Group theoretical study of LOCC-detection of maximally entangled state using hypothesis testing

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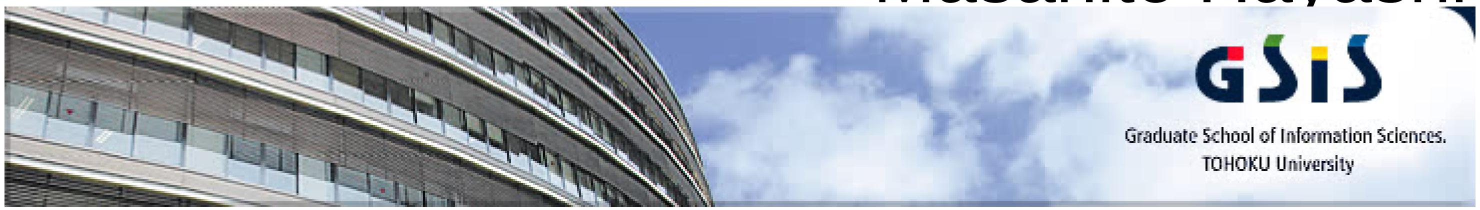

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DEX-SMI

Contents

- Motivation
- Formulation
- Testing of binomial distributions
- Global tests
- A-B locality
- Two different information sources
- A-B locality and sample locality

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Problem

- You find a Bell pair source in a shop.

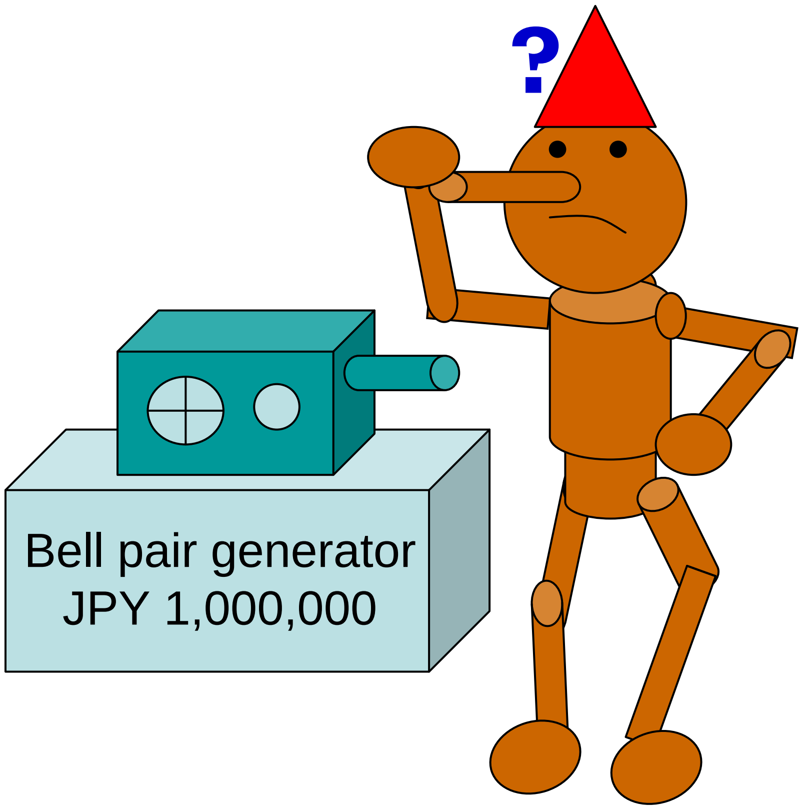
- Shall you buy this?
  - Real Bell pair source ?
  - Precision?

How to check ?

Tomography?, witness?

Bell's inequality?

Coincidence count?



By K. Matsumoto

NO .. !!! PLEASE USE OUR METHOD

Problems with existing methods

Careless treatment of the error.

Often, the conclusion is meaningless

Here, we introduce notion of hypothesis test which is used in statistics.

These methods are not optimal. There are more precise methods, which is easy to implement

We search for optimal test.

Conventional visibility and witness have bias.

(The quality depends not only on the fidelity but also on the angle.)

We propose symmetric testing.

Who needs precise and efficient test?

Huge number of data is available. → Even poor test gives a precise decision!

However, this is not always true.

1. Bell pair sharing in the long distance (repeater, entanglement distillation)
2. At the development stage, some new Bell pair sources may be very unstable.

