Local Asymptotic Normality in Quantum Statistics

Mădălin Guță

School of Mathematical Sciences
University of Nottingham

Richard Gill (Leiden)
Jonas Kahn (Paris XI)
Bas Janssens (Utrecht)
Anna Jencova (Bratislava)
Luc Bouten (Caltech)

Outline:

- Quantum state estimation and optimality
- Local Asymptotic Normality in classical statistics
- Local Asymptotic Normality for qubits
- Local Asymptotic Normality for d-dimensional state

Quantum state estimation

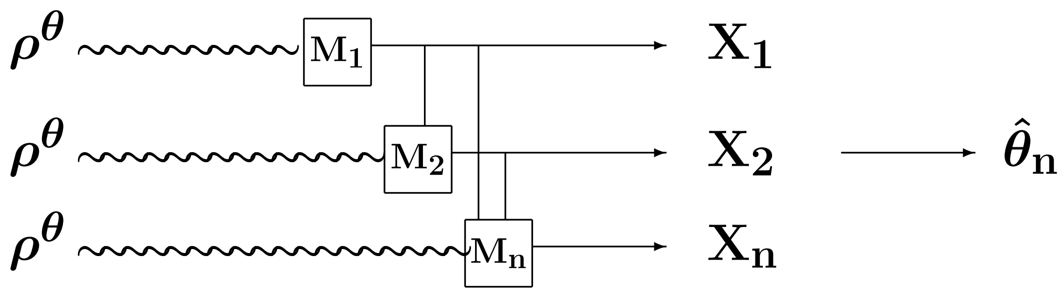
Problem: given n identically prepared systems in the state ρ^θ with $\theta \in \Theta$, perform a measurement $M^{(n)}$ and construct an estimator $\hat{\theta}_n$ of θ from the result $X^{(n)}$.

$$\rho^\theta \otimes \rho^\theta \otimes \cdots \otimes \rho^\theta \longmapsto X^{(n)} \longmapsto \hat{\theta}_n$$

Quantum state estimation

Problem: given n identically prepared systems in the state ρ^θ with $\theta \in \Theta$, perform a measurement $M^{(n)}$ and construct an estimator $\hat{\theta}_n$ of θ from the result $X^{(n)}$.

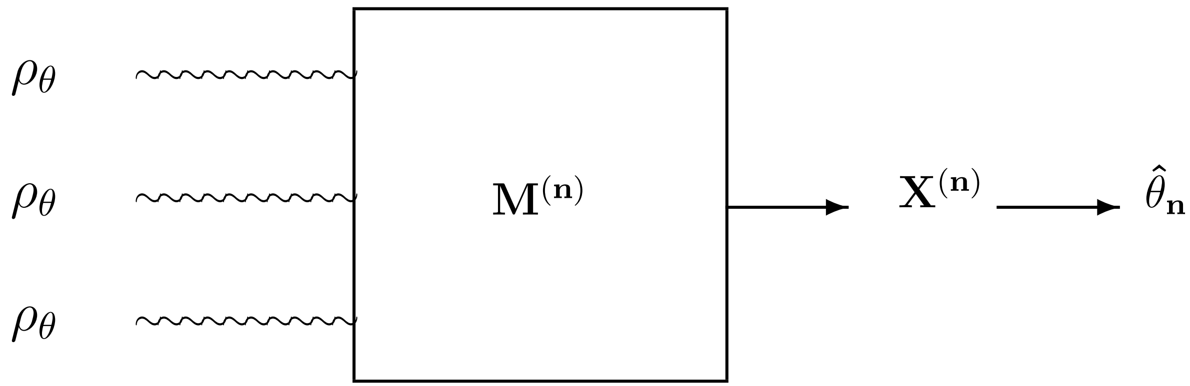
$$\rho^\theta \otimes \rho^\theta \otimes \cdots \otimes \rho^\theta \longmapsto X^{(n)} \longmapsto \hat{\theta}_n$$



Quantum state estimation

Problem: given n identically prepared systems in the state ρ^θ with $\theta \in \Theta$, perform a measurement $M^{(n)}$ and construct an estimator $\hat{\theta}_n$ of θ from the result $X^{(n)}$.

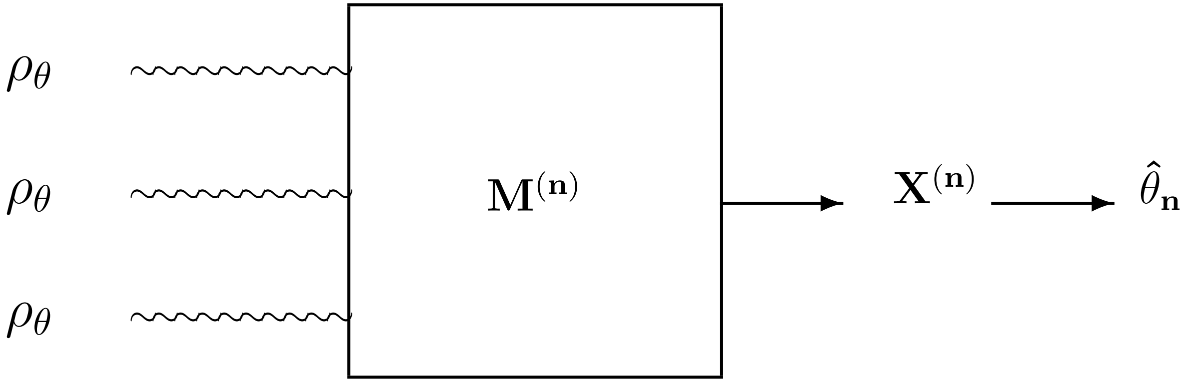
$$\rho^\theta \otimes \rho^\theta \otimes \cdots \otimes \rho^\theta \longmapsto X^{(n)} \longmapsto \hat{\theta}_n$$



Quantum state estimation

Problem: given n identically prepared systems in the state ρ^θ with $\theta \in \Theta$, perform a measurement $M^{(n)}$ and construct an estimator $\hat{\theta}_n$ of θ from the result $X^{(n)}$.

$$\rho^\theta \otimes \rho^\theta \otimes \cdots \otimes \rho^\theta \longmapsto X^{(n)} \longmapsto \hat{\theta}_n$$



Risk and Optimality: The quality of the estimation strategy $(M, \hat{\theta}_n)$ is given by the risk

$$R(\theta, \hat{\theta}_n) = \mathbb{E}\|\rho^\theta - \rho^{\hat{\theta}_n}\|_1^2 \quad \text{or} \quad R(\theta, \hat{\theta}_n) = 1 - \mathbb{E}\, F(\rho^\theta,\, \rho^{\hat{\theta}_n})$$

Bayesian vs frequentist optimality

Bayesian: prior $\pi(d\theta)$

$$
\begin{aligned}
R_\pi(\hat\theta_n) &:= \int R(\theta,\hat\theta_n)\pi(d\theta) \\
R_{\pi,n} &:= \inf_{M_n} R_\pi(\hat\theta_n) \\
R_\pi &:= \lim_{n\to\infty} nR_{\pi,n}
\end{aligned}
$$

