G. M. D'Ariano, arXiv:0807.4383

Probabilistic Theories: what is special about Quantum Mechanics

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BACKROUND

QM is a probabilistic theory + *something more*

OBJECTIVE

to understand what is the *something more*, and derive QM solely from operational principles

Operational framework

Axioms:
(primitive notions)

- probability ...
- events
- independent systems
- ...

General principles for mathematical representation

- all mathematical objects must be defined operationally
- mathematical completion is for convenience (e.g. algebraic closure, norm closure, linear span, etc.)

In this talk w.l.g. we consider only:

- finite dimension
- only one kind of “system”

Postulates

- **NSF:** No signaling from the future.
- **NS:** No signaling (=existence of independent systems)
- **PFAITH:** There exists preparationally faithful states

- **AE:** Atomicity of evolution
- **CJ:** Choi-Jamiolkowski isomorphism ← mathematical

Postulates under exploration

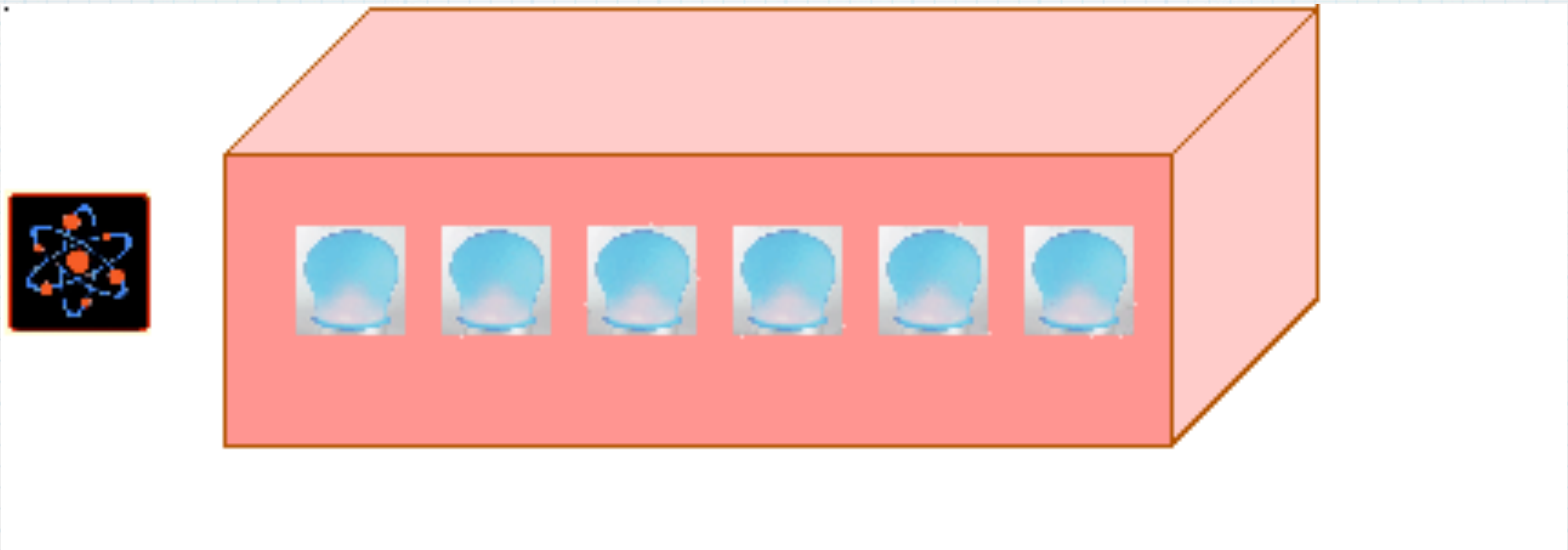
- **FAITHE:** There exists a faithful effect
- **PURIFY:** There exists a purification for each state

Probabilistic theories

Test:= set of probabilistic events

TESTS

- Test/experiment: $\mathbb{A} \equiv \{\mathscr{A}_j\}$ set of possible events $\mathscr{A}_j$



(deterministic test/transformation: $\mathbb{D} = \{\mathscr{D}\}$)

Notice: <u>the same event can occur in different tests</u>

TESTS

Unions of events: $\mathscr{A} \cup \mathscr{B}$

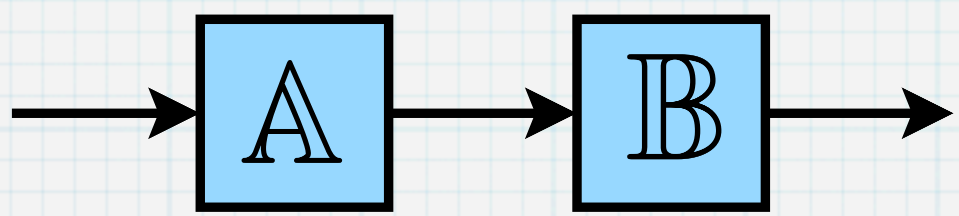
$$\mathscr{D}_{\mathbb{A}} := \bigcup_{\mathscr{A}_i \in \mathbb{A}} \mathscr{A}_i$$

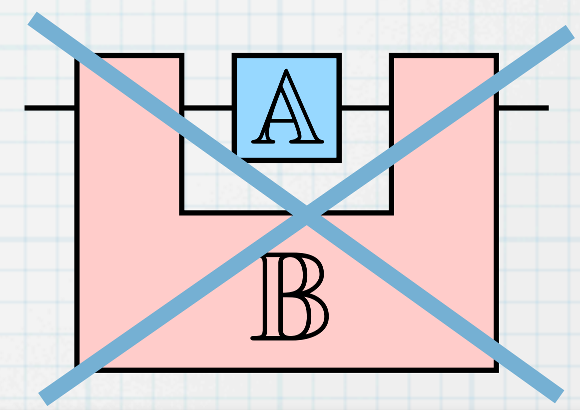
$$\mathbb{A} = \{\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\} \quad \xrightarrow{\text{coarse-graining}} \quad \mathbb{A}' = \{\mathscr{A}_1, \mathscr{A}_2 \cup \mathscr{A}_3\}$$

refinement (←)

Atomic: an event that cannot be refined in any test

Time-cascade:





$\mathbb{B} \circ \mathbb{A} = \{\mathscr{B}_j \circ \mathscr{A}_i\}$ cascade of tests $\mathbb{A} = \{\mathscr{A}_i\}$, $\mathbb{B} = \{\mathscr{B}_j\}$,

composition of events: $\mathscr{B} \circ \mathscr{A}$

SYSTEM

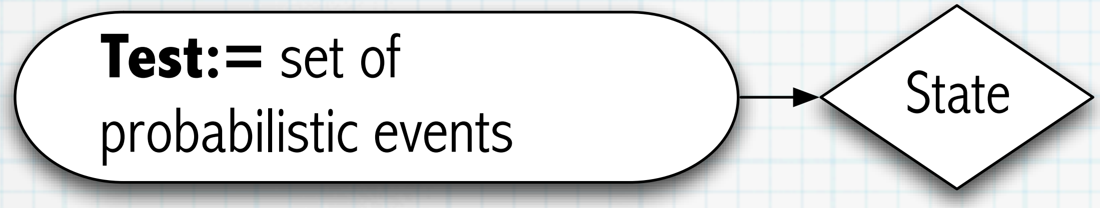
Copenhagen

$$\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$$

collection of tests closed under

- coarse-graining
- conditioning
- cascading (mono-systemic)
- (convex combination)

Probabilistic theories



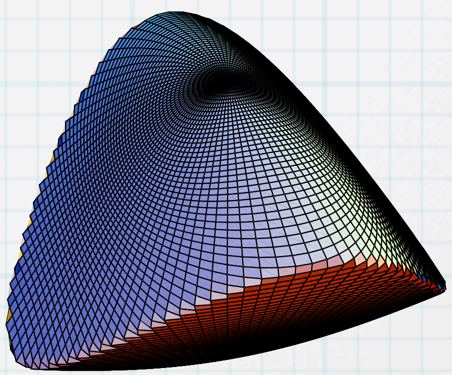
QUit quantum information theory group

STATES

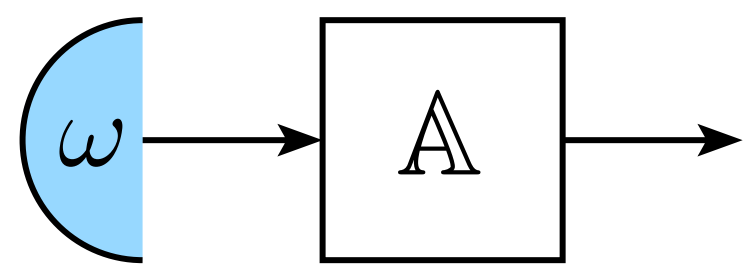
State ω: probability rule $\omega(\mathscr{A})$ for any possible event $\mathscr{A}$ in any test