Huge number of data is not available. → Poor test gives a bad decision! → **Efficient test is needed!**

Contents

Quantum Hypothesis Testing I

- Two Hypotheses

$H_0 : \rho \in \mathcal{S}_0$ **null hypothesis**

$H_1 : \rho \in \mathcal{S}_1$ **alternative hypothesis**

- Two-valued measurement $M = \{T, I - T\}$

T : **Accept** H_0

$I - T$: **Accept** H_1 **(Reject** H_0**)**

- Two error probabilities

First error probability

$$\alpha(T,\rho) \triangleq 1 - \mathrm{Tr}\rho T \quad (\rho \in \mathcal{S}_0)$$

Second error probability

$$\beta(T,\rho) \triangleq \mathrm{Tr}\rho T \quad (\rho \in \mathcal{S}_1)$$

Quantum Hypothesis Testing II

- Test T is called level-α if $\alpha(T,\rho) \le \alpha \ \forall \rho \in \mathcal{S}_0$

$$\mathcal{T}_{\alpha,\mathcal{S}_0} \triangleq \left\{ T \middle| 0 \le T \le I, \forall \rho \in \mathcal{S}_0, \alpha(T,\rho) \le \alpha \right\}$$

$$\beta_\alpha\left(\mathcal{S}_0 \| \rho\right) \triangleq \min_{T \in \mathcal{T}_{\alpha,\mathcal{S}_0}} \beta(T,\rho)$$

- Test T is called a UMP level-α test if $\beta(T,\rho) = \beta_\alpha\left(\mathcal{S}_0 \| \rho\right), \forall \rho \in \mathcal{S}_1$
- Test T is called a UMP C_1, C_2 level-α test if $\beta(T,\rho) = \beta^{C_2}_{\alpha,C_1}\left(\mathcal{S}_0 \| \rho\right), \forall \rho \in \mathcal{S}_1$

$$\beta^{C_2}_{\alpha,C_1}\left(\mathcal{S}_0 \| \rho\right) \triangleq \min_{T \in \mathcal{T}_{\alpha,\mathcal{S}_0}} \left\{ \beta(T,\rho) \middle| \begin{matrix} T \text{ satisfies} \\ C_1 \text{ and } C_2 \end{matrix} \right\}$$

C_1, C_2 : conditions

Our setting

- Is the state close to the maximally entangled state $\phi^0_{A,B} \triangleq \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B$ on $\mathcal{H}_{A,B}$?
- We assume that n i.i.d. condition *i.e.,* $\rho = \sigma^{\otimes n}$
- Our hypotheses are

$$H_0 : \sigma \in \mathcal{S}_{\leq \varepsilon} \triangleq \left\{ \sigma \middle| 1 - \left\langle \phi^0_{A,B} \middle| \sigma \middle| \phi^0_{A,B} \right\rangle \leq \varepsilon \right\}$$

$$H_1 : \sigma \in \mathcal{S}^c_{\leq \varepsilon}$$

- The set of level-α tests

$$\mathcal{T}^n_{\alpha,\leq\varepsilon} \triangleq \left\{ T \middle| 0 \leq T \leq I, \forall \sigma \in \mathcal{S}_{\leq\varepsilon}, 1 - \mathrm{Tr}\,\sigma^{\otimes n} T \leq \alpha \right\}$$

Group invariance on $\mathcal{H}_{A,B}$

- U(1)-action

$$U_\theta \triangleq e^{i\theta}\left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right| + \left(I - \left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right|\right)$$

- SU(d)-action

$$U(g) \triangleq g \otimes \overline{g}, \forall g \in SU(d)$$

- SU(d)xU(1)-action

$$U(g,\theta) \triangleq U(g)U_\theta, \forall (g,\theta) \in SU(d) \times U(1)$$

