Fourier Analytic Method in Phase Estimation Problem

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Outline

Estimation of Unknown Unitary Operation (preliminary)

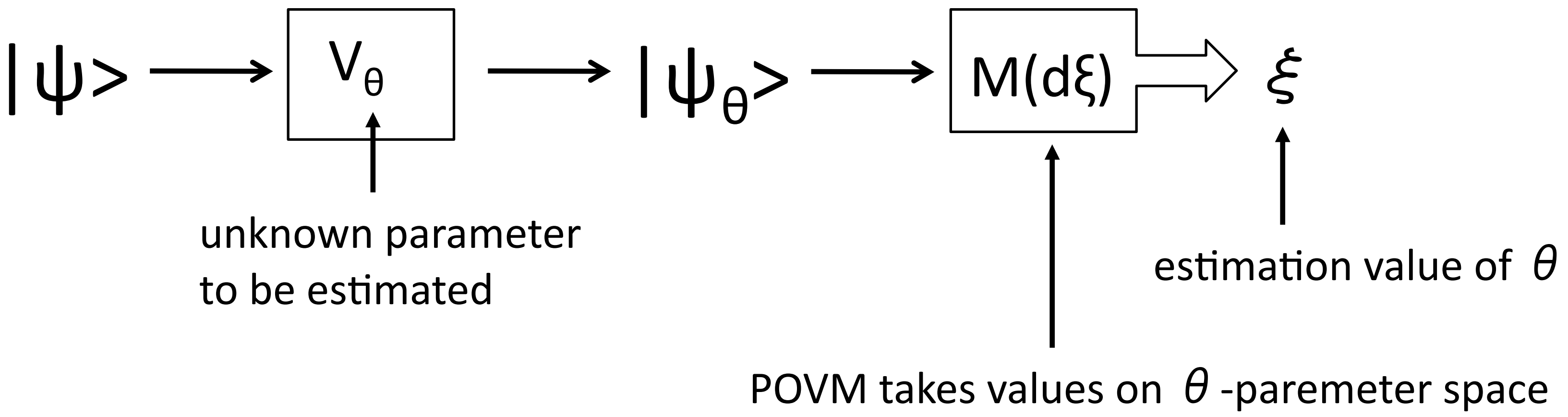
Phase Estimation Problem (main problem)

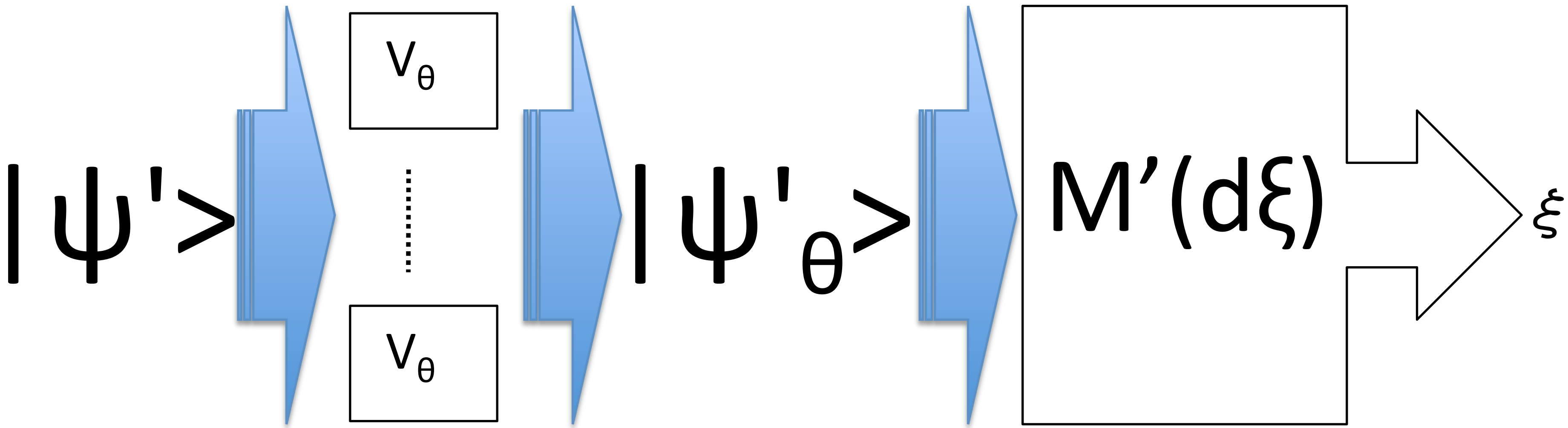
Limiting Distribution of Phase Estimation (main result)