Normalization: $$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$$

Convex set of states of a system: $\mathfrak{S}$

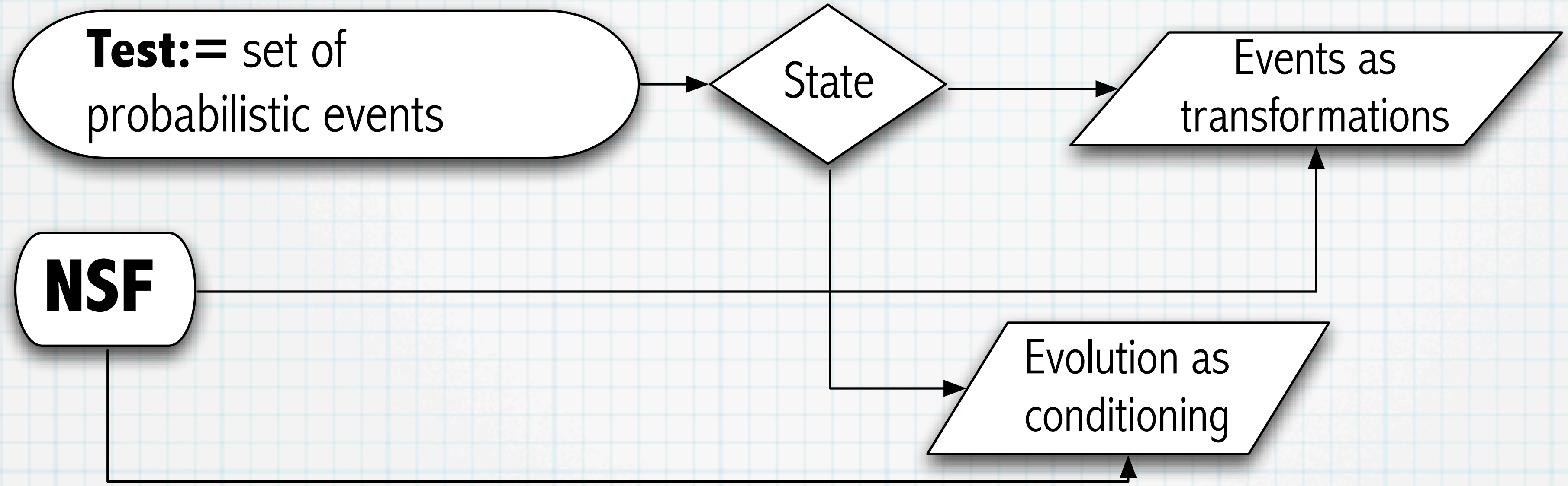


States will also be regarded as tests themselves "preparation-tests".



$$\mathrm{S} = \{\omega_1, \omega_2, \ldots, \mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$$

Probabilistic theories



Events ≡ transformations

Cascade: Event $\mathscr{B}\circ\mathscr{A}$: event $\mathscr{B}\in\mathbb{B}$ following $\mathscr{A}\in\mathbb{A}$

NSF (Ozawa) $$\sum_{\mathscr{B}_j\in\mathbb{B}}\omega(\mathscr{B}_j\circ\mathscr{A})=\omega(\mathscr{A}),\quad\forall\mathbb{B},\ \forall\mathscr{A},\ \forall\omega$$

$\Longrightarrow$ conditional probability: $p(\mathscr{B}|\mathscr{A})=\omega(\mathscr{B}\circ\mathscr{A})/\omega(\mathscr{A})$

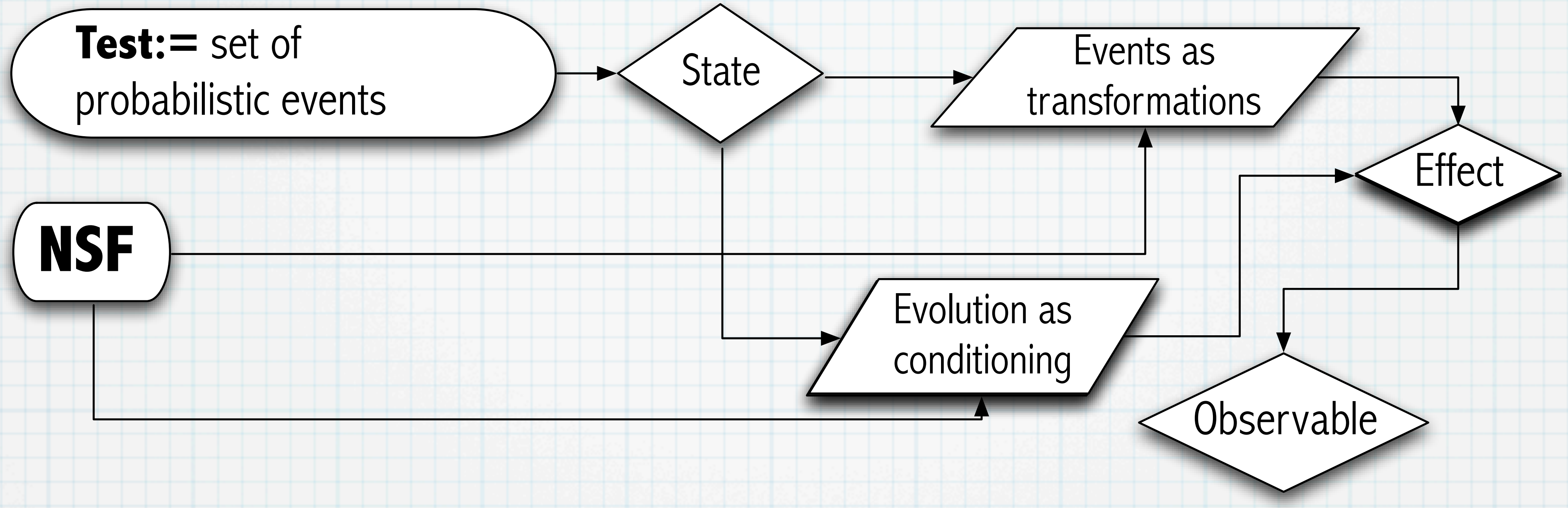
$\Longrightarrow$ **conditional state:** $\omega_{\mathscr{A}}:=\omega(\cdot\circ\mathscr{A})/\omega(\mathscr{A})$

$\Longrightarrow$ **evolution ≡ state conditioning:** $\mathscr{A}\omega:=\omega(\cdot\circ\mathscr{A})$

$\Longrightarrow$ events ≡ transformations

Convex monoid of transformations: $\mathfrak{T}$

Probabilistic theories



2 equivalence classes for transformations

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are conditioning equivalent if

$$\omega_{\mathscr{A}} = \omega_{\mathscr{B}} \qquad \forall \omega \in \mathfrak{S}$$

Conditioning-equivalence class

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are probabilistically equivalent if

$$\omega(\mathscr{A}) = \omega(\mathscr{B}) \quad \forall \omega \in \mathfrak{S}$$

Probabilistic equivalence class

2 equivalence classes for transformations

A transformation is completely specified by the two classes:

$$\mathscr{A}\omega = \omega(\mathscr{A})\omega_{\mathscr{A}}$$

probabilistic

conditioning

variable

$$\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$$

Effects

Effect $\underline{\mathscr{A}}$: equivalence class of transformations occurring with the same probability as $\mathscr{A}$ for all states.

$$\forall \omega \in \mathfrak{S} : \quad \omega(\mathscr{A}) \equiv \omega(\underline{\mathscr{A}})$$

a effect ➡ $\mathscr{A} \in a$ means $\omega(\mathscr{A}) \equiv \omega(a)$

$\mathfrak{E}$:= convex set of effects

Duality: effects $\mathfrak{E}$ positive linear functionals over states (bounded by 1)

$$a \in \mathfrak{E}, \ \omega \in \mathfrak{S}, \quad \omega(a) \equiv a(\omega)$$

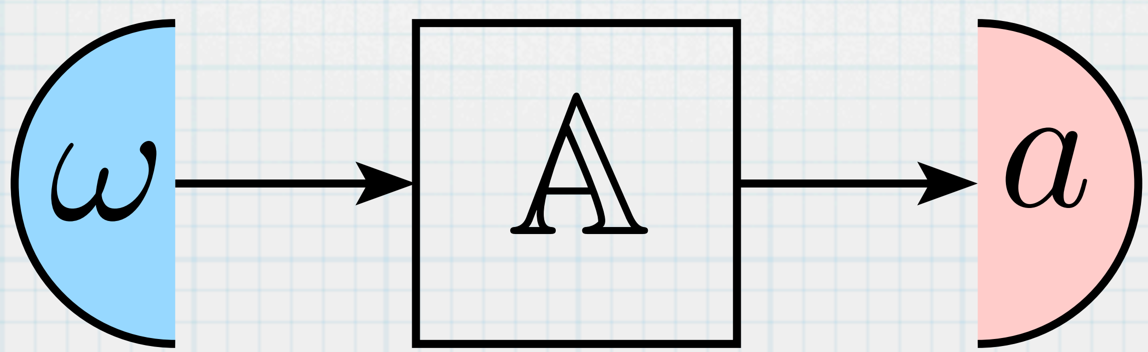
e deterministic effect i.e. $\omega(e) = 1 \quad \forall \omega \in \mathfrak{S}$