- U(d²-1)-action

$$V(g) \triangleq \left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right| + g\left(I - \left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right|\right),$$

$$\forall g \in U(d^2 - 1)$$

Locality restriction

$$\beta^{C}_{\alpha,n,G}\left(\varepsilon\|\sigma\right) \triangleq \min_{T\in\mathcal{T}_{\alpha,\mathcal{S}_{\leq\varepsilon}}}\left\{\beta\left(T,\sigma^{\otimes n}\right)\middle| \begin{array}{l} T: G\text{-inv.} \\ T \text{ satisfies } C \end{array}\right\}$$

where $G=U(1), SU(d), SU(d)\times U(1), U(d^2-1)$

$\varnothing$: **No condition**

$S(A,B)$: A-B **separable**

$L(A \rightleftarrows B)$: **2-way LOCC**

$L(A\to B)$: **1-way LOCC** $A\to B$

$S(A_1,\ldots,A_n,B_1,\ldots,B_n)$: **separable on** $A_1,\ldots,A_n,B_1,\ldots,B_n$

$L(A_1,\ldots,A_n,B_1,\ldots,B_n)$:

2-way LOCC on $A_1,\ldots,A_n,B_1,\ldots,B_n$

$L(A_1,\ldots,A_n\to B_1,\ldots,B_n)$:

1-way LOCC $A_1,\ldots,A_n\to B_1,\ldots,B_n$

Contents

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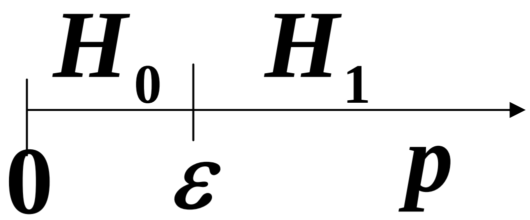
Binomial distribution

- The data k obeys the distribution

$$P_p^n(k) \triangleq \binom{n}{k}(1-p)^{n-k}\,p^k$$



- The test is described by a map $\tilde{T}$ from $\{0,1,\dots,n\}$ to $[0,1]$

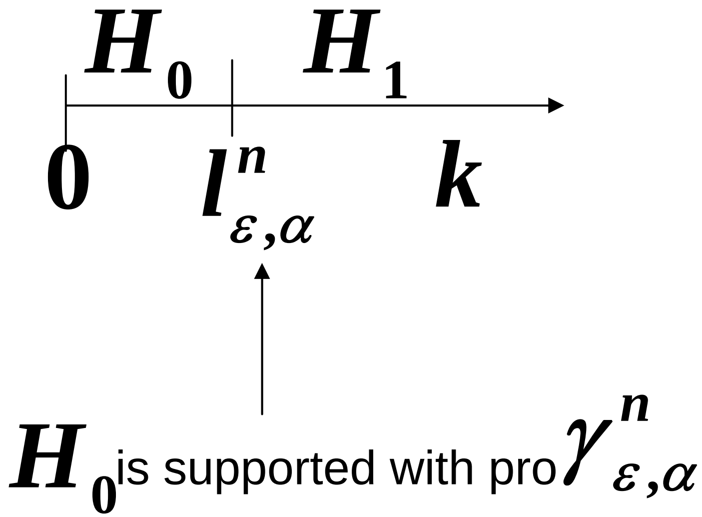
$$\beta_\alpha^n(\varepsilon\|q) \triangleq \min_{\tilde{T}}\left\{P_q^n(\tilde{T})\,\middle|\,\forall p\in[0,\varepsilon],1-P_p^n(\tilde{T})\le\alpha\right\}$$

$$P_p^n(\tilde{T}) \triangleq \sum_{k=0}^n P_p^n(k)\tilde{T}(k)$$

- The test $\tilde{T}_{\varepsilon,\alpha}^n$ (will be fined in the next slide) is UMP test, *i.e.*, $\beta(\tilde{T}_{\varepsilon,\alpha}^n, P_q^n) = \beta_\alpha^n(\varepsilon\|q),\ \forall p>\varepsilon.$

Definition of $\tilde{T}^{n}_{\varepsilon,\alpha}$

$$\tilde{T}^{n}_{\varepsilon,\alpha}(k) \triangleq \begin{cases} 1 & \text{if } k < l^{n}_{\varepsilon,\alpha} \\ \gamma^{n}_{\varepsilon,\alpha} & \text{if } k = l^{n}_{\varepsilon,\alpha} \\ 0 & \text{if } k > l^{n}_{\varepsilon,\alpha}, \end{cases}$$



where $\gamma^{n}_{\varepsilon,\alpha}$ and $l^{n}_{\varepsilon,\alpha}$ are defined as

$$\sum_{k=0}^{l^{n}_{\varepsilon,\alpha}-1} P^{n}_{\varepsilon}(k) < 1-\alpha \le \sum_{k=0}^{l^{n}_{\varepsilon,\alpha}} P^{n}_{\varepsilon}(k),$$

$$\gamma^{n}_{\varepsilon,\alpha} P^{n}_{\varepsilon}(l^{n}_{\varepsilon,\alpha}) = 1-\alpha-\sum_{k=0}^{l^{n}_{\varepsilon,\alpha}-1} P^{n}_{\varepsilon}(k).$$

