $\widetilde{\Lambda}$.

## Properties of spinor universal characters $s'_{[\lambda]}$ (cont.)

- Duality :

$$\omega(s'_{[\lambda]}) = s'_{[{}^t\lambda]}.$$

- Dual Jacobi–Trudi type identity

$$s'_{[\lambda]} = \det\left(e_{{}^t\lambda_i - i + j} - \varepsilon e_{{}^t\lambda_i - i - j + 1}\right).$$

## Specialization

By the dual Jacobi–Trudi type identity, we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = \det\left(E_{{}^t\lambda_i - i + j} - E_N E_{{}^t\lambda_i - i - j + 1}\right)_{1 \le i,j \le r},$$

where $r = l({}^t\lambda)$.

We put

$$E'_k = E_k - E_N E_{k-1},$$

and define a row vector $\overrightarrow{E}'_k$ by

$$\overrightarrow{E}'_k = \left(E'_k, E'_{k+1} + E'_{k-1}, \cdots, E'_{k+(r-1)} + E'_{k-(r-1)}\right).$$

Then the above determinant can be rewritten as

$$\widetilde{\pi}_N(s'_{[\lambda]}) = \det {}^t\!\left(\overrightarrow{E}'_{\alpha_1}, \overrightarrow{E}'_{\alpha_2}, \cdots, \overrightarrow{E}'_{\alpha_r}\right)$$

where

$$\alpha = ({}^t\lambda_1, {}^t\lambda_2 - 1, \cdots, {}^t\lambda_r - (r-1)).$$

By using the relations

$$\overrightarrow{E}'_k = 0 \quad \text{for } k \geq N + r, \qquad \overrightarrow{E}'_k + \overrightarrow{E}'_{N+1-k} = 0,$$

we can compute

$$\widetilde{\pi}_N(s'_{[\lambda]}) = \det {}^t\!\left(\overrightarrow{E}'_{\alpha_1}, \overrightarrow{E}'_{\alpha_2}, \cdots, \overrightarrow{E}'_{\alpha_r}\right)$$

for a partition $\lambda$ with length $> N/2$.

(1) If $\alpha_i \geq N + r$ for some $i$, then we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = 0.$$

(2) If $\alpha_i + \alpha_j = N + 1$ for some $i$ and $j$, then we have

$$\widetilde{\pi}_N(s'_{[\lambda]}) = 0.$$

(3) Otherwise we can find an index $p$, a permutation $\sigma$ and a partition $\mu$ of length $\leq N/2$ such that

$$\Delta_N \cdot \widetilde{\pi}_N(s'_{[\lambda]}) = (-1)^p \operatorname{sgn}(\sigma) S_{[\mu+1/2]}.$$

Here $p$, $\sigma$ and $\mu$ are given as follows. Let $p$ be an index such that

$$\alpha_1 > \cdots > \alpha_p > \frac{N+1}{2} \geq \alpha_{p+1} > \cdots > \alpha_r,$$

and define a new sequence $\beta$ by

$$\beta = (N+1-\alpha_1, \cdots, N+1-\alpha_p, \alpha_{p+1}, \cdots, \alpha_r).$$

Let $\gamma$ be the sequence obtained from $\beta$ by rearranging components in decreasing order, and $\sigma$ be a permutation such that $\gamma = \sigma(\beta)$. Finally a partition $\mu$ is given by

$$\gamma = ({}^t\mu_1, {}^t\mu_2 - 1, \cdots, {}^t\mu_r - (r-1))$$

**Example** Let $\lambda = (4,3,3,3,2,2,1,1)$ and $N = 8$. Then ${}^t\lambda = (8,6,4,1)$ and

$$\alpha = (8, 6-1, 4-2, 1-3) = (8,5,2,-2).$$

There are two components larger than $(N+1)/2 = 9/2$, so $p = 2$ and

$$\beta = (9-8, 9-5, 2, -2) = (1,4,2,-2).$$

Hence

$$\gamma = (4,2,1,-2), \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

and

$${}^t\mu = (4, 2+1, 1+2, -2+3) = (4,3,3,1), \quad \mu = (4,3,3,1).$$

Hence we have

$$\Delta \cdot \widetilde{\pi}_8(s'_{[4,3,3,3,2,2,1,1]}) = (-1)^2 \cdot (-1)^2 \cdot S_{[(4,3,3,1)+1/2]}.$$

