

Games & Algorithms with Hook Structures

Shoji Conference, March 2012

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Most part of this talk

to appear in :

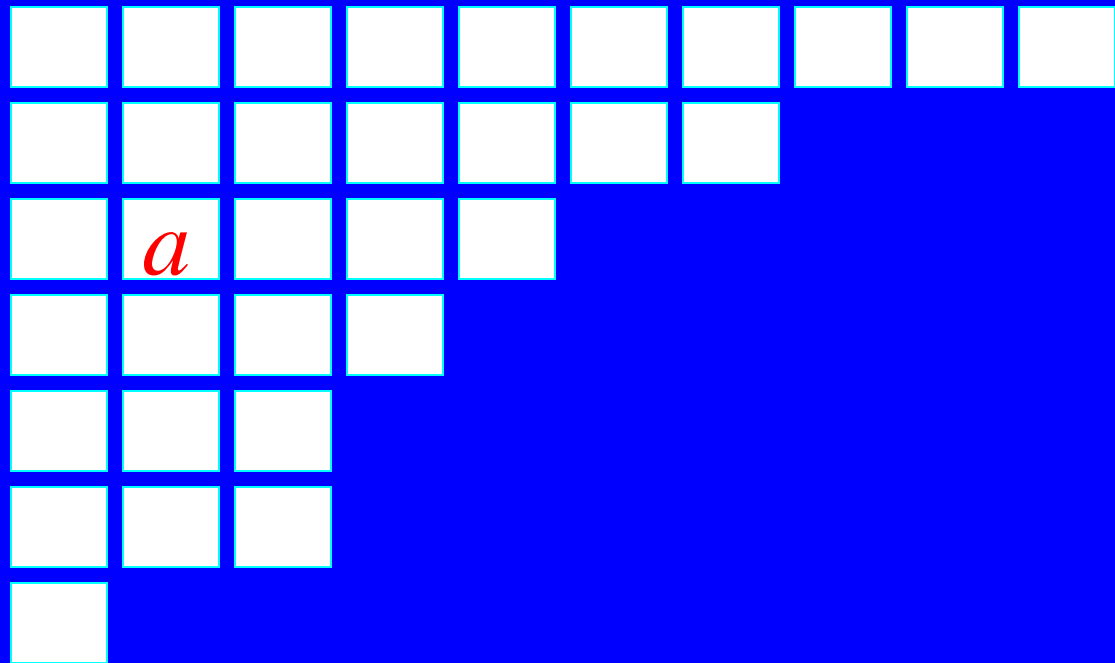
Sugaku Expositions (AMS)

(appeared in :

Sugaku (Iwanami))