Asymptotic theory (small deviation)

- When the true parameter is close to 0, the distribution goes to Poisson distribution.

$$\lim_{n\to\infty} P_{t/n}^{n}(k) = P_t(k) \triangleq e^{-t}\frac{t^k}{k!}$$

$$\lim_{n\to\infty} \beta_\alpha^{n}(\delta/n \| t'/n) = \beta_\alpha(\delta \| t')$$

$$\beta_\alpha(\delta \| t') \triangleq \min_{\tilde{T}} \left\{ P_{t'}(\tilde{T}) \middle| \begin{array}{l} \forall t \in [0,\delta], \\ 1 - P_t(\tilde{T}) \le \alpha \end{array} \right\}$$

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Global Test (n samples)

Theorem

$$\beta_{\alpha,n,G}(\varepsilon\|\sigma)=\beta_\alpha^n(\varepsilon\|p)$$

$$=\beta\left(T_\alpha^n\left(\left|\phi_{A,B}^0\right\rangle\left\langle\phi_{A,B}^0\right|,\varepsilon\right),\sigma^{\otimes n}\right)$$

where $G=U(1),SU(d)\times U(1),U(d^2-1)$

$$T_\alpha^n(T,\varepsilon)\triangleq\sum_{k=0}^{l_{\varepsilon,\alpha}^n-1}P_k^n(T)+\gamma_{\varepsilon,\alpha}^n P_{l_{\varepsilon,\alpha}^n}^n(T)$$

$$P_k^n(T)\triangleq\underbrace{(I-T)\otimes\cdots\otimes(I-T)}_{n-k}\otimes\underbrace{T\otimes\cdots\otimes T}_{k}$$

$$+\cdots$$

$$+\underbrace{T\otimes\cdots\otimes T}_{k}\otimes\underbrace{(I-T)\otimes\cdots\otimes(I-T)}_{n-k}.$$

Contents

A-B locality (One sample)

- For a rank-one POVM $M = \left\{ p_i \left| u_i \right\rangle \left\langle u_i \right| \right\}$ the test $T(M) \triangleq \sum_i p_i \left| u_i \otimes \overline{u_i} \right\rangle \left\langle u_i \otimes \overline{u_i} \right|$ is constructed as follows:
  1. Alice performs POVM $M = \left\{ p_i \left| u_i \right\rangle \left\langle u_i \right| \right\}$.
  2. Bob performs two-valued POVM $\left\{ \left| \overline{u_i} \right\rangle \left\langle \overline{u_i} \right|, I - \left| \overline{u_i} \right\rangle \left\langle \overline{u_i} \right| \right\}$
  3. If $\left| \overline{u_i} \right\rangle \left\langle \overline{u_i} \right|$ is observed, accept.
     Otherwise, reject

A-B locality (One sample)

- Covariant POVM $M^1_{\mathbf{cov}}$

$$M^1_{\mathbf{cov}}(d\varphi) \triangleq d \left|\varphi\right\rangle\left\langle\varphi\right| \nu(d\varphi)$$

where ν is invariant measure.

- The POVM $M^1_{\mathbf{cov}}$ can be realized by randomly choosing measuring basis

$$T(M^1_{\mathbf{cov}}) = \left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right| + 1/(d+1)\left(I - \left|\phi^0_{A,B}\right\rangle\left\langle\phi^0_{A,B}\right|\right)$$

$$\mathrm{Tr}\,T(M^1_{\mathbf{cov}})\sigma = 1 - dp/(d+1),$$

$$p \triangleq 1 - \left\langle\phi^0_{A.B}\right|\sigma\left|\phi^0_{A.B}\right\rangle$$

- The test $T^{1,A-B}_{\varepsilon,\alpha} \triangleq T^1_{\alpha}(T(M^1_{\mathbf{cov}}), d\varepsilon/(d+1))$ is level- α.