Frequentist

$$
\begin{aligned}
R_{\theta_0}(\hat\theta_n) &:= \sup_{\theta\in B(\theta_0,n^{-1/2})} R(\theta,\hat\theta_n) \\
R_{\theta_0,n} &:= \inf_{M_n} R_{\theta_0}(\hat\theta_n) \\
R_{\theta_0} &:= \lim_{n\to\infty} nR_{\theta_0,n} = C^H(\theta_0)
\end{aligned}
$$

$$
R_\pi = \int R_\theta \, \pi(d\theta)
$$

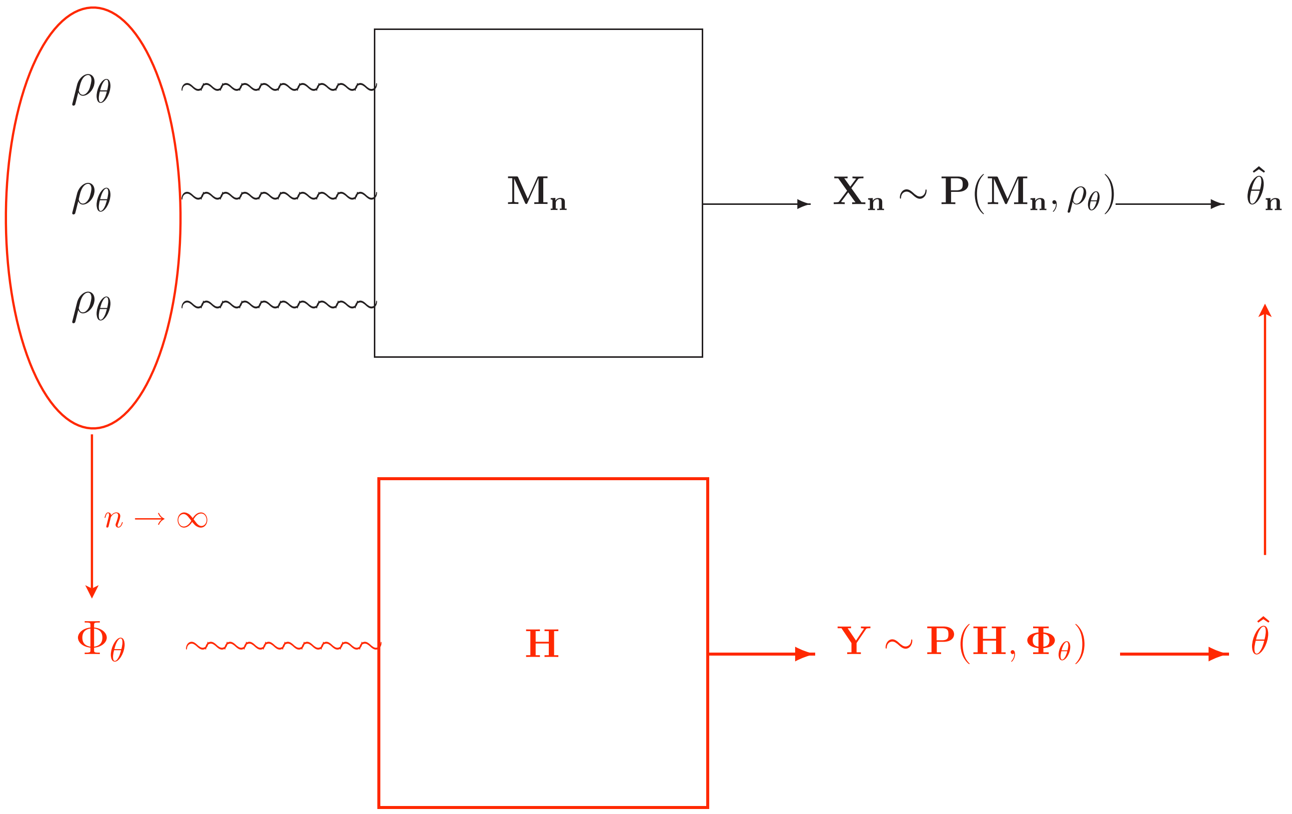
A rough classification of state estimation problems

	Separate measurements	Joint measurements
Parametric	Practically feasible Optimal for pure states Optimal for one parameter $R_n \approx C_{\rm sep}/n$	More difficult to implement Optimal for mixed states $R_n \approx C_{\rm joint}/n$
Non-parametric	Q. Homodyne Tomography, Direct detection of Wigner fct... non-parametric rates for estimation of state as a whole $R_n = O(\,(\log n)^k/n,\ n^{-\alpha}, ...)$	Conjecture/Program: L.A.N. = convergence to model of displaced quasifree states of infinite dimensional CCR alg.

Asymptotically things become easier...

Idea of using (local) asymptotic normality in optimal estimation:

- as $n \to \infty$ the n particle model gets 'close' to a Gaussian shift model Φ_θ
- the latter has fixed, known variance and unknown mean (related to) θ,
- the mean can be estimated optimally by simple measurements (heterodyne)
- the measurement can be 'pulled back' to the n systems



Motivation / earlier work

- **Classical L.A.N. theory of Le Cam**

  asymptotic equivalence of statistical models
  optimal estimation rates

- **Central Limit behaviour for quantum systems**

  Coherent spin states
  Gaussian description of atoms-light interaction (Mabuchi, Polzik experiments)

- **Work by Hayashi and Matsumoto on asymptotics of state estimation**

  M. Hayashi,
  Quantum estimation and the quantum central limit theorem (in Japanese),
  Bull. Math. Soc. Japan 55 (2003)
  English translation: quant-ph/0608198

  M. Hayashi, K. Matsumoto,
  Asymptotic performance of optimal state estimation in quantum two level system
  arXiv:quant-ph/0411073

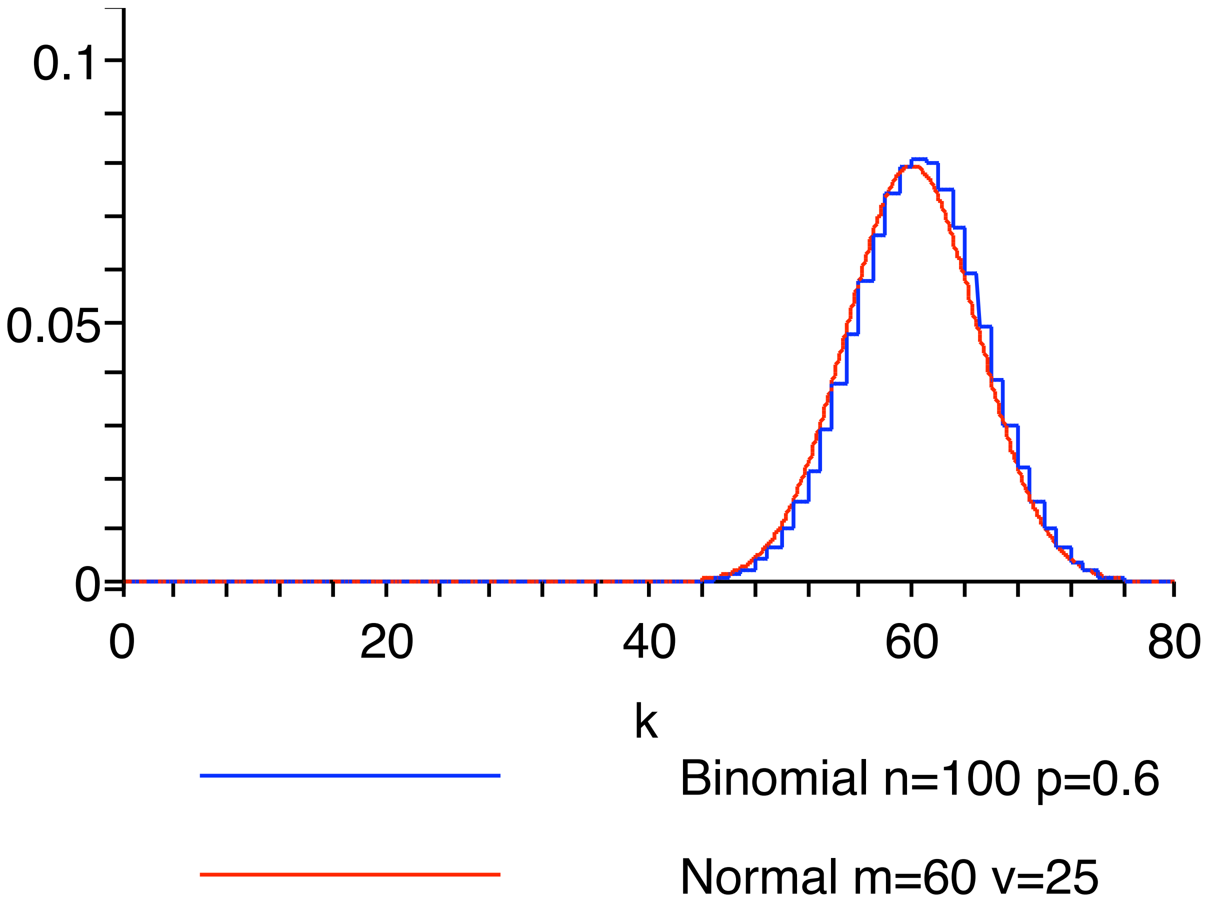
Local Asymptotic Normality for coin toss