Effects

State-conditioning ⇒ Transformations act linearly over effects:

$$\underline{\mathscr{B}} \circ \mathscr{A} \in \underline{\mathscr{B} \circ \mathscr{A}} \qquad \text{(Heisenberg picture)}$$

Effects will also be regarded as tests themselves: “effect-tests”



Observables

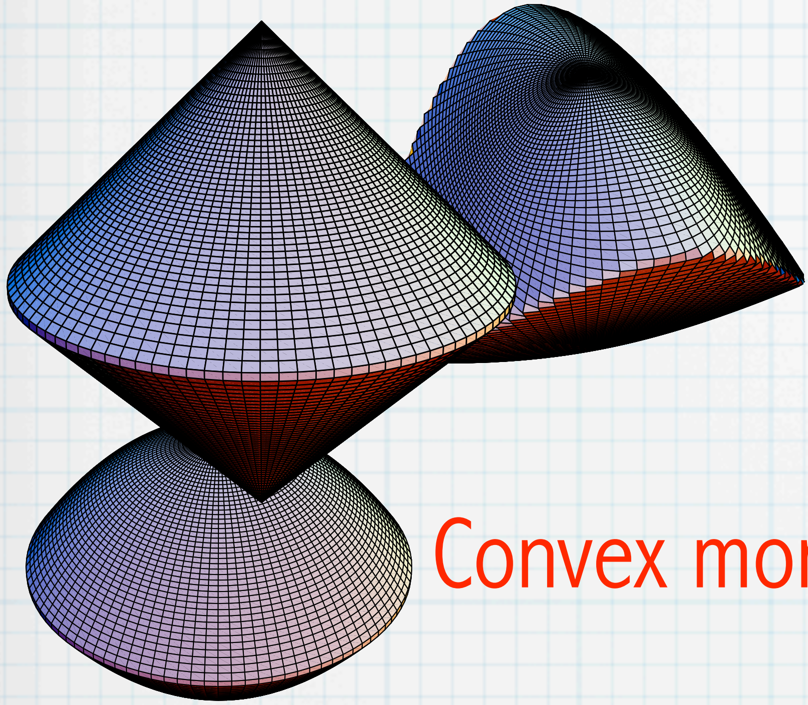
Observable $\mathbb{L} = \{l_i\}$: complete set of effects of a test

Normalization: $\sum_{i\in\mathbb{L}} l_i = e$

Informationally complete observable: $\mathbb{L}$

$$\mathfrak{E}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathbb{L})$$

Convex sets, Cones and Linear spaces

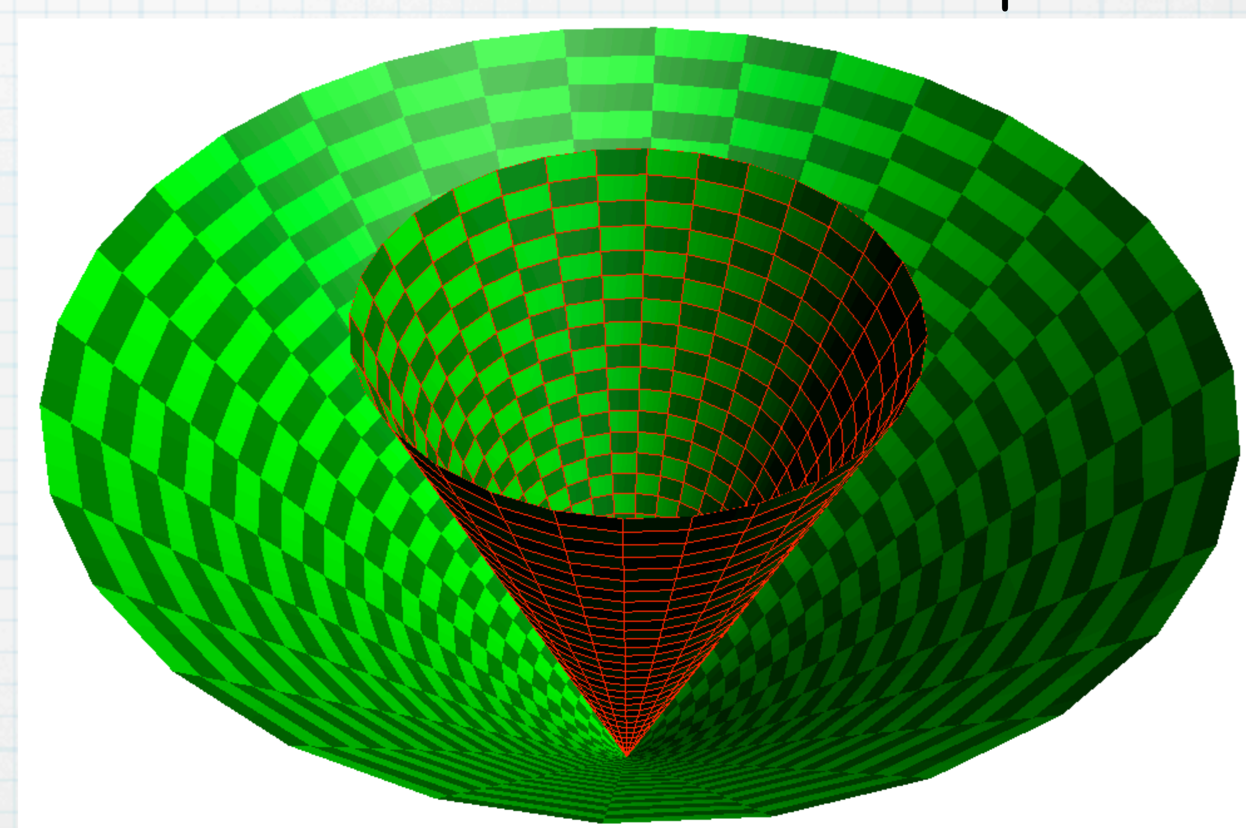


Convex set of states: $\mathfrak{S}$, cone: $\mathfrak{S}_+$

Convex set of effects: $\mathfrak{E}$, cone: $\mathfrak{E}_+$

Convex monoid of transformations: $\mathfrak{T}$, cone: $\mathfrak{T}_+$

Linear spaces:



$$\mathfrak{S}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}} \mathfrak{S}$$

$$\mathfrak{S}_{\mathbb{C}} = \mathsf{Span}_{\mathbb{C}} \mathfrak{S}$$

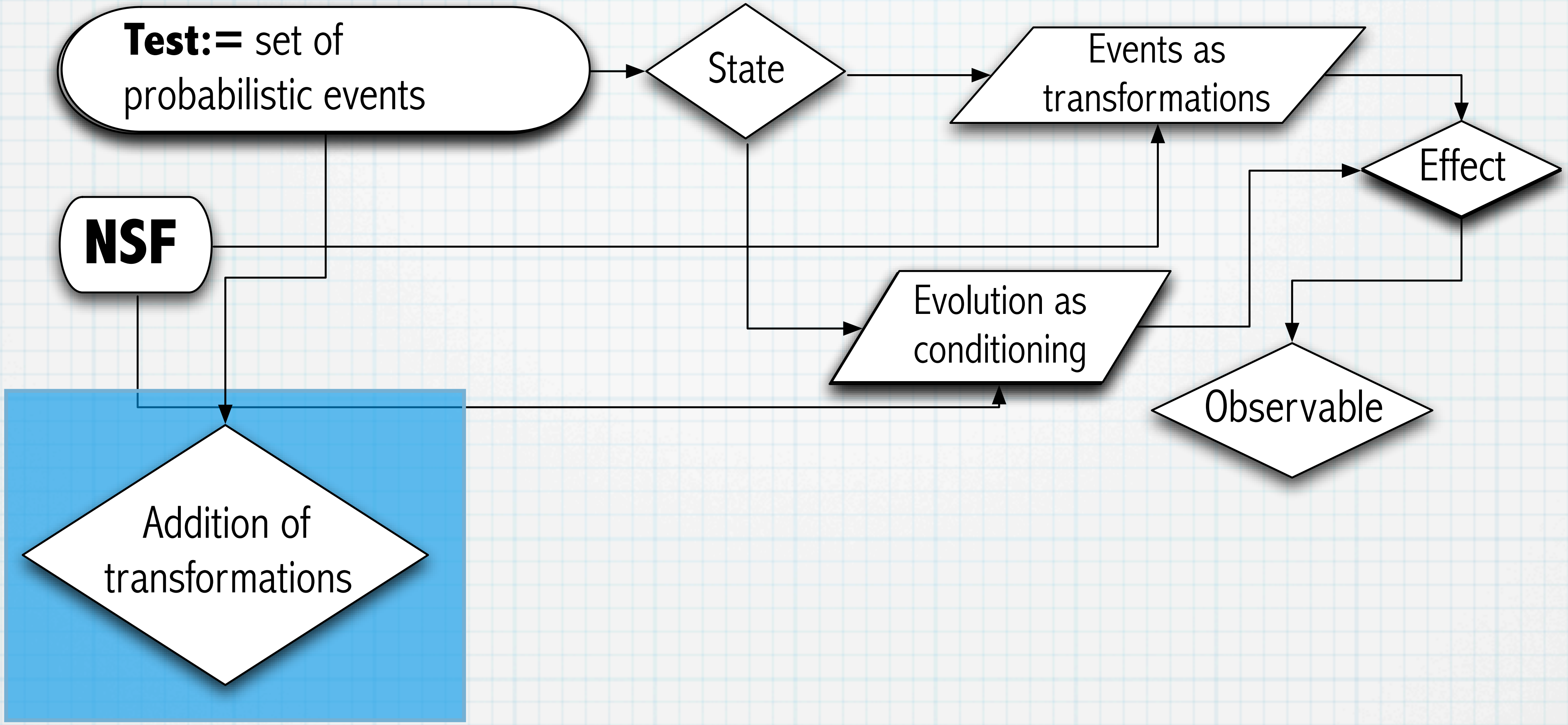
$$\mathfrak{E}_{\mathbb{R}}, \mathfrak{E}_{\mathbb{C}}, \mathfrak{T}_{\mathbb{R}}, \mathfrak{T}_{\mathbb{C}}$$