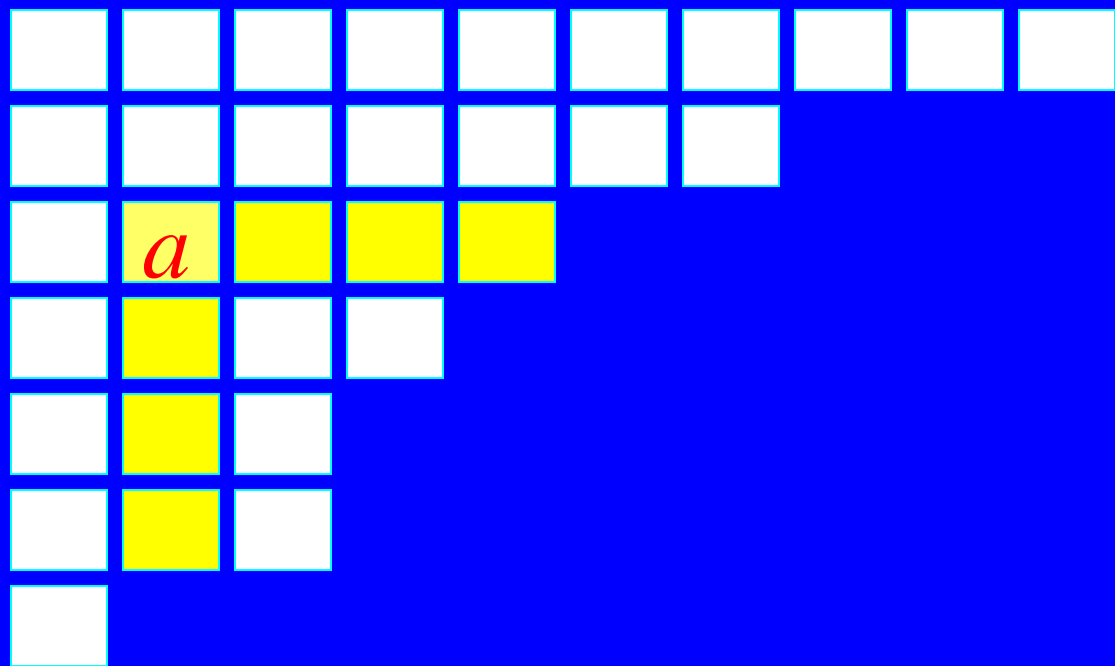
Young diagram Y

$Y =$

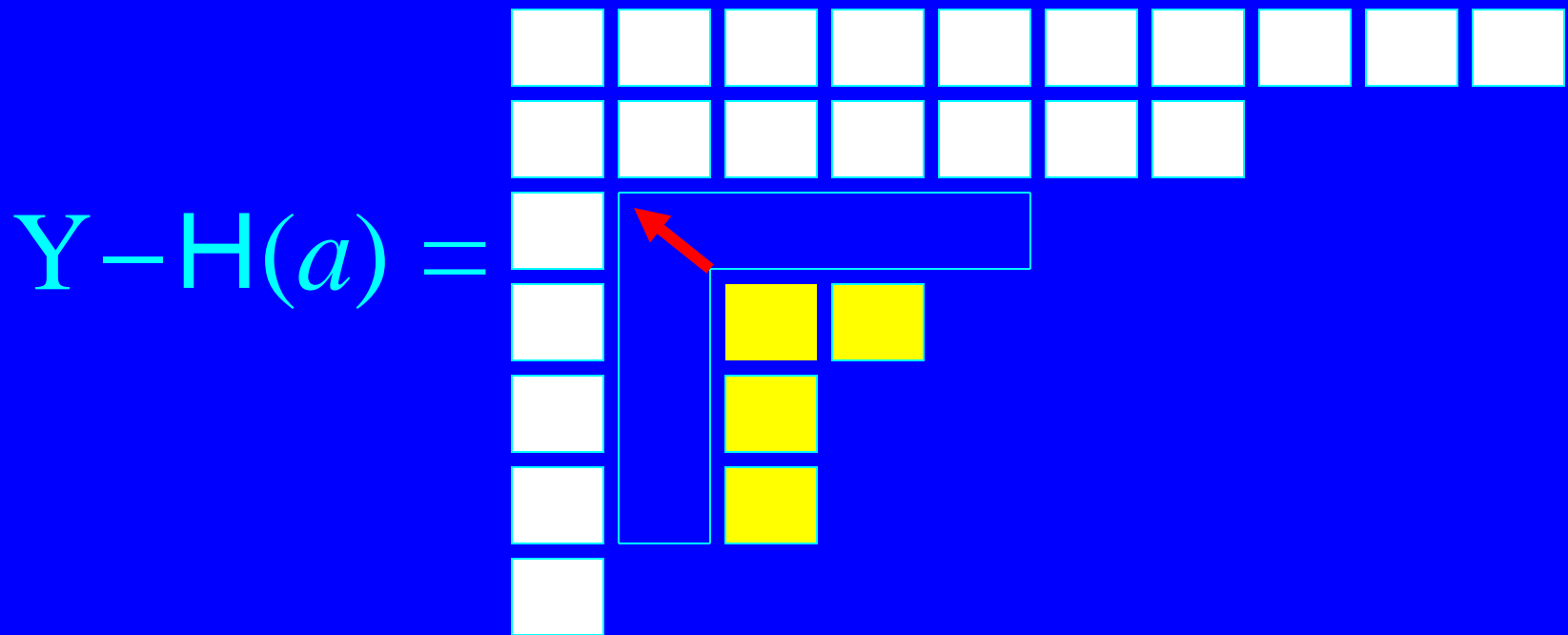


the hook $H(a)$ of $a \in Y$

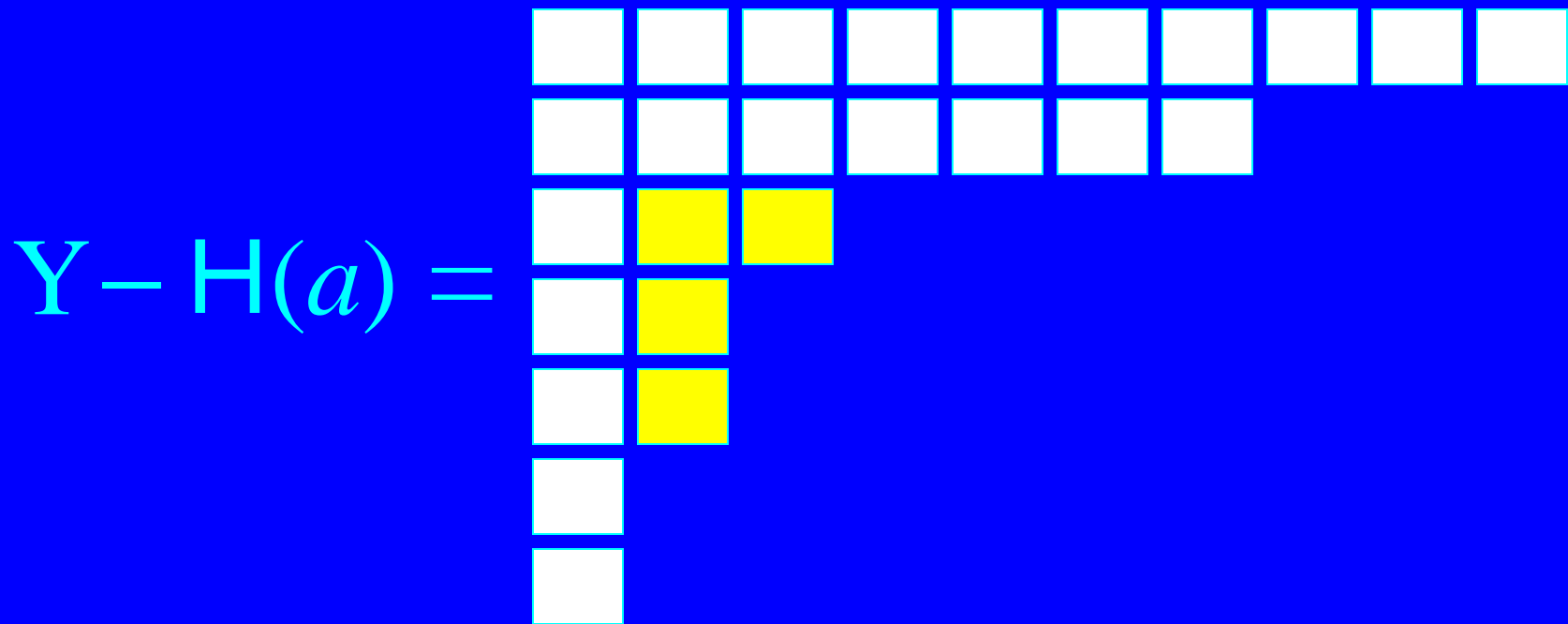
$Y =$



Subtraction $Y-H(a)$ (1)

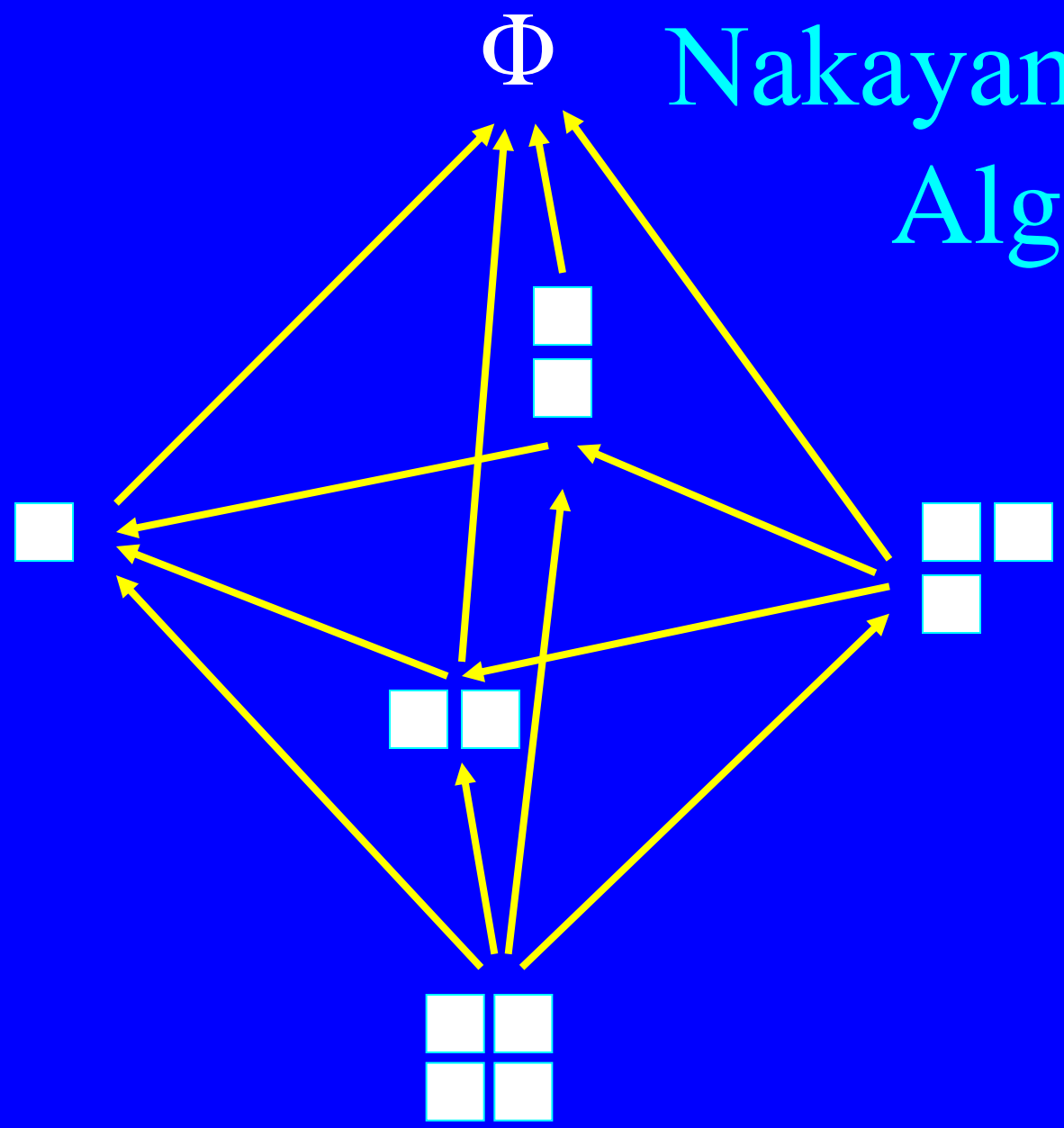


Subtraction $Y-H(a)$ (2)



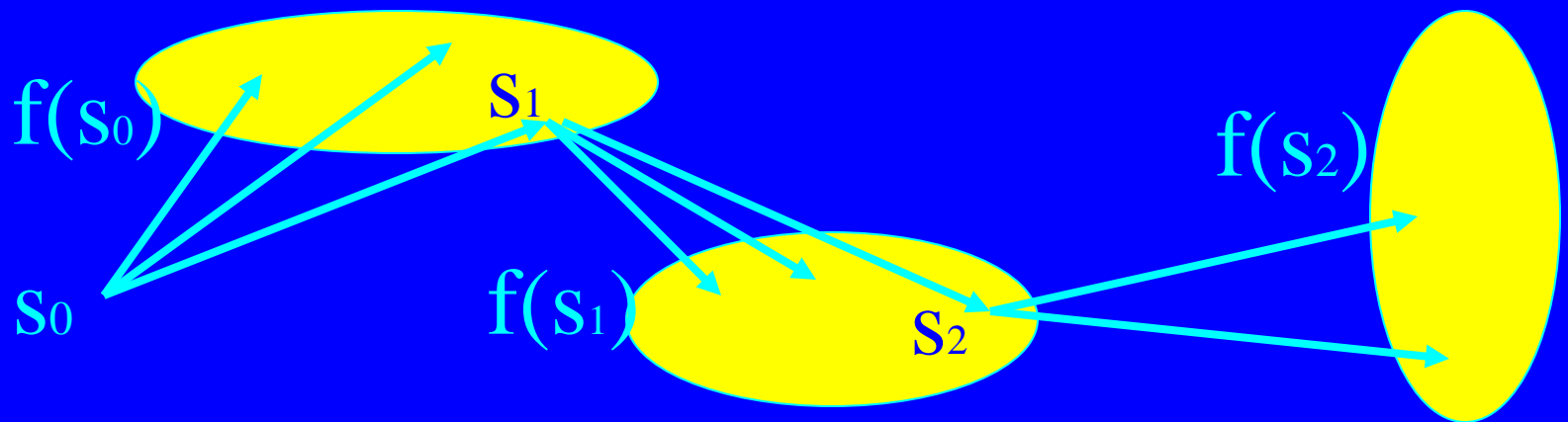
Introduced by Nakayama 1940

Nakayama's Algorithm



$(S, f) = (\text{abstract})$ algorithm

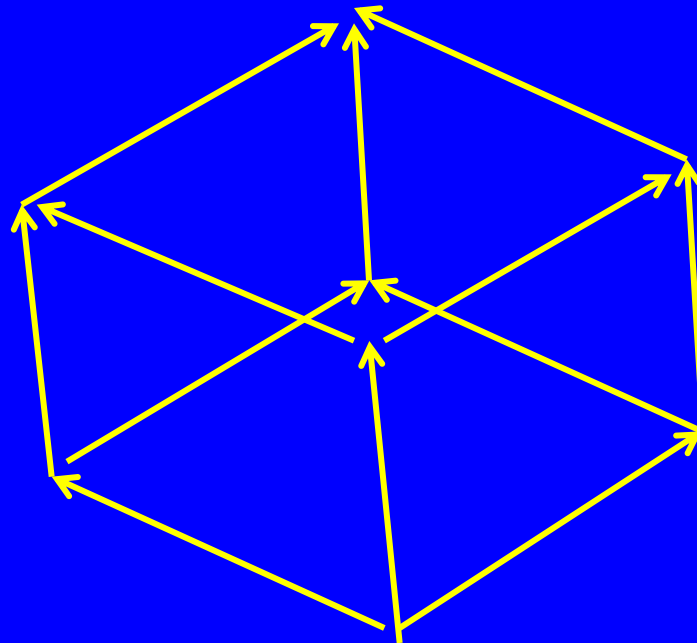
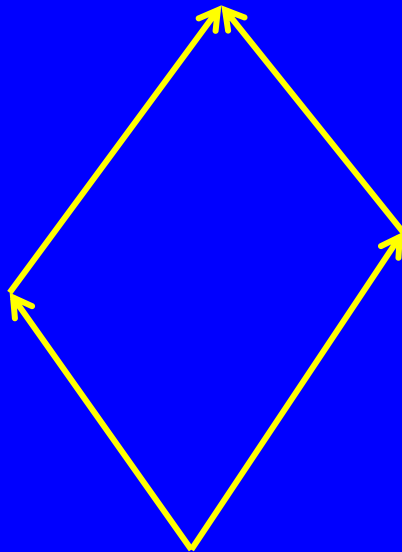
$S : \text{set}, \quad f : S \rightarrow 2^S$



