Parabolic Deligne-Lusztig varieties and Broué's conjectures for reductive groups

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Motivation

To simplify we consider just the case of the principal block.

Conjecture (Broué)

If the ℓ-Sylow S of the finite group G is abelian, then the principal ℓ-block B of $\overline{\mathbb{Z}}_\ell G$ is derived-equivalent to the principal block b of $\overline{\mathbb{Z}}_\ell N_G(S)$.

By Rickard's theorem there exists then a *tilting complex* T, a complex in $D^b(B)$ of finitely generated and projective B-modules, such that

- $\mathrm{Hom}_{D^b(B)}(T, T[k]) = 0$ for $k \neq 0$.
- $\mathrm{End}_{D^b(B)}(T) \simeq b$.

For a reductive group in characteristic $p \neq \ell$, T should be the ℓ-adic cohomology complex of some Deligne-Lusztig variety, and $\mathrm{End}_{D^b(B)(T)} \simeq b$ should come through the action on that cohomology of a cyclotomic Hecke algebra associated to $N_G(S)/C_G(S)$, which is a complex reflection group.

Finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over the algebraic closure of a finite field of characteristic p, and F an isogeny such that some power F^δ is a split Frobenius for an $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$; this defines δ and a positive real number q which is a power of $\sqrt{p}$ if $\mathbf{G}$ is simple.
We choose an F-stable pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup;

- the Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ acts as a reflection group on the complex vector space $V = X(\mathbf{T}) \otimes \mathbb{C}$,
- The action of F on V is of the form $q\phi$ where q is a power of $\sqrt{p}$ and $\phi \in N_{\mathrm{GL}(V)}(W)$ is of finite order. The action of $W\phi$ is defined on $X(\mathbf{T}) \otimes \mathbb{Q}_{W\phi}$ where $\mathbb{Q}_{W\phi} = \mathbb{Q}$ except for the Suzuki and Ree groups where $\mathbb{Q}_{W\phi} = \mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$ (we denote $\mathbb{Z}_{W\phi}$ the ring of integers of $\mathbb{Q}_{W\phi}$).
- The polynomial invariants $S(V)^W$ are a polynomial algebra $\mathbb{C}[f_1, \ldots, f_n]$ where $n = \dim V$. We denote by d_i the degree of f_i; they can be chosen eigenvectors of ϕ, and we denote ε_i the corresponding eigenvalues.

Polynomial order

We have $|\mathbf{G}^F| = q^{\sum_{i=1}^n (d_i-1)} \prod_{i=1}^n (q^{d_i} - \varepsilon_i) = q^{\sum_{i=1}^n (d_i-1)} \prod_\Phi \Phi(q)^{|a(\Phi)|}$ *where* Φ *runs over the irreducible cyclotomic polynomials over* $\mathbb{Q}_{W\phi}$, *and* $a(\Phi) = a(\zeta) = \{d_i \mid \zeta^{d_i} = \varepsilon_i\}$, *where* ζ *is a root of* Φ.

In $\mathbb{Q}(\sqrt{2})[x]$ we have $\Phi_8 = x^4 + 1 = (x^2 - \sqrt{2}q + 1)(x^2 + \sqrt{2}q + 1)$. We have the following theorem for an arbitrary complex reflection group W

(Springer)

- $|a(\zeta)|$ *is the dimension of a maximal* ζ*-eigenspace of an element of* $W\phi \subset \mathrm{GL}(V)$.
- *Two maximal* ζ*-eigenspaces are* W*-conjugate.*
- *For a maximal* ζ*-eigenspace* V_ζ, *the group* $W_\zeta := N_W(V_\zeta)/C_W(V_\zeta)$, *acting on* V_ζ, *is a complex reflection group with reflection degrees* $a(\zeta)$. *(Lehrer-Springer)*

Geometric Sylows

Assume that $w\phi \in W\phi$ has an eigenspace of dimension $a = |a(\zeta)|$. Let $(\mathbf{T}_w, F)$ be $\mathbf{G}$-conjugate to $(\mathbf{T}, wF)$. The factor Φ^a of the characteristic polynomial of $w\phi$ defines a wF-stable sublattice of $X(\mathbf{T}) \otimes \mathbb{Z}_{W\phi}$, thus an F-stable subtorus $\mathbf{S}$ of $\mathbf{T}_w$ such that $|\mathbf{S}^F| = \Phi(q)^a$. In the previous situation

We say that $\mathbf{S}$ is a Φ-Sylow.