No-restriction hypotesis:

(no limitations to preparability)

$$\mathfrak{S}_+ = (\mathfrak{E}_+)^*$$

Probabilistic theories



Addition of transformations

Two transformations* $\mathscr{A}$ and $\mathscr{B}$ are test-compatible if for every state ω one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

For any two test-compatible transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ we define the transformation $\mathscr{A}_1 + \mathscr{A}_2$ as the union event $\mathscr{A}_1 \cup \mathscr{A}_2$ (the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred)

$$\omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \qquad \text{(probabilistic class)}$$

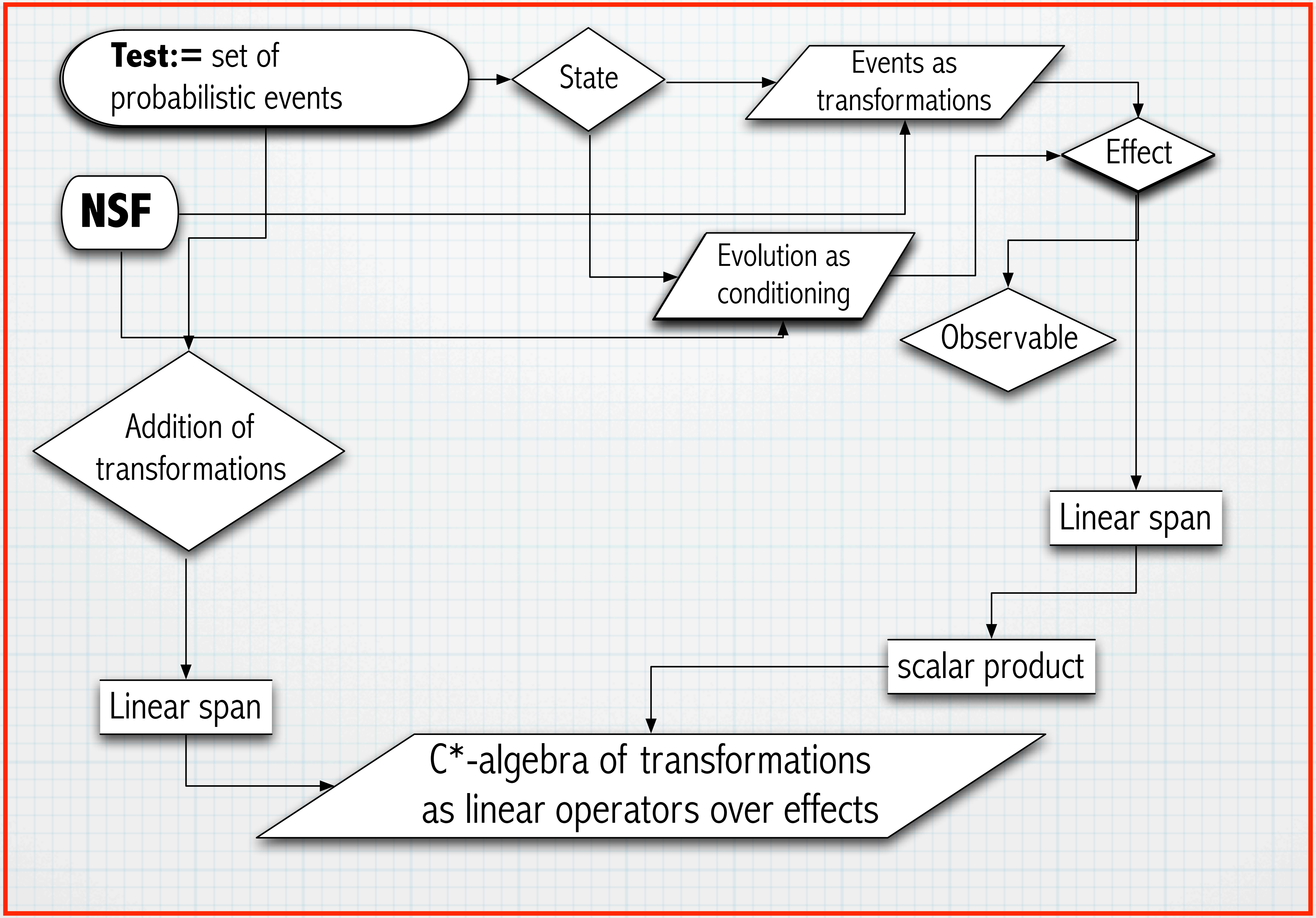
$$(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega \qquad \text{(conditioning class)}$$

$$\omega_{\mathscr{A}_1+\mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)}\omega_{\mathscr{A}2}$$

$$\omega(b \circ (\mathscr{A}_1 + \mathscr{A}_2)) = \omega(b \circ \mathscr{A}_1) + \omega(b \circ \mathscr{A}_2), \quad \forall b \in \mathfrak{E},\ \forall \omega \in \mathfrak{S}$$

(*) occurring also in different tests

Probabilistic theories



C^*-algebra of transformations (finite dim.)

Transformations/events are linear maps over effects, i.e. they make a matrix algebra over effects

One can introduce a scalar product over effects ...
$\Rightarrow$ transformations become a C^*-algebra ...

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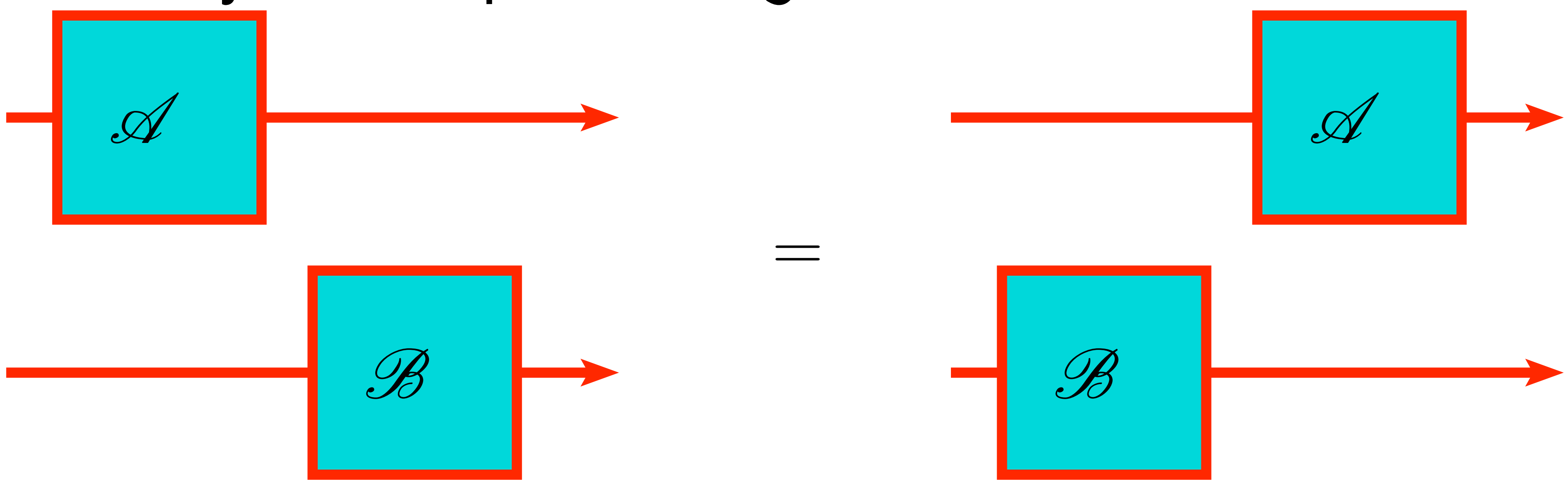
INDEPENDENT SYSTEMS

Two systems are independent if on each system it is possible to perform local tests for which on every joint state one has the commutativity of the pertaining transformations