Results about Different Criteria (coloraries)

- Minimize Variance
- Minimize Tail Prob. for Fixed Interval
- Interval Estimation

Parameter Estimation of An Unknown Unitary





We discuss the optimal |ψ'> and the variance or tail probability of $P_\theta^{M'}$

Asymptotic Behavior

	Variance	Limiting Distribution
Classical $P_\theta^{\times n}$	With $O(n^{-1})$ $\to J_\theta^{-1}$	$\to N(\mu, J_\theta^{-1})$
Quantum $\rho_\theta^{\otimes n}$	With $O(n^{-1})$ $\to J^S{}_\theta^{-1}$	$\to N(\mu, J^S{}_\theta^{-1})$
Phase estimation $V_\theta^{\otimes n}\|\psi'>$	With $O(n^{-2})$ $\to J^S{}_\theta^{-1}$	$\to$?

for optimal input ψ'

under the parameter translation $y=n^{1/2}(\xi-\theta)$

Why Limiting Distribution?

➡ We can treat uniformly the optimization under different criteria

Why The Phase Estimation Problem?

➡ singularity of variance, appl. to quantum comp., exp. realizability

Phase Estimation

$$V_\theta = \begin{bmatrix} e^{(i/2)\theta} & 0 \\ 0 & e^{-(i/2)\theta} \end{bmatrix}$$

← We want to estimate phase trans. θ

$|\phi^{(n)}\rangle$: sequence of input states

$$|\phi^{(n)}\rangle \longrightarrow \boxed{V_\theta^{\otimes n}} \longrightarrow |\psi_\theta^{(n)}\rangle \longrightarrow \boxed{M(d\xi)} \Rightarrow P_{\theta,n}^{M}(d\xi)$$

From now on,

$$V_\theta^{\otimes n} \cong \Sigma_k e^{i(k-n/2)\theta} |k\rangle\langle k| =: U_\theta$$