A-B locality (One sample)

$$\beta^{C}_{\alpha,1,G}(\varepsilon \| \sigma) = \mathrm{Tr}\,\sigma T^{1,A-B}_{\varepsilon,\alpha}$$

$$= \begin{cases} (1-\alpha)\left(1-\dfrac{dp}{d+1}\right) \Big/ \left(1-\dfrac{d\varepsilon}{d+1}\right) & \text{if } \dfrac{d\varepsilon}{d+1} \leq \alpha \\ 1-\dfrac{\alpha p}{\varepsilon} & \text{if } \dfrac{d\varepsilon}{d+1} > \alpha \end{cases}$$

where $G = SU(d), SU(d)\times U(1), U(d^2-1)$

$C = S(A,B), L(A \rightleftarrows B), L(A \to B)$

A-B locality (Two samples)

- Covariant POVM M^2_{cov}

$$M^2_{\mathrm{cov}}(dg_1 dg_2)$$

$$\triangleq d^2 (g_1 \otimes g_2)|u\rangle\langle u|(g_1 \otimes g_2)^\dagger \nu(dg_1)\nu(dg_2)$$

where ν is invariant measure, and u is maximally entangled on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$.

$$T(M^2_{\mathrm{cov}}) = |\phi^0_{A,B}\rangle\langle\phi^0_{A,B}| \otimes |\phi^0_{A,B}\rangle\langle\phi^0_{A,B}| + 1/(d^2-1)\left(I - |\phi^0_{A,B}\rangle\langle\phi^0_{A,B}|\right) \otimes \left(I - |\phi^0_{A,B}\rangle\langle\phi^0_{A,B}|\right)$$

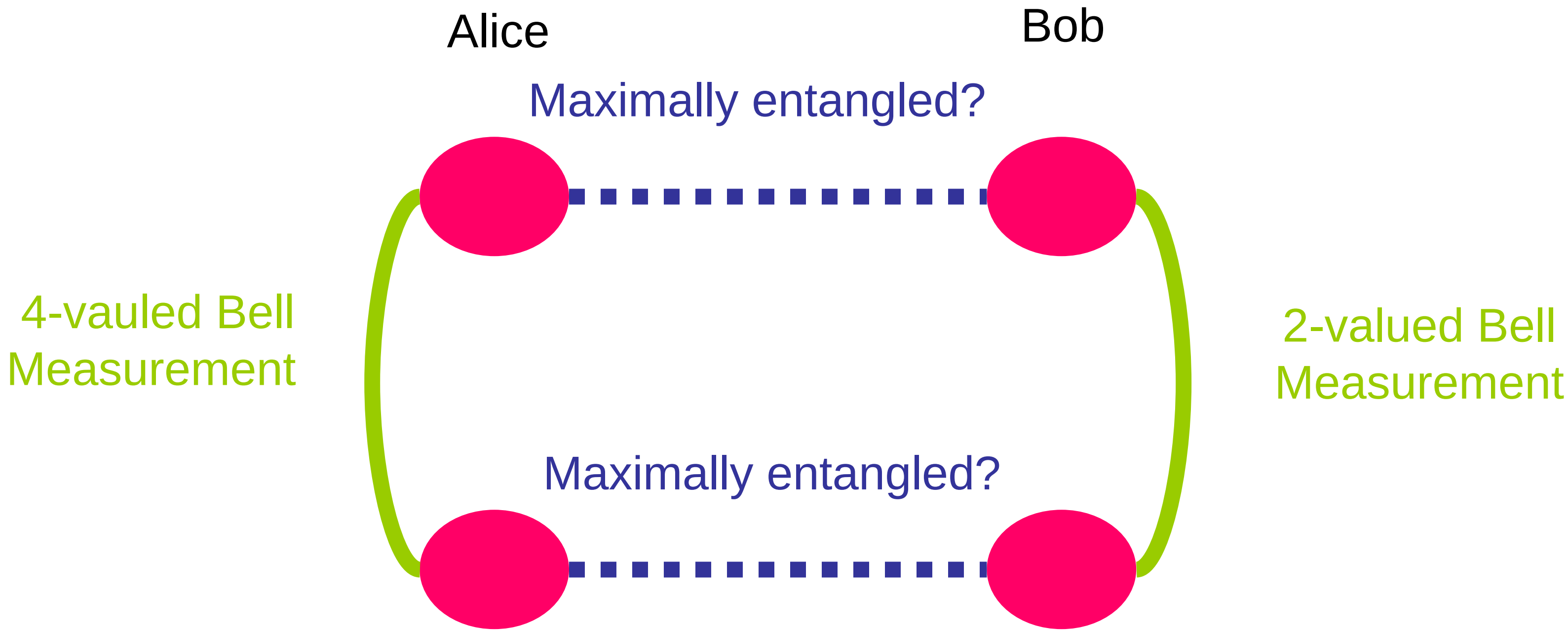
$$\mathrm{Tr}\,T(M^2_{\mathrm{cov}})\sigma^{\otimes 2} = 1 - 2p + d^2 p^2/(d^2-1)$$