n-cube

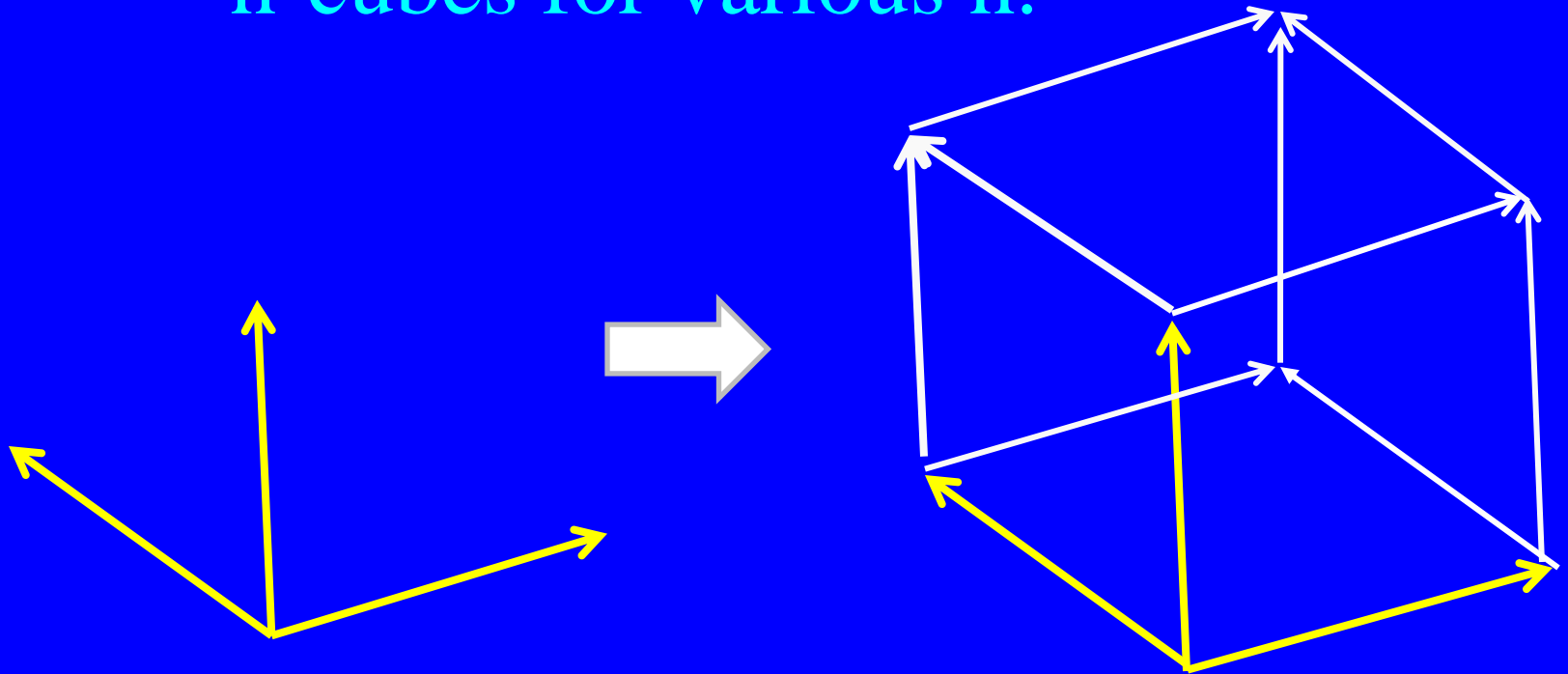
square = 2-cube

3-cube ...



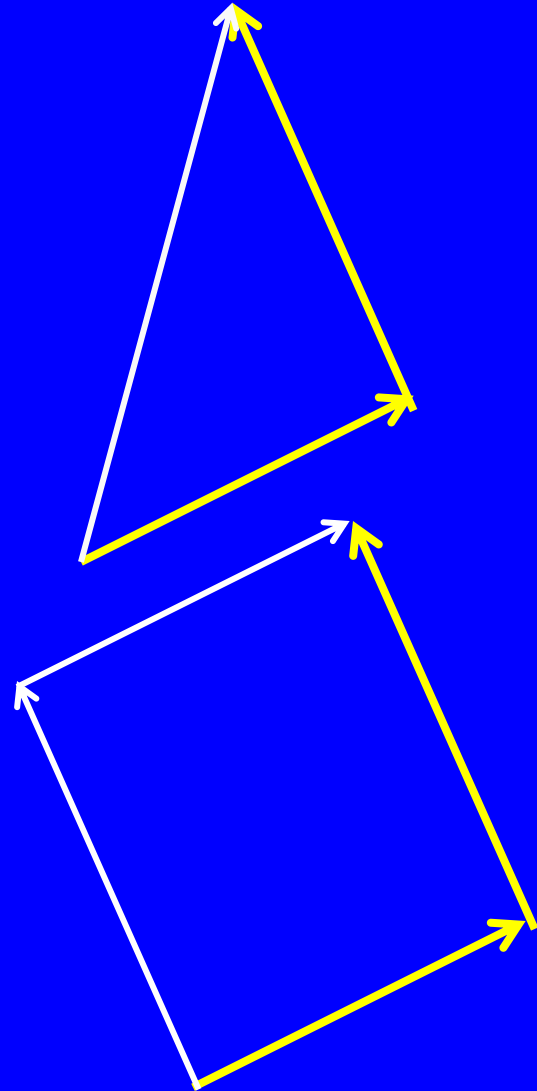
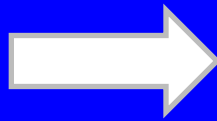
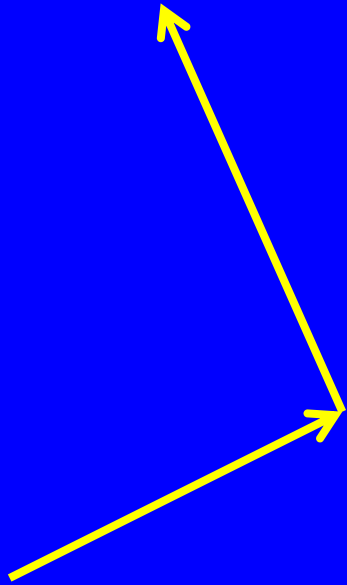
Plain algorithm (1)

(P1) A plain algorithm contains a lot of n-cubes for various n.



Plain algorithm (2)

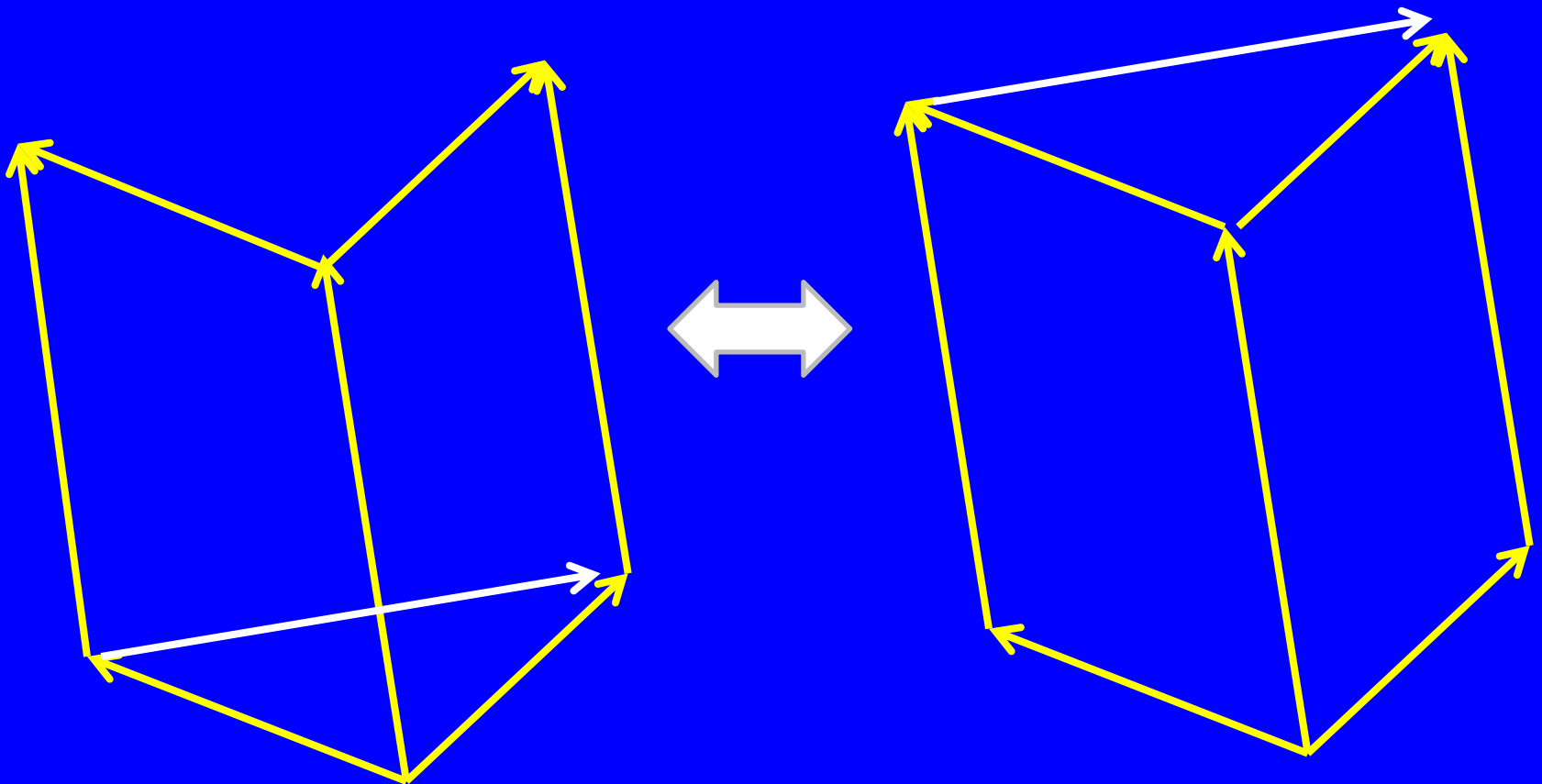
(P2)



2 consecutive arrows can
be completed in 2 ways.