$$|\phi^{(n)}\rangle = \Sigma_k a_k^{(n)} |k\rangle$$

M : Holevo's group covariant POVM

Problem is reduced to optimization of $a_k^{(n)}$

Limiting Distribution of Phase Estimation

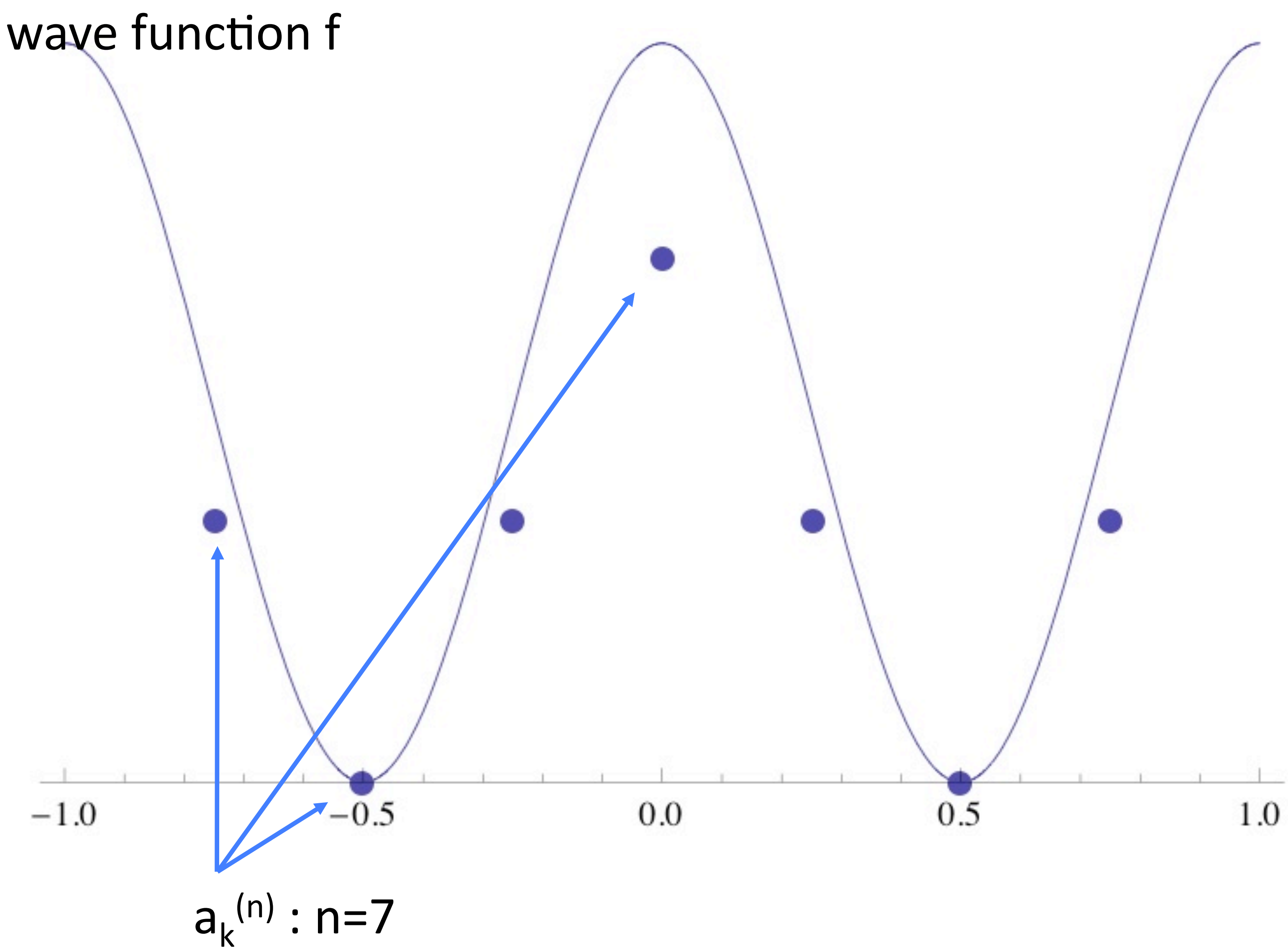
To analyze limiting dist. we change the parameter : $y=(n+1)(\xi-\theta)/2$

$$
\begin{aligned}
P^{M}_{\theta,n}(\xi)\cdot\frac{1}{2\pi}\frac{\mathrm{d}\xi}{\mathrm{dy}}\mathrm{dy} &= \left|\langle\phi_\theta^{(n)}|\,U_\xi\,{}^{\exists}t\rangle\right|^2\frac{1}{2\pi}\frac{\mathrm{d}\xi}{\mathrm{dy}}\mathrm{dy} \\
&= \frac{1}{\pi(n+1)}\left|\sum_k \overline{a_k^{(n)}}e^{i(k-n/2)(\xi-\theta)}\right|^2\mathrm{dy} \\
&\to \frac{1}{2\pi}\left|\int_{-1}^{1} f(x)e^{ixy}\mathrm{dx}\right|^2\mathrm{dy}
\end{aligned}
$$

Here f satisfies $f(x_k)/c_k=\overline{a_k}^{(n)}$, $x_k=2k/n-1$

and f is a square integrable function.