Repeated coin toss: $X_1, \ldots, X_n$ i.i.d. with $\mathbb{P}[X_i = 1] = \theta,\ \mathbb{P}[X_i = 0] = 1 - \theta$

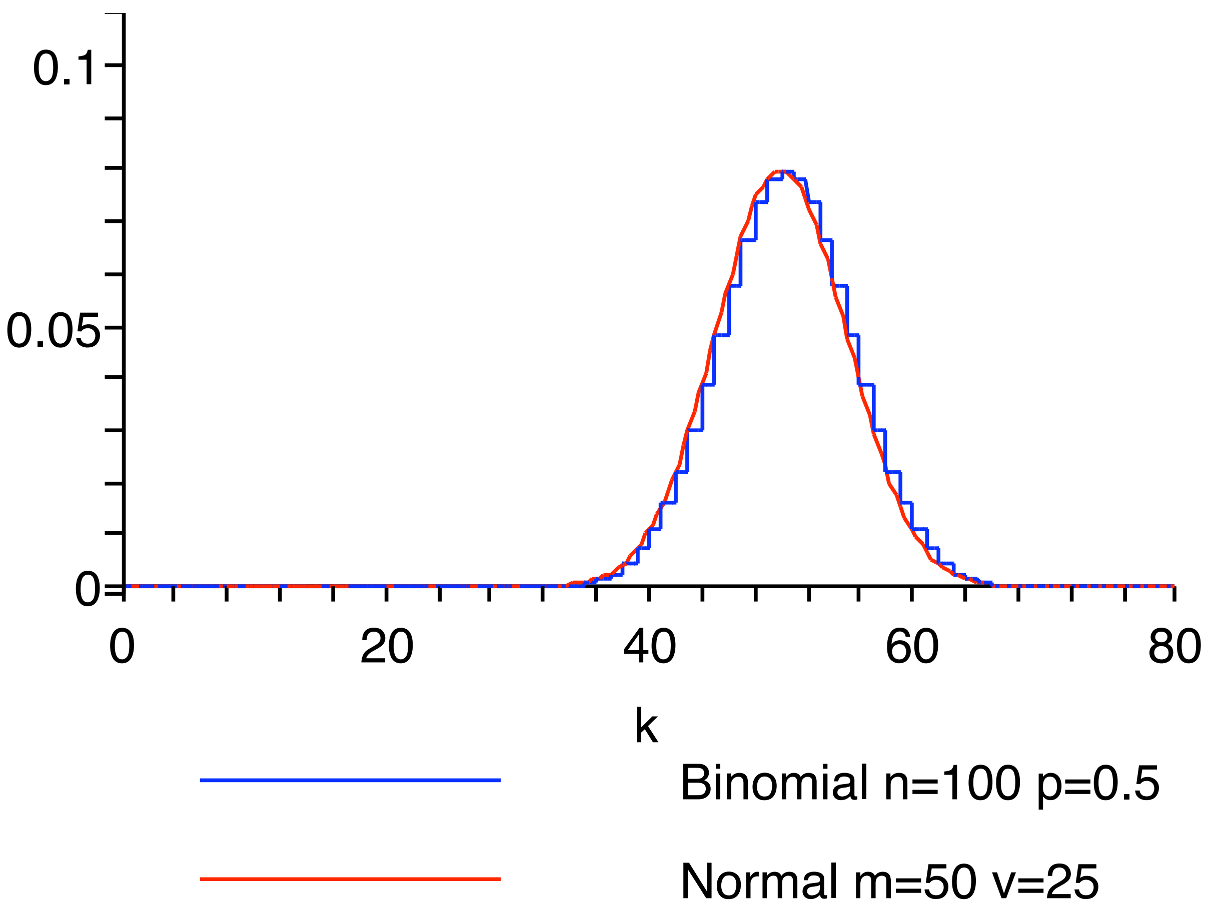
Sufficient statistic: $\hat{\theta}_n := \frac{1}{n}\sum_{i=1}^n X_i$ unbiased estimator since $\mathbb{E}(X) = \theta$

Central Limit Theorem: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \theta(1-\theta))$

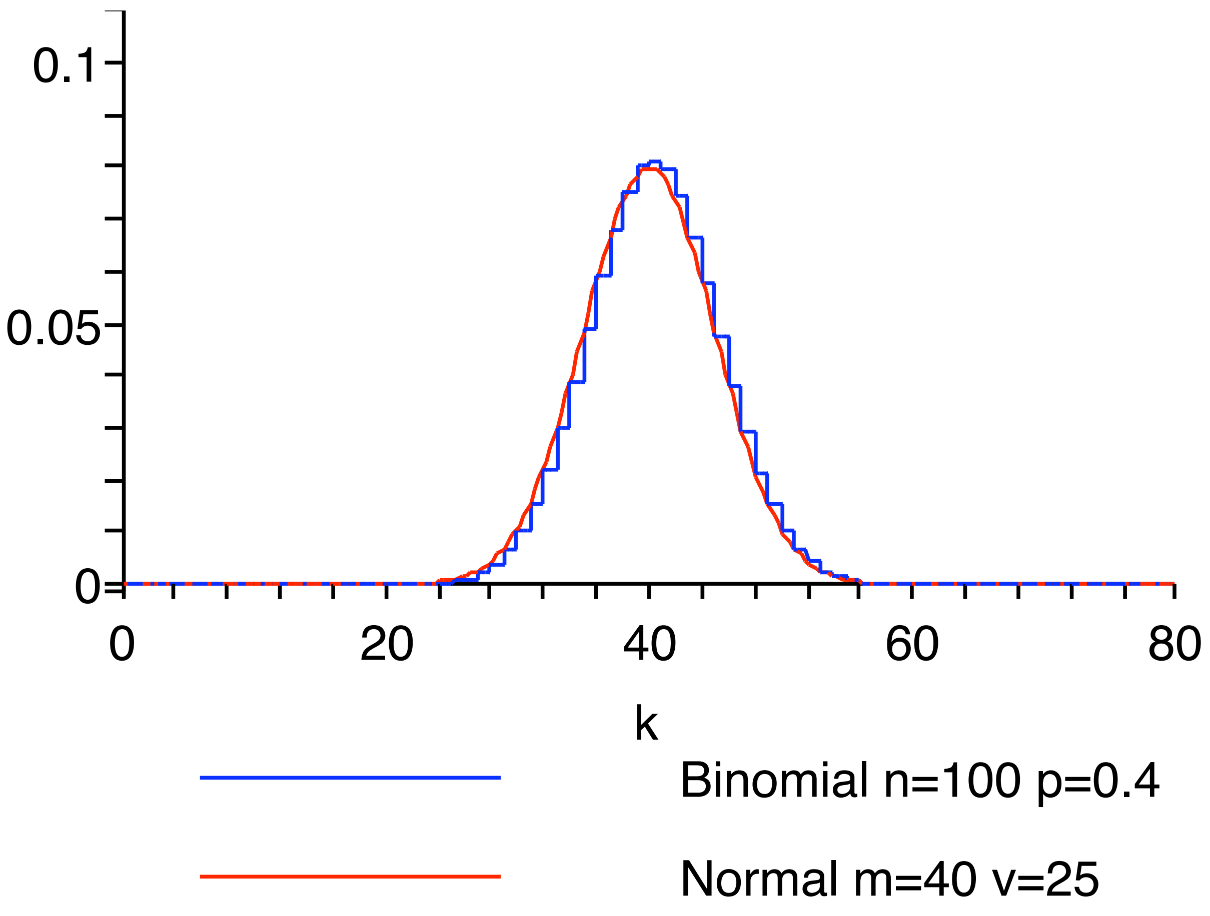
Local Asymptotic Normality for coin toss



