Visualization of Quantum Operations

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Outline

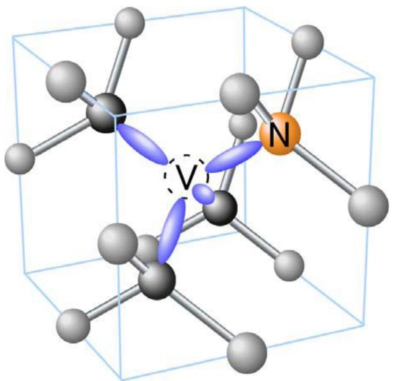
- Introduction and motivation
- Idea for visualization of quantum operations
  - Definition of information flow
- Examples
  - Single-qubit system $\longrightarrow$ Single-qubit system
  - Single-qubit system $\longrightarrow$ Multi-qubit system
- Summary and Future Issues

Introduction

Experiments of transferring quantum states between different systems

- Nuclear Spin ⟺ Electron Spin
  - NV Center

    M.V.Dutt *et al*., Science **315**, 1312 (2007).



- Laser-Cooled Atoms ⟺ Photon
  - Electromagnetically Induced Transparency (EIT)

    K.Honda *et al*., Phys. Rev. Lett. **100**, 093601 (2008).

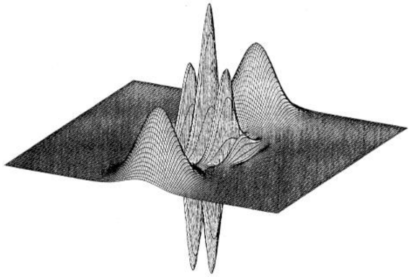
Motivation

How to characterize quantum operations ?

Motivation

- Quantum state tomography (QST) and the Wigner function

Unknown states (i.i.d.) $\hat{\rho}^{\otimes n}$ → QST → Probability distributions → the Radon transformation → the Wigner function



We can visually understand the results of quantum state tomography by plotting the Wigner function.

E. P. Wigner, Phys. Rev. **40**, 749 (1932).
K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).

Motivation

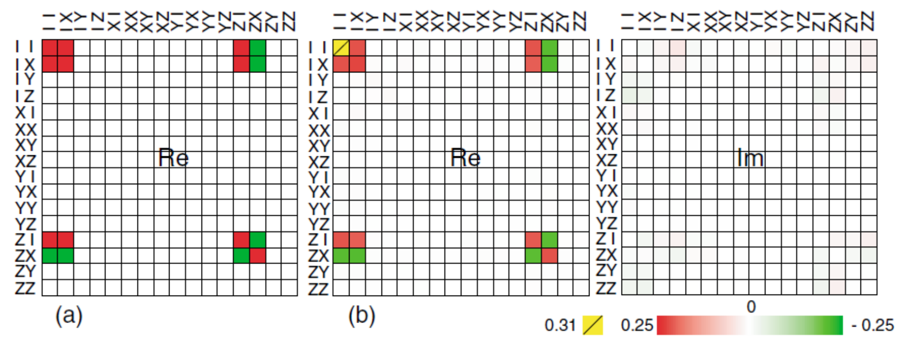
- Quantum process tomography (QPT)
  and visualization of quantum operation

Unknown quantum operation $\mathcal{E}$

QPT

Results of QPT

Ex. Process matrix of CNOT gate



J. L. O'Brien *et al.*, Phys. Rev. Lett. **93**, 080502 (2004).

How to understand the results of quantum process tomography ?
How to extract characteristics of quantum operations ?

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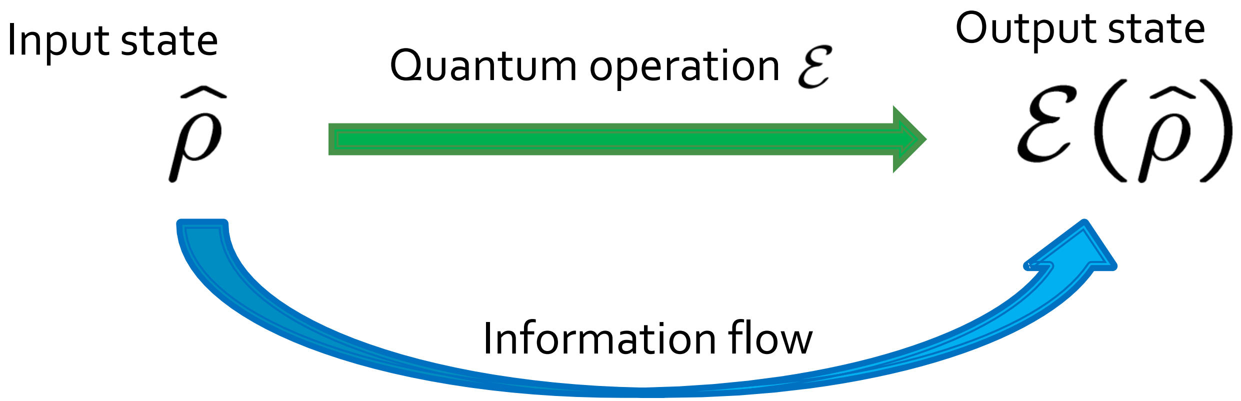
Idea for Characterizing Quantum Operations

- Characterization of quantum operation $\mathcal{E}$ by information flow.
  Information flow :
  Information about input state transferred to output state.



The information flow is large
$\longrightarrow$ We can estimate the input state from the output state accurately.
The information flow is small
$\longrightarrow$ Accuracy of the estimation is low.

Estimation of the Expectation Value of an Observable

- Estimation of $\langle \hat{X} \rangle$ from the output states of $\mathcal{E}$
  - $\langle \hat{X} \rangle = \mathrm{Tr}\left[\hat{X}\hat{\rho}\right]$: the expectation value of an observable $\hat{X}$
  - $\mathcal{E}$: an quantum operation

POVM Measurement

M M $\cdots$ M

Input states (i.i.d.) $\hat{\rho}^{\otimes n}$ $\xrightarrow{\mathcal{E}}$ Output states

$$\mathcal{E}(\hat{\rho})^{\otimes n} = \mathcal{E}(\hat{\rho}) \otimes \mathcal{E}(\hat{\rho}) \otimes \cdots \otimes \mathcal{E}(\hat{\rho})$$

$\hat{\rho} \in \mathcal{S}(\mathcal{H}_N)$

: N Level Quantum State

Maximum likelihood estimation

X^* : Estimated value of $\langle \hat{X} \rangle$

$$\lim_{n \to \infty} X^* = \langle \hat{X} \rangle$$

Accuracy of X^* is measured by the Fisher information.