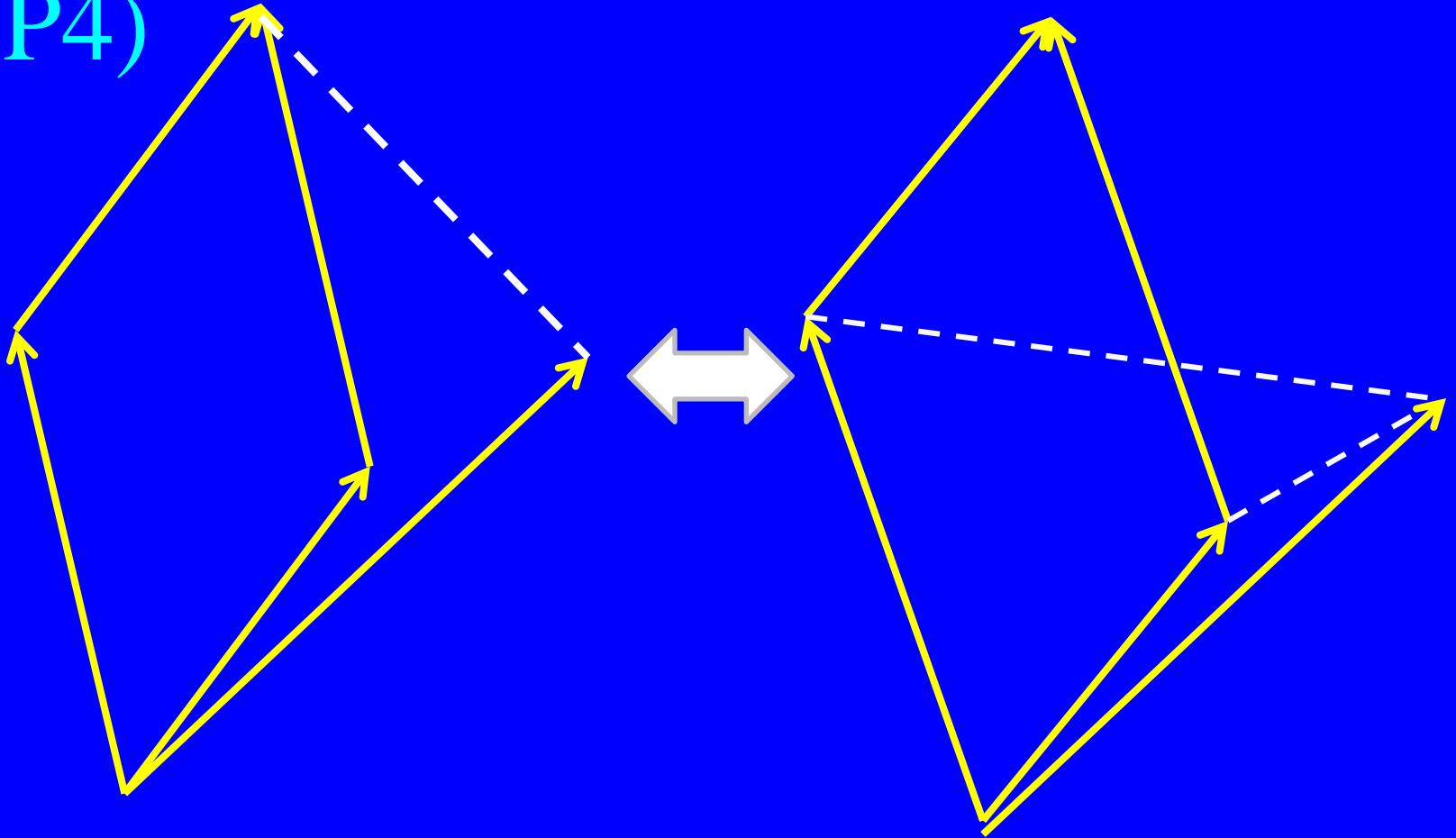
Plain algorithm (3)

(P3)



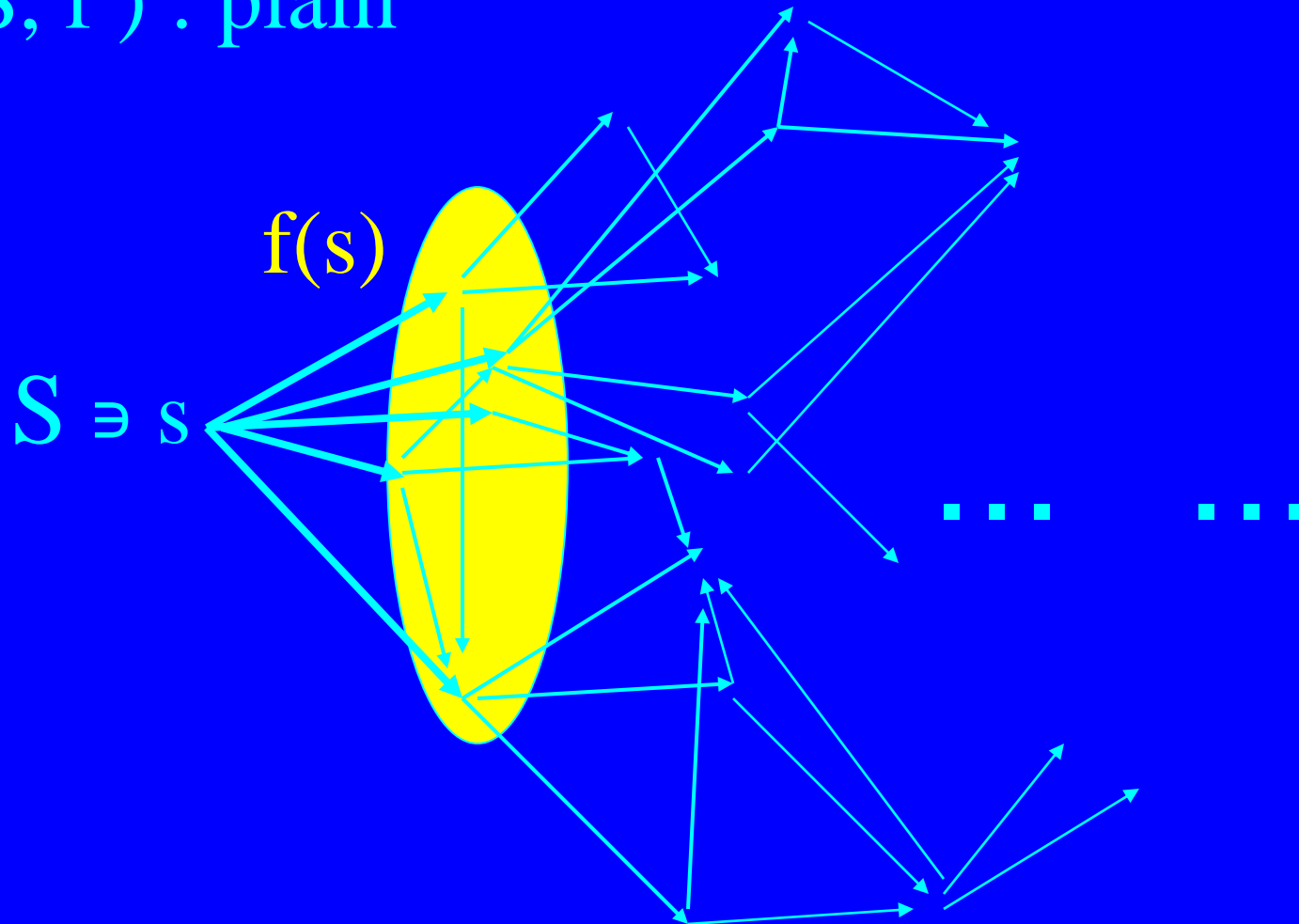
Plain algorithm (4)

(P4)