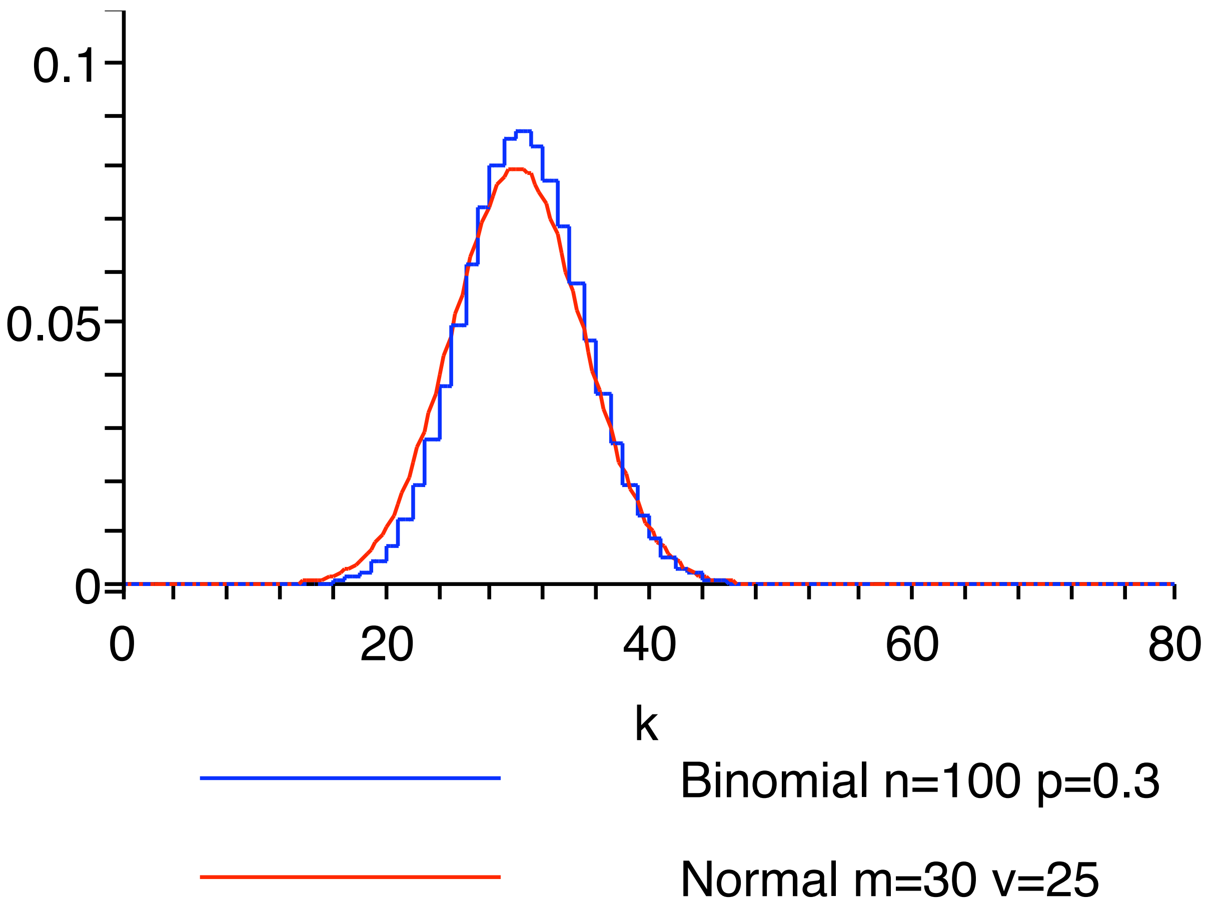
Local Asymptotic Normality for coin toss



Local Asymptotic Normality for coin toss



Local Asymptotic Normality for coin toss



Local Asymptotic Normality for coin toss

Repeated coin toss: $X_1, \ldots, X_n$ i.i.d. with $\mathbb{P}[X_i = 1] = \theta,\ \mathbb{P}[X_i = 0] = 1 - \theta$

Sufficient statistic: $\hat{\theta}_n := \frac{1}{n}\sum_{i=1}^n X_i$ unbiased estimator since $\mathbb{E}(X) = \theta$

Central Limit Theorem: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \theta(1-\theta))$

Local parameter: let $\theta = \theta_0 + u/\sqrt{n}$ for a fixed known θ_0, then

$$\hat{u}_n := \sqrt{n}(\hat{\theta}_n - \theta_0) \approx N(u, \theta_0(1-\theta)_0)$$

Gaussian shift model

Local Asymptotic Normality for coin toss

Repeated coin toss: $X_1, \ldots, X_n$ i.i.d. with $\mathbb{P}[X_i = 1] = \theta,\ \mathbb{P}[X_i = 0] = 1 - \theta$

Sufficient statistic: $\hat{\theta}_n := \frac{1}{n}\sum_{i=1}^n X_i$ unbiased estimator since $\mathbb{E}(X) = \theta$

Central Limit Theorem: $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \theta(1-\theta))$

Local parameter: let $\theta = \theta_0 + u/\sqrt{n}$ for a fixed known θ_0, then

$$\hat{u}_n := \sqrt{n}(\hat{\theta}_n - \theta_0) \approx N(u, \theta_0(1-\theta)_0)$$

Gaussian shift model

Why can we restrict to a local neighbourhood ?

You can construct a θ_0 from the data and the true θ will be in a

'$1/\sqrt{n}$-neighbourhood' with high probability

Local Asymptotic Normality: general case

Let $(Y_1, \ldots, Y_n)$ be i.i.d. with $\mathbb{P}^{\theta_0+u/\sqrt{n}}$ a 'smooth' family with $u \in \mathbb{R}^k$. Then

$$\left\{ \left(\mathbb{P}^{\theta_0+u/\sqrt{n}} \right)^n : u \in \mathbb{R}^k \right\} \rightsquigarrow \{ N(u, I_{\theta_0}^{-1}) : u \in \mathbb{R}^k \}$$

Local Asymptotic Normality: general case

Let $(Y_1, \ldots, Y_n)$ be i.i.d. with $\mathbb{P}^{\theta_0 + u/\sqrt{n}}$ a 'smooth' family with $u \in \mathbb{R}^k$. Then

$$\left\{ \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n : u \in \mathbb{R}^k \right\} \rightsquigarrow \{ N(u, I_{\theta_0}^{-1}) : u \in \mathbb{R}^k \}$$

Strong convergence: there exist randomizations (Markov kernels) T_n, S_n such that

$$\lim_{n \to \infty} \sup_{\|u\| < a} \left\| T_n \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n - N(u, I_{\theta_0}^{-1}) \right\|_{\text{tv}} = 0$$

and

$$\lim_{n \to \infty} \sup_{\|u\| < a} \left\| \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n - S_n N(u, I_{\theta_0}^{-1}) \right\|_{\text{tv}} = 0$$

Local Asymptotic Normality: general case

Let $(Y_1, \ldots, Y_n)$ be i.i.d. with $\mathbb{P}^{\theta_0 + u/\sqrt{n}}$ a 'smooth' family with $u \in \mathbb{R}^k$. Then

$$\left\{ \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n : u \in \mathbb{R}^k \right\} \rightsquigarrow \{ N(u, I_{\theta_0}^{-1}) : u \in \mathbb{R}^k \}$$