The Fisher Information and the Cramer-Rao Inequality

- The Cramer-Rao inequality

$$\mathrm{Var}(X^*) \geq \frac{1}{n\,J} + o\left(\frac{1}{n}\right)$$

For the maximum likelihood estimator X^*

$$\lim_{n\to\infty} n\mathrm{Var}(X^*) = J^{-1}$$

$\mathrm{Var}(X^*)$: Variance of the estimated value

n : Number of the samples

J : the Fisher information

- The Fisher information is large. $\rightarrow$ Accuracy of X^* is high.
- The Fisher information is small. $\rightarrow$ Accuracy of X^* is low.

Definition of the Fisher Information (1) --- The Fisher Information Matrix ---

- The Fisher information matrix G

$$G(\boldsymbol{M}) \equiv \sum_i p_i (\nabla_{\boldsymbol{\theta}} \log p_i)(\nabla_{\boldsymbol{\theta}} \log p_i)^{\mathsf{T}}$$

$\boldsymbol{\theta}$: Parameter of the input state $\hat{\rho} = \hat{\rho}_{\boldsymbol{\theta}} = \frac{1}{N}\hat{I} + \frac{1}{2}\boldsymbol{\theta}\cdot\hat{\boldsymbol{\lambda}}$

$\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{N^2-1})^{\mathsf{T}}$: Generators of the Lie algebra $\mathfrak{su}(N)$

In the case of N=2, $\hat{\lambda}_i$ are the Pauli matrices.

In the case of N=3, $\hat{\lambda}_i$ are the German matrices.

M. S. Byrd and N. Khaneja, Phys. Rev. A **68**, 062322 (2003).
G. Kimura, Phys. Lett. A **314**, 339 (2003).

$p_i = \mathrm{Tr}\left[\mathcal{E}(\hat{\rho})\hat{M}_i\right]$: Probability of getting the ith result

$\hat{M}_i$: An element of the POVM $\boldsymbol{M} = \{\hat{M}_1, \hat{M}_2, \ldots\}$

Definition of the Fisher Information (2) --- The Fisher Information ---

- The Fisher information J about an observable $\hat{X}$

$$J(\boldsymbol{x}; M) \equiv \frac{1}{\boldsymbol{x} \cdot [G(M)]^{-1} \boldsymbol{x}}$$

$\boldsymbol{x}$: Parameter of the observable $\hat{X} = x_0 \hat{I} + \boldsymbol{x} \cdot \hat{\boldsymbol{\lambda}}$

$\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{N^2-1})^\top$: Generators of the Lie algebra $\mathfrak{su}(N)$

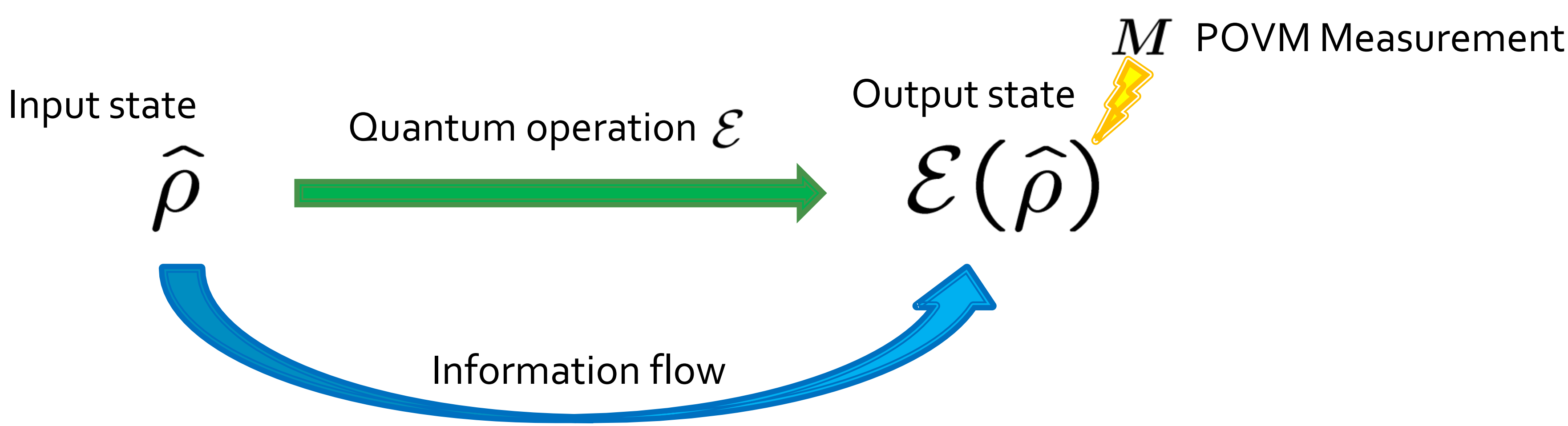
Definition of the Information Flow

- The information flow about an observable $\hat{X}$

$$\chi(\boldsymbol{x}) \equiv \max_{\boldsymbol{M}} \{J(\boldsymbol{x}; \boldsymbol{M})\}$$

$\boldsymbol{x}$: Parameter of the observable $\hat{X}$

  - We define information flow as maximal fisher information about the input state obtained from the output state.