# Tensor Products and Restrictions

## Tensor product of a spinor repr. and a tensor repr.

In order to compute the product

$$S_{[\mu+1/2]} \cdot S_{[\nu]} = \Delta \cdot \widetilde{\pi}_N(s'_{[\mu]} s_{[\nu]}) \quad \text{in } \mathrm{Rep}(\mathbf{Pin}_N),$$

it is enough to compute

$$s'_{[\mu]} \cdot s_{[\nu]} \quad \text{in } \widetilde{\Lambda}.$$

**Theorem 2** (See [King, 1975].) In the ring $\widetilde{\Lambda}$, we have

$$s'_{[\mu]} \cdot s_{[\nu]} = \sum_{\lambda} \left( \sum_{\substack{\xi,\eta,\tau \\ \nu/\sigma\ :\ \text{v-strip}}} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\sigma}_{\tau,\eta} \, \varepsilon^{|\nu|-|\sigma|} \right) s'_{[\lambda]},$$

where $\xi$, $\eta$, $\tau$ run over all partitions and $\sigma$ runs over all partitions such that $\nu/\sigma$ is a vertical strip.

**Proof of Theorem 2** Consider the generating function with respect to Schur functions.

$$\sum_{\mu,\nu} s'_{[\mu]}(X) s_{[\nu]}(X) s_\mu(U) s_\nu(V)$$

$$= \frac{\prod_i (1-\varepsilon u_i) \prod_{i<j}(1-u_iu_j)}{\prod_{i,j}(1-x_iu_j)} \cdot \frac{\prod_{i\le j}(1-v_iv_j)}{\prod_{i,j}(1-x_iv_j)}$$

$$= \prod_i (1+\varepsilon v_i) \cdot \frac{1}{\prod_{i,j}(1-u_iv_j)}$$

$$\cdot \frac{\prod_i(1-\varepsilon u_i)\prod_i(1-\varepsilon v_i)\prod_{i<j}(1-u_iu_j)\prod_{i,j}(1-u_iv_j)\prod_{i<j}(1-v_iv_j)}{\prod_{i,j}(1-x_iu_j)\prod_{i,j}(1-x_iv_j)}$$

$$= \left(\sum_{k\ge 0} \varepsilon^k e_k(V)\right) \cdot \left(\sum_\tau s_\tau(U) s_\tau(V)\right) \cdot \left(\sum_\lambda s'_{[\lambda]}(X) s_\lambda(U\cup V)\right).$$

Now we expand

$$e_k(V) \cdot s_\tau(U)s_\tau(V) \cdot s_\lambda(U \cup V)$$

as a linear combination of the product of Schur functions in $U$ and $V$. Finally we get

$$\sum_{\mu,\nu} s'_{[\mu]}(X)s_{[\nu]}(X)s_\mu(U)s_\nu(V)$$
$$= \sum_{\mu,\nu}\sum_{\lambda}\left(\sum_{\xi,\eta,\tau,\sigma} \varepsilon^{|\nu|-|\sigma|}\,\mathrm{LR}^{\lambda}_{\xi,\eta}\,\mathrm{LR}^{\mu}_{\tau,\xi}\,\mathrm{LR}^{\sigma}_{\tau,\eta}\right) s'_{[\lambda]}(X)s_\mu(U)s_\nu(V),$$

where $\nu$ runs over all partitions such that $\nu/\sigma$ is a vertical strip. By comparing the coefficient of $s_\mu(U)s_\nu(V)$, we obtain the desired identity.

By applying the specialization $\widetilde{\pi}_N$, we obtain

**Corollary** If $l(\mu) + l(\nu) \le N/2$, then we have

$$S_{[\mu+1/2]} \cdot S_{[\nu]} = \sum_{\lambda} \left( \sum_{\xi,\eta,\tau,\sigma} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\sigma}_{\tau,\eta} \, E_N^{|\nu|-|\sigma|} \right) S_{[\lambda+1/2]},$$

where $\lambda$ runs over all partitions of length $\le N/2$.

In this case, the decomposition depends only on $\mu$ and $\nu$ (and the parity of $N$).