From the Springer theorem they form a single orbit under $\mathbf{G}^F$-conjugacy.

For $\ell \neq p$ assume that a ℓ-Sylow S of $\mathbf{G}^F$ is abelian. Then $|S|$ divides a unique factor $\Phi(q)^{|a(\Phi)|}$ of $\mathbf{G}^F$, and there is a unique Φ-Sylow $\mathbf{S} \supset S$.

It follows that

- $C_{\mathbf{G}}(\mathbf{S}) = C_{\mathbf{G}}(S)$ is a Levi subgroup $\mathbf{L}$.
- $N_{\mathbf{G}^F}(S)/C_{\mathbf{G}^F}(S) = N_W(V_\zeta)/C_W(V_\zeta) = W_\zeta$, attached to the ζ-eigenspace V_ζ of some element of $W\phi$.

The principal block b of $N_{\mathbf{G}^F}(S)$ is isomorphic to $\overline{\mathbb{Z}}_\ell(S \rtimes W_\zeta)$.

Deligne-Lusztig varieties

Let $\mathbf{P}$ be a parabolic subgroup with F-stable Levi $\mathbf{L}$ and unipotent radical $\mathbf{V}$. The Deligne-Lusztig variety

$$\mathbf{Y_V} = \{g\mathbf{V} \in \mathbf{G}/\mathbf{V} \mid g\mathbf{V} \cap F(g\mathbf{V}) \neq \emptyset\}$$

has a left action of $\mathbf{G}^F$ and a right action of $\mathbf{L}^F$. The (virtual)$\mathbf{G}^F$-module-$\mathbf{L}^F$ given by $\sum_i (-1)^i H^i_c(\mathbf{Y_V}, \overline{\mathbb{Z}}_\ell)$ defines the *Deligne-Lusztig induction* $R_{\mathbf{L}^F}^{\mathbf{G}^F}$. If $F(\mathbf{V}) = \mathbf{V}$ the variety $\mathbf{X_V}$ reduces to the discrete variety $\mathbf{G}^F/\mathbf{V}^F$ and the alternating sum reduces to $\overline{\mathbb{Z}}_\ell[\mathbf{G}^F/\mathbf{V}^F]$, giving *Harish-Chandra induction*.

Conjecture (Geometric version)

There exists $\mathbf{P}$ of Levi $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ such that $R\Gamma_c(\mathbf{Y_V}, \overline{\mathbb{Z}}_\ell)$ considered as an object of $D^b(\overline{\mathbb{Z}}_\ell\mathbf{G}^F \otimes (\overline{\mathbb{Z}}_\ell\mathbf{L}^F)^{opp})$, and restricted to B, is tilting between B and $\overline{\mathbb{Z}}_\ell[S \rtimes W_\zeta]$.

Shadow on unipotent characters

The map $g\mathbf{V} \mapsto g\mathbf{P}$ makes $\mathbf{Y_V}$ an $\mathbf{L}^F$-torsor over the variety

$$\mathbf{X_P} = \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g\mathbf{P} \cap F(g\mathbf{P}) \neq \emptyset\}$$

- For any $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ we have $H^i_c(\mathbf{X_V}, \overline{\mathbb{Z}}_\ell)_\chi = H^i_c(\mathbf{X_P}, \mathcal{F}_\lambda)$.
- The conjecture can be mostly reduced to the study of $\mathbf{X_P}$ with sheaves $\mathcal{F}_\lambda$ associated to unipotent characters. We will look at the case $\chi = \mathrm{Id}$, and further, discard any torsion by going from $\overline{\mathbb{Z}}_\ell$ to $\overline{\mathbb{Q}}_\ell$.

Conjecture (Restricted)

1. $\langle H^i_c(\mathbf{X_P}, \overline{\mathbb{Q}}_\ell), H^j_c(\mathbf{X_P}, \overline{\mathbb{Q}}_\ell)\rangle \mathbf{G}^F = 0$ *for* $i \neq j$.
2. $\mathrm{End}_{\mathbf{G}^F} \oplus_i H^i_c(\mathbf{X_P}, \overline{\mathbb{Q}}_\ell) \simeq \overline{\mathbb{Q}}_\ell W_\zeta$.