Visualization of Quantum Operation by Information Flow

- Normalization
  - $|\boldsymbol{x}| = \sqrt{N/2}$
  - $\hat{\rho} = \hat{I}/N$

$$\hat{X} = x_0 \hat{I} + \boldsymbol{x} \cdot \hat{\boldsymbol{\lambda}}$$

$$\Longrightarrow \quad 0 \leq \chi(\boldsymbol{x}) \leq 1$$

No information about $\langle \hat{X} \rangle$ is transferred to the output state

Information about $\langle \hat{X} \rangle$ is perfectly transferred to the output state

T. Sagawa and M. Ueda, Phys. Rev. A **77**, 012313 (2008).

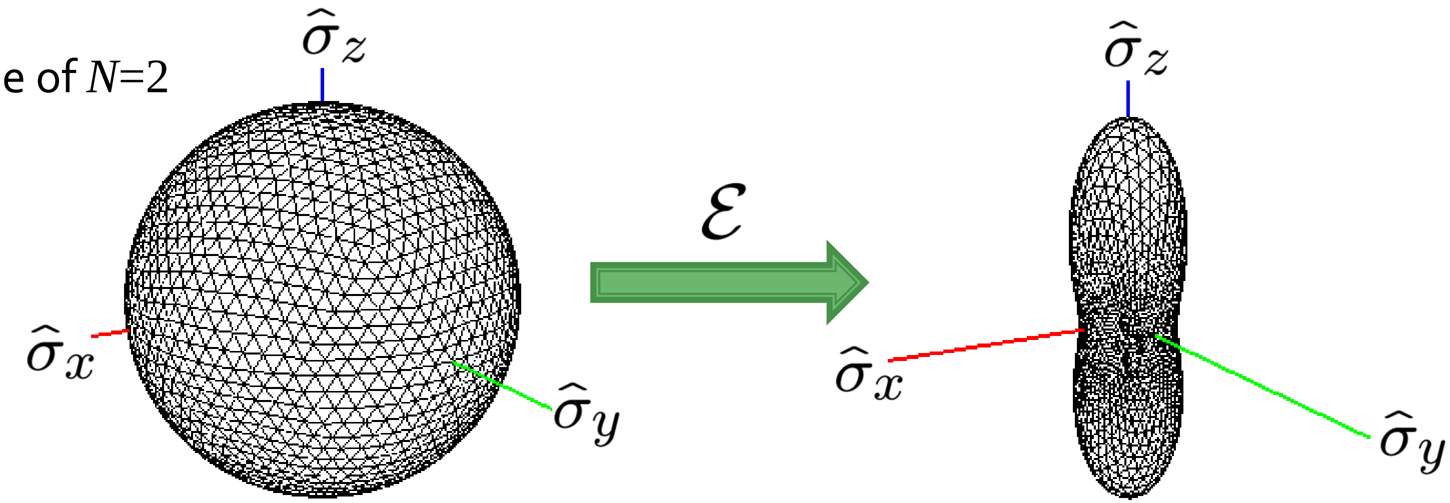
Visualization of Quantum Operation by Information Flow

- Normalization
  - $|\boldsymbol{x}| = \sqrt{N/2}$
  - $\hat{\rho} = \hat{I}/N$

$$\hat{X} = x_0 \hat{I} + \boldsymbol{x} \cdot \hat{\boldsymbol{\lambda}}$$

$$\Longrightarrow \quad 0 \leq \chi(\boldsymbol{x}) \leq 1$$

The case of N=2



Before the quantum operation

After the quantum operation

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Example (1)
--- Unitary Operations ---

- A unitary operation

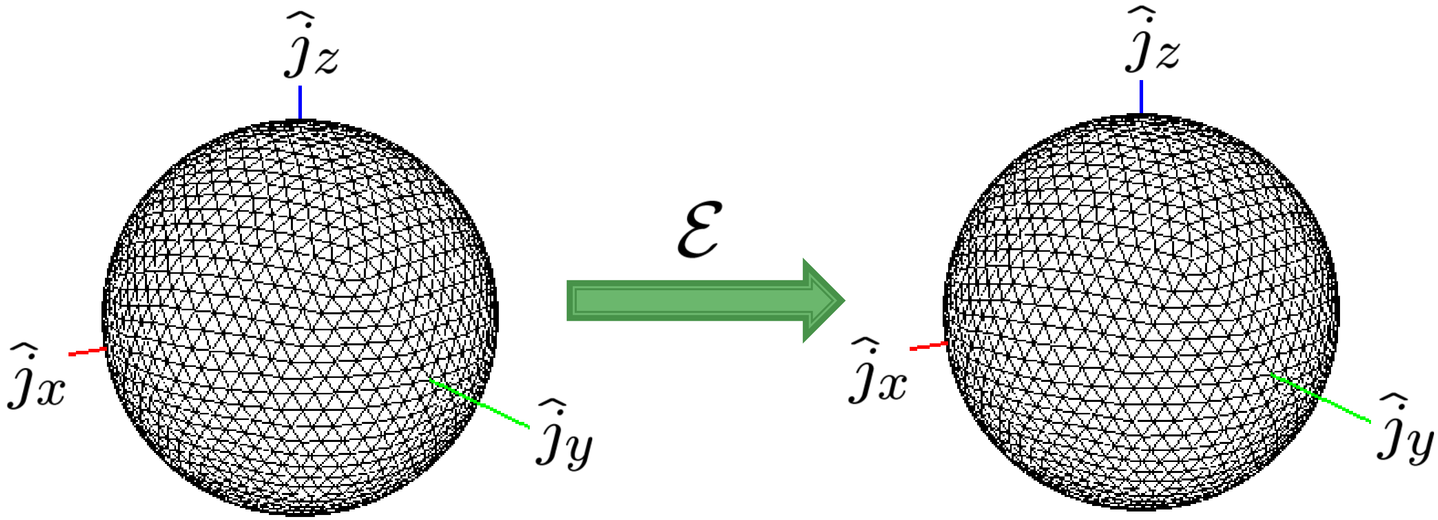
$$\mathcal{E}(\hat{\rho}) = \hat{U}\hat{\rho}\hat{U}^{\dagger}$$

$\hat{U}$: a unitary operator

For all $\boldsymbol{x}$, such that $|\boldsymbol{x}| = \sqrt{N/2}$

$$\chi(\boldsymbol{x}) = 1$$

Any unitary operations conserve information.