Input States Constructed from A Wave Function



Input States Constructed from A Wave Function

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Conversely, from a square integrable f whose supp. is included in [-1,1], we can construct coefficients $a_k^{(n)}$:

$$\overline{\mathrm{supp} f} \subset [-1, 1], \int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = 1$$

⇕

$$f(x_k)/c_k = \overline{a_k^{(n)}}, \quad x_k = 2k/n-1$$

Limiting Distribution and Fourier Transform

$$P_{\theta,n}^{M}(\xi)\cdot\frac{1}{2\pi}\frac{\mathrm{d}\xi}{\mathrm{d}y}\mathrm{d}y \to P^{f}(\mathrm{d}y) := |F_1(f)(y)|^2\,\mathrm{d}y$$

where $F_1(f)(y) := \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(x)e^{ixy}\mathrm{d}x$

Optimization of input states＝Optimization of wave function f

Asymptotic Behavior

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Quantum $\rho_\theta^{\otimes n}$	With $O(n^{-1})$ $\rightarrow {J^S_\theta}^{-1}$	$\rightarrow N(\mu, {J^S_\theta}^{-1})$
Phase estimation $V_\theta^{\otimes n}\|\psi'>$	With $O(n^{-2})$ $\rightarrow {J^S_\theta}^{-1}$	$\rightarrow \|F_1(f)(y)\|^2 dy$

under the parameter trans. $y=(n+1)(\xi-\theta)/2$

Optimization of An Input-state

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P^f: optimize f in what criterion?

- variance

- decreasing order of tail probability

- tail probability for fixed interval

$\Rightarrow$ We can treat them systematically because:

$$P^f(\mathrm{d}y) = |F_1(f)(y)|^2 \, \mathrm{d}y$$

Time-limited Fourier analytic method can be used.

Minimize Variance

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$$V(f) := \int_{-\infty}^{\infty} y^2 P^f(\mathrm{d}y) \Rightarrow \text{minimize w.r.t. } f$$

This problem is reduced to Dirichlet problem

Optimal wave function is

$$f_1 = 2\sqrt{2}\sin\pi\left(\frac{x+1}{2}\right)$$

Corresponding minimum variance is

$$V(f_1) = \pi^2$$

Minimize Variance

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Does f_1 makes tail prob. decreasing rapidly?

➡ The answer is NO!