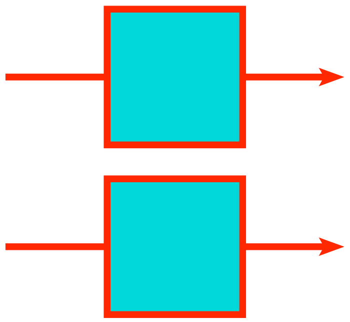
$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

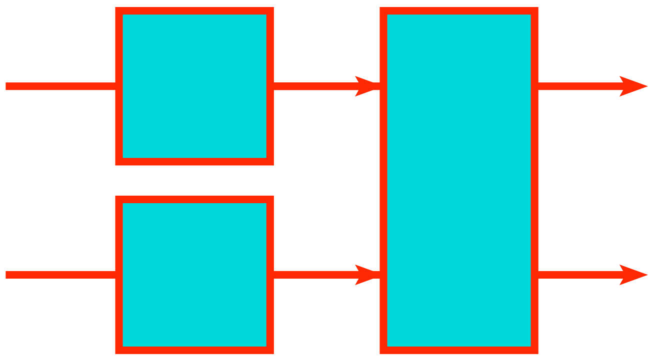
$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) \doteq \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} \circ \mathscr{C}^{(3)} \circ \ldots$$

$$[(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots)]_{\mathrm{eff}} \equiv (\underline{\mathscr{A}}, \underline{\mathscr{B}}, \underline{\mathscr{C}}, \ldots)$$

COMPOSTING SYSTEMS

We compost the two systems S_1 and S_2 into the bipartite system $\mathsf{S}_1 \odot \mathsf{S}_2$ by embedding the local tests $\mathsf{S}_1 \times \mathsf{S}_2$ into the bipartite system $\mathsf{S}_1 \odot \mathsf{S}_2$ as $\mathsf{S}_1 \odot \mathsf{S}_2 \supseteq \mathsf{S}_1 \times \mathsf{S}_2$ and closing w.r.t. coarse graining, convex combination and cascading.





Nonlocal tests: $\mathsf{S}_1 \odot \mathsf{S}_2 \setminus \mathsf{S}_1 \times \mathsf{S}_2$

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MARGINAL STATE

For a multipartite system we define the marginal state $\Omega|_n$ of the n-th system the state that gives the probability of any local transformation $\mathscr{A}$ on the n-th system with all other systems untouched, namely

$$\Omega|_n(\mathscr{A}) := \Omega(\mathscr{I},\ldots,\mathscr{I},\underbrace{\mathscr{A}}_{n-th},\mathscr{I},\ldots)$$

$$\Omega|_n(a) \doteq \Omega(e,\ldots,e,\underbrace{a}_{n\mathrm{th}},e,\ldots)$$

NS: (no-signaling) any local test on a system is equivalent to no-test on an another independent system.

Probabilistic theory?

Matrix algebra of transformations over effects!

Independent systems = no-signaling

Review of notation

Convex sets:

$\mathfrak{S}$ states

$\mathfrak{E}$ effects

$\mathfrak{T}$ transformations

cones: $\mathfrak{S}_+$ $\mathfrak{E}_+$ $\mathfrak{T}_+$

$\mathsf{S} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots\}$

System tests

$\mathscr{A}\omega = \omega(\cdot \circ \mathscr{A})$

variable

$\mathscr{A} \in a$ ← effect

transformation

e deterministic effect

Bipartite: $\mathfrak{E}(\mathsf{S}_1 \odot \mathsf{S}_2)$

$\mathfrak{E}^{\odot 2} := \mathfrak{E}(\mathsf{S}^{\odot 2})$

$\omega, \sigma \in \mathfrak{S}$
$a, b \in \mathfrak{E}$

$\Omega, \Phi \in \mathfrak{S}^{\odot 2}$
$E \in \mathfrak{E}^{\odot 2}$

$\mathscr{U}_{12}, (\mathscr{A}, \mathscr{I}) \in \mathfrak{T}^{\odot 2}$

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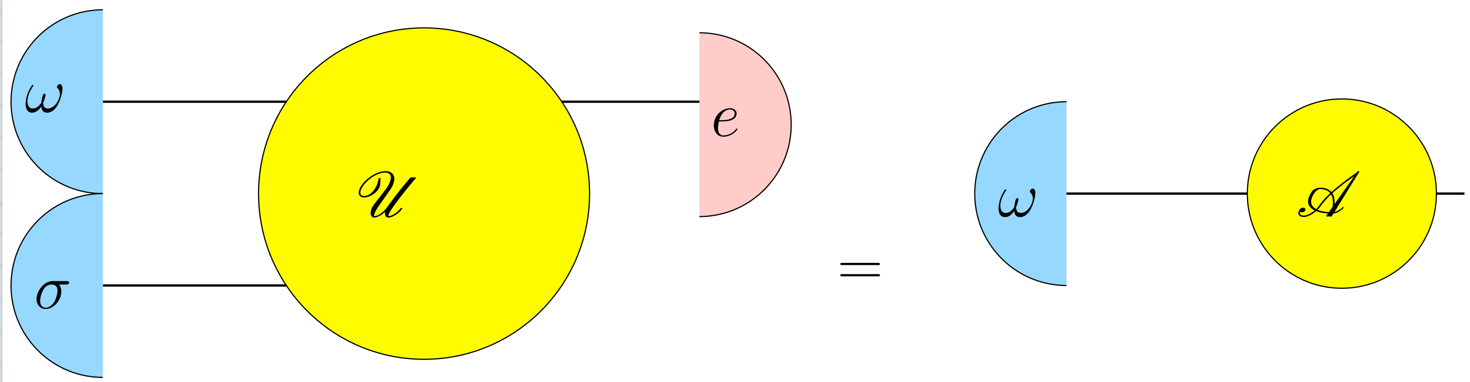
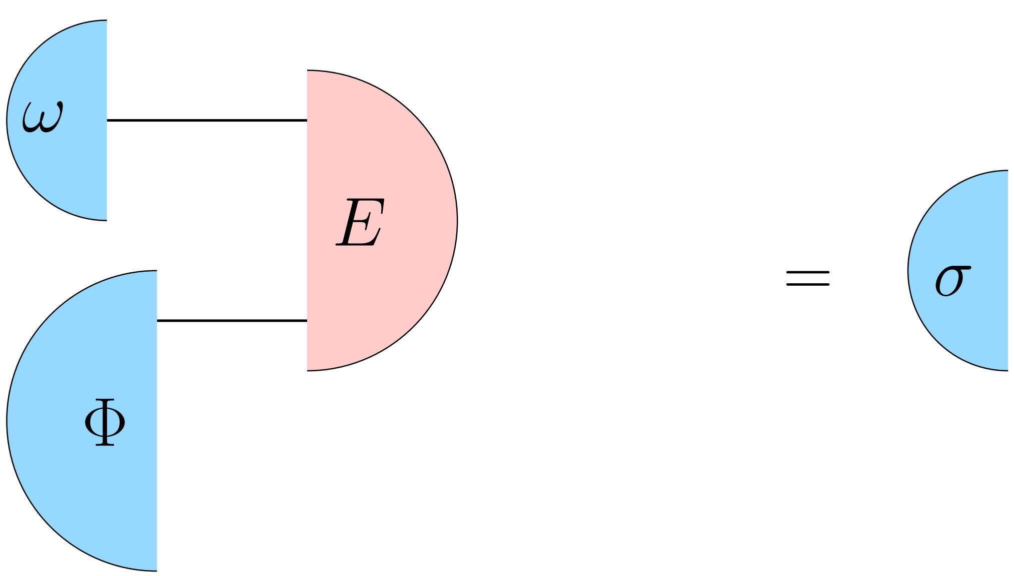
Review of notation

$$\omega(a) \equiv a(\omega)$$

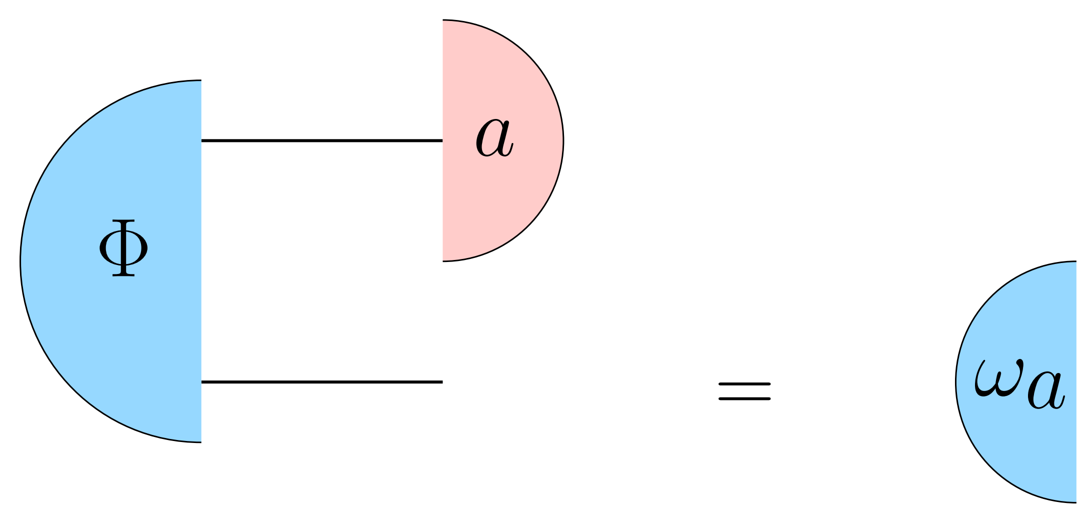
$$\mathscr{U}_{12}(\sigma,\omega)(e,\cdot) = \mathscr{A}\omega$$