Before the quantum operation

After the quantum operation

Example (2)
--- Bit Flip and Phase Flip Channel ---

- Bit flip with probability p

$$\mathcal{E}(\hat{\rho}) = p\,\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x + (1-p)\hat{\rho}$$

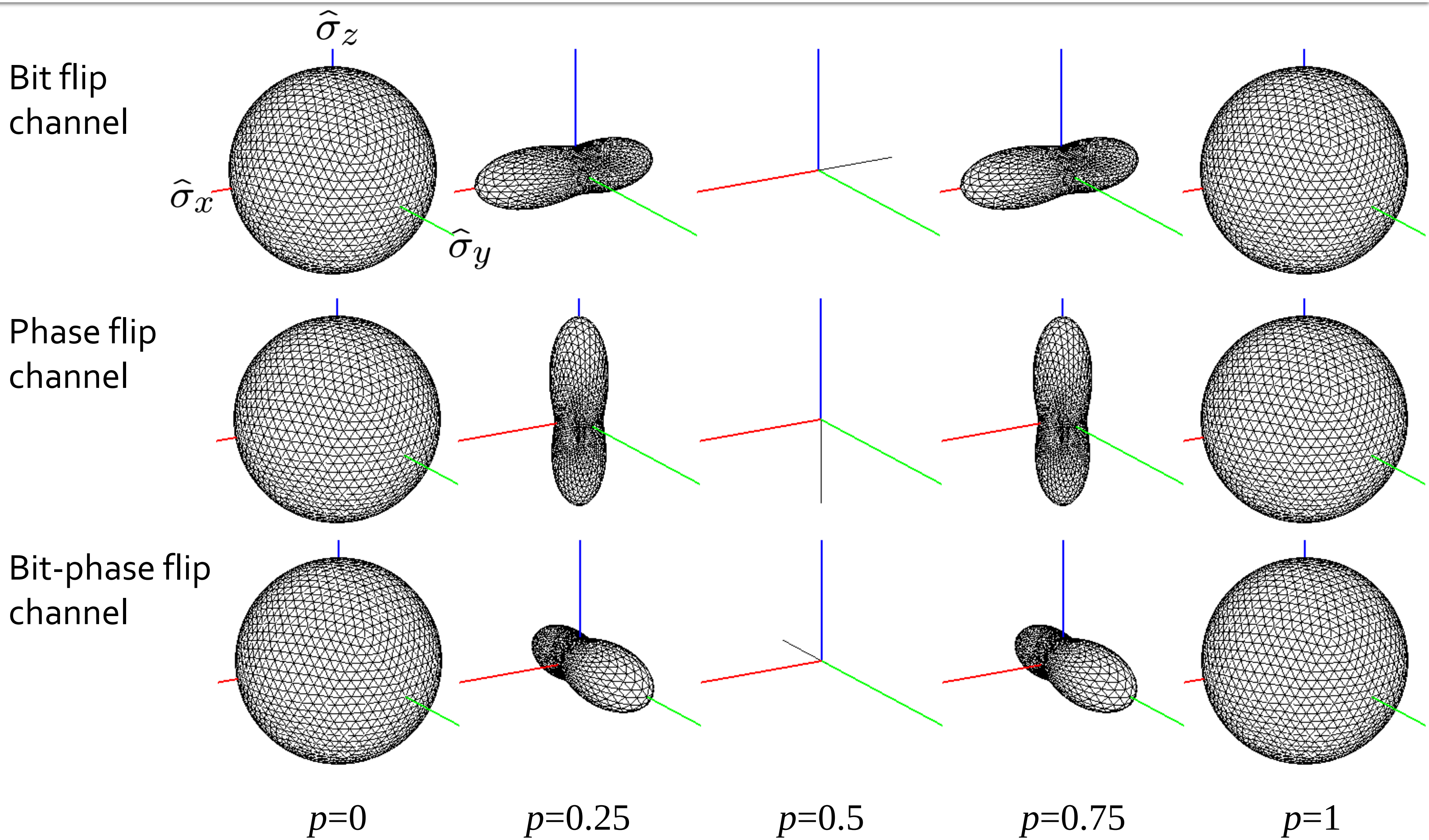
- Phase flip with probability p

$$\mathcal{E}(\hat{\rho}) = p\,\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z + (1-p)\hat{\rho}$$

- Bit-phase flip with probability p

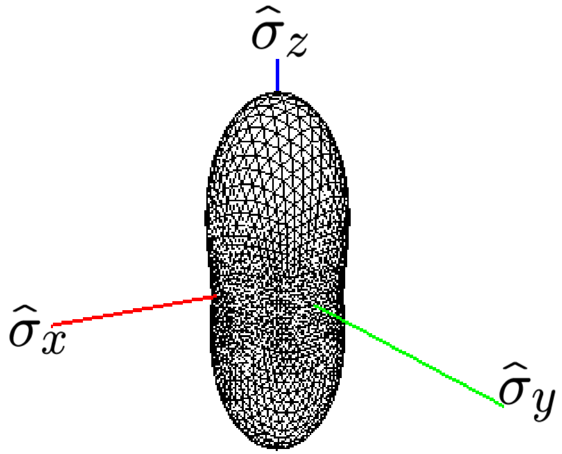
$$\mathcal{E}(\hat{\rho}) = p\,\hat{\sigma}_y\hat{\rho}\hat{\sigma}_y + (1-p)\hat{\rho}$$