A-B locality (Two samples)

- Bell POVM M^2_{Bell} on $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$

$$1 - 2p + p^2 \le \mathrm{Tr} T(M^2_{Bell})\sigma^{\otimes 2}$$

$$\le 1 - 2p + 2p^2$$



A-B locality ($2n$ samples)

- The test $T_{\varepsilon,\alpha}^{2n,A-B} \triangleq T_{\alpha}^{n}\left(T(M_{\text{cov}}^{2}), 2\varepsilon - \frac{d^2\varepsilon^2}{d^2-1}\right)$ is level-α .

$$\lim_{n\to\infty} \beta_{\alpha,n,G}^{C}(\delta/n \| \sigma_n) = \lim_{n\to\infty} \beta\left(T_{\varepsilon,\alpha}^{2n,A-B}, \sigma_n^{\otimes 2n}\right)$$

$$= \beta_{\alpha}(\delta \| t)$$

where $\left\langle \phi_{A,B}^{0} \middle| \sigma_n \middle| \phi_{A,B}^{0} \right\rangle = 1 - t/n$

$$G = U(1), SU(d)\times U(1), U(d^2-1)$$
$$C = \emptyset, S(A,B), L(A \rightleftarrows B), L(A \to B)$$

Hence, $T_{\varepsilon,\alpha}^{2n,A-B}$ is asymptotically UMP C G-inv. Test.

A-B locality ($2n$ samples)

- The test $T^{2n,A-B}_{\varepsilon,\alpha,Bell} \triangleq T^n_\alpha\left(T(M^2_{Bell}), 2\varepsilon - \frac{d^2\varepsilon^2}{d^2-1}\right)$ is asymptotically level- α .

$$\lim_{n\to\infty} \beta^C_{\alpha,n,G}(\delta/n \| \sigma_n) = \lim_{n\to\infty} \beta\left(T^{2n,A-B}_{\varepsilon,\alpha,Bell}, \sigma_n^{\otimes 2n}\right)$$