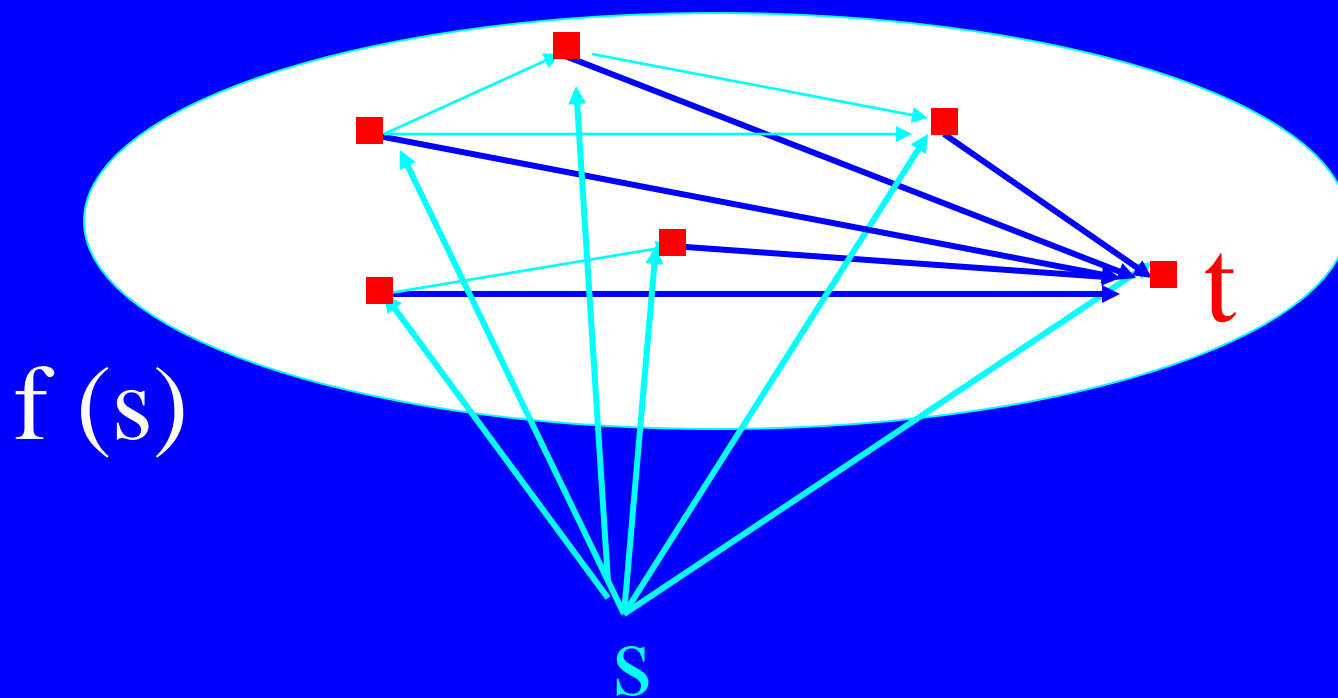
$f(s) = \text{diagram at } s$

$(S, f) : \text{plain}$

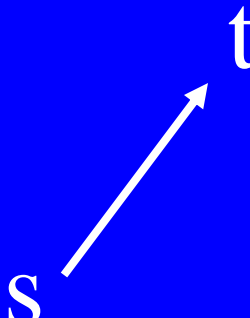


$$H_s(t) = \{f(s) \cap f^{-1}(t)\} \cup \{t\}$$

the hook of t



subtraction of a hook

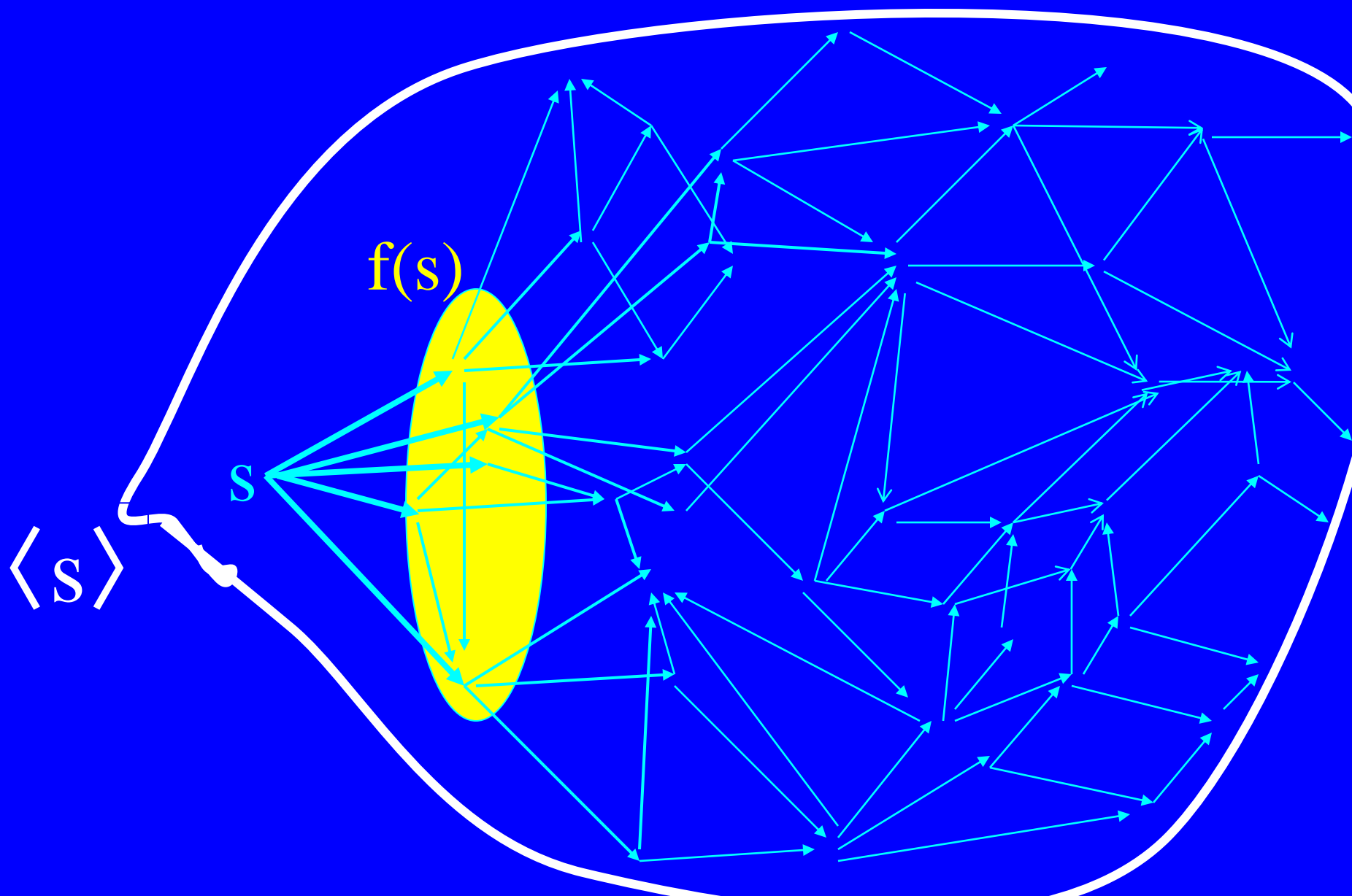
If , then

$$f(t) \cong f(s) - H_s(t).$$

(Each arrow represents
a hook subtraction process.)

Fundamental Theorem of Plain Algorithms

The diagram $f(s)$ equipped
with the hook structure
knows everything
about $\langle s \rangle$.



In what follows, we assume:

(S, f) is a plain algorithm
such that :

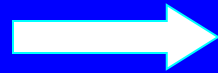
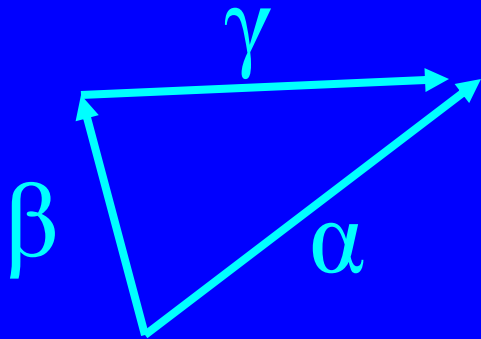
$$|f(s)| < \infty$$

for any $s \in S$.

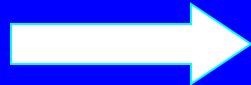
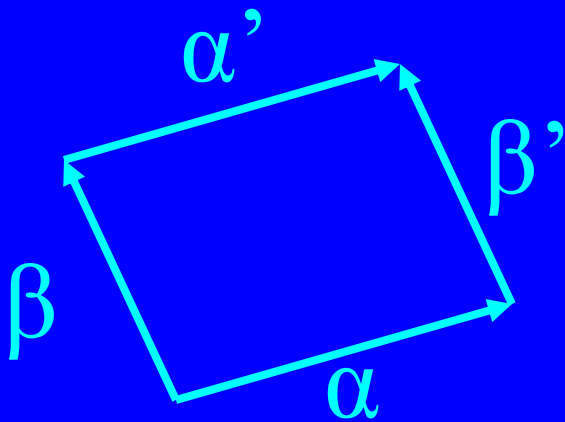
Hook Formula (1)

With an arrow $\alpha = (s \rightarrow t)$,
we associate an element
 $F(\alpha)$ of a field K
in such a way that :

Hook Formula (2)



$$F(\alpha) = F(\beta) + F(\gamma)$$



$$F(\alpha) = F(\alpha')$$

$$F(\beta) = F(\beta')$$

Hook Formula (3)

With a path of finite length

$$p = (s \xrightarrow{\alpha} t \xrightarrow{\beta} u \xrightarrow{\gamma} \cdots),$$

we associate

$$F(p) = F(\alpha) + F(\beta) + F(\gamma) + \cdots$$

($F(p) = 1$ if p is of length 0.)

Hook Formula (4)

Let $v \in \langle s \rangle$, $\langle s \setminus v \rangle = \{\langle s \rangle \setminus \langle v \rangle\} \cup \{v\}$,

$P(s \setminus v) = \{s \rightarrow t \rightarrow \cdots \text{ in } \langle s \setminus v \rangle\}$

$$\rightarrow \sum_{p \in P(s \setminus v)} F(p)^{-1} = \prod_{x \in f(s) \setminus f(v)} (1 + F(s \rightarrow x))^{-1}$$

: a skew version of Nakada's formula

(Osaka J.Math. 2008)

(generalized) Sato's game

(Rule of the game)

Starting from a given diagram,

2 players alternatively subtract a hook.

A player who produces the empty diagram Φ is a winner.

An equivalent game is known as Welter's game.