$$P^{f_1}(y)=O(y^{-4})$$

We want to find a wave function f which makes tail prob. decreasing rapidly

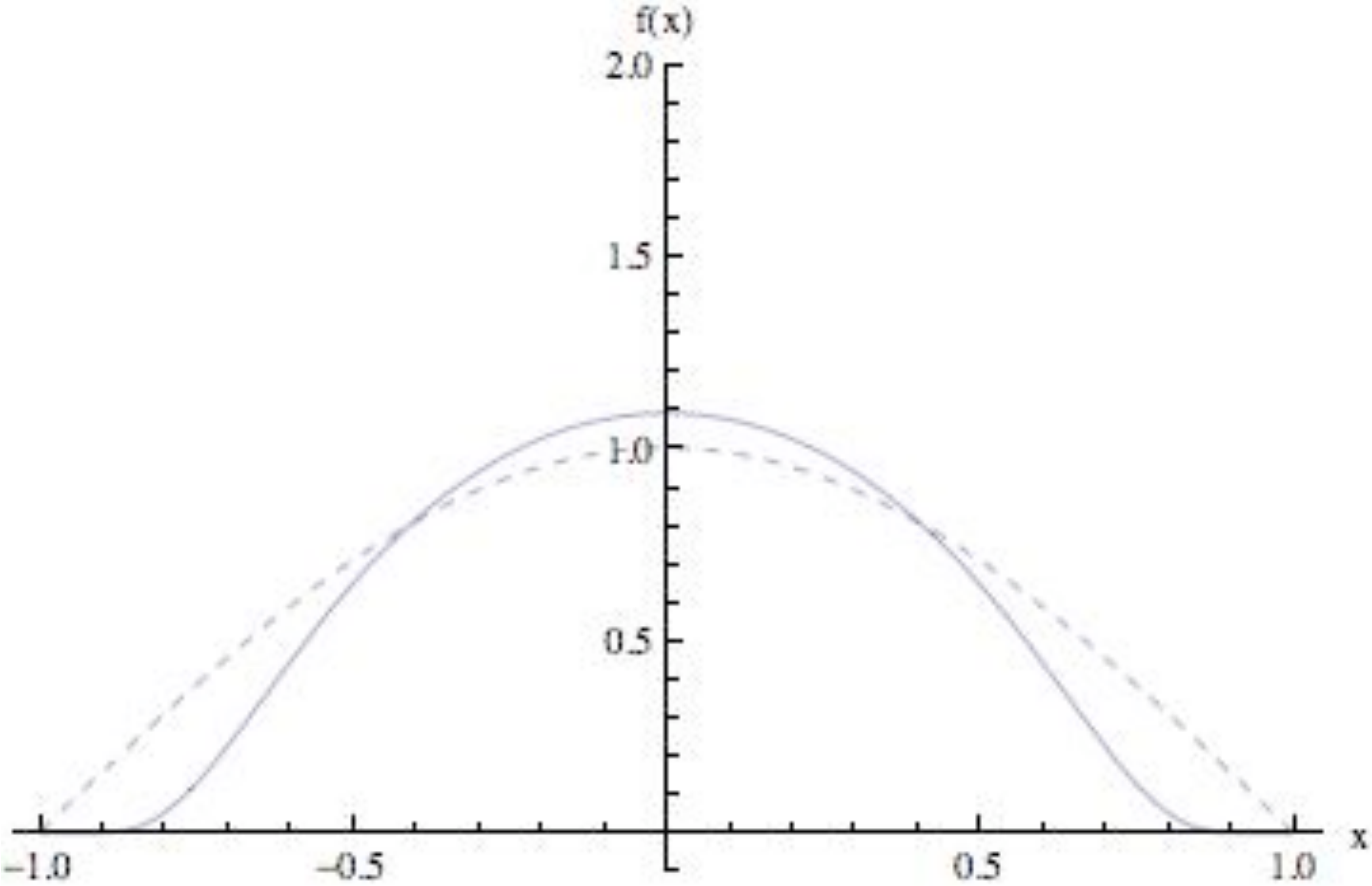
➡ It suffices to construct a rapidly decreasing func. with $\overline{\text{supp}}\, f \subset [-1,1]$.

(Fourier trans. of rapidly decreasing func. decreases rapidly.)

