Squeezing the limit: Quantum benchmarks for the teleportation and storage of squeezed states

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Introduction

Quantum teleportation and Quantum memory.

Both processing can be written down the following processing:

- Alice and Bob are spatially or temporally separated.
- Alice wants to send an unknown quantum state to Bob.
- They known an unknown state is in $\{\rho_\omega\}_{\omega\in\Omega}$.
- They may be also know the prior probability $\{p_\omega\}_{\omega\in\Omega}$.
- An error is caused by an inevitable noise, and Bob gets $\Gamma(\rho_\omega) \neq \rho_\omega$.

ρ_ω $\xrightarrow{\quad\Gamma\quad}$ $\Gamma(\rho_\omega)$

Alice

Bob

Ideal case: Γ is an identity channel → Impossible in a real experiment

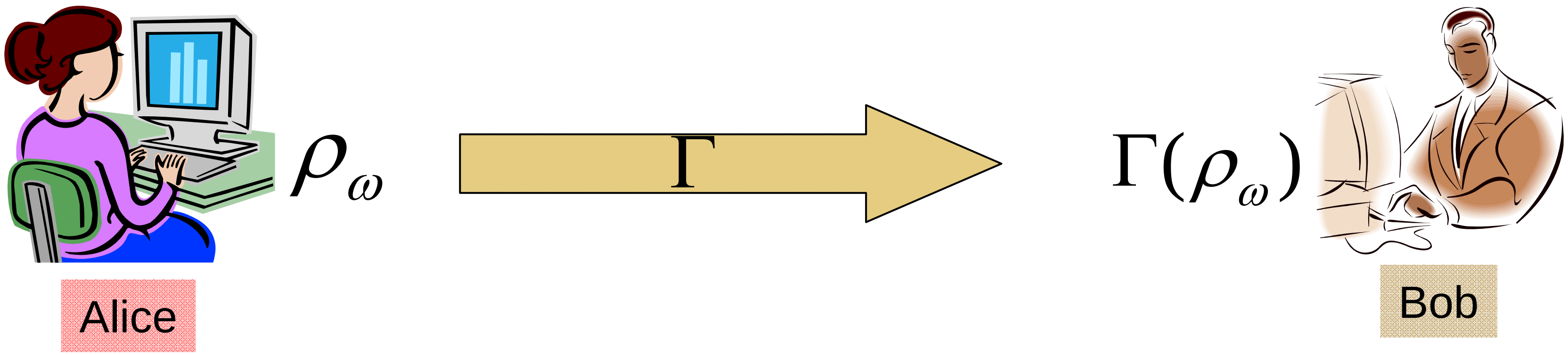
Introduction

Quantum teleportation and Quantum memory.

Suppose an experiment is done, and we have data of ρ_ω and $\Gamma(\rho_\omega)$.
However, Γ looks far from the identity channel.

Question: Is this process really "quantum"?

At least, it should not be simulated by a "classical" scheme.



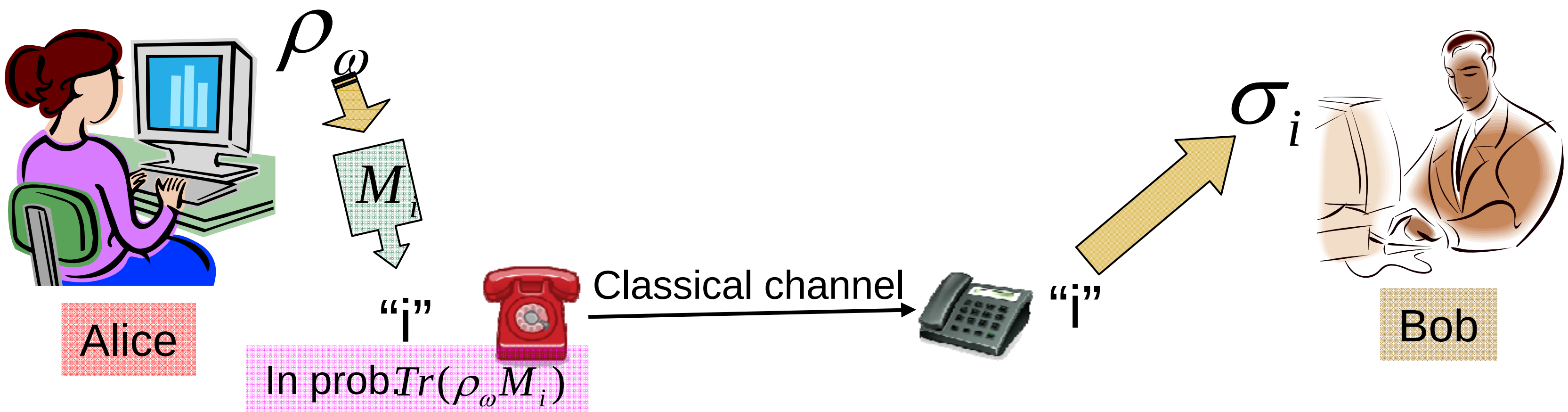
Introduction

Quantum teleportation and Quantum memory.

Classical scheme (or Measure and Preparing scheme):

(also called Entanglement breaking channel)

1. Alice measure ρ_ω by POVM $\{M_i\}_{i=1}^N$.
2. Alice send a result of the measurement "i" to Bob.
3. Bob choose a state σ_i depending on a classical information "i".