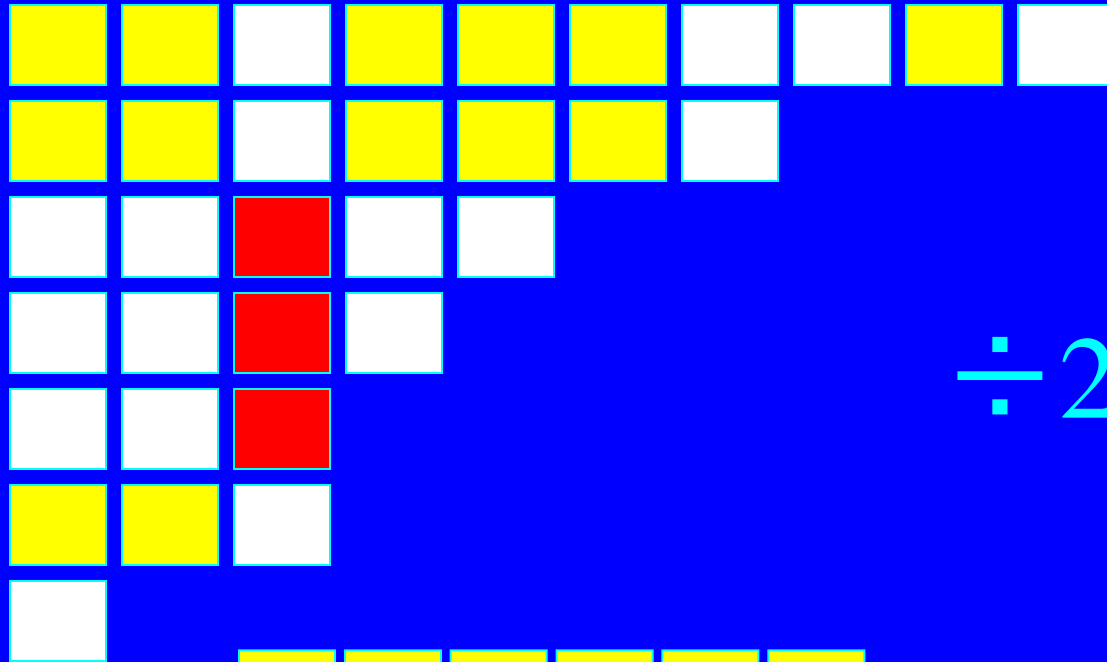
(J.H. Conway : On Numbers and Games, Ch.13.)

2-adic expansion of Y (1)

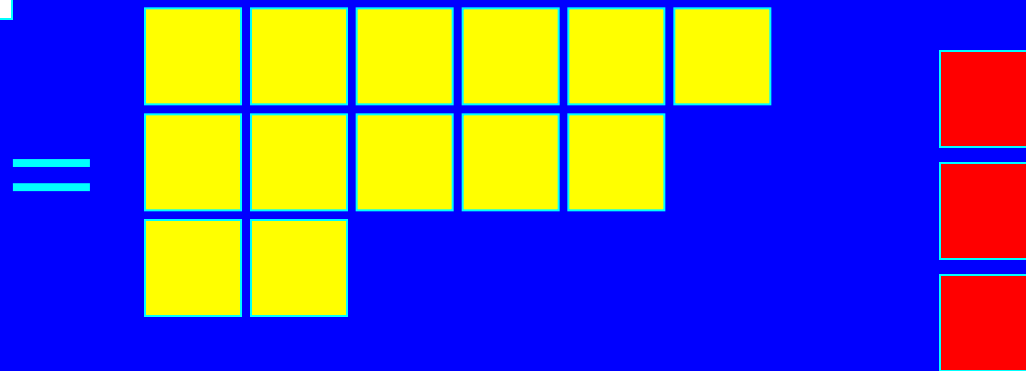
$$Y = \begin{array}{cccccccccccc} \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & & & & & & \\ \square & \square & \square & \square & \square & & & & & & & & \\ \square & \square & \square & \square & & & & & & & & & \\ \square & \square & \square & & & & & & & & & & \\ \square & \square & \square & & & & & & & & & & \\ \square & & & & & & & & & & & & \\ & & & & & & & & & & & & \end{array} \equiv 1 \pmod{2}$$

$\therefore e_0 = 1$

The quotient of

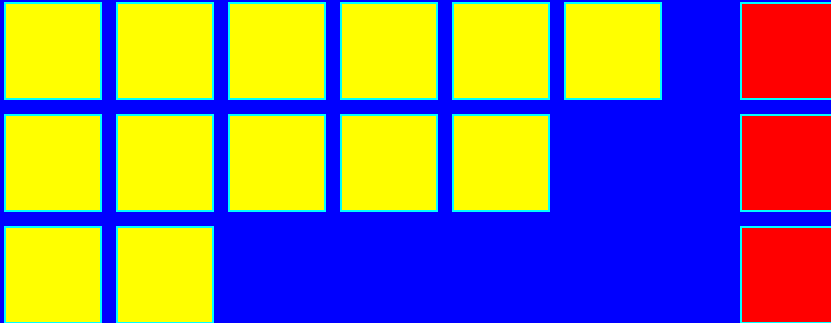


$\div 2$



introduced by G. de Robinson

2-adic expansion of Y (2)

The quotient
of $Y \div 2$ = 

$$\equiv 0 \pmod{2}$$

$$\therefore e_1 = 0$$

2-adic expansion of Y (3)

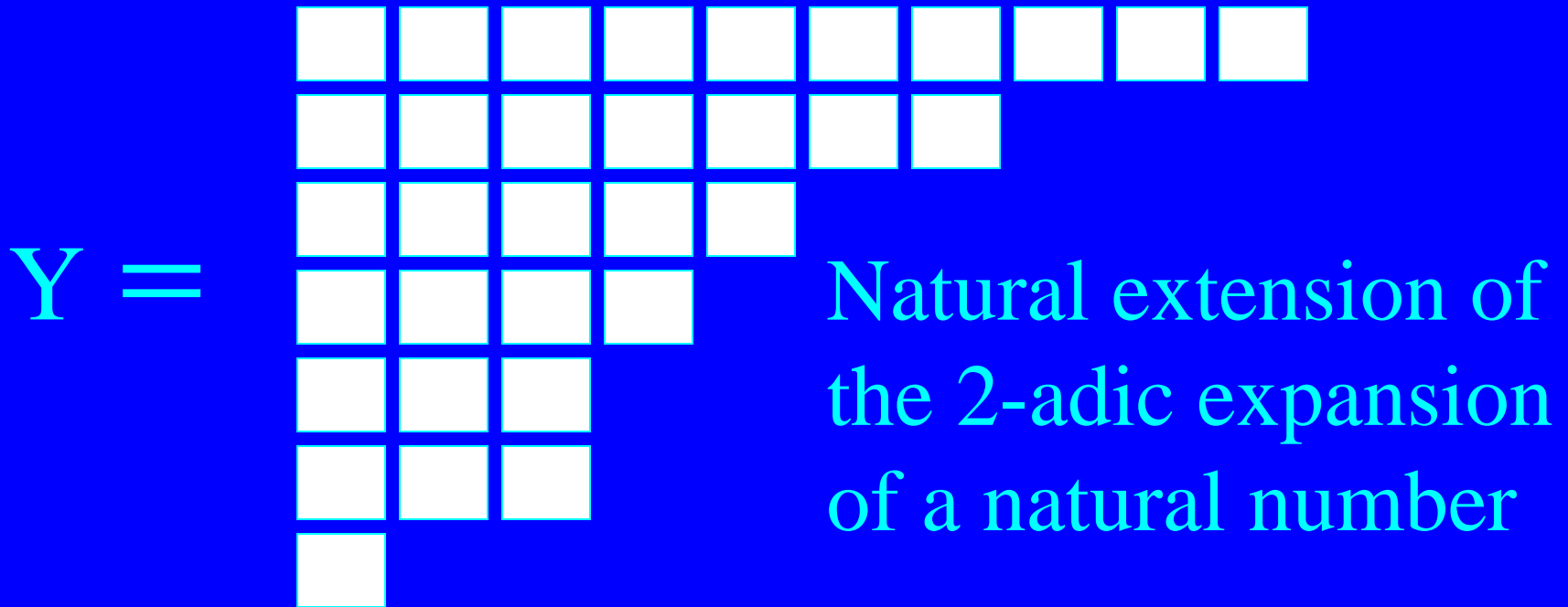
the quotient of

$$\left(\begin{array}{cccccc|c} \text{yellow} & \text{white} & \text{white} & \text{yellow} & \text{white} & \text{white} & \text{white} \\ \text{yellow} & \text{white} & \text{white} & \text{yellow} & \text{white} & & \text{red} \\ \text{yellow} & \text{white} & & & & & \text{white} \end{array} \right) \div 2$$

$$= \begin{array}{cc} \text{yellow} & \text{yellow} \\ \text{yellow} & \text{yellow} \\ \text{yellow} & \end{array} \begin{array}{c} \text{red} \\ \end{array} \equiv 0 \pmod{2}$$

$$\therefore e_2 = 0$$

2-adic expansion of Y (4)