A braid monoid attached to the complex reflection group W_ζ acts on $\mathbf{X_P}$ as $\mathbf{G}^F$-endomorphisms, such that on the cohomology the action factors through a cyclotomic Hecke algebra for W_ζ.

Choice of P

Let (W, S) be the Coxeter system associated to the BN-pair $(\mathbf{B}, N_{\mathbf{G}}(\mathbf{T}))$. We can conjugate $\mathbf{P}$ to a standard parabolic subgroup $\mathbf{P}_I$. This conjugates the ζ-eigenspace to a V_ζ such that $C_W(V_\zeta) = W_I$, the Weyl group of $\mathbf{L}_I$. The $w\phi \in W\phi$ with ζ-eigenspace V_ζ form a class $W_I w\phi$. We choose $w\phi$ to be I-reduced.

$\mathbf{X_P}$ *is isomorphic to* $\{\mathbf{P} \mid \mathbf{P} \xrightarrow{I,w,{}^\phi I} {}^F\mathbf{P}\}$ *which means that* $(\mathbf{P}, {}^F\mathbf{P}) \sim_{\mathbf{G}} (\mathbf{P}_I, {}^w\mathbf{P}_{\phi I})$ *(we have* ${}^{w\phi}I = I$*).*

We denote this variety $\mathbf{X}(I \xrightarrow{w} {}^\phi I)$.
The choice of a parabolic subgroup with Levi $C_{\mathbf{G}}(\mathbf{S})$ corresponds to the choice of a class $W_I w\phi$ up to W-conjugacy, or to the choice of an I-reduced element w such that ${}^{w\phi}I = I$ up to W-conjugacy of such pairs (w, I); for such an element we have $\dim \mathbf{X}(I \xrightarrow{w} {}^\phi I) = l(w)$.

Craven's formula

Block theory and the work of Rouquier and Craven in constructing "perverse equivalences" led to a very specific conjecture for the cohomology of the variety $\mathbf{X_P}$ we are looking for.
Let ρ be a unipotent character which occurs in $H^i(\mathbf{X_P}, \mathcal{F}_\lambda)$, where $\mathbf{P}$ is the "right" parabolic subgroup of Levi $C_\mathbf{G}(\mathbf{S})$, where $\mathbf{S}$ is a Φ-Sylow. Choose ζ as the root of Φ with minimal argument and write $\zeta = e^{2ik\pi/d}$. Let $P = \deg\rho/\deg\lambda$, a polynomial in q.
Then

Conjecture (Craven)

$$i = k/d(\mathrm{degree}(P) + \mathrm{valuation}(P)) + \{\textit{number of roots of } P \textit{ of argument less than that of } \zeta\} - 1/2\{\textit{number of times 1 is a root of } P\}$$

Further, we should have $\dim \mathbf{X_P} = 2k/d(l(w_0) - l(w_I))$ where w_0 (resp. w_I) is the longest element of W (resp W_I).

$\mathbf{G}^F$-endomorphisms of $\mathbf{X}(I \xrightarrow{w} {}^\phi I)$

The idea for constructing $\mathbf{G}^F$-endomorphisms of $\mathbf{X}(I \xrightarrow{w} {}^\phi I)$ is:

- If $w = xy$ with $l(w) = l(x) + l(y)$,
- and $I^x = J \subset S$,

when $\mathbf{P} \xrightarrow{I,w,{}^\phi I} {}^F\mathbf{P}$ there is a unique $\mathbf{P}'$ such that $\mathbf{P} \xrightarrow{I,x,J} \mathbf{P}' \xrightarrow{J,y,{}^\phi I} {}^F\mathbf{P}$.

- If we have also $l(y\phi(x)) = l(y) + l(\phi(x))$,

then since $\mathbf{P}' \xrightarrow{J,y,{}^\phi I} F(\mathbf{P}) \xrightarrow{{}^\phi I,\phi(x),{}^\phi J} F(\mathbf{P}')$, we have $\mathbf{P}' \in \mathbf{X}(J \xrightarrow{y\phi(x)} {}^\phi J)$, thus $\mathbf{P} \mapsto \mathbf{P}'$ defines a map $\mathbf{X}(I \xrightarrow{w} {}^\phi I) \xrightarrow{D_x} \mathbf{X}(J \xrightarrow{y\phi(x)} {}^\phi J)$ which is $\mathbf{G}^F$-equivariant.