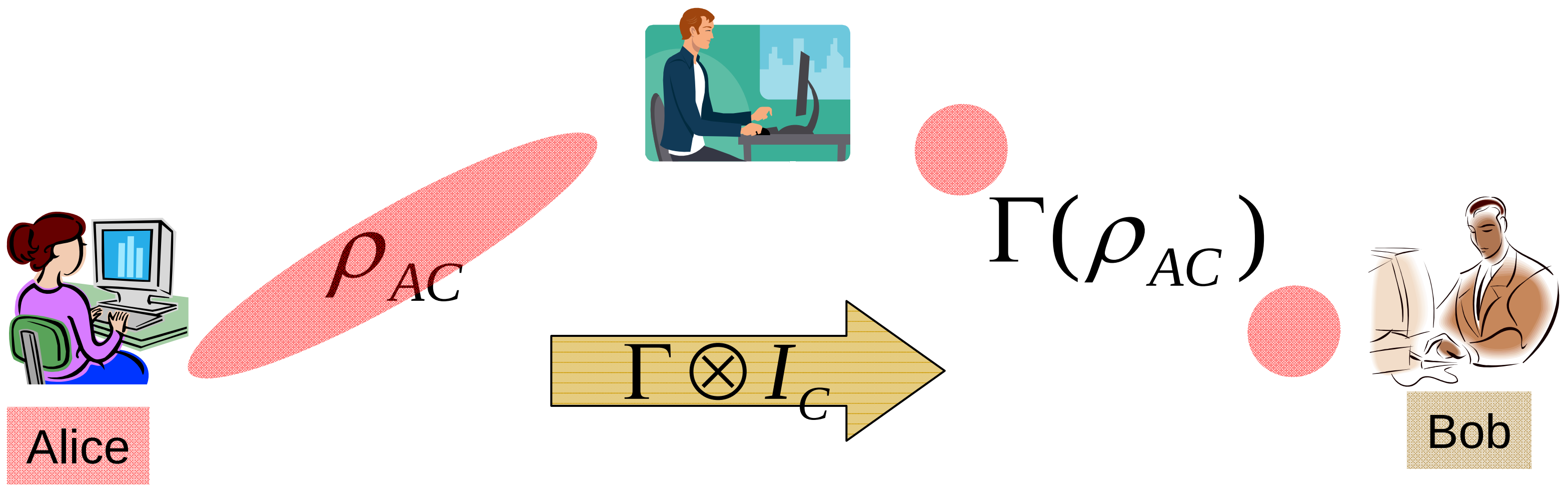
They want an average output state $\sum_{i=1}^{N} Tr(\rho_\omega M_i)\cdot\sigma_i$ to be similar to ρ_ω.

Introduction

Quantum teleportation and Quantum memory.

Classical scheme ⟺ iff ⟺ Entanglement breaking (EB) channel

Suppose there exists another system $\rho_{AC} \in H_A \otimes H_C$
For all $\rho_{AC} \in H_A \otimes H_C$, $\Gamma \otimes I_C(\rho_{AC})$ is separable.



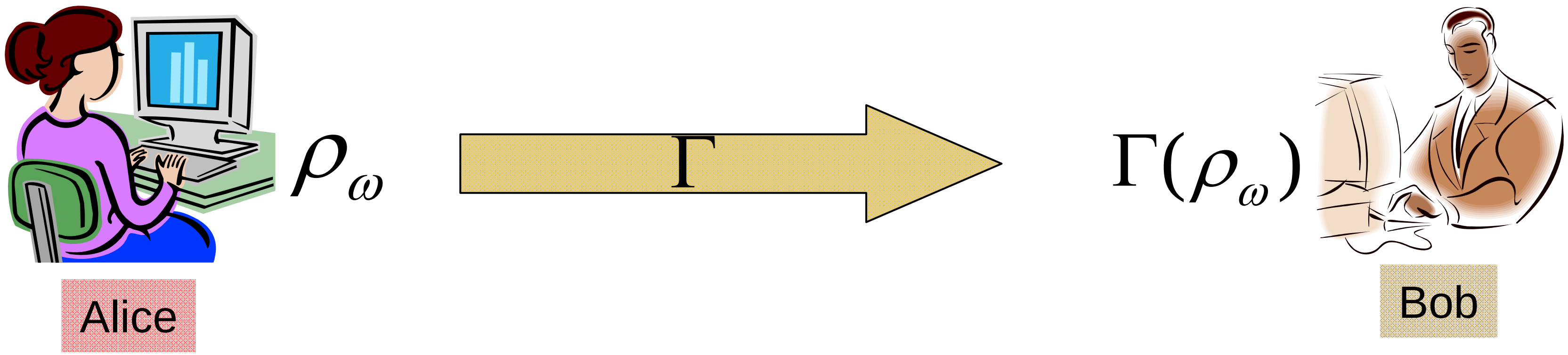
Such a channel is useless: e.g. Repeater, Computation, etc

Introduction

Quantum teleportation and Quantum memory.

Our aim:

By using experimental data (data of input ρ_ω and output states $\Gamma(\rho_\omega)$),
we want to show "a given channel can not simulated by classical scheme".