$$E(Y) = \cdots e_4 e_3 e_2 e_1 e_0 = \cdots 010001$$

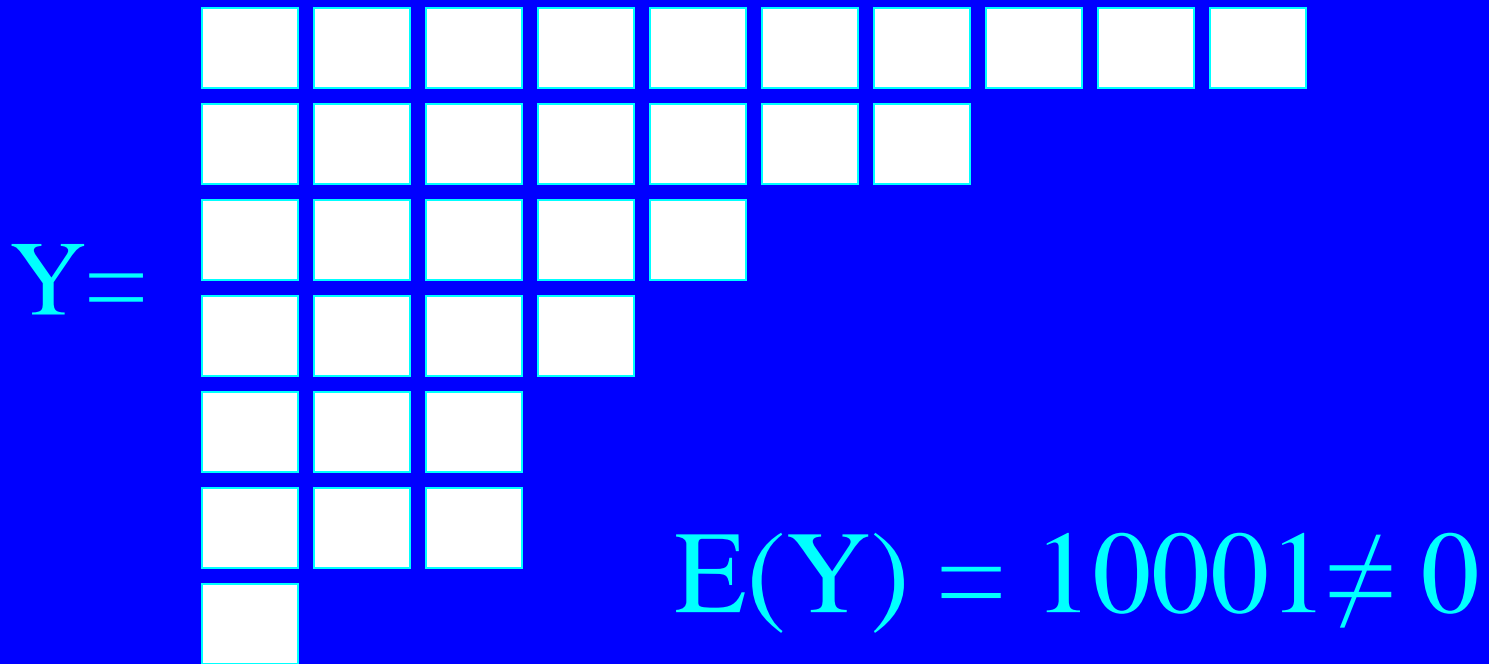
(generalized) Sato's game (2)

This game can be analyzed completely :

The diagram Y is a winning position
for the 2nd player (resp. 1st player)

 $E(Y) = 0$ (resp. $E(Y) \neq 0$)

Find out good moves.