- If in addition $I^x = I$
- and x commutes to $w\phi$

we get an endomorphism. There are too many conditions so this do not construct enough endomorphisms.

The ribbon category

If $W = \langle S \mid s^2 = 1, \underbrace{st\ldots}_{m_{s,t}} = \underbrace{ts\ldots}_{m_{s,t}} \text{ for } s,t \in S\rangle$

The braid monoid is $B^+ = \langle \mathbf{S} \mid \underbrace{\mathbf{st}\ldots}_{m_{s,t}} = \underbrace{\mathbf{ts}\ldots}_{m_{s,t}} \text{ for } \mathbf{s},\mathbf{t} \in \mathbf{S}\rangle$.

There is a natural section $w \mapsto \mathbf{w} : W \xrightarrow{\sim} \mathbf{W}$ obtained by replacing each s by $\mathbf{s}$ in a reduced expression of w.

Let $\mathcal{I}$ be the set of conjugates in $\mathbf{S}$ of $\mathbf{I} \subset \mathbf{S}$. We define a category $B(\mathcal{I})$ whose objects are the elements of $\mathcal{I}$, and morphisms $\mathbf{I} \xrightarrow{\mathbf{b}} \mathbf{J}$ are $\mathbf{b} \in B^+$ such that

- $\mathbf{I}^{\mathbf{b}} = \mathbf{J}$.
- No element of $\mathbf{I}$ divides $\mathbf{b}$ on the left (we say $\mathbf{b}$ is $\mathbf{I}$-reduced).

Varieties associated to ribbons

The morphisms of $B(\mathcal{I})$ are generated by those where $\mathbf{b} \in \mathbf{W}$.

To the variety $\mathbf{X}(I \xrightarrow{w} {}^{\phi}I)$ we associate the map $\mathbf{I} \xrightarrow{\mathbf{w}} {}^{\phi}\mathbf{J}$.

Conversely, to a map $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{J} = \mathbf{I} \xrightarrow{\mathbf{w}_1} \mathbf{I}_1 \ldots \mathbf{I}_{n-1} \xrightarrow{\mathbf{w}_n} {}^{\phi}\mathbf{I}$ with $\mathbf{w}_i \in \mathbf{W}$ we associate the variety $\{\mathbf{P}, \mathbf{P}_1, \ldots \mathbf{P}_n \mid \mathbf{P} \xrightarrow{I,w,I_1} \mathbf{P}_1 \ldots \mathbf{P}_n \xrightarrow{I_{n-1},w_n,{}^{\phi}I} {}^{F}\mathbf{P}\}$.

By extending to the category $B(\mathcal{I})$ a theorem of Deligne on representations of the braid monoid in a category, one can show that there is a canonical isomorphism between the varieties attached to two decompositions of $\mathbf{b}$. This allows to attach "parabolic Deligne-Lusztig varieties" $\mathbf{X}(\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I})$ to morphisms in $B(\mathcal{I})$.

Now, whenever we have a divisor $\mathbf{x}$ of $\mathbf{b}$ in $B(\mathcal{I})$ there is a well-defined morphism $\mathbf{X}(\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}) \xrightarrow{D_{\mathbf{x}}} \mathbf{X}(\mathbf{I}^{\mathbf{x}} \xrightarrow{\mathbf{x}^{-1}\mathbf{b}^{\phi}\mathbf{x}} {}^{\phi}\mathbf{I}^{\mathbf{x}})$.

Let $\mathcal{D}(\mathcal{I})$ be the category with objects the morphisms of $B(\mathcal{I})$ and morphisms compositions of the $D_{\mathbf{x}}$.

In $\mathcal{D}(\mathcal{I})$ there will be enough endomorphisms of $\mathbf{I} \xrightarrow{\mathbf{b}} {}^{\phi}\mathbf{I}$.

Eigenspaces and roots in the braid group

For any finite Coxeter group W, with diagram automorphism ϕ.