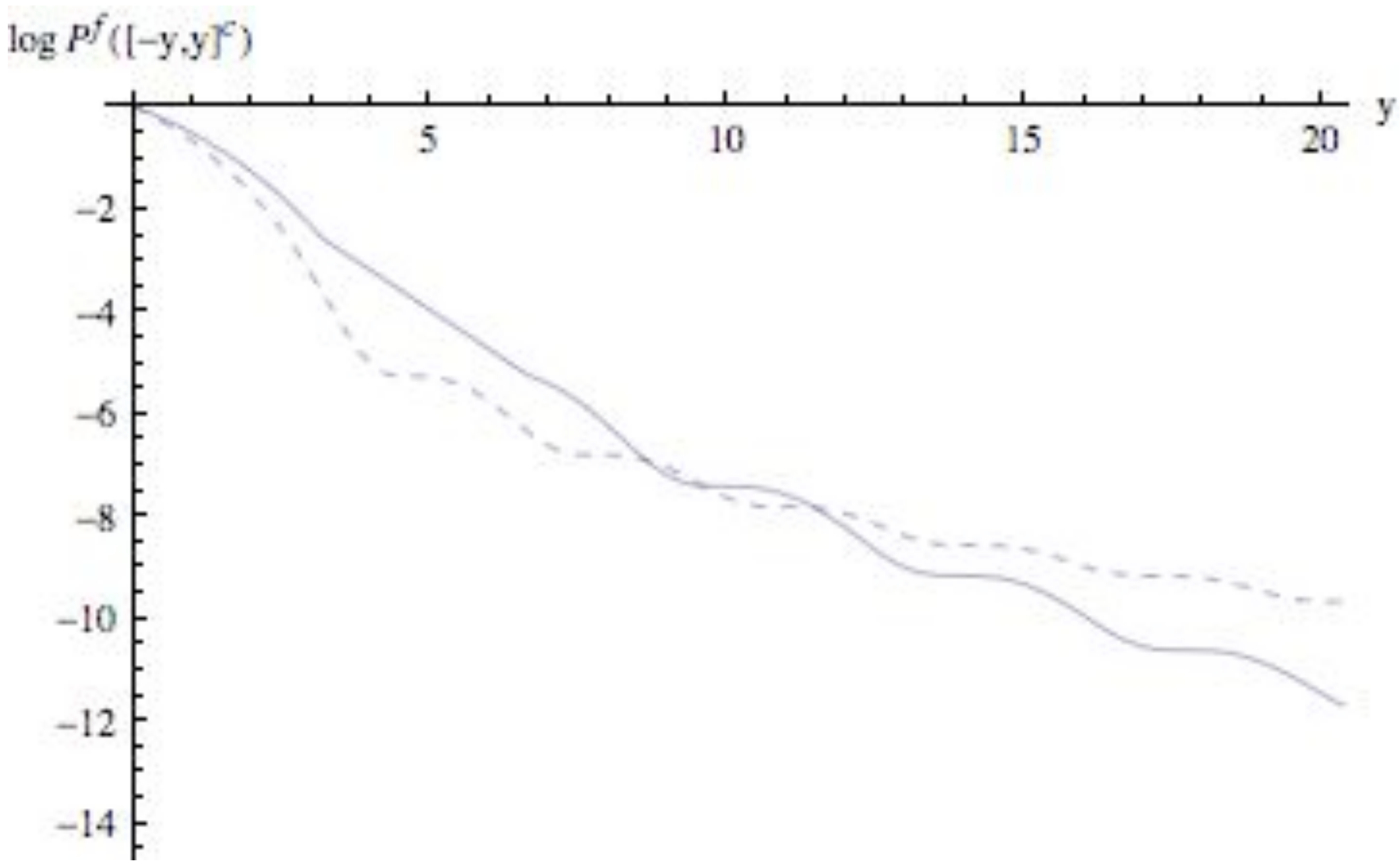
$$\lim_{y\to\infty} \frac{-1}{\sqrt{|y|}} \log |F(f)(y)|^2 \geq 2\sqrt{2}$$

decrease exponentially!

Construct A Rapidly Decreasing Wave Function



Construct A Rapidly Decreasing Wave Function



Minimize Tail Prob. for Fixed Interval

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[-R, R] : fixed closed interval

We want to evaluate $\min_f P^f([-R,R]^c) = 1 - \max_f P^f([-R,R])$

Define D_R, F_R as follows:

$$D_R \quad : \quad L^2(\mathbb{R}) \to L^2[-R, R] : \text{projection}$$
$$B_R \quad := \quad F^* D_R F$$

$$\max_f P^f([-R, R]) = \max_f \langle f \mid B_R \mid f \rangle = \max_{g \in L^2(\mathbf{R})} \frac{\langle g \mid D_1 B_R D_1 \mid g \rangle}{\|g\|^2}$$

It suffices to evaluate maximum eigenvalue of the bounded linear operator $D_1 B_R D_1$

Minimize Tail Prob. for Fixed Interval

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Slepian showed the maximum eigenvalue λ(R) satisfies