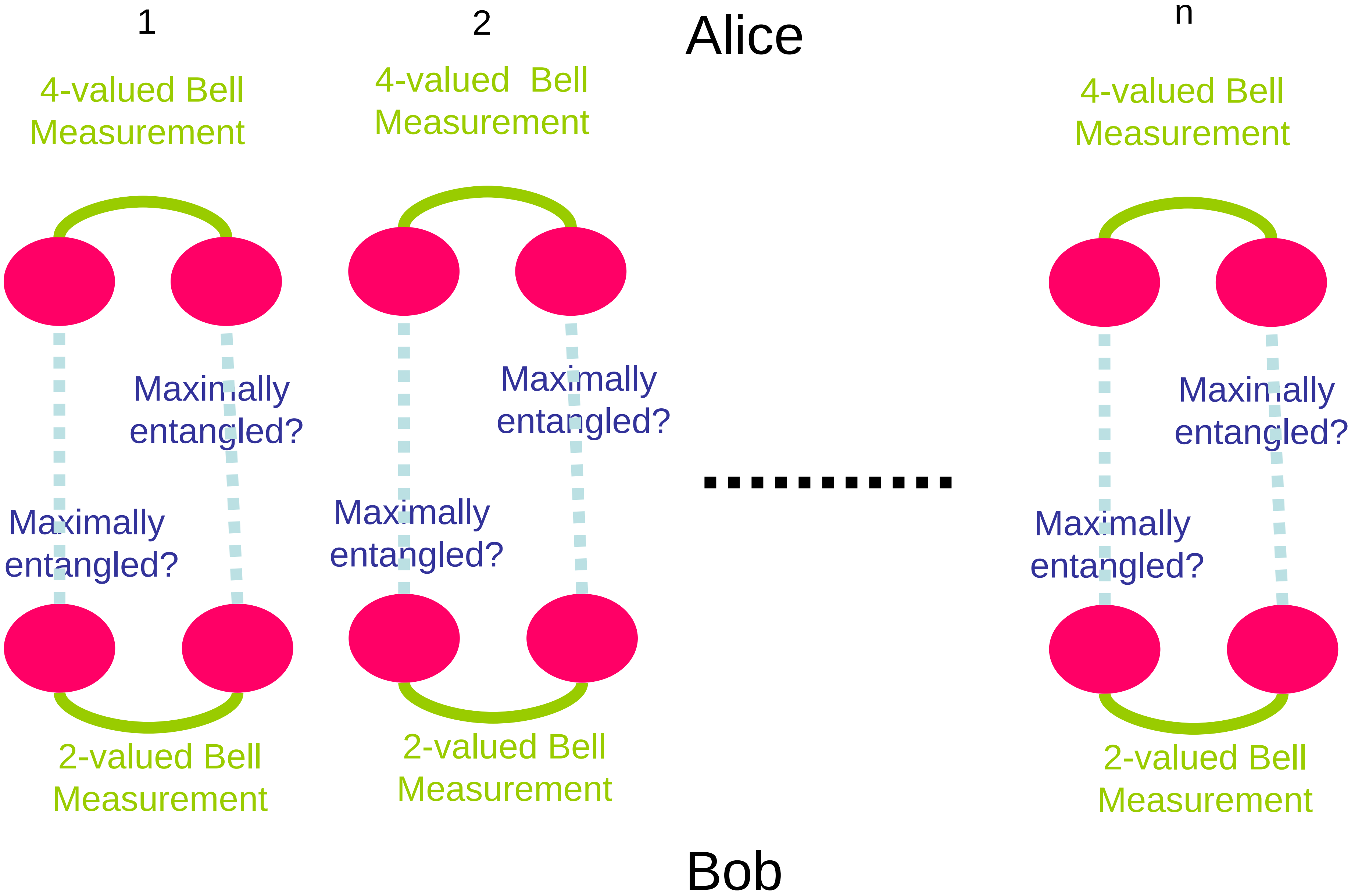
$$= \beta_\alpha(\delta \| t)$$

where $\left\langle \phi^0_{A,B} \middle| \sigma_n \middle| \phi^0_{A,B} \right\rangle = 1 - t/n$

$$G = U(1), SU(d)\times U(1), U(d^2-1)$$
$$C = \emptyset, S(A,B), L(A \rightleftarrows B), L(A \to B)$$

Hence, $T^{2n,A-B}_{\varepsilon,\alpha,Bell}$ is asymptotically UMP C G-inv. Test.

Experiment


1
2
Alice
n
4-valued Bell Measurement
4-valued Bell Measurement
4-valued Bell Measurement
Maximally entangled?
Maximally entangled?
Maximally entangled?
Maximally entangled?
Maximally entangled?
Maximally entangled?
2-valued Bell Measurement
2-valued Bell Measurement
2-valued Bell Measurement
Bob