Example (2)
--- Bit Flip and Phase Flip Channel ---

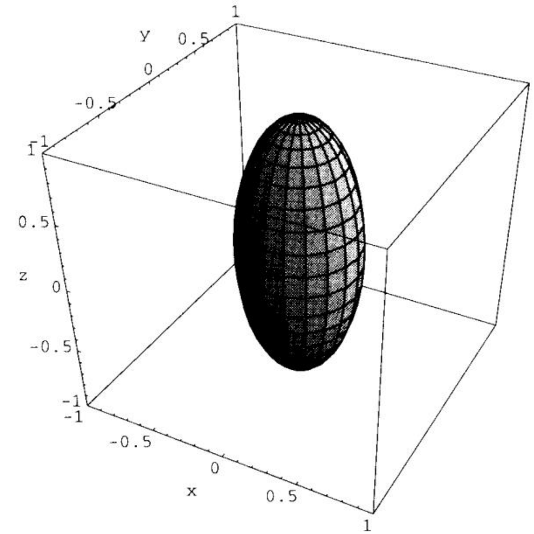


Comparison with Bloch Sphere

Ex. The phase flip channel with probability p=0.3



Our method



Bloch sphere

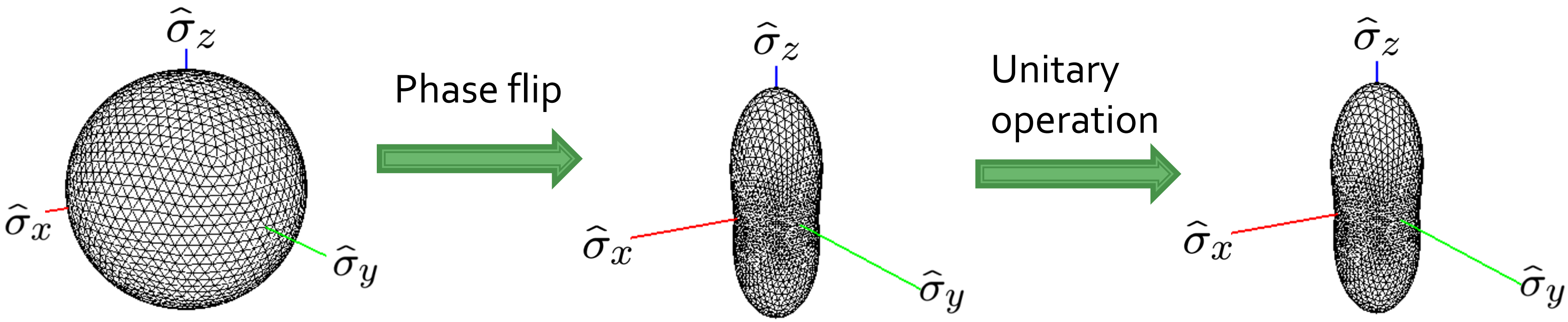
- What is the deference between
  Bloch sphere and our visualizing method ?

Advantages of our method

- A unitary operation after the phase flip channel

$$\mathcal{E}'(\hat{\rho}) = \hat{U}\mathcal{E}(\hat{\rho})\hat{U}^\dagger$$

$\mathcal{E}$: the phase flip channel



・Our visualizing method can extract effects of decoherence and noise by quantum operations.

・Our method can be also applied to the case that input system and output system are different.

Example (3)
--- Decoherence by a Head Bath ---

- Single qubit interacting with a heat bath
  - Hamiltonian

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}$$

$$\hat{H}_S = \hbar\omega_0 \frac{\hat{\sigma}_z}{2} \qquad \hat{H}_B = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k$$

$$\hat{H}_{SB} = \sum_k \hbar\hat{\sigma}_z (g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k)$$

  - The initial state of the heat bath

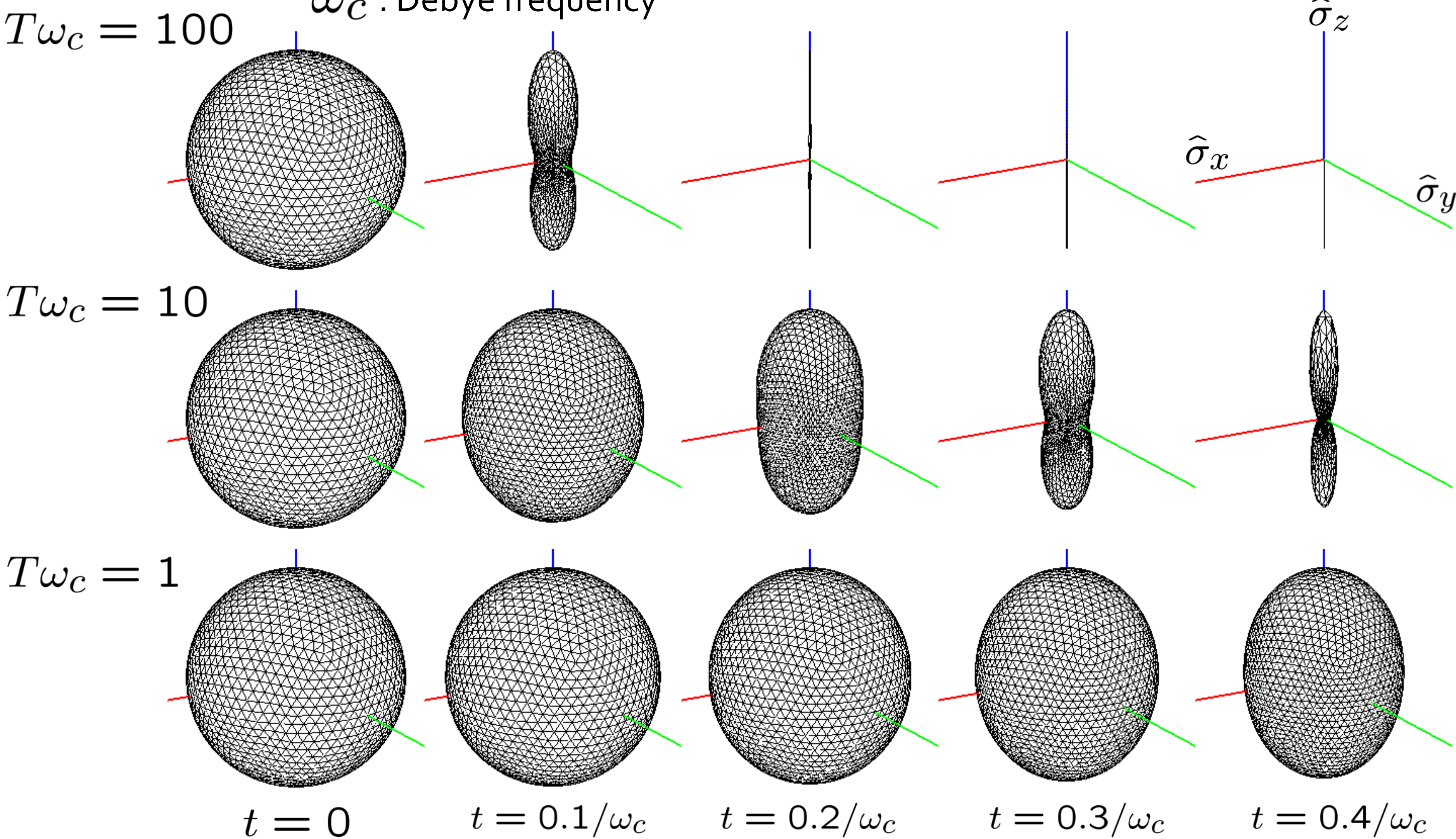
$$\hat{\rho}_B = \prod_k (1 - e^{-\beta\hbar\omega_k}) e^{-\beta\hbar\omega_k \hat{b}_k^\dagger \hat{b}_k}$$