Introduction

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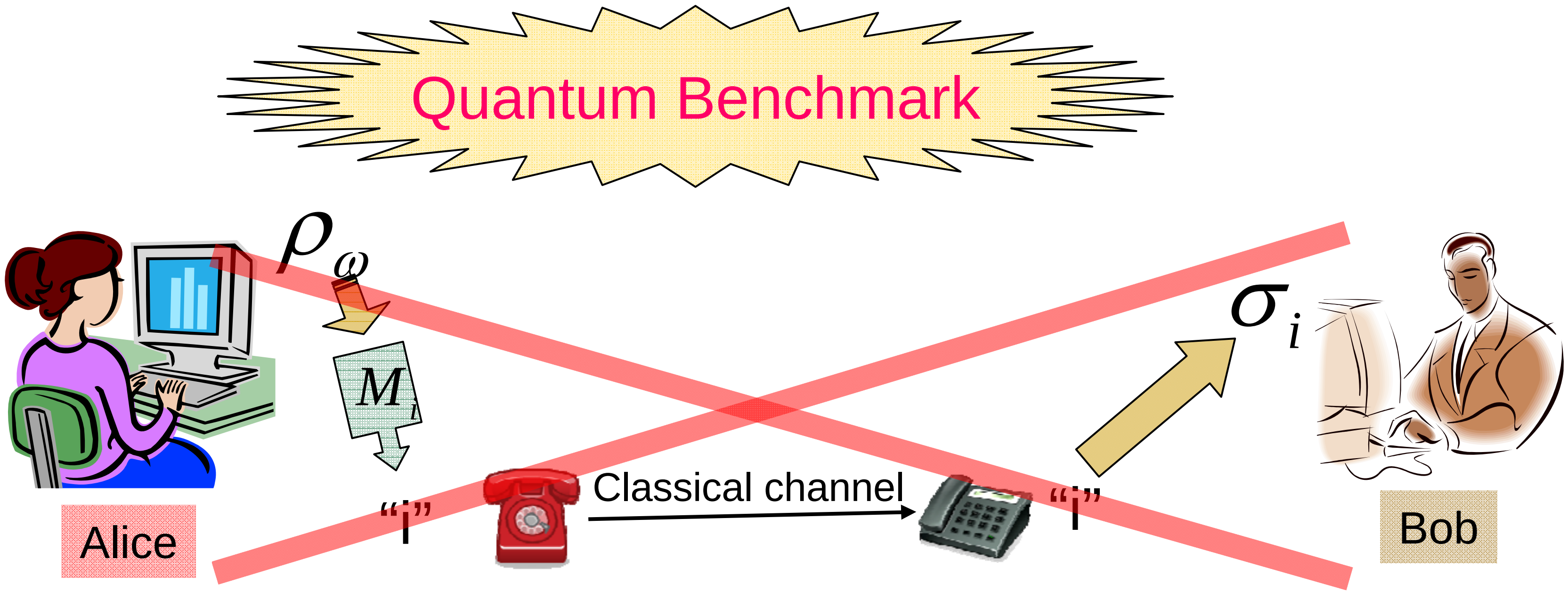


The optimal average fidelity

Most natural quantum benchmark is the optimal average fidelity between input and output states.

For a given channel Γ and an input ensemble $\{\rho_\omega, p_\omega\}_{\omega\in\Omega}$,
an average fidelity $\overline{F}(\Gamma)$ is given as:

$$\overline{F}(\Gamma) \equiv \int_{\omega\in\Omega} F(\rho_\omega \,|\, \Gamma(\rho_\omega))\, d\omega$$

Then, the optimal average fidelity is derived as

$$\overline{F} \equiv \sup_{\Gamma\in \mathrm{E}_b} \overline{F}(\Gamma),$$ E_b : a set of all EB channels

$\overline{F}$ is a legitimate quantum benchmark:

1. $\overline{F}(\Gamma)$ can be calculated by only experimental data of ρ_ω and $\Gamma(\rho_\omega)$.
2. If $\overline{F}(\Gamma) \geq \overline{F}$, then, Γ is not EB channel.

   ⇒ This experiment can not simulated by a classical scheme.

The optimal worst fidelity

Another popular quantum benchmark is the optimal worst fidelity between input and output states.

For a given channel Γ and an input ensemble $\{\rho_\omega\}_{\omega\in\Omega}$,
an worst fidelity $F_0(\Gamma)$ is given as:

$$F_0(\Gamma) \equiv \inf_{\omega\in\Omega} F(\rho_\omega \mid \Gamma(\rho_\omega))$$

Then, the optimal worst fidelity is defined as

$$F_0 \equiv \sup_{\Gamma\in \mathrm{E}_b} F_0(\Gamma)$$

The optimal average fidelity F_0 is a legitimate quantum benchmark, too.

F_0 is not depend on prior probability.
Therefore, even in the case where we cannot define a reasonable prior probability,
We can use F_0.

By definition, $\bar{F}(\Gamma) \geq F_0(\Gamma)$, and thus, $\bar{F} \geq F_0$.

Known results: finite dimension

Driving $\overline{F}$ or F_0 is equal to solving a normal estimation problem of $\{\rho_\omega\}_{\omega\in\Omega}$.
Many results have been derived as the state estimation problem.

(example)
For an ensemble of pure states $\{U|\psi\rangle, dU\}_{U\in SU(D)}$ distributed according to Haar measure dU in a D-dimensional system.

$$\overline{F} = F_0 = \frac{2}{D+1}$$

(Werner 98, Horodecki×3 99)

In this talk, I concentrate on a infinite dimensional system.

Known results: infinite dimension

Of course, quantum benchmark in an infinite dimensional system is also really important as an technological application.

Difference between infinite and finite dimensional systems:

- A set of pure states is non-compact.
- It is impossible to make all pure states in an experiment.

We are interested in a particular set of states.

Quantum benchmark for a set of coherent states

- For an ensemble of coherent states $\{|\alpha\rangle, p_\alpha\}_{\alpha\in\mathbb{C}}$
  where $p(\alpha) = \frac{\lambda}{\pi}\exp(-\lambda|\alpha|^2)$:

$$\bar{F} = \frac{1+\lambda}{2+\lambda}$$

(Braunstein et al. 2000, Hammerer et al. 2005)

Especially, in the limit of flat distribution $\lambda \to \infty$, $\bar{F} = \frac{1}{2}$

However, a coherent state is a "classical" state.
People are interested in a quantum teleportation and quantum memory for more quantum states

So, we want to derive a quantum benchmark for squeezed states.