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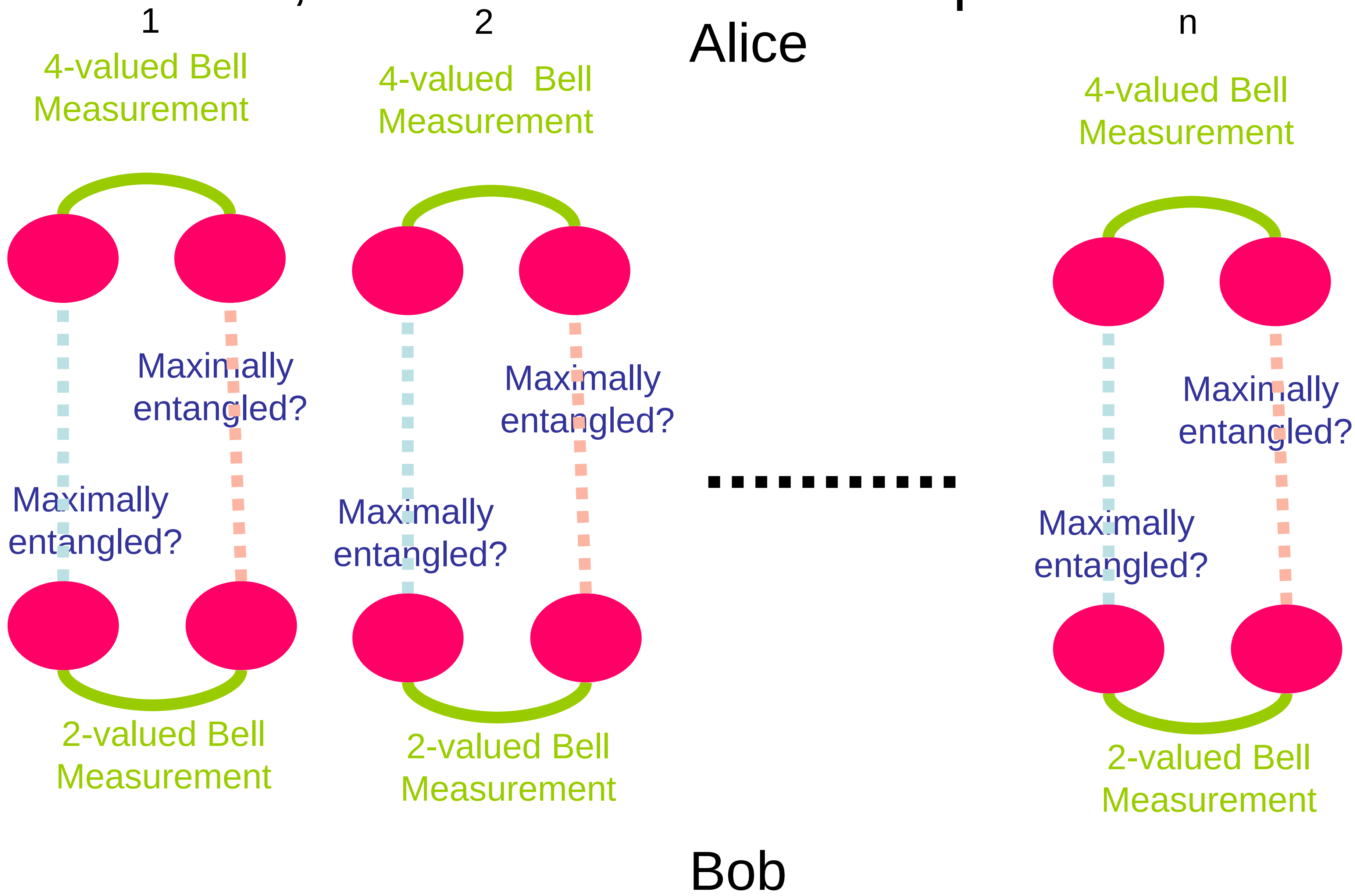
Even if the two independent states are different, this test has the same performance.



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A-B locality and sample locality (n samples)

$$T'^{n,A-B}_{\varepsilon,\alpha} \triangleq T^{n}_{\alpha}\left(T(M^{1}_{\mathrm{cov}}), \frac{d\varepsilon}{d+1}\right)$$

$$\lim_{n\to\infty} \beta^{C}_{\alpha,n,G}(0\|\sigma_n) = \lim_{n\to\infty} \beta\left(T'^{n,A-B}_{0,\alpha}, \sigma_n^{\otimes n}\right)$$

$$= \alpha e^{-dt/(d+1)} < \alpha e^{-t} = \beta_{\alpha}(0\|t)$$

where $\left\langle \phi^{0}_{A,B} \middle| \sigma_n \middle| \phi^{0}_{A,B} \right\rangle = 1 - t/n$

$$G = SU(d), SU(d)\times U(1), U(d^2-1)$$

$$C = S(A_1,\ldots,A_n,B_1,\ldots,B_n),$$

$$L(A_1,\ldots,A_n,B_1,\ldots,B_n),$$

$$L(A_1,\ldots,A_n \to B_1,\ldots,B_n)$$

Hence, $T'^{n,A-B}_{0,\alpha}$ is asymptotically UMP C G-inv. Test.