  - Quantum operation

$$\mathcal{E}(\hat{\rho}) = \mathrm{Tr}_B \left[e^{-i\hat{H}t/\hbar} (\hat{\rho} \otimes \hat{\rho}_B) e^{i\hat{H}t/\hbar} \right]$$

Example (3)
--- Decoherence by a Head Bath ---

ω_c : Debye frequency



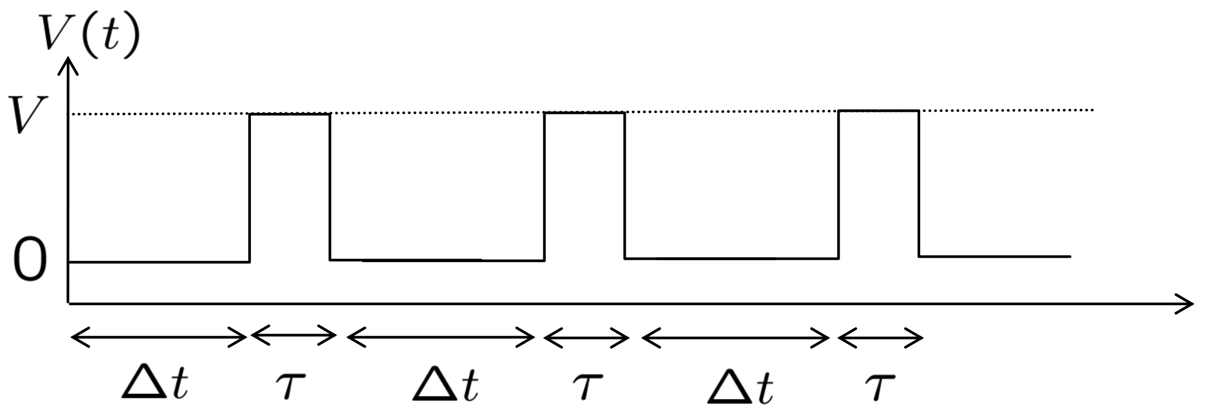
Example (4)
--- Dynamical Decoupling (Spin Echo) ---

- Time evolution with sequence of pulses
  - Hamiltonian

L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
L. Viola *et al.*, Phys. Rev. Lett. **82**, 2417 (1999).

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB} + \hat{H}_{\text{rf}}(t)$$

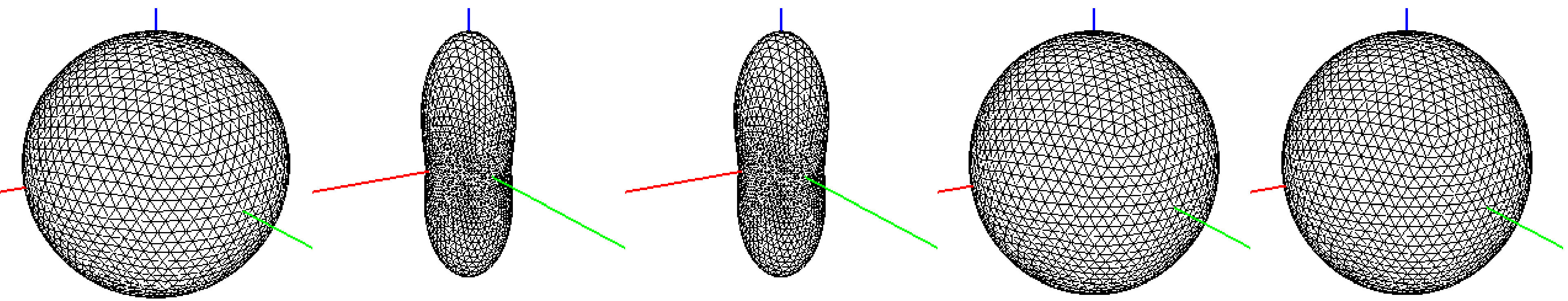
$$\hat{H}_{\text{rf}}(t) = V(t)(\hat{\sigma}_x \cos \omega_0 t + \hat{\sigma}_y \sin \omega_0 t)$$



$$\frac{V\tau}{\hbar} = \frac{\pi}{2}$$

Example (4)
--- Dynamical Decoupling (Spin Echo) ---

- The sequence of pulses, decoherence is suppressed.



$t = 0$ $\quad$ $t = \Delta t$ $\quad$ $t = \Delta t + \tau$ $\quad$ $t = 2\Delta t + \tau_p$ $\quad$ $t = 2(\Delta t + \tau)$

Using our method, effects of decoherence and its suppression by the sequence of pulses are visually understood.