## Tensor product of two spinor repr.

We consider the product

$$S_{[\mu+1/2]} \cdot S_{[\nu+1/2]} = \Delta_N^2 \cdot \widetilde{\pi}_N(s'_{[\mu]} s'_{[\nu]}) \quad \text{in } \mathrm{Rep}(\mathbf{Pin}_N).$$

It is known that

$$\Delta_N^2 = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{r=0}^{N} E_N^r E_r & \text{if } N \text{ is odd,} \\ \displaystyle\sum_{r=0}^{N} E_N^r E_r & \text{if } N \text{ is even.} \end{cases}$$

Hence the tensor product of two spinor representations can be computed by using the following two formulae.

**Theorem 3** (See [King, 1975].) In the ring $\widetilde{\Lambda}$, we have

$$s'_{[\mu]} \cdot s'_{[\nu]} = \sum_{\lambda} \left( \sum_{\xi,\eta,\tau} \mathrm{LR}^{\lambda}_{\xi,\eta} \, \mathrm{LR}^{\mu}_{\tau,\xi} \, \mathrm{LR}^{\nu}_{\tau,\eta} \right) s'_{[\lambda]}.$$

Also we have

$$\sum_{k \geq 0} \varepsilon^k e_k \cdot s'_{[\mu]} = \sum_{\lambda} \varepsilon^{|\lambda|-|\mu|} s_{[\lambda]},$$

where $\lambda$ runs over all partitions such that $\lambda/\mu$ is a vertical strip.

By applying the specialization $\widetilde{\pi}_N$, we obtain

**Corollary** If $N$ is odd and $l(\mu)+l(\nu) \le N/2$, then we have

$$S_{[\mu+1/2]} \cdot S_{[\nu+1/2]}$$

$$= \sum_{l(\lambda) \le N/2} \left\{ \sum_{\lambda/\sigma\ :\ \text{even v-strip}} \left( \sum_{\xi,\eta,\tau} \mathrm{LR}^{\sigma}_{\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi} \mathrm{LR}^{\nu}_{\tau,\eta} \right) S_{[\lambda]} \right.$$

$$\left. + \sum_{\lambda/\sigma\ :\ \text{odd v-strip}} \left( \sum_{\xi,\eta,\tau} \mathrm{LR}^{\sigma}_{\xi,\eta} \mathrm{LR}^{\mu}_{\tau,\xi} \mathrm{LR}^{\nu}_{\tau,\eta} \right) E_N \cdot S_{[\lambda]} \right\}.$$

In this case, the decomposition itself is not stable in $N$, but the tensor product multiplicities are stable.

**Corollary** If $N$ is even and $l(\mu)+l(\nu)\le N/2$, then we have

$$S_{[\mu+1/2]}\cdot S_{[\nu+1/2]}$$
$$=\sum_{l(\lambda)=N/2}\sum_{\lambda/\sigma\ :\ \text{v-strip}}\left(\sum_{\xi,\eta,\tau}\mathrm{LR}^{\sigma}_{\xi,\eta}\,\mathrm{LR}^{\mu}_{\tau,\xi}\,\mathrm{LR}^{\nu}_{\tau,\eta}\right)S_{[\lambda]}$$
$$+\sum_{l(\lambda)<N/2}\sum_{\lambda/\sigma\ :\ \text{v-strip}}\left(\sum_{\xi,\eta,\tau}\mathrm{LR}^{\sigma}_{\xi,\eta}\,\mathrm{LR}^{\mu}_{\tau,\xi}\,\mathrm{LR}^{\nu}_{\tau,\eta}\right)(S_{[\lambda]}+E_N S_{[\lambda]}).$$

## Restriction

We consider the restriction from $\mathbf{Pin}_{M+N}$ to the subgroup

$$\mathbf{Pin}_M * \mathbf{Pin}_N = \pi^{-1}(\mathbf{O}_M \times \mathbf{O}_N)$$

corresponding to $\mathbf{O}_M \times \mathbf{O}_N \subset \mathbf{O}_{M+N}$.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathbf{Pin}_{M+N} & \xrightarrow{\pi} & \mathbf{O}_{M+N} & \longrightarrow & 1 \\ & & & & \bigcup & & \bigcup & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathbf{Pin}_M * \mathbf{Pin}_N & \xrightarrow{\pi} & \mathbf{O}_M \times \mathbf{O}_N & \longrightarrow & 1 \end{array}$$

Note that the subgroup $\mathbf{Pin}_M * \mathbf{Pin}_N$ is not the direct product of $\mathbf{Pin}_M$ and $\mathbf{Pin}_N$, but it is the twisted central product.