Proposition (Digne-M., He-Nie)

Let $\zeta = e^{2ik\pi/d}$, where $2k \leq d$ (k prime to d). Let $V_\zeta \subset V$ be a subspace on which $w\phi \in W\phi$ acts by ζ. Then, up to W-conjugacy we have

- $C_W(V_\zeta) = W_I$ *for some* $I \subset S$ *(thus* ${}^{w\phi}I = I$*).*
- *For the I-reduced element w the lift $\mathbf{w}$ to the braid monoid satisfies* $(\mathbf{w}\phi)^d = \phi^d(\mathbf{w}_0^2/\mathbf{w}_I^2)^k$

If w is as above, we have $l(w) = 2k/d(l(w_0) - l(w_I))$, the length predicted in Craven's formula.

The theorems of He and Nie

Given an element $w\phi$ with eigenvalue $\zeta = e^{2ik\pi/d}$, the proposition gives a conjugate $vw_1\phi$ where

- $l(w_1) = 2\frac{k}{d}(l(w_0 - l(w_I))$.
- $w_1\phi$ gives a diagram automorphism of $W_I = C_W(V_\zeta)$.
- $v \in W_I$.

If we pick another eigenvalue, we can apply again the proposition to the element $vw_1\phi$ of the coset $W_I w_1\phi$.
Let $\theta_0 < \theta_1 \ldots < \theta_r$ be the arguments $\leq \pi$ of eigenvalues of $w\phi$.

Theorem (He and Nie)

- *If we apply the proposition taking the θ_i in increasing order, we end up with an element of minimal length in the conjugacy class of $w\phi$.*
- *If we take the θ_i in decreasing order, we end up with an element of maximal length in the conjugacy class of $w\phi$.*

Further they show the lifts in $\mathbf{W}$ of minimal length (resp. maximal length) elements in the class are conjugate in B.

Roots

We have a kind of converse

Let $\mathbf{w} \in B^+$ *and* d *such that* $(\mathbf{w}\phi)^d = \phi^d(\mathbf{w}_0^2/\mathbf{w}_\mathbf{I}^2)$ *for some* ϕ^d*-stable* $\mathbf{I} \subset \mathbf{S}$*. Then*

- ${}^{\mathbf{w}\phi}\mathbf{I} = \mathbf{I}$, *thus* $\mathbf{w}$ *defines a morphism* $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}) \in B(\mathcal{I})$*.*
- *Let* V_d *be the* $\zeta_d = e^{2i\pi/d}$*-eigenspace of* $w\phi$*. Then* $C_W(V) \subset W_I$

Further, the following conditions are equivalent

- $\mathbf{w}\phi$ *is "not extendible", that is there does not exist a* ϕ^d*-stable* $\mathbf{J} \subset \mathbf{I}$ *and* $\mathbf{v} \in B_\mathbf{I}^+$ *such that* $(\mathbf{vw}\phi)^d = \phi^d(\mathbf{w}_0^2/\mathbf{w}_\mathbf{J}^2)$*.*
- $C_W(V) = W_I$*, and* V_d *is a maximal* ζ_d*-eigenspace of* $W\phi$*.*

Varieties for roots

The previous results suggest that one should take varieties attached to roots. One more result

Assume that $(\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}) \in B(\mathcal{I})$ *is such that some power of* $(\mathbf{w}\phi)^d\phi^{-d}$ *is divisible by* $\mathbf{w}_0/\mathbf{w}_\mathbf{I}$*. Then* $\mathrm{End}_{\mathcal{D}(\mathcal{I})}(\mathbf{I} \xrightarrow{\mathbf{w}} {}^\phi\mathbf{I}) = \{D_\mathbf{x} \mid \mathbf{x} \in C_{B^+}(\mathbf{w}\phi)\}$ *(and the other conditions:* $\mathbf{I}^\mathbf{x} = \mathbf{I}$ *and* $\mathbf{x}$ *is* $\mathbf{I}$*-reduced).*

When $\mathbf{I} = \emptyset$ the above is $C_{B^+}(\mathbf{w}\phi)$ and by Lusztig (case by case) and He and Nie the morphism $C_{B^+}(\mathbf{w}\phi) \to C_W(w\phi)$ is surjective. When $(\mathbf{w}\phi)^d = \phi^d\mathbf{w}_0^2$ then $W_\zeta = C_W(w\phi)$. Thus we get closer to have enough endomorphisms.