Quantum benchmark for squeezed states

(Difficulty)

- Hammerer et al.'s trick does not work for squeezed states.
- In experiment, a pure squeezed states rapidly becomes mixed, because of attenuation of light fields.

  Therefore, we should treat mixed states

  However, the fidelity for mixed states is non-linear!

$$F(\rho\|\sigma)=Tr\left[\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right]$$

Under two restrictions, we will give a way to calculate a benchmark!

(Restriction)

- States became mixed by a fixed rotationally covariant noisy channel.

$\{\rho_\omega, p_\omega\}_{\omega\in\Omega}=\{\mathrm{N}(|\psi_\omega\rangle\langle\psi_\omega|), p_\omega\}_{\omega\in\Omega}$ for a noisy channel Λ.

- The ensemble is rotationally invariant.

Discussion about the first restriction

(The first restriction)

: States became mixed by a fixed rotationally covariant noisy channel.

$$\{\rho_\omega, p_\omega\}_{\omega\in\Omega} = \{\mathrm{N}(|\psi_\omega\rangle\langle\psi_\omega|), p_\omega\}_{\omega\in\Omega}$$

for a noisy channel N s.t. $\mathrm{N}\left(U_\theta \rho U_\theta^*\right) = U_\theta \mathrm{N}(\rho) U_\theta^*$.

This is a natural assumption for experiment (e.g. attenuation channel).

Under this restriction, we can redefine a quantum benchmark as follows:

The optimal average fidelity between an ideal input pure state and a output state:

$$\overline{F}(\mathbf{T}) \equiv \int_{\omega\in\Omega} F(|\psi_\omega\rangle\langle\psi_\omega| \,\|\, \Gamma(\rho_\omega))\, d\omega = \int_{\omega\in\Omega} Tr\left(|\psi_\omega\rangle\langle\psi_\omega| \cdot \Gamma(\rho_\omega)\right) d\omega$$

$$\overline{F} \equiv \sup_{\mathbf{T}\in \mathrm{E}_b} \overline{F}(\mathbf{T})$$

$\overline{F}$ is still a legitimate quantum benchmark.

We succeeded to remove non-linearity from the definition of benchmark!

Discussion about the rotational invariance

(The second restriction) The ensemble is rotationally invariant.
=We should rotate a input state randomly in the phase space.

↓

But, this is easily done in an experiment.
We do not need to do anything, but just wait for a short time!
(Rotation in the phase space is just a natural time evolution.)

↓

However, the rotational invariance makes the problem much simpler!

Group invariance of an ensemble ⟷ Group covariance of the optimal strategy

Group invariance and Group covariance

Suppose $\{\rho_\omega, p_\omega\}_{\omega\in\Omega}$ is invariant under the action of a symmetric group G.

That is, $\forall g \in G,\ p_\omega = p_{g(\omega)}$

and $\exists$ unitary representation s.t. $\rho_{g(\omega)} = U_g \rho_\omega U_g^*$.

⇓

Then, we can choose an group covariant optimal strategy.

Γ is covariant w.r.t. G ⟺ (define) $\forall \rho, U_g \Gamma(\rho) U_g^* = \Gamma(U_g \rho U_g^*)$

(Proof for a compact group)

Suppose Γ is a optimal classical strategy.

Define a covariant $\overline{\Gamma}$ by $\overline{\Gamma}(\rho) = \int_g dg\, U_g^* \Gamma(U_g \rho U_g^*) U_g$.

Then, $\overline{F}(\overline{\Gamma}) = \int d\omega\ p_\omega F(\rho_\omega \| \overline{\Gamma}(\rho_\omega))$

$$\geq \iint d\omega dg\ p_\omega F(\rho_{g(\omega)} \| \Gamma(\rho_{g(\omega)}))$$

$$= \iint d\omega dg\ p_{g^{-1}(\omega)} F(\rho_\omega \| \Gamma(\rho_\omega)) = \overline{F}(\Gamma)$$

We can do the same discussion for F_0 .

Even for a “non-compact” group this statement is valid!

Main results

Under the two restriction,

- States became mixed by a fixed rotationally covariant noisy channel.
- The ensemble is rotationally invariant.

We derive the following results for an ensemble of squeezed states:

1. For input states with uniform rotations and displacement,
 we derive an analytical formula of F_0.

2. For input states with uniform rotations and general displacement,
we derive an upper-bound of $\overline{F}$ described as a finite dimensional SDP.
So, we can efficiently calculate it by a numerical calculation.

Notations

- An ensemble of squeezed states $\{\rho_\omega, p_\omega\}_{\omega\in\Omega}$,

  $\omega = (\xi, \theta)$, $\Omega = C \times [0, 2\pi]$ (Fixed squeezing)

  ρ_ω is characterized by the covariant matrix (CM) γ_{ρ_ω} and the displacement vector d_{ρ_ω} :