Strong convergence: there exist randomizations (Markov kernels) T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| T_n \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n - N(u, I_{\theta_0}^{-1}) \right\|_{\mathrm{tv}} = 0$$

and

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| \left(\mathbb{P}^{\theta_0 + u/\sqrt{n}} \right)^n - S_n N(u, I_{\theta_0}^{-1}) \right\|_{\mathrm{tv}} = 0$$

Importance:

- Shows that for large n the statistical model is 'locally easy': Gaussian shift model
- Asymptotically, we only need to solve the statistical problem for the Gaussian shift model

L. A. N. for finite dimensional quantum systems

Let $\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n}$ be the joint state of n i.i.d. systems with $\rho_\theta \in M(\mathbb{C}^d)$ 'smooth'. Then

$$\left\{\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} : u \in \mathbb{R}^{d^2-1}\right\} \rightsquigarrow \left\{\Phi(u, H_{\theta_0}^{-1}) : u \in \mathbb{R}^{d^2-1}\right\}$$

L. A. N. for finite dimensional quantum systems

Let $\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n}$ be the joint state of n i.i.d. systems with $\rho_\theta \in M(\mathbb{C}^d)$ 'smooth'. Then

$$\left\{\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} : u \in \mathbb{R}^{d^2-1}\right\} \rightsquigarrow \left\{\Phi(u, H_{\theta_0}^{-1}) : u \in \mathbb{R}^{d^2-1}\right\}$$

Strong convergence: there exist quantum channels T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| T_n \left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} - \Phi(u, H_{\theta_0}^{-1}) \right\|_1 = 0$$

and

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| \left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} - S_n \Phi(u, H_{\theta_0}^{-1}) \right\|_1 = 0$$

L. A. N. for finite dimensional quantum systems

Let $\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n}$ be the joint state of n i.i.d. systems with $\rho_\theta \in M(\mathbb{C}^d)$ 'smooth'. Then

$$\left\{\left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} : u \in \mathbb{R}^{d^2-1}\right\} \rightsquigarrow \left\{\Phi(u, H_{\theta_0}^{-1}) : u \in \mathbb{R}^{d^2-1}\right\}$$

Strong convergence: there exist quantum channels T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| T_n \left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} - \Phi(u, H_{\theta_0}^{-1}) \right\|_1 = 0$$

and

$$\lim_{n\to\infty} \sup_{\|u\|<a} \left\| \left(\rho_{\theta_0+u/\sqrt{n}}\right)^{\otimes n} - S_n \Phi(u, H_{\theta_0}^{-1}) \right\|_1 = 0$$

Importance:

- Shows that for large n the statistical model is 'locally easy': Gaussian shift model
- Provides a two step adaptive measurement strategy which is asymptotically optimal

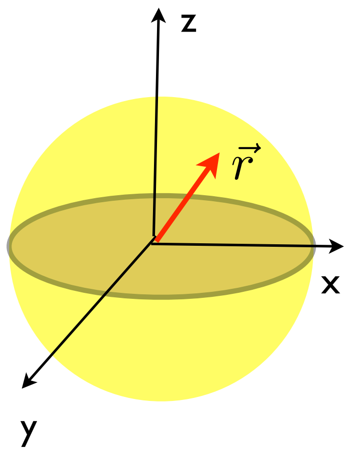
Outline:

- Quantum state estimation and optimality
- Local Asymptotic Normality in classical statistics
- Local Asymptotic Normality for qubits

L. A. N. for qubit states (d=2)

An arbitrary qubit (spin) state:

$$\rho_{\vec{r}} := \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix} = \frac{1}{2}\left(\mathbf{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right), \quad \|\vec{r}\| \leq 1$$



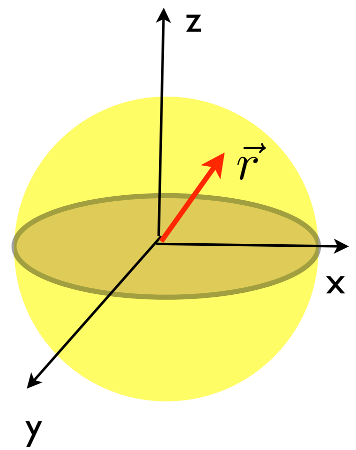
Non-commuting spin components: $\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$

Probability distributions : $\begin{cases} \mathbb{P}([\sigma_a = +1]) = (1+r_a)/2 \\ \mathbb{P}([\sigma_a = -1]) = (1-r_a)/2 \end{cases}$

L.A.N. for qubit states (d=2)

An arbitrary qubit (spin) state:

$$\rho_{\vec{r}} := \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} = \frac{1}{2}\left(\mathbf{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right), \qquad \|\vec{r}\| \leq 1$$

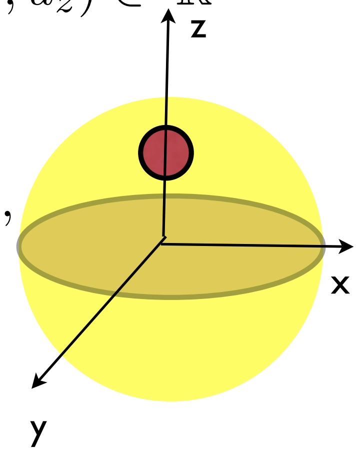


Non-commuting spin components: $\sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z$

Probability distributions : $\begin{cases} \mathbb{P}([\sigma_a = +1]) = (1+r_a)/2 \\ \mathbb{P}([\sigma_a = -1]) = (1-r_a)/2 \end{cases}$

A local neighborhood of $\rho_0 := \begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix}$ is parametrised by $\mathbf{u} = (u_x, u_y, u_z) \in \mathbb{R}^3$

$$\rho_{\mathbf{u}/\sqrt{n}} := U\left(\frac{\mathbf{u}}{\sqrt{n}}\right)\begin{pmatrix} \mu + \frac{u_z}{\sqrt{n}} & 0 \\ 0 & 1-\mu-\frac{u_z}{\sqrt{n}} \end{pmatrix} U\left(\frac{\mathbf{u}}{\sqrt{n}}\right)^*,$$

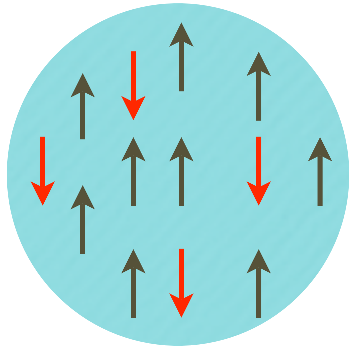


where $U(\mathbf{u}) \in SU(2)$ is the unitary $U(\mathbf{u}) := \exp(i(u_x\sigma_x + u_y\sigma_y))$

LAN for qubits: the big ball picture

'Quantum coin toss': $\rho_0 = \mu \left|\uparrow\right\rangle \left\langle\uparrow\right| + (1-\mu) \left|\downarrow\right\rangle \left\langle\downarrow\right| \quad \Longrightarrow$