$$E_{23}(\Phi,\omega) = \sigma \in \mathfrak{S}$$

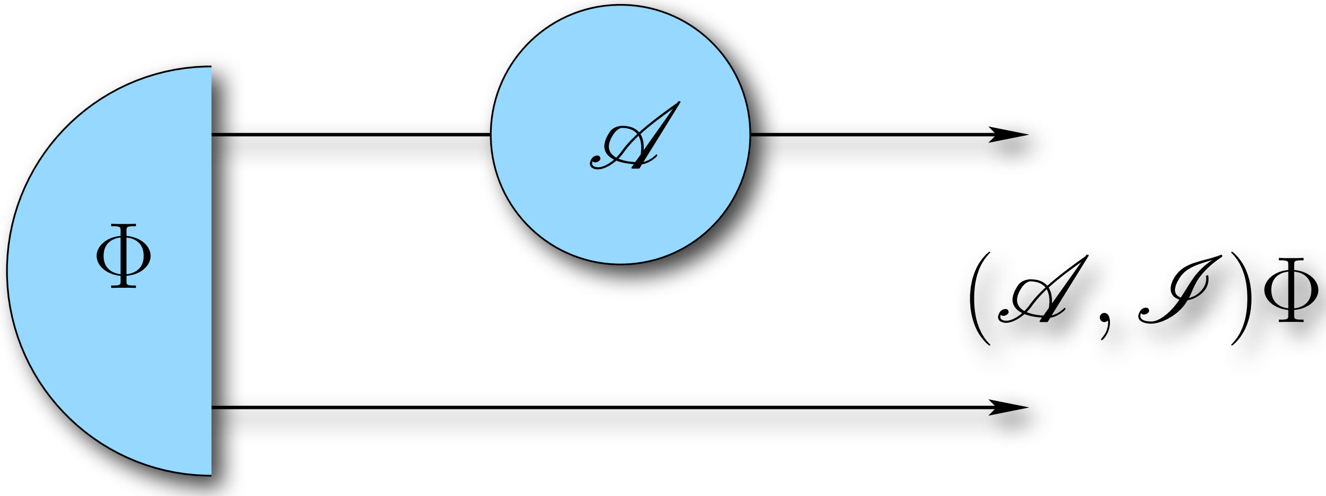


$$\Phi(a,\cdot) = \omega_a \in \mathfrak{S}_+$$

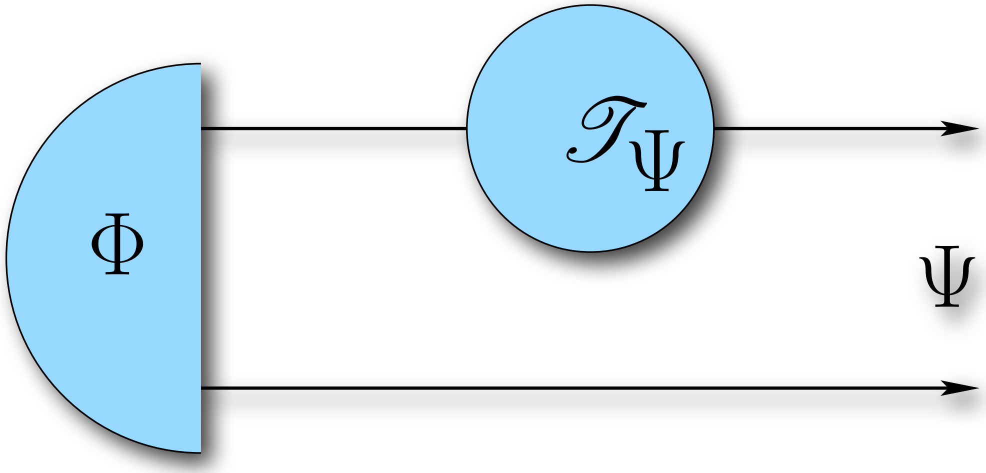


FAITHFUL STATES

A state Φ of a bipartite system is dynamically faithful when the output state $(\mathscr{A},\mathscr{I})\Phi$ from a local transformation $\mathscr{A}$ on one system is in 1-to-1 correspondence with the transformation $\mathscr{A}$



A state Φ of a bipartite system is preparationally faithful if every joint state Ψ can be achieved by a suitable local transformation $\mathscr{T}_\Psi$ on one system occurring with nonzero probability



Postulate PFAITH

PFAITH: For any couple of identical systems, there exist a symmetric* state Φ that is preparationally faithful.

(*) under permutation of the two systems

Theorem: Φ is also dynamically faithful.

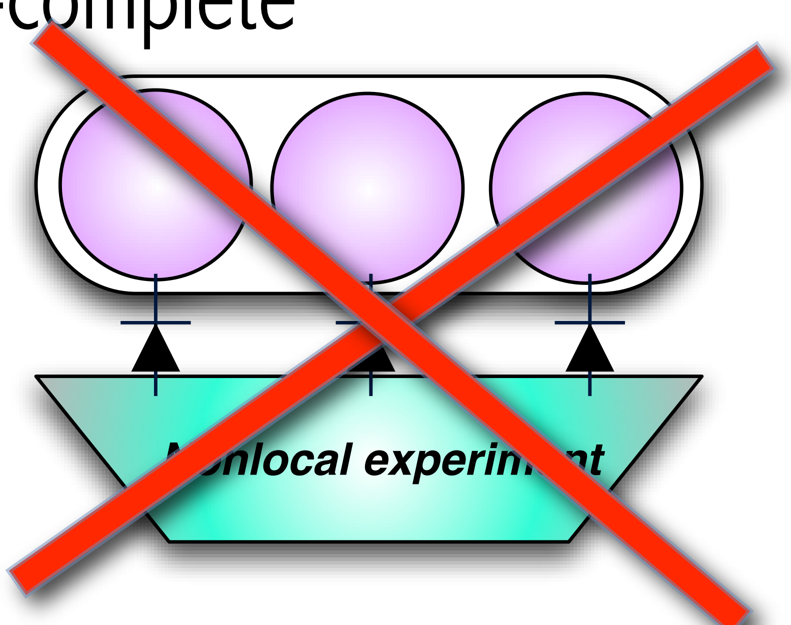
Consequences of PFAITH

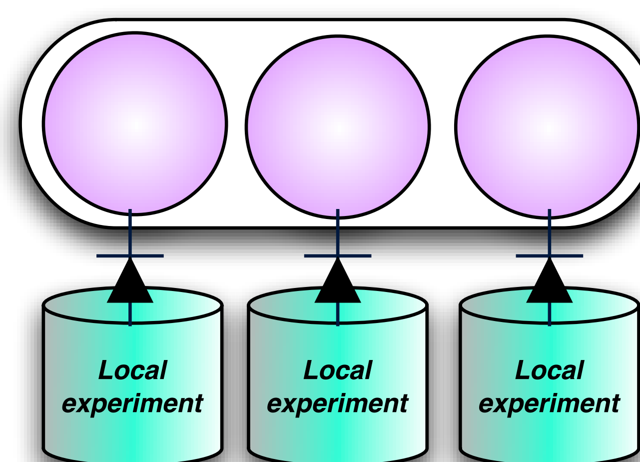
Calibrability & Preparability

Impossibility of secure bit commitment

Marginal state $\chi = \Phi(e, \cdot)$ internal and invariant under a transposed channel

Local observability: There exist global info-complete observables made of local info-complete





Holism  **Reductionism**

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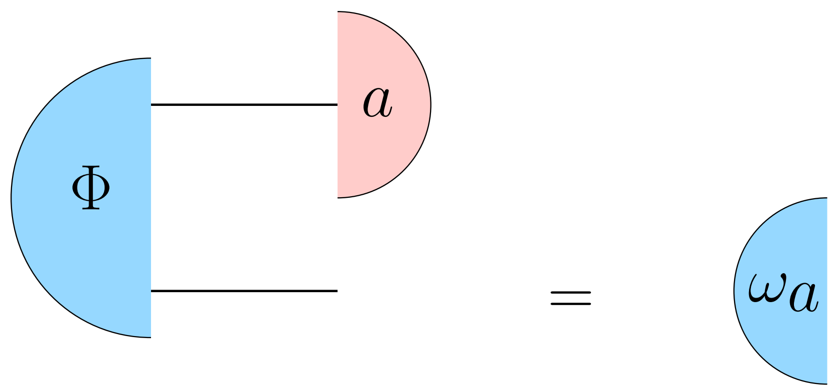
Consequences of PFAITH