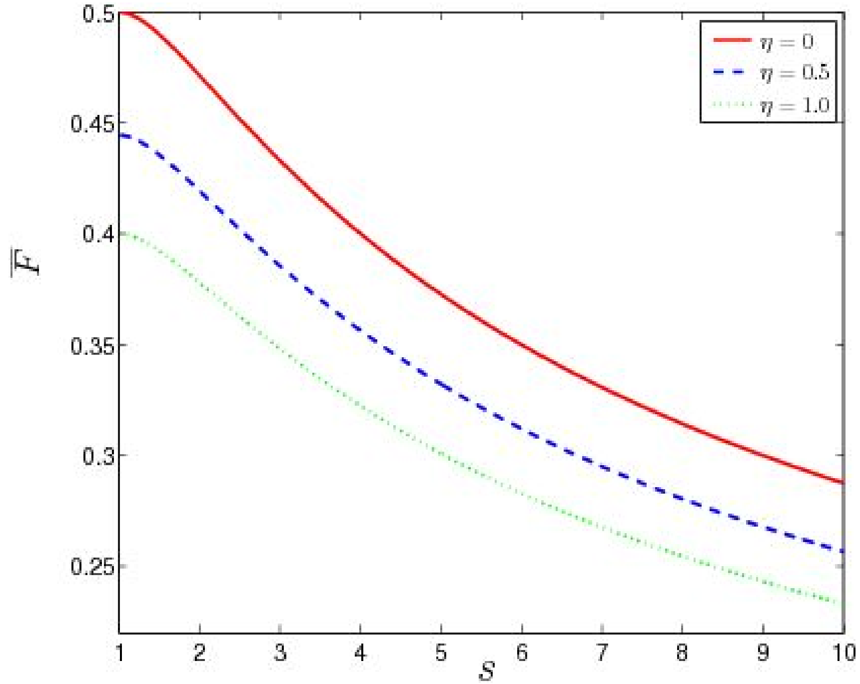
$$\gamma_{\rho_\omega} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$d_{\rho_\omega} = \xi$$

In other words, ρ_ω is derived from a squeezed vacuum ρ_s as

$\rho_\omega = W_\xi U_\theta \rho_s U_\theta^* W_\xi^*$, where W_ξ is a displacement (Weyl) operator, and U_θ is a phase-rotation operator.

Main theorem: Uniform displacement



Main theorem (pure states):

$$F_0(s) = \sup_{\Gamma \in E_b} \inf_{\omega \in \Omega} Tr[\rho_\omega \Gamma(\rho_\omega)] = \frac{\sqrt{s}}{1+s}$$

Main theorem (mixed states):

For a noisy channel N s.t. $\gamma_{\mathrm{N}(\rho)} = \gamma_\rho + \eta I$ and $d_{N(\rho)} = d_\rho$,

$$F_0(s) = \sup_{\Gamma \in E_b} \inf_{\omega \in \Omega} Tr[\rho_\omega \Gamma(\mathrm{N}(\rho_\omega))] = \left[\left(1 + \frac{\eta}{2} + \frac{1}{s}\right)\left(1 + \frac{\eta}{2} + s\right)\right]^{-1/2}$$

For pure squeezing states, $F_0(s) < 1/2$ for any $s \neq 1$.

Without the rotation, $F_0(s) = 1/2$ for any s .

Proof of the main theorem

(Proof for a pure ensemble)

From the phase space invariance of the ensemble,

$$F_0(s) = \sup_{\Gamma \in \mathrm{E}_b} \inf_{(\xi,\theta)\in C\times[0,2\pi]} Tr\left[W_\xi U_\theta \rho_s U_\theta^* W_\xi^* \, \Gamma\left(W_\xi U_\theta \rho_s U_\theta^* W_\xi^*\right)\right]$$

$$= \sup_{\overline{\Gamma} \in \overline{\mathrm{E}}_b} \inf_{\theta\in[0,2\pi]} Tr\left[U_\theta \rho_s U_\theta^* \, \overline{\Gamma}\left(U_\theta \rho_s U_\theta^*\right)\right]$$

($\overline{\mathrm{E}}_b$ is a set of all covariant EBs)

We just need to optimize over squeezed vacuums.

Then, we use the following lemma.

Lemma (Holevo 96) For a phase space covariant channel Γ, $\Gamma \circ \vartheta$ is completely positive, iff there exist a state τ s.t. it has the form $\Gamma^*\left(W_\xi\right) = Tr\left(\tau W_{\sqrt{2}\xi}\right) W_\xi$,

where ϑ is the time reversal operator defined by $\vartheta\left(W_\xi\right) = W_{Z\xi}$.

All EB channel Γ satisfies the complete positivity of $\Gamma \circ \vartheta$.

Proof of the main theorem

By using the lemma and the Parseval relation:

$$
\begin{aligned}
Tr\left(\rho\,\overline{\Gamma}(\rho)\right) &= \frac{1}{2\pi}\int d^2\xi\; Tr\left(\rho W_\xi\right) Tr\left(\overline{\Gamma}(\rho) W_\xi\right) \\
&= \frac{1}{2\pi}\int d^2\xi\; Tr\left(\rho W_\xi\right)^2\; Tr\left(\tau W_{\sqrt{2}\xi}\right) \\
&= \frac{1}{2\pi}\int d^2\xi\; Tr\left(\rho W_{\sqrt{2}\xi}\right) Tr\left(\tau W_{\sqrt{2}\xi}\right) = \frac{1}{2} Tr(\rho\,\tau)
\end{aligned}
$$

Thus, $F_0 \le \sup_\tau Tr\left(\frac{1}{4\pi}\left(\int_0^{2\pi} U_\theta \rho_s U_\theta^* d\theta\right)\right) = \left\| \frac{1}{4\pi}\left(\int_0^{2\pi} U_\theta \rho_s U_\theta^* d\theta\right)\right\|_{op}$

Since $\frac{1}{2\pi}\int_0^{2\pi} U_\theta \rho_s U_\theta^* d\theta$ consists of just diagonal elements of ρ_s,
we can conclude $F_0 \le \frac{1}{2}\langle 0|\rho_s|0\rangle = \frac{\sqrt{s}}{1+s}$.

Moreover, the following strategy can achieve this upperbound!

(Optimal strategy)
Bob Prepares $W_\xi|0\rangle$ after Alice's heterodyne measurement $\{W_\xi|0\rangle\langle 0|W_\xi / 2\pi\}$.

The same optimal strategy w.r.t. the coherent states case.