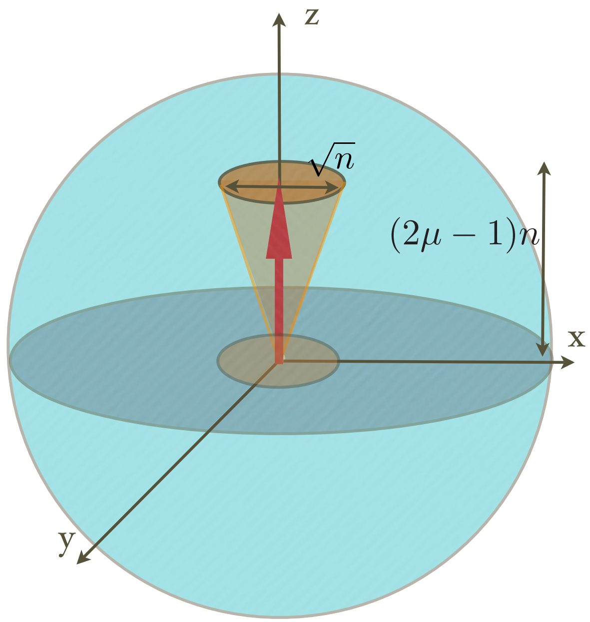
$$\mathbb{P}([\sigma_x = \pm 1]) = \mathbb{P}([\sigma_y = \pm 1]) = 1/2, \quad \mathbb{P}([\sigma_z = 1]) = \mu$$



n identically prepared systems: $\rho_0 \otimes \cdots \otimes \rho_0$

Central Limit Theorem... Collective spin $L_{x,y,z} := \sum_{i=1}^{n} \sigma_{x,y,z}^{(i)}$

$$\frac{1}{\sqrt{n}} L_x \xrightarrow{\mathcal{D}} N(0,1),$$
$$\frac{1}{\sqrt{n}} L_y \xrightarrow{\mathcal{D}} N(0,1),$$



...with a quantum twist

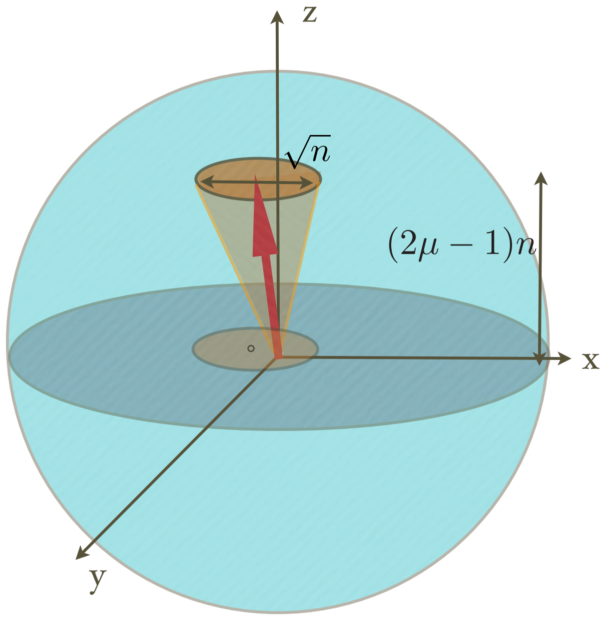
$$\left[\frac{1}{\sqrt{n}} L_x \ , \ \frac{1}{\sqrt{n}} L_y\right] = 2i \frac{1}{n} L_z \approx 2i(2\mu - 1)\mathbf{1}$$

Gaussian states

Quantum particle (harmonic oscillator)

$$\left.\begin{array}{l}\frac{1}{\sqrt{2n(2\mu-1)}}L_x \longrightarrow Q \\ \frac{1}{\sqrt{2n(2\mu-1)}}L_y \longrightarrow P\end{array}\right\} \quad \Longrightarrow \quad [Q,P]=i\mathbf{1}$$

Heisenberg commutation relation



Thermal equilibrium state: $< Q^2 >=< P^2 >= \frac{1}{2(2\mu-1)}$

$$\phi^{\mathbf{0}} := (1-p)\sum_{k=0}^{\infty} p^k \, |k\rangle \langle k| \, , \qquad p = \frac{1-\mu}{\mu} < 1$$

Quantum Gaussian shift: spin rotations become displacements

$$\phi^{\mathbf{u}} := D(\mathbf{u})\, \phi^{\mathbf{0}}\, D(\mathbf{u})^*, \qquad D(\mathbf{u}) := \exp\left(i\sqrt{2(2\mu-1)}(u_x Q + u_y P)\right)$$

Classical Gaussian shift: diagonal parameter behaves like in coin toss

$$N^{\mathbf{u}} := N(u_z, \mu(1-\mu))$$

Local Asymptotic Normality for qubit states

Classical Gaussian shift gives info about the eigenvalues of ρ

$$N^{\mathbf{u}} := N(u_z, \mu(1-\mu))$$

Quantum Gaussian shift gives info about the eigenvectors of ρ

$$\phi^{\mathbf{u}} := D(\mathbf{u})\,\phi^{\mathbf{0}}\,D(\mathbf{u})^*, \qquad \begin{cases} Q & \sim N\left(-\sqrt{2(2\mu-1)}u_y,\ 1/2(2\mu-1)\right) \\ P & \sim N\left(\sqrt{2(2\mu-1)}u_x,\ 1/2(2\mu-1)\right) \end{cases}$$

Theorem: Let $\rho_{\mathbf{u}}^{(n)} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$. Then there exist q. channels (randomizations) T_n, S_n such that for any $\eta < 1/4$

$$\lim_{n\to\infty} \sup_{\|u\|<n^\eta} \left\| T_n\left(\rho_{\mathbf{u}}^{(n)}\right) - N^{\mathbf{u}} \otimes \phi^{\mathbf{u}} \right\|_1 = 0,$$

and

$$\lim_{n\to\infty} \sup_{\|u\|<n^\eta} \left\| \rho_{\mathbf{u}}^{(n)} - S_n\left(N^{\mathbf{u}} \otimes \phi^{\mathbf{u}}\right) \right\|_1 = 0.$$

Localisation + Local Asymptotic Normality = optimal estimation

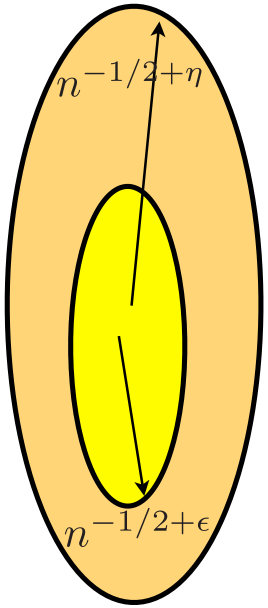
- localise the state within the small local neighborhood of ρ_0 while waisting $\tilde{n} \ll n$ qubits

- use local asymptotic normality on the bigger ball to design the second stage measurement

$$N^{\mathbf{u}} \otimes \phi^{\mathbf{u}}$$

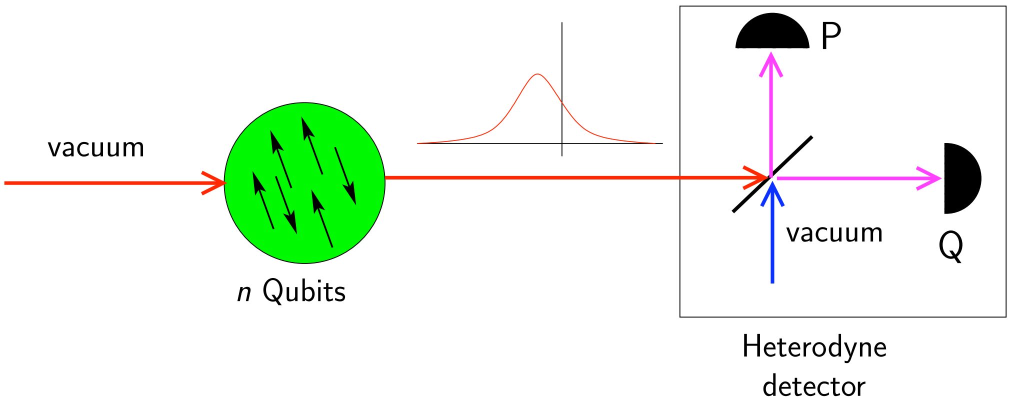
"Observe" $N^{\mathbf{u}}$

Do heterodyne on $\phi^{\mathbf{u}}$



Implementation with continuous time measurements

Couple the qubits with a Bosonic field, let the state leak into the field and do heterodyne (quantum part) followed by a L_z measurement (classical part)