Conjecture

$\{\mathbf{x} \in C_{B^+}(\mathbf{w}\phi) \mid \mathbf{I}^\mathbf{x} = \mathbf{I}$ *and* $\mathbf{x}$ *is* $\mathbf{I}$*-reduced*$\}$ *is a monoid for the braid group of* W_ζ*.*

This is true by the work of David Bessis when $\mathbf{I} = \emptyset$ and $\phi = \mathrm{Id}$.

The variety $\mathbf{X}_\pi$

The conjectures on the cohomology are already interesting for $\zeta = 1$. This corresponds to the case $\ell | q - 1$, and to the variety $\mathbf{X}(\emptyset \xrightarrow{\mathbf{w}_0^2} \emptyset)$. We set $\boldsymbol{\pi} = \mathbf{w}_0^2$; the variety is

$$\mathbf{X}_\pi = \{\mathbf{B}_1, \mathbf{B}_2 \mid \mathbf{B}_1 \xrightarrow{w_0} \mathbf{B}_2 \xrightarrow{w_0} F(\mathbf{B}_1)\}$$

If $\mathbf{B} = \mathbf{UT}$ is the Levi decomposition of $\mathbf{B}$, by Puig(1985) there is a Morita equivalence through $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{U}^F] = H_c^*(\mathbf{Y_U})$ between the principal block of $\mathbf{G}^F$ and that of $N_{\mathbf{G}^F}(\mathbf{T})$. This uses the isomorphism with the ordinary Hecke algebra $W^\phi \simeq H_q(W^\phi) = \mathrm{End}_{\mathbf{G}^F}(H_c^*(\mathbf{X_B}))$.

The conjecture says instead to consider $\mathbf{X}_\pi$, *i.e.* that its cohomology is only in even degrees with $\mathbf{G}^F$-endomorphisms a graded version of $H_q(W^\phi)$.

(Broué-M. 1995)

$(B+)^\phi$ acts on $\mathbf{X}_\pi$, factoring on $H_c^*(\mathbf{X}_\pi, \overline{\mathbb{Q}}_\ell)$ through $H_q(W^\phi)$.

The variety $\mathbf{X}_\pi$ (continued)

(Digne-M. 2005)

For GL_n and small rank groups, for $T \in H_q(W^\phi)$

- $\sum_i (-1)^i \,\mathrm{Trace}(T \mid H_c^i(\mathbf{X}_\pi, \overline{\mathbb{Q}}_\ell))$ *is the canonical trace of* $H_q(W^\phi)$.

This means that the above virtual module is isomorphic as a representation of $\mathbf{G}^F \times H_q(W^\phi)$ to $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbf{B}^F]$.
Actually the isomorphism is Galois-twisted; for irrational characters of $H_q(E_7)$ and $H_q(E_8)$ the correspondence with unipotent characters is through the specialization $q^{1/2} \mapsto -1$ (instead of $q^{1/2} \mapsto 1$ for the correspondence in $\overline{\mathbb{Q}}_\ell[\mathbf{G}^F/\mathbb{F}]$).

- In [Digne-M.-Rouquier] we prove that the cohomology of $\mathbf{X}_\pi$ is concentrated in even degree for groups $\mathbf{G}$ of rank 1 or 2.

By Craven's formula a character χ_q of the Hecke algebra should occur in $H_c^{4l(w_0)-2A_\chi}(\mathbf{X}_\pi)$ where A_χ is the degree of the generic degree.

Dudas on GL_n

Proposition (Dudas)

Let $\mathbf{G} = \mathrm{GL}_n$ and assume

- *The cohomology of $\mathbf{X}_\pi$ is concentrated in even degrees.*

then the geometric version of the Broué conjectures hold for GL_n over $\overline{\mathbb{Q}}_\ell$.

That is, Dudas proves that for every $\zeta = e^{2i\pi/d}$ and any unipotent sheaf $\mathcal{F}_\lambda$ on the associated variety, the cohomology is as predicted by Craven's formula.

If $\mathbf{X}_{n,d}$ is the variety in GL_n associated to a $e^{2i\pi/d}$, he does it by relating the cohomology of $\mathbf{X}_{n,d}$ to that of $\mathbf{X}_{n-1,d-1}$ and $\mathbf{X}_{n-1,d}$.

The extreme cases needed for the induction are $\mathbf{X}_{n,n+1}$ which is the Coxeter variety whose cohomology is known by Lusztig, and $\mathbf{X}_{n,1}$ which is $\mathbf{X}_\pi$.