Finite displacement

So far, we have derived an analytical formula of the benchmark in the case of uniform rotation and uniform displacement in phase space.

⇩

However, uniform displacement is impossible in an experiment!

We need to find a way to calculate a value of benchmark for an ensemble with finite (or exponentially dumping) displacement.

⇩

Here, we give an upper-bound of $\overline{F}$ which is in the form of a finite dimensional SDP. (Therefore, efficiently calculable)

Main theorem: finite displacement

For an ensemble of squeezed states $\{U_\theta \rho_{s,\xi} U_\theta^*, q(\xi)\}_{(\xi,\theta)\in C\times[0,2\pi]}$,
where $\rho_{s,\xi} = W_\xi \rho_s W_\xi^*$, we derived the following theorem:

Main theorem

For any probability density $q(\xi)$ and rotationally covariant noise channel N , we have

$$\overline{F} \overset{def}{=} \sup_{\mathbf{T}\in \mathrm{E}_b} \int_{\theta\in[0,2\pi]} \int_{\xi\in C} q(\xi)\, Tr\left[\mathbf{T}\left(\mathrm{N}\left(U_\theta \rho_{s,\xi} U_\theta^*\right)\right)\cdot U_\theta \rho_{s,\xi} U_\theta^*\right] \frac{d\theta}{2\pi} d\xi$$

$$\leq \sup_{\Omega\in B(supp R_c)} \left\{ Tr\left(\Omega P_c \eta P_c\right) \middle| \Omega \geq 0, \Omega^\Gamma \geq 0, Tr_B \Omega = I_A \right\} + 1 - Tr\left(P_c \eta P_c\right)$$

where $\eta \overset{def}{=} \int_{\theta\in[0,2\pi]} \int_{\xi\in C} q(\xi)\, U_\theta \mathrm{N}\left(\rho_{s,\xi}\right) U_\theta^* \otimes U_\theta \rho_{s,\xi} U_\theta^* \frac{d\theta}{2\pi} d\xi$

$$P_c = \sum_{i=0}^{c} \sum_{k+l=i} |k\rangle\langle k| \otimes |l\rangle\langle l| ,$$

and $R_c = \left(\sum_{i=0}^{c} |i\rangle\langle i|\right) \otimes \left(\sum_{i=0}^{c} |i\rangle\langle i|\right)$.

This bound only includes a finite dimensional SDP.
$P_c \eta P_c$ can be also derived by a numerical calculation.

Relation with Miguel's talk

Suppose

$$\overline{F}_{c,N} = \sup_{\Omega \in B(Ran R_c)} \left\{ Tr\left(\Omega P_c \eta P_c\right) \middle| \Omega \in S_{N,ppt}, Tr_B \Omega = I_A \right\} + 1 - Tr(P_c \eta P_c),$$

where $S_{N,ppt}$ is the set of all PPT N - extendible positive operators.

Then, we derive $\overline{F}_{c,N} \geq \overline{F}$ and $\overline{F} = \lim_{c,N\to\infty} \overline{F}_{c,N}$.

However, from our experience, when our memory is limited,

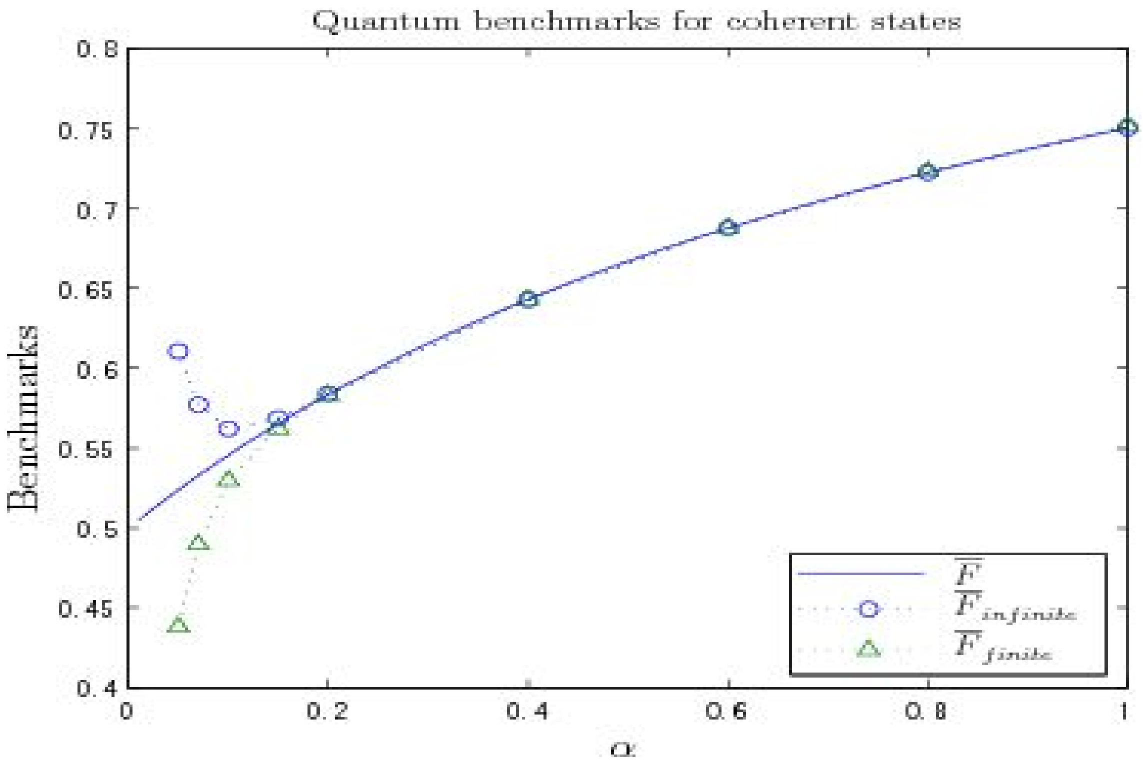
by increasing c , we can decrease $\overline{F}_{c,N}$ more than by increasing N.