Klein's 4-group G

$$G = \langle A, B \rangle \quad AB = BA$$

$$A^2 = B^2 = E$$

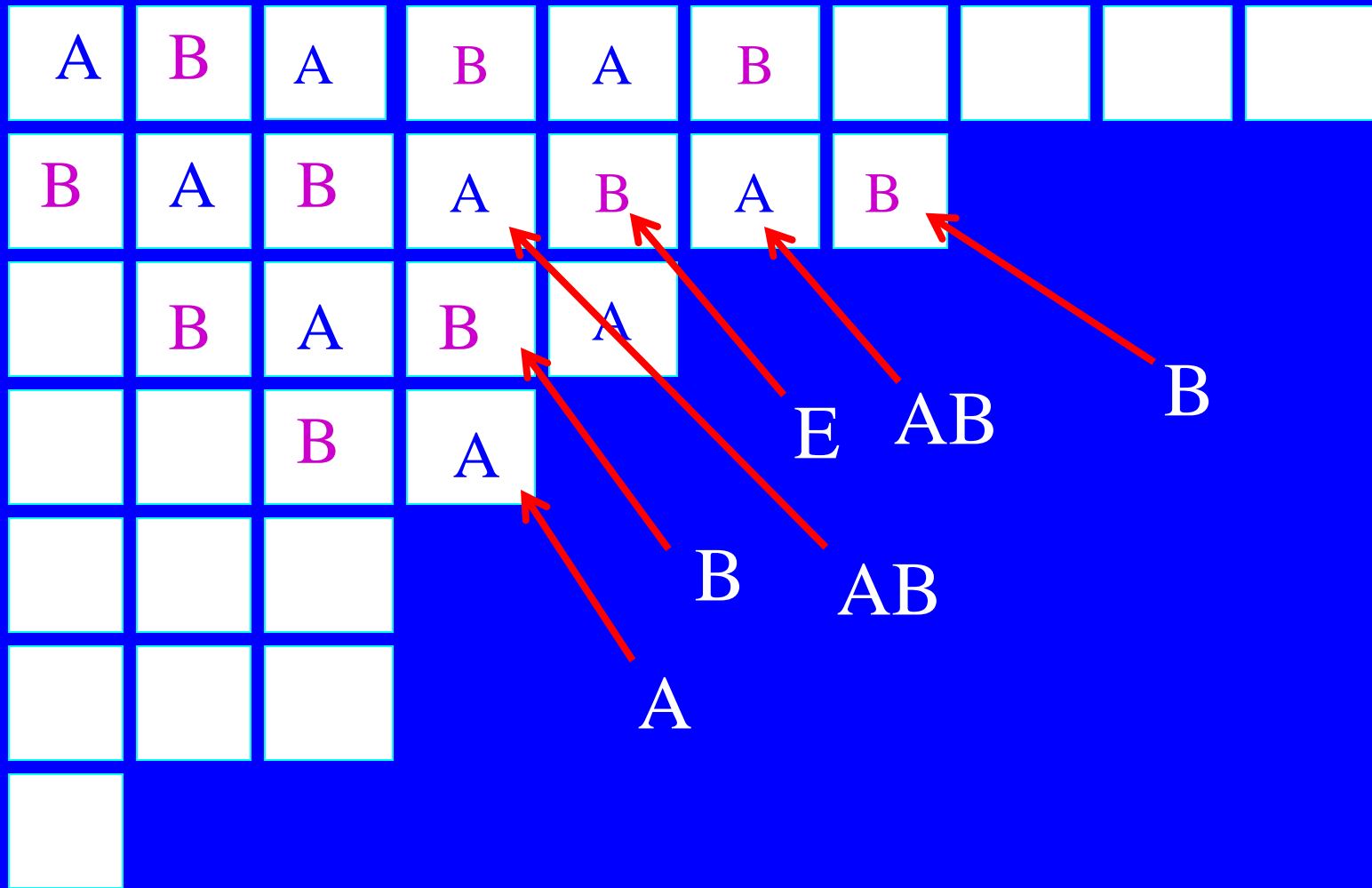
$$\langle AB \rangle = \{ E, AB \}$$

$$\langle A \rangle = \{ E, A \}$$

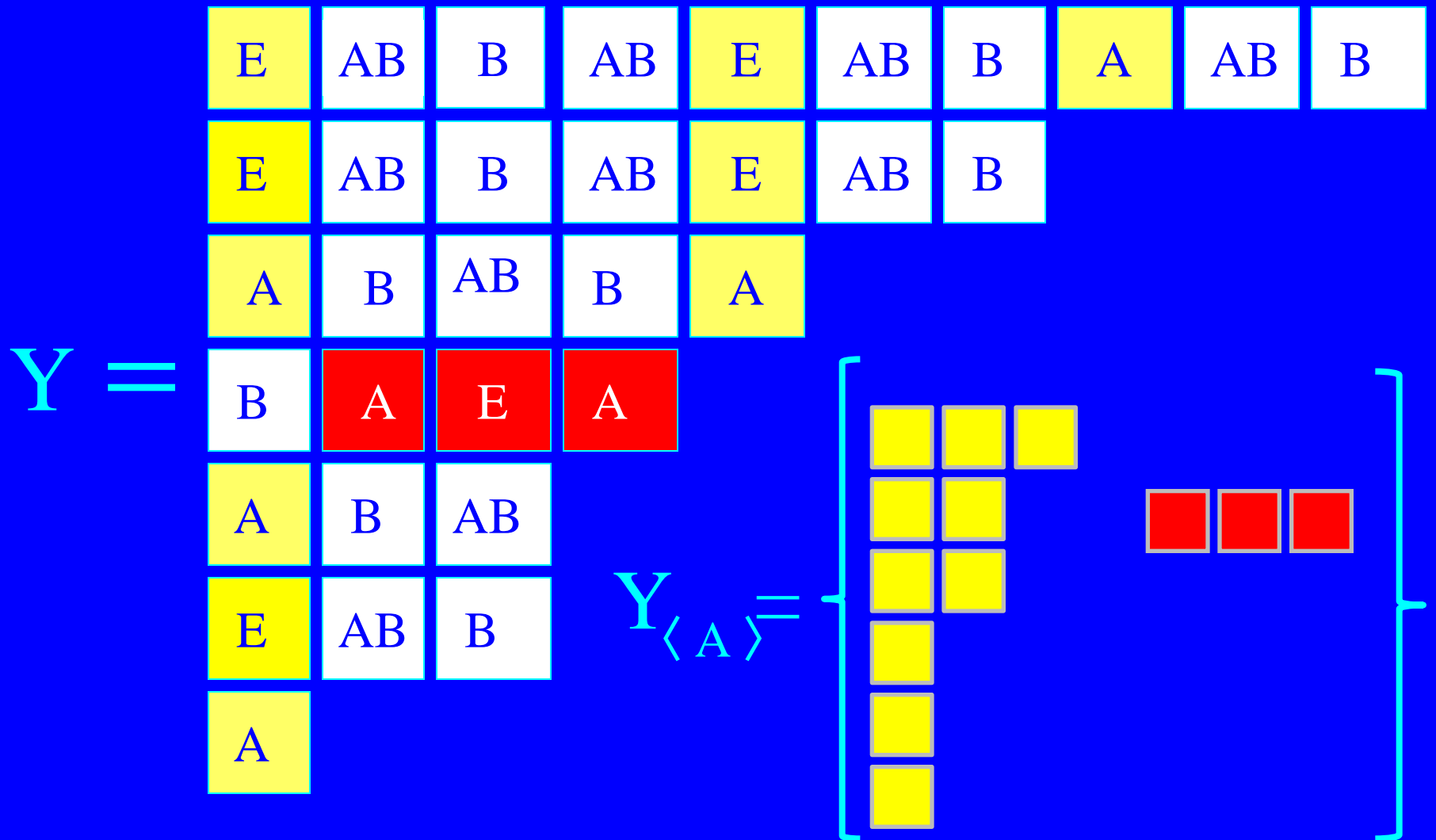
$$\langle B \rangle = \{ E, B \}$$

Multiply A & B's in a hook

Y =



$\langle A \rangle$ -quotient $Y_{\langle A \rangle}$

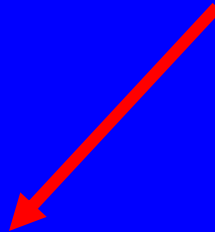
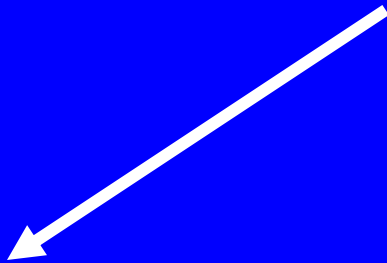


$$E(Y) = \underbrace{01\dots10}_{j \text{ 1's}} \underbrace{\dots010}_{i \text{ 0's}} \underbrace{\dots0}_{h \text{ 0's}}$$

j 1's

i 0's

h 0's



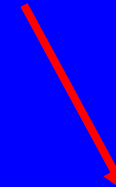
$\langle B \rangle$ -quotient
 j times

$\langle A \rangle$ -quotient
 i times

$\langle AB \rangle$ -quotient
 h times

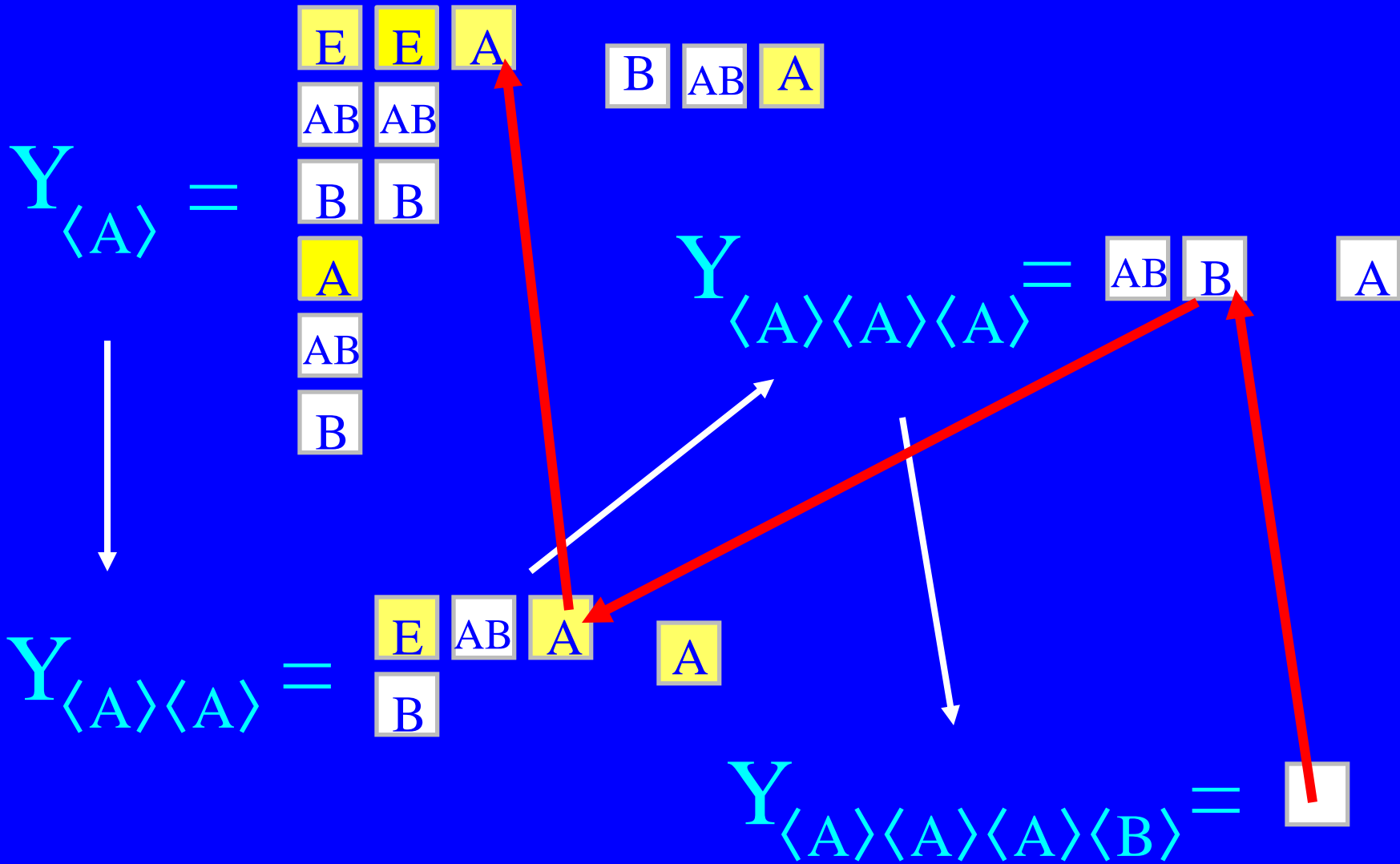
In the above example,

$$E(Y) = \underline{10001}$$



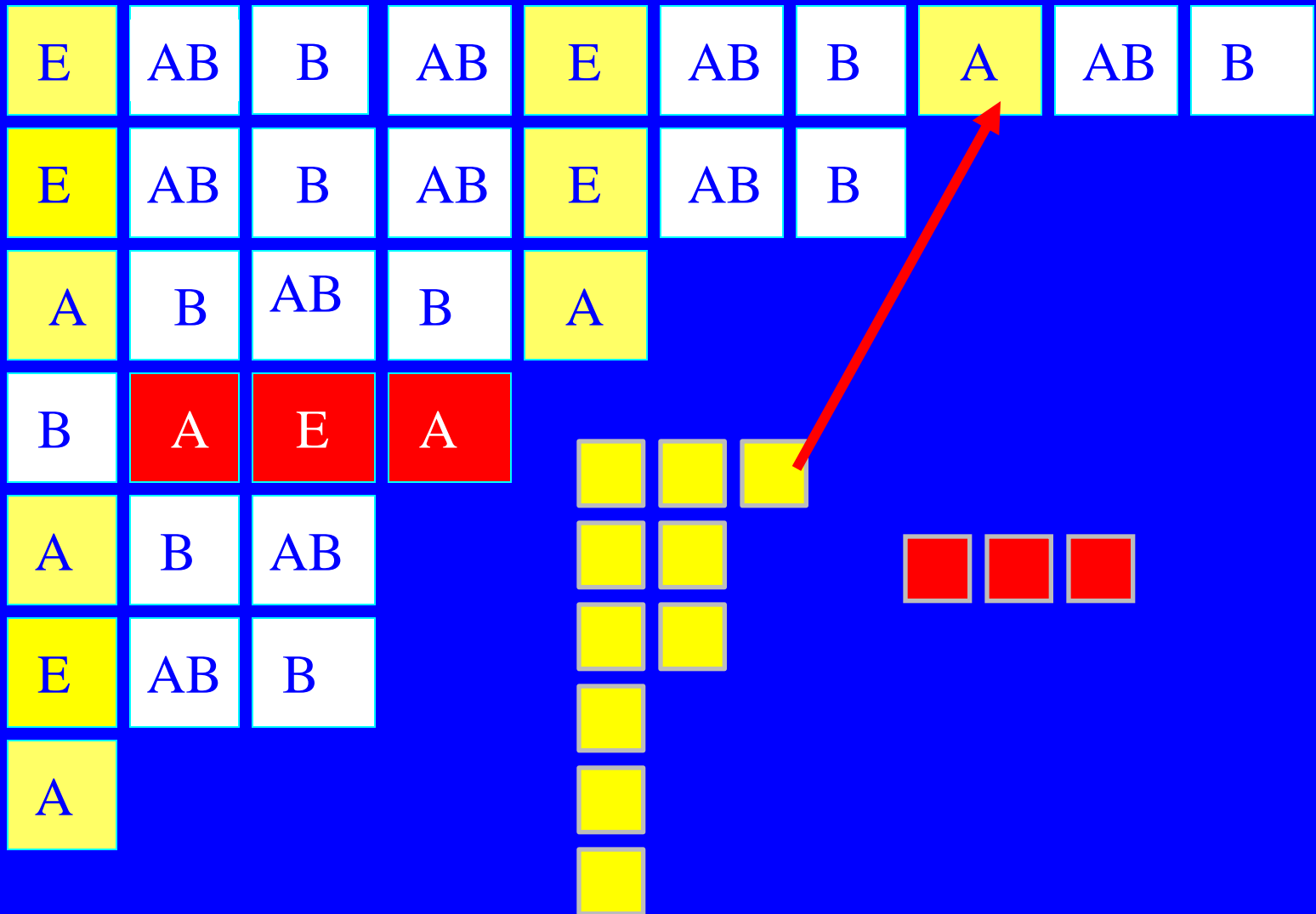
< A >-quotient

< B >-quotient



Subtracting the hook of this box is the unique good move.

Y =



Thank you.