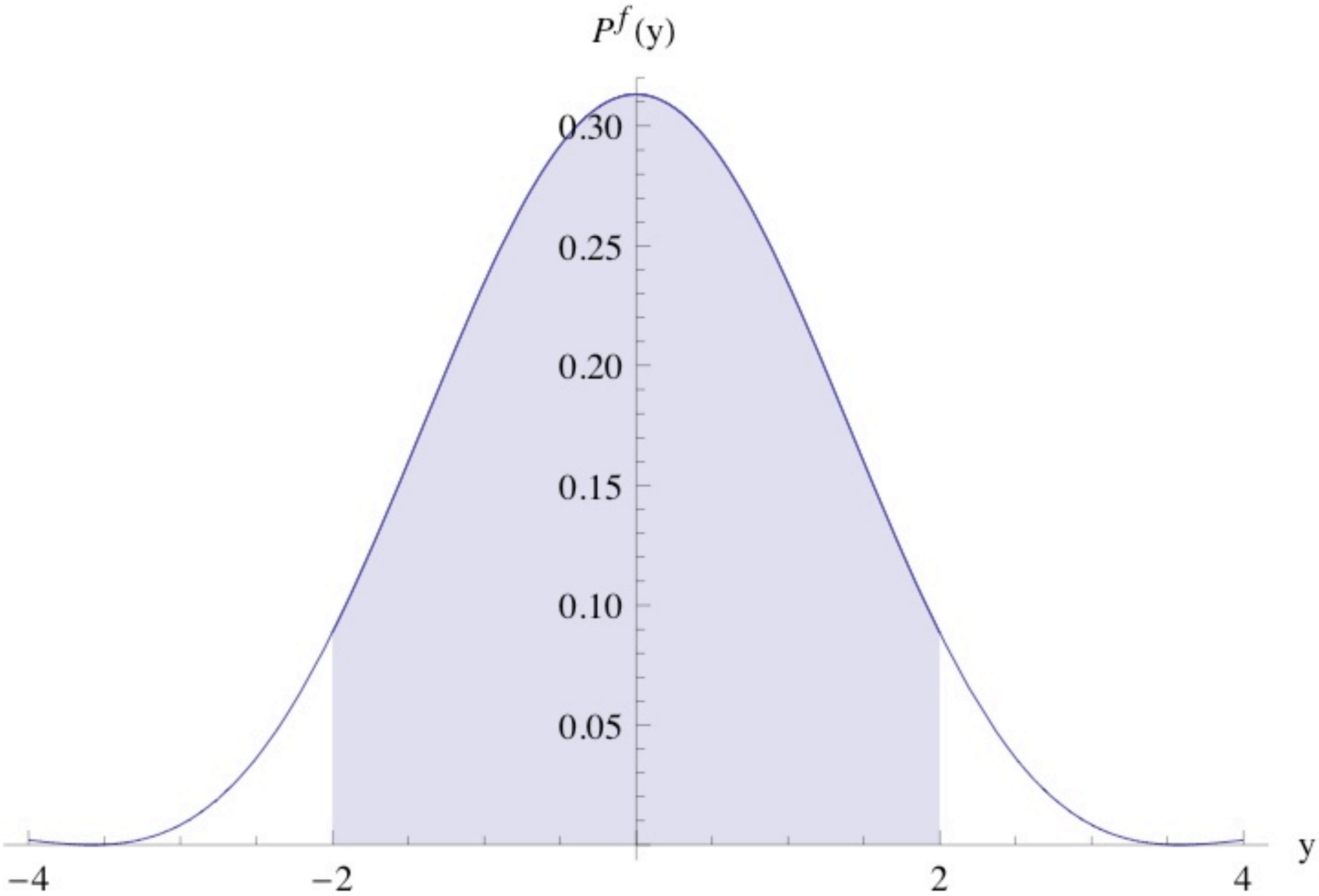
$$1 - \lambda(R) \simeq 4\sqrt{\pi R} e^{-2R} \left(1 - \frac{3}{32R} + O(R^{-2})\right)$$