Practically, we should choose $N=1$.

($S_{1,ppt}$ is the set of all PPT positive operators.)

Numerical result 1

When $s = 1$ and $q(\xi) = \frac{\alpha}{\pi}\exp\left(-\alpha\|\xi\|^2\right)$, our ensemble coincides Hammerer et al.'s ensemble of coherent states. → We can compare them.



(We chose $c = 35$.)

$$\overline{F} \leq \overline{F}_{infinite} \overset{def}{=} \overline{F}_{finite} + 1 - Tr\left(P_c \eta P_c\right)$$

$$\overline{F}_{finite} = \sup_{\Omega \in B(Ran\ R_c)} \left\{ Tr\left(\Omega P_c \eta P_c\right) \middle| \Omega \geq 0, \Omega^{\Gamma} \geq 0, Tr_B \Omega = I_A \right\}$$

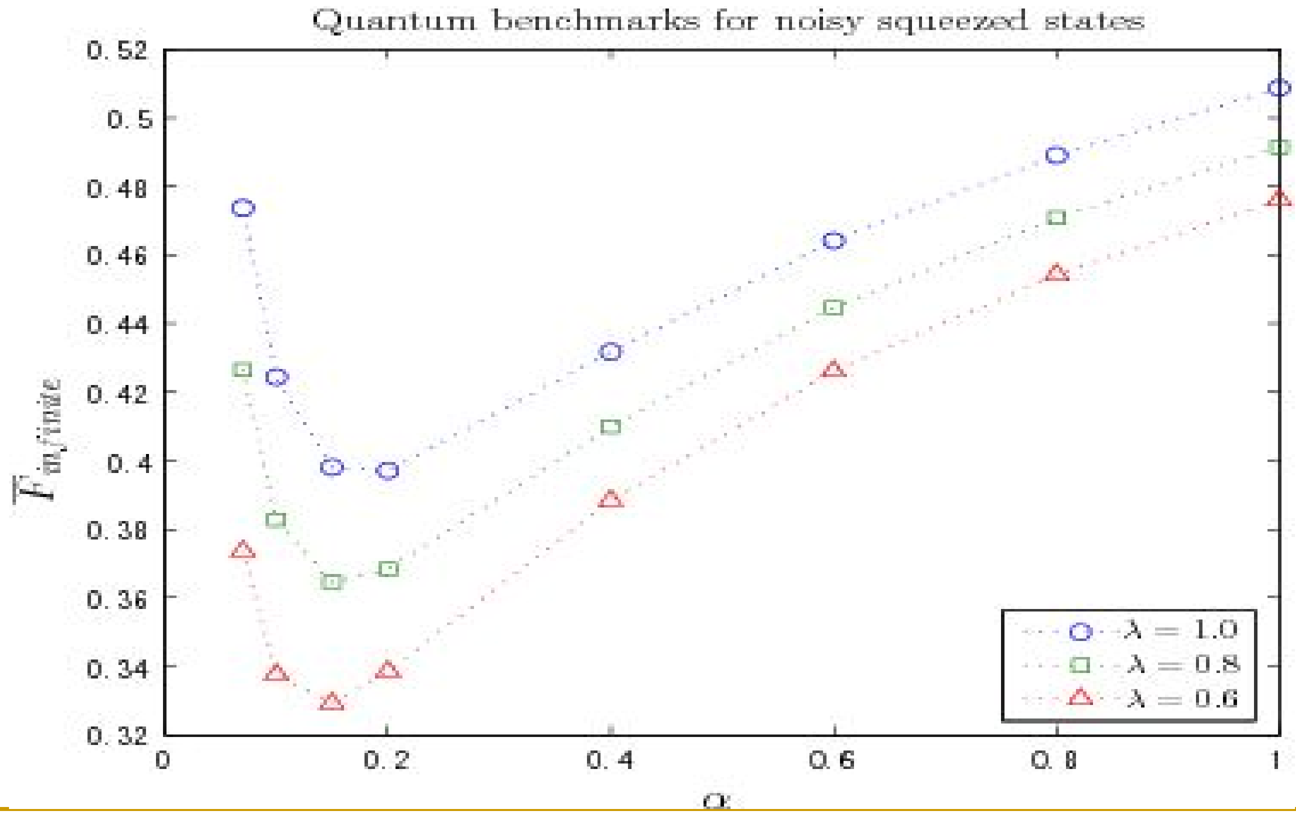
Our method efficiently recover's the known result for coherent states!

Numerical result 2:

The numerical results for $q(\xi) = \frac{\alpha}{\pi} \exp\left(-\alpha \|\xi\|^2\right)$ and $s = 8$.

A noisy channel N is chosen as an attenuation channel: $\gamma_{N(\rho)} = \gamma_\rho + \lambda I$

$$d_{N(\rho)} = \sqrt{\lambda}\, d_\rho$$



(We chose $c = 30$.)

In the case of squeezed states, $\overline{F}$ can be less than $\frac{1}{2}$ for a finite displacement.

Proof of the main theorem

First, we use the following well-known lemma about the Jamiolkowski isomorphism of EB channels:

Lemma

A channel Γ on H is entanglement breaking,

if and only if

there exist a unique separable positive operator $\Omega(\Gamma)$ on H $\otimes$H such that

$$Tr_{B}\left(\Omega(\Gamma)\right)=\mathbf{I}_{B} \quad \text{and} \quad Tr\left(B\Gamma(A)\right)=Tr\left(\Omega(\Gamma)A\otimes B\right)$$

for all $A\in C_1(H)$ and $B\in B(H)$.