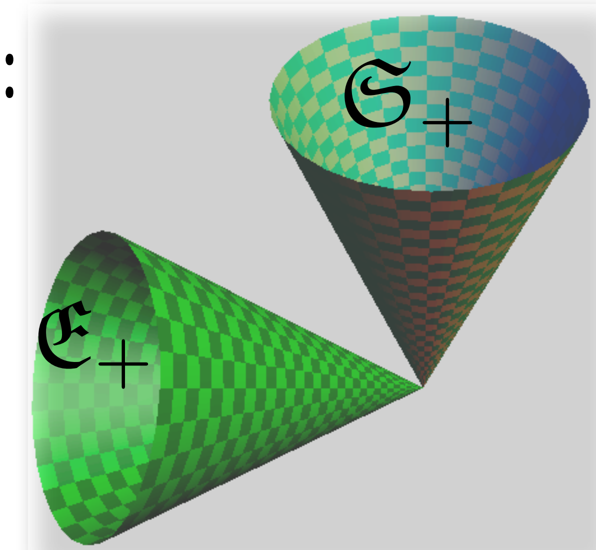
$$\mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_{\mathbb{F}}(\mathsf{S})$$

$$\mathbb{F} = \mathbb{R}, \mathbb{C}$$

Weak self-duality: State and effect cones are isomorphic:

$$\mathfrak{E}_+ \ni a \mapsto \omega_a = \Phi(a, \cdot) \in \mathfrak{S}_+$$





Tensor product representation:

$$\mathfrak{E}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{E}_{\mathbb{F}}(\mathsf{S})^{\otimes 2} \qquad \mathfrak{S}_{\mathbb{F}}(\mathsf{S}^{\odot 2}) = \mathfrak{S}_{\mathbb{F}}(\mathsf{S})^{\otimes 2}$$

Space of transformations is complete: $\mathfrak{T}_{\mathbb{F}} = \mathsf{Lin}(\mathfrak{E}_{\mathbb{F}})$

There exist states that are purifiable (e.g. $\mathscr{A}\chi$, with $\mathscr{A}$ atomic)

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Consequences of PFAITH

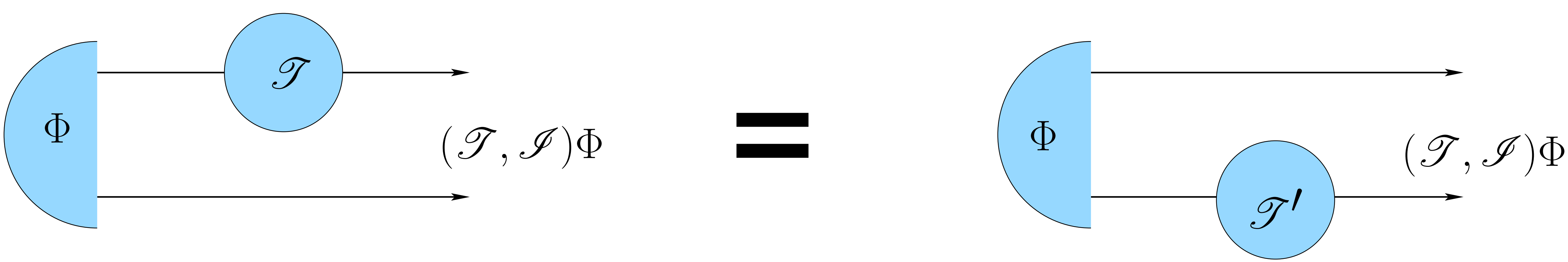
The faithful state Φ provides a non-degenerate scalar product over effects via its Jordan form (ς Jordan involution):

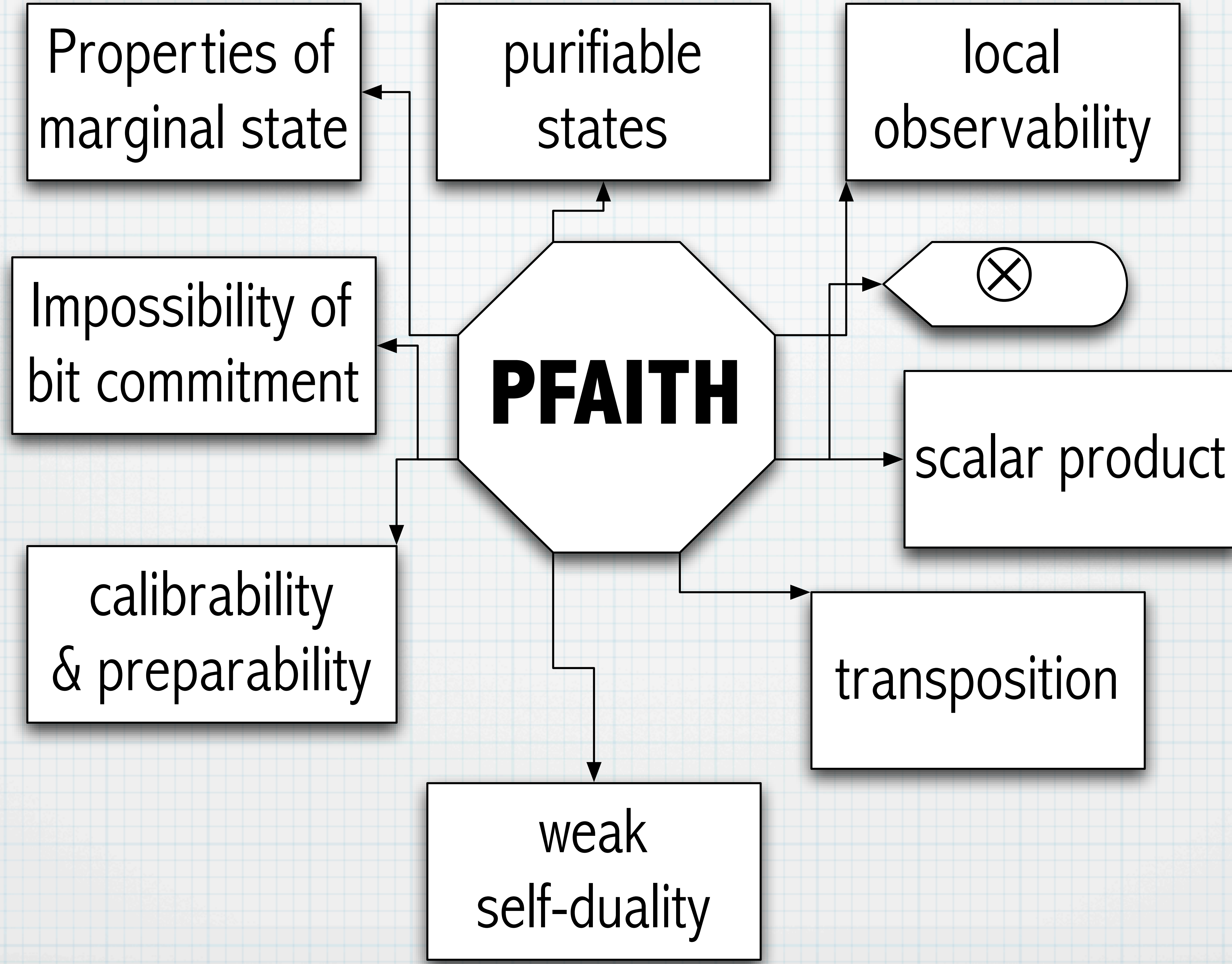
$$\forall a, b \in \mathfrak{E}_{\mathbb{R}}, \qquad {}_{\Phi}(b|a)_{\Phi} := |\Phi|(b, a) = \Phi(\varsigma(b), a)$$

It allows to introduce an operational notion of transposition for transformations:

$$(\mathscr{T}, \mathscr{I})\Phi = (\mathscr{I}, \mathscr{T}')\Phi$$

1. $(\mathscr{A} + \mathscr{B})' = \mathscr{A}' + \mathscr{B}'$
2. $(\mathscr{A}')' = \mathscr{A}$,
3. $(\mathscr{A} \circ \mathscr{B})' = \mathscr{B}' \circ \mathscr{A}'$