$$\Delta t = 0.25/\omega_c$$
$$\tau = 0.1\Delta t$$
$$T = 10\omega_c$$

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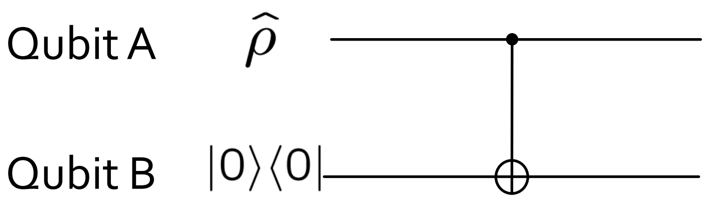
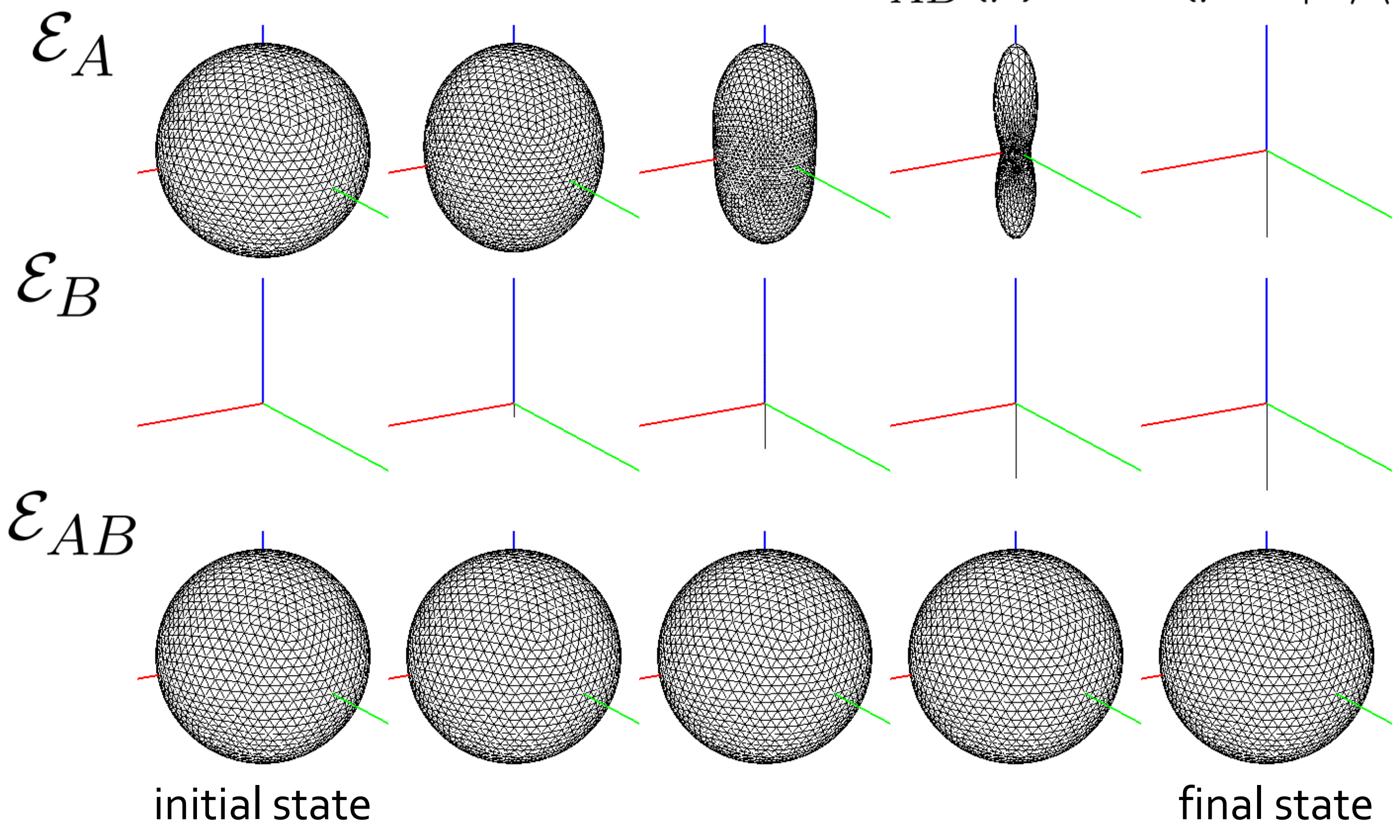
Example (5)
--- the CNOT Gate ---



$$\mathcal{E}_A(\hat{\rho}) = \mathrm{Tr}_B\left[\hat{U}(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}^\dagger\right]$$

$$\mathcal{E}_B(\hat{\rho}) = \mathrm{Tr}_A\left[\hat{U}(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}^\dagger\right]$$

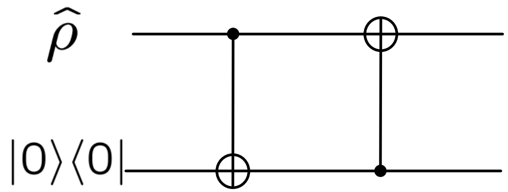
$$\mathcal{E}_{AB}(\hat{\rho}) = \hat{U}(\hat{\rho}\otimes|0\rangle\langle 0|)\hat{U}^\dagger$$



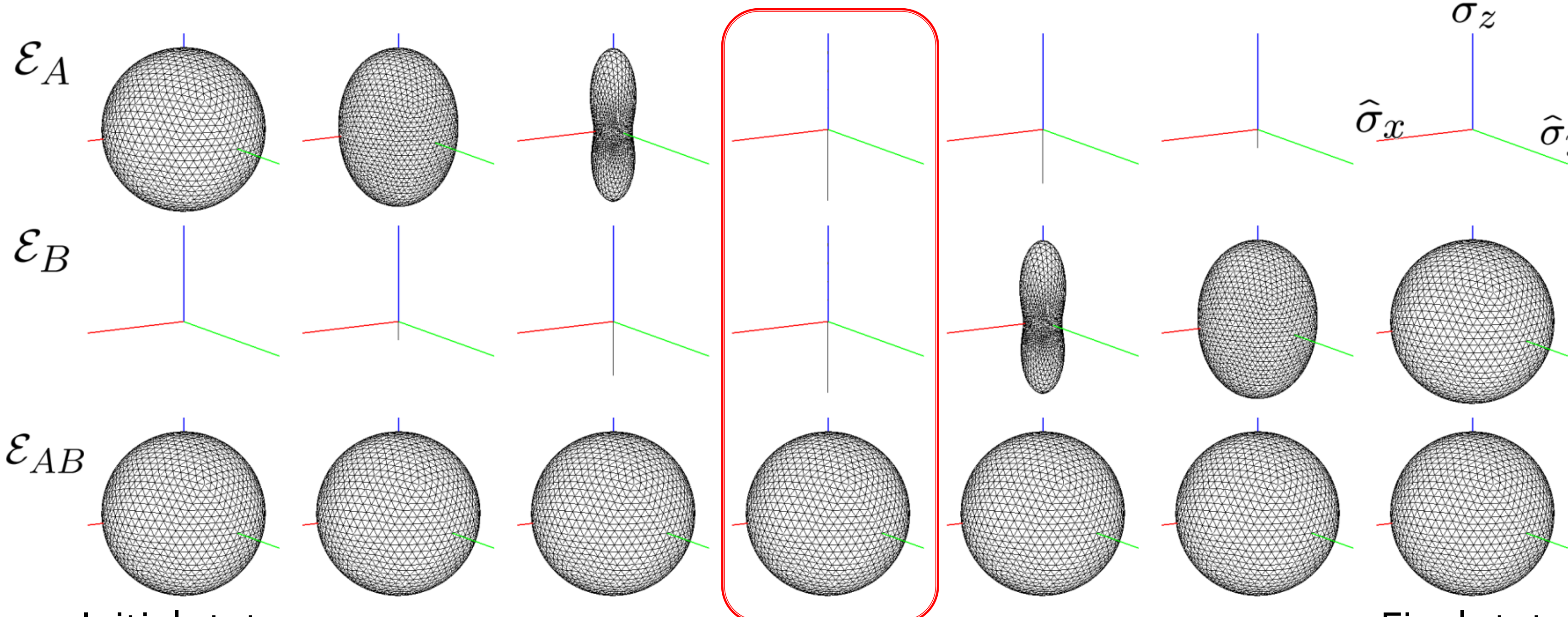
Example (6)
--- Double CNOT Gate ---

Qubit A

Qubit B



States of qubit A and B are swapped by two CNOT gates.