Unitary evolution on $\left(\mathbb{C}^2\right)^{\otimes n} \otimes \mathcal{F}(L^2(\mathbb{R}))$ given by the QSDE:

$$dU_n(t) = (a_n dA_t^* - a_n^* dA_t - \frac{1}{2} a_n^* a_n dt) U_n(t), \qquad a_n := \frac{1}{\sqrt{n(2\mu - 1)}} \sum_{i=1}^{n} \sigma_+^{(i)}$$

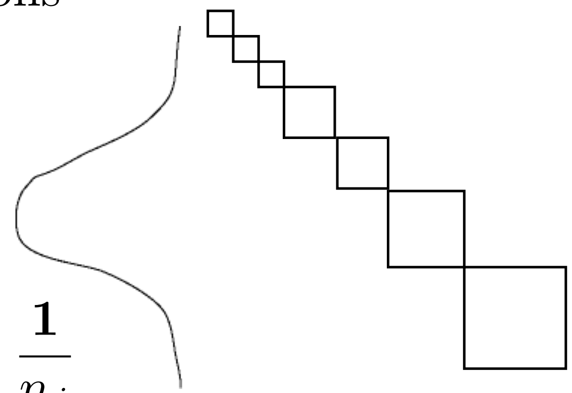
Idea of the proof: typical SU(2) representations

$$\text{“} \quad \rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} \rightsquigarrow N^{\mathbf{u}} \otimes \phi^{\mathbf{u}} \quad \text{”}$$

Two commuting group actions: $SU(2)$ rotations and permutations

$$(\mathbb{C}^2)^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} \mathcal{H}_j \otimes \mathcal{K}_j$$

$$\rho^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} p_n(j) \rho_{n,j} \otimes \frac{\mathbf{1}}{n_j}$$



Classical part: measuring j (‘which block’) does not disturb the state

Distribution $p_n^{\mathbf{u}}(j)$ converges to $N^{\mathbf{u}}$ as in coin toss $(L \cong L_z \sim \mathrm{Bin}(\mu + u_z/\sqrt{n}, n))$

$$\tau_n(j) := (j - (\mu - 1/2)n)/\sqrt{n} \rightsquigarrow N^{\mathbf{u}} = N(u_z, \mu(1-\mu))$$

Idea of the proof: typical SU(2) representations

Quantum part: conditional on j we remain with a typical block state $\rho^{\mathbf{u}}_{n,j}$

Structure of π_j:

Spin operators $L_\pm = L_x \pm iL_y$ act as ladder operators on basis vectors

$$\mathcal{H}_j = \mathrm{Lin}\{|m, j\rangle : m = -j, \ldots, j\}$$

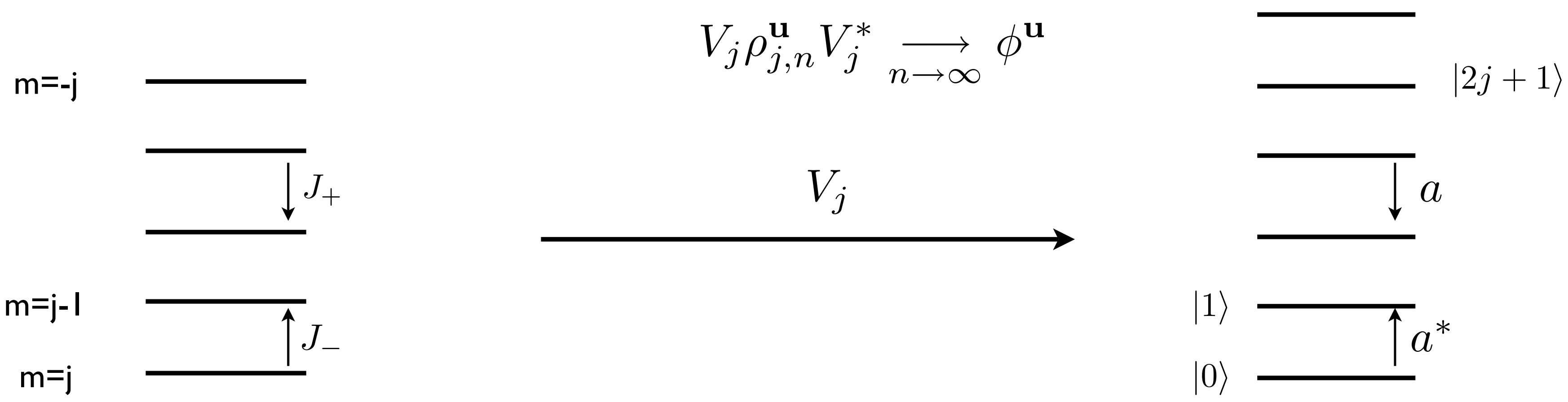
Creation and annihilation operators a^*, a act similarly on the number basis $\{|k\rangle : k \geq 0\}$

Embed irrep $\mathcal{H}_j$ into the harmonic oscillator $\ell^2(\mathbb{N})$ by isometry

$$V_j : |m, j\rangle \mapsto |j - m\rangle$$

By Quantum Central Limit Theorem, collective observables $J_\pm$ become Gaussian and

$$V_j \rho^{\mathbf{u}}_{j,n} V_j^* \underset{n\to\infty}{\longrightarrow} \phi^{\mathbf{u}}$$



Outline:

- Quantum state estimation and optimality
- Local Asymptotic Normality in classical statistics
- Local Asymptotic Normality for qubits
- Local Asymptotic Normality for d-dimensional state

Local asymptotic normality for d-dimensional states

Local neighbourhood around $\rho_0 := \mathrm{Diag}(\mu_1, \ldots, \mu_d)$ with $\mu_1 > \mu_2 > \cdots > \mu_d > 0$

$$\rho_\theta = \begin{bmatrix} \mu_1 + u_1 & \zeta_{1,2}^* & \cdots & \zeta_{1,d}^* \\ \zeta_{1,2} & \mu_2 + u_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \zeta_{d-1,d}^* \\ \zeta_{1,d} & \cdots & \zeta_{d-1,d} & \mu_d - \sum_{i=1}^{d-1} u_i \end{bmatrix}, \quad \theta = (\vec{u}, \vec{\zeta}) \in \mathbb{R}^{d-1} \times \mathbb{C}^{d(d-1)/2}$$

In first order,

$$\rho_{\theta/\sqrt{n}} := U\left(\frac{\vec{\zeta}}{\sqrt{n}}\right) \begin{bmatrix} \mu_1 + u_1/\sqrt{n} & 0 & \cdots & 0 \\ 0 & \mu_2 + u_2/\sqrt{n} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_d - \sum_{i=1}^{d-1} u_i/\sqrt{n} \end{bmatrix} U^*\left(\frac{\vec{\zeta}}{\sqrt{n}}\right),$$

where

$$U(\vec{\zeta}) := \exp\left[\sum_{1 \le j < k \le d} \frac{\zeta_{j,k}^* E_{k,j} - \zeta_{j,k} E_{j,k}}{\mu_j - \mu_k}\right] \in SU(d)$$

L. A. N. Theorem

- Diagonal parameters give rise to a classical $(d-1)$-dimensional Gaussian

$$N^{\vec{u}} := N(\vec{u}, V_\mu)$$

- Off-diagonal parameters decouple from the diagonal ones *and* from each other

$$\Phi^{\vec{\xi}} := \bigotimes_{j<k} \phi^{\xi_{j,k}}$$

where $\phi^{\xi_{j,k}}$ is a displaced thermal equilibrium state with $\beta = \ln \mu_j / \mu_k$

- Total classical-quantum limit model: $\Phi^\theta := N^{\vec{u}} \otimes \Phi^{\vec{\xi}}$

L. A. N. Theorem

- Diagonal parameters give rise to a classical $(d-1)$-dimensional Gaussian

$$N^{\vec{u}} := N(\vec{u}, V_\mu)$$

- Off-diagonal parameters decouple from the diagonal ones *and* from each other

$$\Phi^{\vec{\xi}} := \bigotimes_{j<k} \phi^{\xi_{j,k}}$$

where $\phi^{\xi_{j,k}}$ is a displaced thermal equilibrium state with $\beta = \ln \mu_j/\mu_k$