## Twisted central product

The twisted central product $G_1 * G_2$ is defined for groups $G_1$ and $G_2$ equipped with

- homomorphisms $p_1 : G_1 \to \mathbb{Z}/2\mathbb{Z}$ and $p_2 : G_2 \to \mathbb{Z}/2\mathbb{Z}$,
- central elements $z_1 \in G_1$ and $z_2 \in G_2$ of order $2$.

The Cartesian product $G_1 \times G_2$ admits a group structure with respect to the multiplication given by

$$(x_1, x_2)(y_1, y_2) = \left( z_1^{p_2(x_2)p_1(y_1)} x_1 y_1, x_2 y_2 \right).$$

Then the twisted central product $G_1 * G_2$ is defined by

$$G_1 * G_2 = (G_1 \times G_2)/Z,$$

where $Z = \{(1,1), (z_1, z_2)\}$ is a central subgroup.

## Irreducible characters of $\mathbf{Pin}_M * \mathbf{Pin}_N$

Let $H_k^{[M]}$, $E_k^{[M]}$, $H_k^{[N]}$ and $E_k^{[N]}$ be the characters of $\mathbf{Pin}_M * \mathbf{Pin}_N$ given by

$$H_k^{[M]} = \text{pull-back of the character of } \mathbf{O}_M\text{-module } S^k(\mathbb{C}^M),$$
$$E_k^{[M]} = \text{pull-back of the character of } \mathbf{O}_M\text{-module } \textstyle\bigwedge^k(\mathbb{C}^M),$$
$$H_k^{[N]} = \text{pull-back of the character of } \mathbf{O}_N\text{-module } S^k(\mathbb{C}^N),$$
$$E_k^{[N]} = \text{pull-back of the character of } \mathbf{O}_N\text{-module } \textstyle\bigwedge^k(\mathbb{C}^N),$$

via the maps $\mathbf{Pin}_M * \mathbf{Pin}_N \to \mathbf{Pin}_M \to \mathbf{O}_M$ and $\mathbf{Pin}_M * \mathbf{Pin}_N \to \mathbf{Pin}_N \to \mathbf{O}_N$.

Note that $E_M^{[M]} \cdot E_n^{[N]}$ is a one-dimensional character and

$$E_M^{[M]} \cdot E_N^{[N]} = \mathrm{Res}^{\mathbf{Pin}_{M+N}}_{\mathbf{Pin}_M * \mathbf{Pin}_N} E_{M+N}.$$

**Theorem 4** For partitions $\mu$ and $\nu$ with $l(\mu) \leq M/2$ and $l(\nu) \leq N/2$, we define $S_{[\mu+1/2]*[\nu+1/2]} \in \mathrm{Rep}(\mathbf{Pin}_M * \mathbf{Pin}_N)$ by putting

$$S_{[\mu+1/2]*[\nu+1/2]} = \left(\mathrm{Res}^{\mathbf{Pin}_{M+N}}_{\mathbf{Pin}_M * \mathbf{Pin}_N} \Delta_{M+N}\right) \times \det\left(H^{[M]}_{\mu_i-i+j} - E^{[M]}_M E^{[N]}_N H^{[M]}_{\mu_i-i-j+1}\right)_{1\leq i,j\leq l(\mu)} \times \det\left(H^{[N]}_{\nu_i-i+j} - E^{[M]}_M E^{[N]}_N H^{[N]}_{\nu_i-i-j+1}\right)_{1\leq i,j\leq l(\nu)}.$$

Then $S_{[\mu+1/2]*[\nu+1/2]}$ is an irreducible character of $\mathbf{Pin}_M * \mathbf{Pin}_N$ and