We can reduce an optimization over entanglement breaking channels to an optimization over separable states.

Proof of the main theorem

By using the lemma, we derive:

$$\overline{F} = \sup_{\Omega\in B(H)} \left\{ Tr(\Omega\eta) \,|\, \Omega \in \mathrm{Sep}, Tr_{\mathrm{B}}\Omega = \mathbf{I}_{\mathrm{A}} \right\}$$

$$\leq \sup_{\Omega\in B(H)} \left\{ Tr(\Omega\eta) \,|\, \Omega \geq 0, \Omega^{\Gamma} \geq 0, Tr_{\mathrm{B}}\Omega = \mathbf{I}_{\mathrm{A}} \right\},$$

where $\eta \overset{def}{=} \int_{\theta\in[0,2\pi]} \int_{\xi\in C} q(\xi)\, U_\theta \mathrm{N}\left(\rho_{s,\xi}\right) U_\theta^* \otimes U_\theta \rho_{s,\xi} U_\theta^* \frac{d\theta}{2\pi} d\xi.$

We succeeded to reduce the problem to an infinite dimensional SDP.

However, we could not numerically solve an infinite dim. SDP.

So, then, we reduce the above infinite dim. SDP to a finite dim. SDP.

Proof of the main theorem

We need the following lemma:

Lemma. If a positive separable operator $\Omega \in B(H \otimes H)$ satisfies $Tr_B \Omega = \mathbf{I}_A$, then, $\|\Omega\|_{op} \leq 1$.

By using the above lemma, we derive

$$\begin{aligned} Tr(\Omega\eta) &= Tr(\Omega P_c \eta P_c) + Tr(\Omega(\eta - P_c \eta P_c)) \\ &\leq Tr(\Omega P_c \eta P_c) + \|\Omega\|_{op} \cdot \|\eta - P_c \eta P\|_{tr} \\ &= Tr(\Omega P_c \eta P_c) + \|\Omega\|_{op} \cdot Tr(\eta - P_c \eta P_c) \\ &\leq Tr(\Omega P_c \eta P_c) + 1 - Tr(P_c \eta P_c) \end{aligned}$$

where $P_c = \sum_{i=0}^{c} Q_c$ and $Q_c = \sum_{k+l=c} |k\rangle\langle k| \otimes |l\rangle\langle l|$.

In the third line, we use $\eta - P_c \eta P_c = \sum_{i=c+1}^{\infty} Q_i \eta Q_i \geq 0$; this can be seem from $\eta = \sum_{i=0}^{\infty} Q_i \left(\int_{\xi \in \mathbb{C}} q(\xi) \mathrm{N}(\rho_{s,\xi}) \otimes \rho_{s,\xi} d\xi \right) Q_i$.

We used the rotationally covariance of N .

Proof of the main theorem

Finally, by using the previous inequality, we derive

$$\begin{aligned}
\overline{F} &\leq \sup_{\Omega \in \mathrm{B}(\mathrm{H})} \left\{ Tr\left(\Omega P_c \eta P_c\right) \mid \Omega \in \mathrm{Sep},\ Tr_{\mathrm{B}}\Omega = \mathbf{I}_{\mathrm{A}} \right\} + 1 - Tr\left(P_c \eta P_c\right) \\
&= \sup_{\Omega \in \mathrm{B}(\mathrm{H})} \left\{ Tr\left(R_c \Omega R_c P_c \eta P_c\right) \mid \Omega \in \mathrm{Sep},\ Tr_{\mathrm{B}}\Omega = \mathbf{I}_{\mathrm{A}} \right\} + 1 - Tr\left(P_c \eta P_c\right) \\
&\leq \sup_{\Omega \in \mathrm{B}(\mathrm{supp} R_c)} \left\{ Tr\left(\Omega P_c \eta P_c\right) \mid \Omega \in \mathrm{Sep},\ Tr_{\mathrm{B}}\Omega \leq \mathbf{I}_{\mathrm{A}} \right\} + 1 - Tr\left(P_c \eta P_c\right) \\
&\leq \sup_{\Omega \in \mathrm{B}(\mathrm{supp} R_c)} \left\{ Tr\left(\Omega P_c \eta P_c\right) \mid \Omega \geq 0,\ \Omega^{\Gamma} \geq 0,\ Tr_{\mathrm{B}}\Omega \leq \mathbf{I}_{\mathrm{A}} \right\} + 1 - Tr\left(P_c \eta P_c\right),
\end{aligned}$$

where $R_c = \left(\sum_{i=0}^{c} |i\rangle\langle i|\right) \otimes \left(\sum_{i=0}^{c} |i\rangle\langle i|\right)$.

We have completed the proof!

Summary

Under the two restriction,

- States became mixed by a fixed rotationally covariant noisy channel.
- The ensemble is rotationally invariant.

We derived the following results for an ensemble of squeezed states:

1. For input states with uniform rotations and displacement,
we derived an analytical formula: $$F_0(s) = \left[\left(1 + \frac{\eta}{2} + \frac{1}{s}\right)\left(1 + \frac{\eta}{2} + s\right)\right]^{-1/2}$$

2. For input states with uniform rotations and general displacement,
we derived an upper-bound of $\overline{F}$ described as a finite dimensional SDP.
So, we can efficiently calculate it by a numerical calculation.

Future works

Now Copenhagen Group is running an experiment of an atomic ensemble quantum memory for a rotationally invariant set of squeezed states.

The experimental result may appear soon.