Initial state

Final state

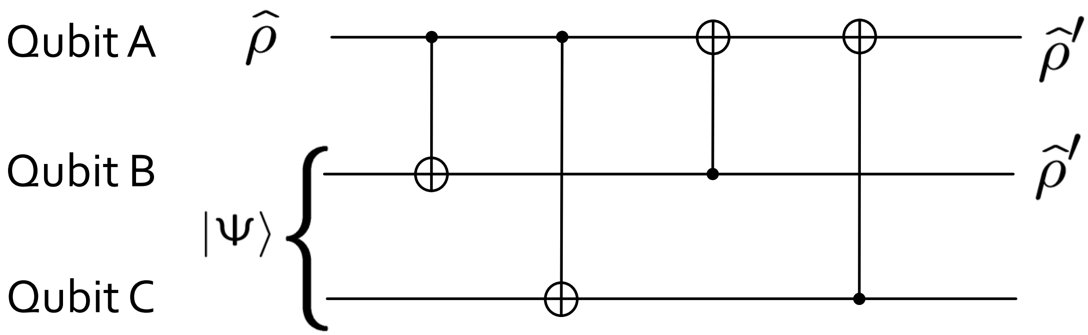
Creation of entanglement

⟺ Erasure of information from local states

Example (7)
--- Quantum State Cloning ---

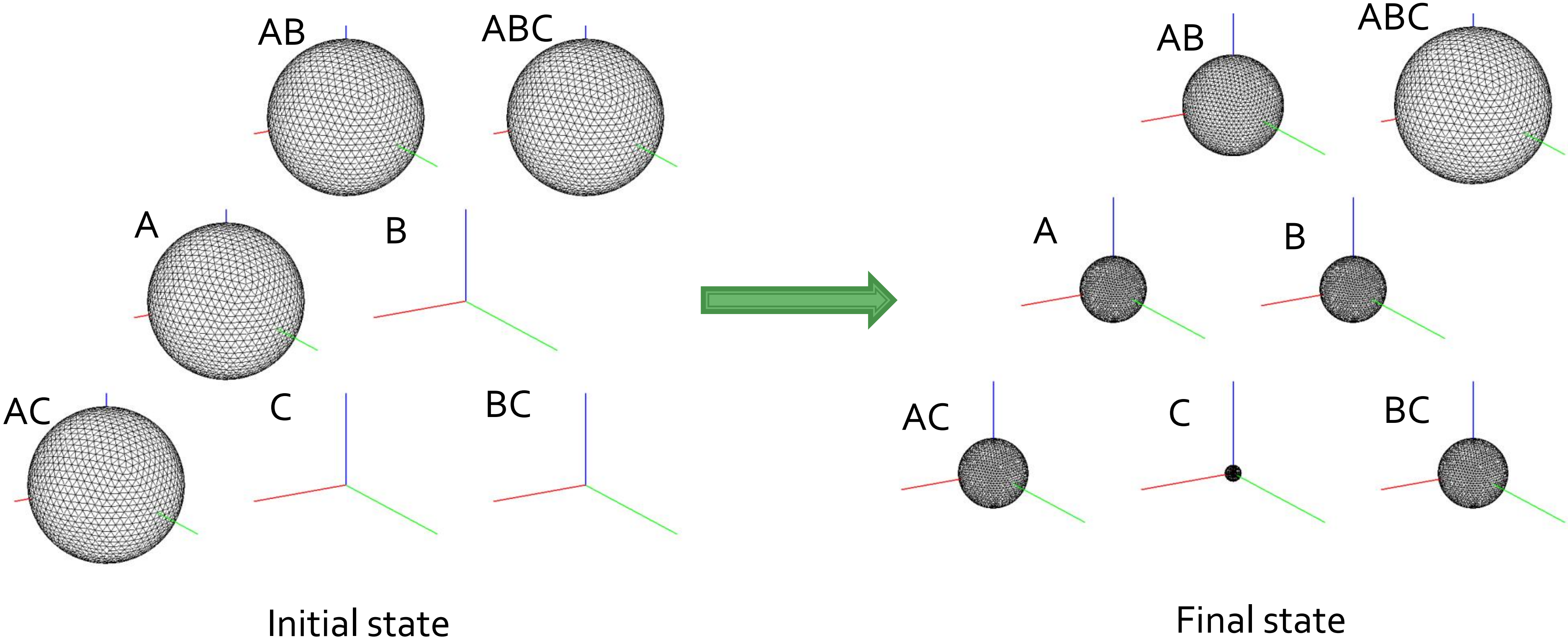
- Optimal symmetric cloning (1 qubit → 2 qubit)

Chi-Sheng Niu and Robert B. Griffiths, Phys. Rev. A **58**, 4377 (1998).
V. Buzek and M. Hillery, Phys. Rev. Lett. **81**, 5003 (1998).
N. J. Cref, Acta Phys. Slov. **48**, 115 (1998).



$$|\Psi\rangle = \frac{1}{\sqrt{6}}\left(2|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|1\rangle\right)$$

Example (7)
--- Quantum State Cloning ---



No cloning theorem = Information can not be distributed perfectly.

Summary and Future Issues

- We define the information flow.
  - Information flow =

    Maximal Fisher information about the input state obtained from the output state of quantum operations.

- We propose a method of visualization of quantum operations by plotting the information flow.
  - Effects of decoherence and noise can be visually understood.
  - Generation of entanglements can be visually understood.

Future Issues :
Are there any relations between entanglement measures and our visualizing method of quantum operations ?