(1) if $M \equiv N \bmod 2$, then $\mathrm{Rep}^-(\mathbf{Pin}_M * \mathbf{Pin}_N)$ has a basis

$$S_{[\mu+1/2]*[\nu+1/2]} \quad (l(\mu) \leq M/2,\ l(\nu) \leq N/2).$$

(2) if $M \not\equiv N \bmod 2$, then $\mathrm{Rep}^-(\mathbf{Pin}_M * \mathbf{Pin}_N)$ has a basis

$$\left(E^{[M]}_M E^{[N]}_N\right)^r S_{[\mu+1/2]*[\nu+1/2]} \quad (r = 0, 1,\ l(\mu) \leq M/2,\ l(\nu) \leq N/2).$$

## Restriction

Let $\widetilde{\pi}_{M,N} : (\Lambda \otimes \Lambda)[\varepsilon]/(\varepsilon^2 - 1) \to \mathrm{Rep}(\mathbf{Pin}_M * \mathbf{Pin}_N)$ be the ring homomorphism defined by

$$\widetilde{\pi}_{M,N}(h_k \otimes 1) = H_k^{[M]},$$
$$\widetilde{\pi}_{M,N}(1 \otimes h_k) = H_k^{[N]},$$
$$\widetilde{\pi}_{M,N}(\varepsilon) = E_M^{[M]} \cdot E_N^{[N]}.$$

Then we have, for partitions $\mu$ and $\nu$ with $l(\mu) \leq M/2$ and $l(\nu) \leq N/2$,

$$S_{[\mu+1/2]*[\nu+1/2]\cdot} = \left(\mathrm{Res}^{\mathbf{Pin}_{M+N}}_{\mathbf{Pin}_M * \mathbf{Pin}_N} \Delta_{M+N}\right) \cdot \widetilde{\pi}_{M,N}(s'_{[\mu]} \otimes s'_{[\nu]}).$$

Let $\widetilde{\Delta} : \Lambda[\varepsilon]/(\varepsilon^2-1) \to (\Lambda\otimes\Lambda)[\varepsilon]/(\varepsilon^2-1)$ be the ring homomorphism defined by

$$\widetilde{\Delta}(h_k) = \sum_{i=0}^{k} h_i \otimes h_{k-i}, \quad \widetilde{\Delta}(\varepsilon) = \varepsilon.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc}
\Lambda[\varepsilon]/(\varepsilon^2-1) & \xrightarrow{\widetilde{\Delta}} & (\Lambda\otimes\Lambda)[\varepsilon]/(\varepsilon^2-1) \\
\widetilde{\pi}_{M+N}\downarrow & & \downarrow\widetilde{\pi}_{M,N} \\
\mathrm{Rep}(\mathbf{Pin}_{M+N}) & \xrightarrow[\mathrm{Res}]{} & \mathrm{Rep}(\mathbf{Pin}_M * \mathbf{Pin}_N) \\
\cdot\Delta_{M+N}\downarrow & & \downarrow\cdot\,\mathrm{Res}\,\Delta_{M+N} \\
\mathrm{Rep}(\mathbf{Pin}_{M+N}) & \xrightarrow[\mathrm{Res}]{} & \mathrm{Rep}(\mathbf{Pin}_M * \mathbf{Pin}_N)
\end{array}$$

## Theorem 5

$$s'_{[\lambda]}(X \cup Y) = \sum_{\mu,\nu} \left( \sum_{\kappa} \mathrm{LR}^{\lambda}_{\mu,\nu,\kappa} \, \varepsilon^{|\kappa|} \right) s'_{[\mu]}(X) s'_{[\nu]}(Y),$$

where $\mu$, $\nu$, $\kappa$ run over all partitions.

**Corollary** If $l(\lambda) \leq \min(M/2, N/2)$, then we have

$$\begin{aligned} &\mathrm{Res}^{\mathbf{Pin}_{M+N}}_{\mathbf{Pin}_M * \mathbf{Pin}_N} S_{[\lambda+1/2]} \\ &\quad = \sum_{\mu,\nu} \left( \sum_{\kappa} \mathrm{LR}^{\lambda}_{\mu,\nu,\kappa} \right) \left( E^{[M]}_{M} E^{[N]}_{N} \right)^{|\lambda|-|\mu|-|\nu|} S_{[\mu+1/2]*[\nu+1/2]}, \end{aligned}$$

where $\mu$ and $\nu$ run over all partitions with $l(\mu) \leq M/2$ and $l(\nu) \leq N/2$.