- Total classical-quantum limit model: $\Phi^\theta := N^{\vec{u}} \otimes \Phi^{\vec{\xi}}$

Theorem. Let $\rho_\theta^{(n)} := \rho_{\theta/\sqrt{n}}^{\otimes n}$. Then there exist q. channels T_n, S_n such that

$$\lim_{n\to\infty} \sup_{\theta \in \Theta_{n,\beta,\gamma}} \left\| \Phi^\theta - T_n(\rho^{\theta,n}) \right\|_1 = 0,$$

$$\lim_{n\to\infty} \sup_{\theta \in \Theta_{n,\beta,\gamma}} \left\| S_n(\Phi^\theta) - \rho^{\theta,n} \right\|_1 = 0,$$

where $\Theta_{n,\beta,\gamma} := \{\theta = (\vec{\xi}, \vec{u}) : \|\vec{\xi}\| \leq n^\beta, \|\vec{u}\| \leq n^\gamma\}, \qquad \beta < 1/9, \gamma < 1/4.$

Local asymptotic normality for d-dimensional states

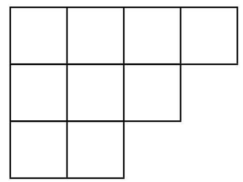
Two commuting group actions: $SU(d)$ rotations and permutations

$$\begin{aligned}\left(\mathbb{C}^d\right)^{\otimes n} &= \bigoplus_\lambda \mathcal{H}_\lambda \otimes \mathcal{K}_\lambda \\ \rho^{\otimes n} &= \bigoplus_\lambda p_n(\lambda)\rho_{n,\lambda} \otimes \frac{\mathbf{1}}{n_\lambda}\end{aligned}$$

$SU(d)$ irreps λ are labelled by Young diagrams with d rows and n boxes

$\lambda_1 \approx n\mu_1$

$\lambda_d \approx n\mu_d$



Classical part: measure 'which Young diagram λ'

Distribution $p_{n,\theta}(\lambda)$ converges to multivariate Gaussian shift

$$\{(\lambda_i - n\mu_i)/\sqrt{n} \ : \ i = 1, \dots d-1\} \rightsquigarrow N(\vec{u}, I_\mu^{-1})$$

same as the multinomial model Mult $\left(\mu_1 + \frac{u_1}{\sqrt{n}}, \dots \mu_d - \sum_i \frac{u_i}{\sqrt{n}}; n\right)$!

Quantum part: conditional on λ we remain with a typical block state $\rho^\theta_{\lambda,n}$

Incursion into SU(d) irreps

Structure of $\mathcal{H}_\lambda$:

- Write tensors into λ-tableaux

$$e_{\mathbf{a}} := e_{a(1)} \otimes \cdots \otimes e_{a(n)} \longmapsto t_{\mathbf{a}}, \qquad \text{e.g. } e_2 \otimes e_1 \otimes e_1 \mapsto \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

- Young symmetriser Y_λ is minimal projection in $\mathrm{Alg}(S(n))$

$$Y_\lambda = Q_\lambda \cdot P_\lambda := \sum_{\tau \in \mathcal{C}(\lambda)} \mathrm{sgn}(\tau)\tau \cdot \sum_{\sigma \in \mathcal{R}(\lambda)} \sigma$$

- Non-orthogonal basis of $\mathcal{H}_\lambda$ indexed by semi-standard Young tableaux, e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$

$$f_{\mathbf{a}} := Y_\lambda e_{\mathbf{a}}$$

- 'Number basis': $t_{\mathbf{a}} \longleftrightarrow \mathbf{m} = \{m_{i,j} = \sharp\mathrm{j}'\mathrm{s} \text{ in row i } : \ i < j\}$

$$|\mathbf{m}, \lambda\rangle := f_{\mathbf{a}} / \|f_{\mathbf{a}}\|$$

Lemma: Not far from 'vacuum' $\mathbf{m} = 0$, basis is almost ON
If $|\mathbf{l}, |\mathbf{m}| = O(n^\eta)$, for some $0 < \eta < 2/9$ then

$$|\langle \mathbf{m}, \lambda | \mathbf{l}, \lambda \rangle| = O(n^{-c(\eta)}), \quad c(\eta) > 0$$

Incursion into SU(d) irreps

Structure of π_λ:

- 'Ladder operators' $L_{i,j} = \pi_\lambda(E_{i,j}),\ L^*_{i,j} = \pi_\lambda(E_{j,i})$ for $1 \leq i < j \leq d$ don't act as ladder...

$$L^*_{2,3} : \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \longrightarrow \begin{array}{ccc} 1 & 1 & \boxed{3} \\ 2 & 2 & \\ 3 & & \end{array} + 2 \begin{array}{ccc} 1 & 1 & 2 \\ 2 & \boxed{3} & \\ 3 & & \end{array}$$

- However they do so on 'typical vectors' $|\mathbf{m}| = O(n^\eta) \ll n$

$$L^*_{2,3} : \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 \end{array} \longrightarrow O(n^\eta) \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \boxed{2} & 3 & 3 \\ 2 & 2 & 2 & 2 & 3 & 3 \\ 3 & 3 & 3 \end{array} + O(n) \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 2 & \boxed{3} & 3 & 3 \\ 3 & 3 & 3 \end{array}$$

After normalisation first term drops and we get an $a^*_{i,j}$ creation operator

$$L^*_{2,3}/\sqrt{n} : |\{m_{1,2}, m_{1,3}, m_{2,3}\},\ \lambda\rangle \xrightarrow{\cong} \sqrt{m_{2,3}+1}\ |\{m_{1,2}, m_{1,3}, m_{2,3}+1\},\ \lambda\rangle \quad !!$$

- Asymptotically, $L_{i,j}/\sqrt{n}$ acts only on row i and they all commute with each other...

We have convergence to a tensor product of harmonic oscillators $(a_{i,j}, a^*_{i,j})$ in the vacuum

Further work:

Local asymptotic normality for i.i.d. infinite dimensional quantum states

- general parametric families of states of light
- optimal estimation rate for states of light

Local asymptotic normality for quantum Markov chains (processes)

- optimal rates for interaction parameters in realistic dynamical models (next talk)
- Central Limit Theorem for quantum Markov chains

Testing with 'non-discrete' hypotheses

- eg $\rho = \rho_0$ vs $\rho \neq \rho_0$

Weak and strong convergence of quantum statistical experiments

$$\mathcal{Q}_n = \{\rho_n^\theta : \theta \in \Theta\} \rightsquigarrow \mathcal{Q} = \{\rho^\theta : \theta \in \Theta\}$$

Quantum statistical decision theory

- quantum experiment $\mathcal{Q} = \{\rho^\theta \in M(\mathbb{C}^d) : \theta \in \Theta\}$
- non-commuting "loss functions" $0 \leq W_\theta \in M(\mathbb{C}^k)$
- "decision" $C : \rho^\theta \mapsto C(\rho^\theta) \in M(\mathbb{C}^k)$ with risk $R(C, \theta) = \mathrm{Tr}(C(\rho^\theta) W_\theta)$
- applications in quantum memory, quantum cloning

References

- M. Guta, J. Kahn
  Local asymptotic normality for qubit states
  P.R.A **73**, 05218, (2006)

- M. Guta, A. Jencova
  Local asymptotic normality in quantum statistics
  Commun. Math. Phys. **276**, 341-379, (2007)

- M. Guta, B Janssens and J. Kahn
  Optimal estimation of qubit states with continuous time measurements
  Commun. Math. Phys. **277**, 127-160, (2008)

- M. Guta, J. Kahn:
  Local asymptotic normality for finite dimensional quantum systems
  arXiv:0804.3876 (Commun. Math. Phys, to appear)