That means,

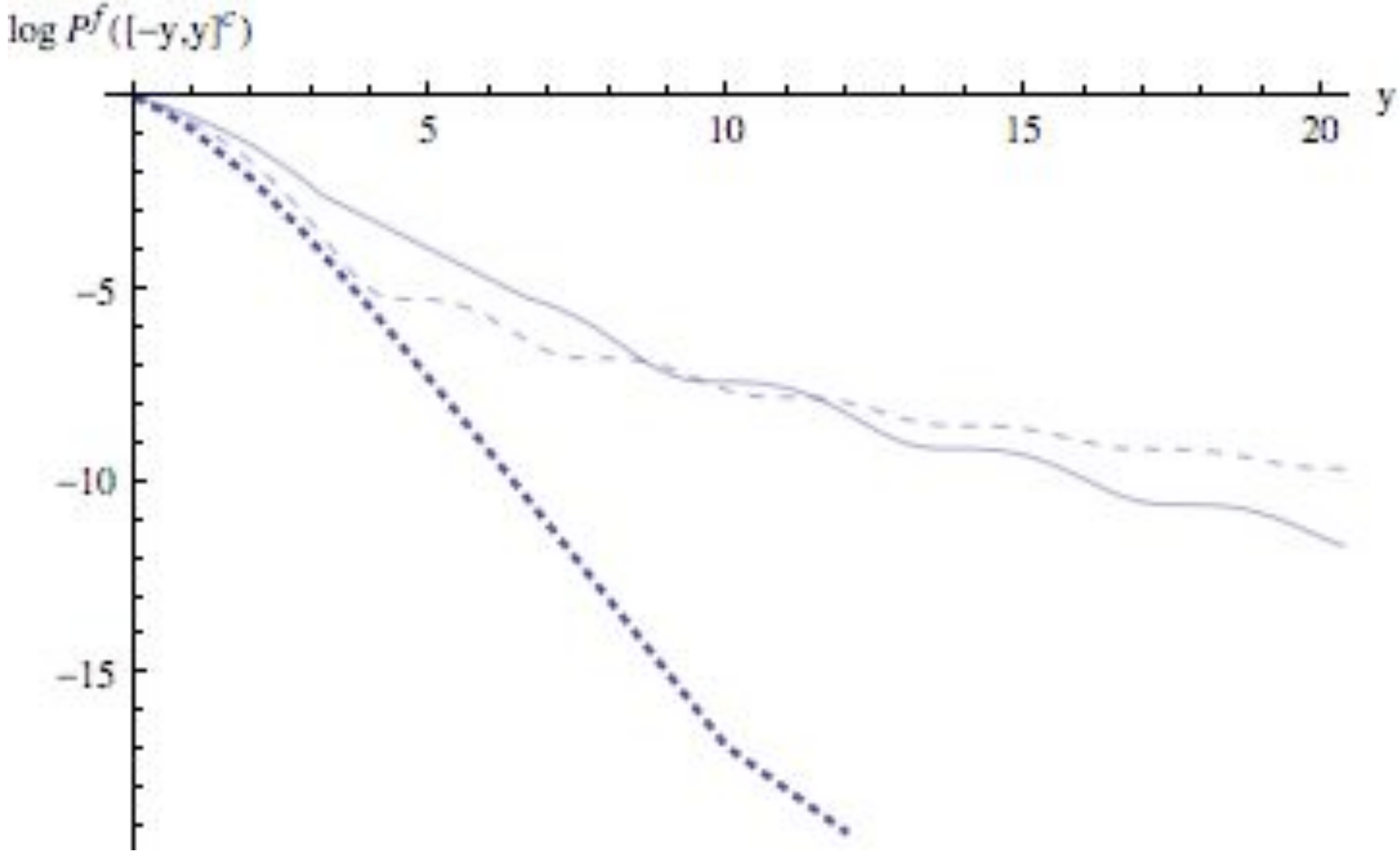
$$\lim_{R \to \infty} \frac{-1}{R} \log \min_{f} P^{f}\left([-R, R]^{c}\right) = 2$$

decrease exponentially!

Minimize Tail Prob. for Fixed Interval



Minimize Tail Prob. for Fixed Interval



Conclusion

We obtained the formula of the Limiting Dist. for Phase Estimation：

$$P^M_{\theta,n}(\xi) \cdot \frac{1}{2\pi} \frac{\mathrm{d}\xi}{\mathrm{d}y} \mathrm{d}y \to P^f(\mathrm{d}y) := |F_1(f)(y)|^2 \, \mathrm{d}y$$

The optimal inputs under each criterion is represented by wave functions. We analyzed them by Fourier analysis.

The optimal wave functions depend of criteria. Thus, we need to employ proper wave function under the situation.