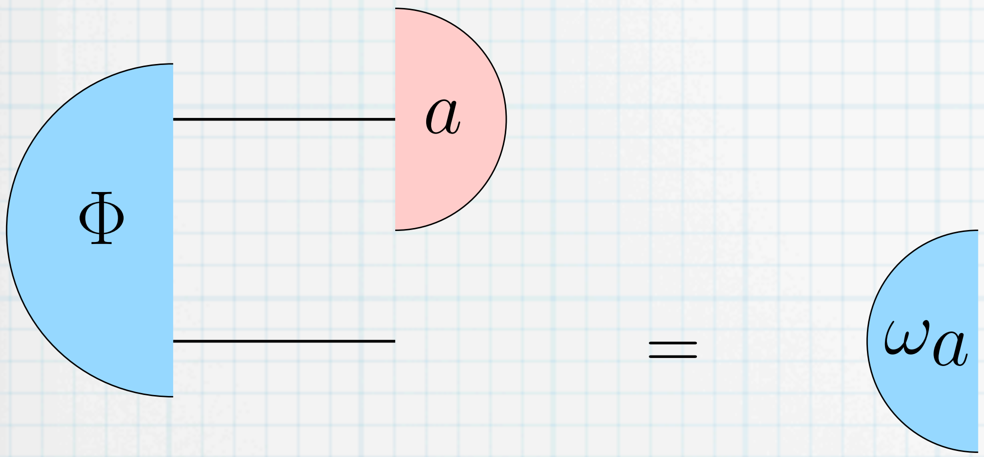



Properties of marginal state
purifiable states
local observability
Impossibility of bit commitment
PFAITH
⊗
scalar product
calibrability & preparability
transposition
weak self-duality

INTERLUDE

Exploring Postulates:
FAITHE and **PURIFY**

Faithful effect

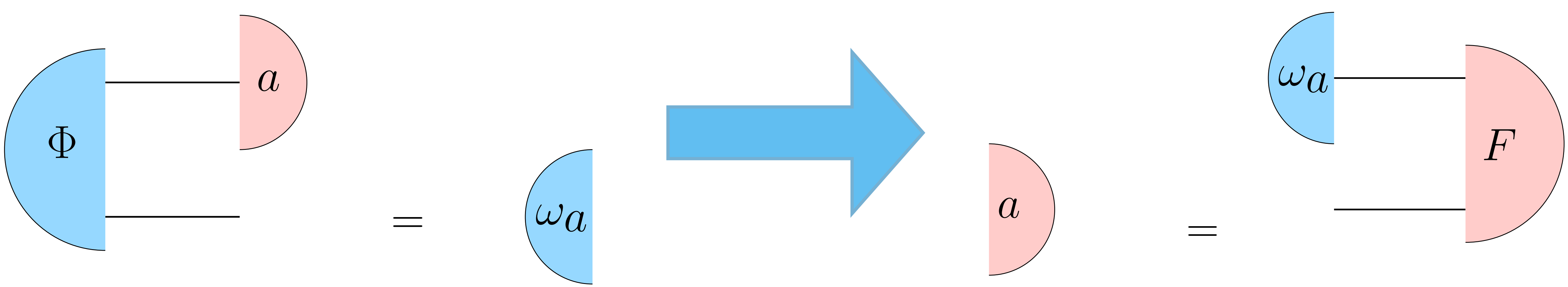


Remind the cone-isomorphism from the faithful state Φ

$$\mathfrak{E}_+ \ni a \mapsto \omega_a = \Phi(a,\cdot) \in \mathfrak{S}_+$$

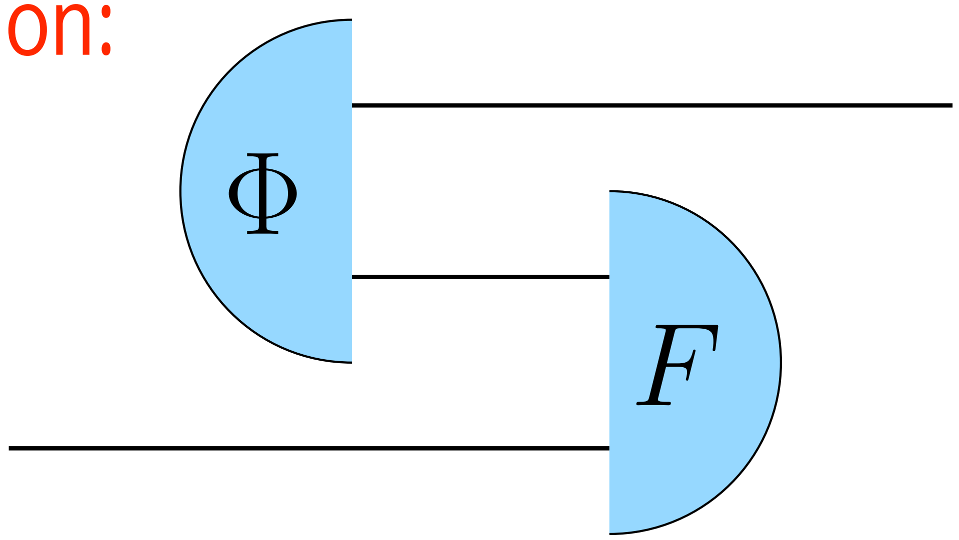
FAITHE: There exist a bipartite effect F achieving the inverse of the isomorphism $a \mapsto \omega_a = \Phi(a,\cdot)$ namely:

$$F_{23}(\omega_a)_2 = F_{23}\Phi_{12}(a,\cdot) = \alpha a_3, \qquad 0 < \alpha \leqslant 1$$

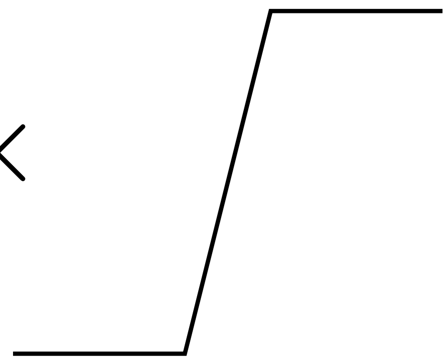


Consequences of FAITHE

Teleportation:



$= \alpha \times$



F is completely faithful, i.e. $F_{\mathscr{A}} := F \circ (\mathscr{I}, \mathscr{A}) \Longleftrightarrow \mathscr{A}$

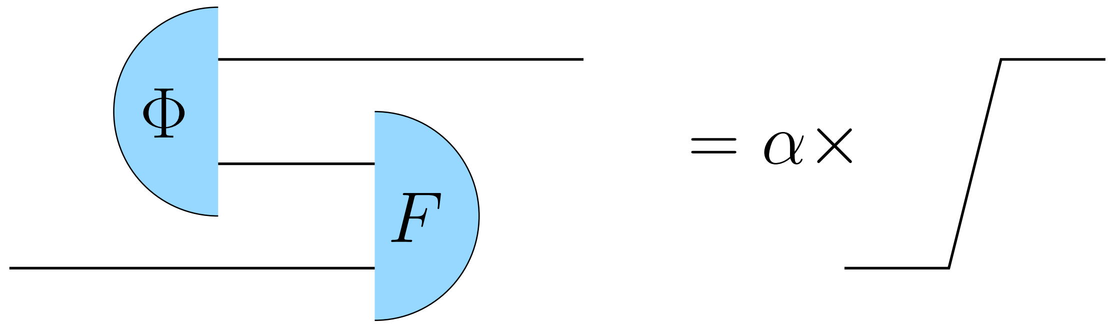
realizes the cone-isomorphism: $\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{T}_+(\mathsf{S})$

$\mathfrak{E}_+(\mathsf{S}^{\odot 2}) \ni A \mapsto \Omega_A := A_{23}(\Phi, \Phi) \in \mathfrak{S}(\mathsf{S}^{\odot 2})$

realizes the cone-isomorphism: $\mathfrak{S}_+(\mathsf{S}^{\odot 2}) \simeq \mathfrak{E}_+(\mathsf{S}^{\odot 2})$

Consequences of FAITHE

Teleportation:



$$\alpha(\mathsf{S}) = \max_{E \in \mathfrak{E}(\mathsf{S}^{\odot 2})} \{(\Phi, \Phi)(e, E, e)\}$$

is a property of the system and depends on the particular probabilistic theory

In Quantum Mechanics: $\alpha = \dim(\mathsf{H})^{-2}$

$$\omega_a = \sqrt{\alpha}\varsigma(a)$$

$$(\cdot, F)(\Phi, \cdot) = \sqrt{\alpha}|\Phi|$$

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Exploring PURIFY

PURIFY: Every state has a purification on two identical systems.

↓

Each state can be obtained by applying an atomic transformation to the marginal state $\chi = \Phi(e, \cdot)$

Each effect contains an atomic transformation.

Φ is pure.

$\mathscr{I}$ is atomic.

What is the *something more*?

PFAITH
+FAITHE
PURIFY?

Maybe...

What is the
something more?

It must give that:
effects make a C-algebra*

Reconstructing QM from probabilities

The axiomatic short-circuit of **CJ+AE**

Quantum Tomography for Measuring Experimentally the Matrix Elements of an Arbitrary Quantum Operation

G. M. D'Ariano and P. Lo Presti

at our disposal a general method for experimentally determining the quantum operation matrix, using any available quantum-tomographic scheme for the system in consideration, and a single fixed state at the input, which is an entangled (not even maximally) state. In the optical domain we show that one can achieve the tomographic reconstruction of the operation using exactly the same apparatus of the recently performed experiment of Ref. [9].

Let us consider for simplicity a "pure" quantum operation in the form (5). Given an orthonormal basis $\{|j\rangle\}$ corresponding to some physical observable, how can we determine the matrix $A_{ij} = \langle i|A|j\rangle$ experimentally? Instead of acting with the contraction A on an "isolated" system, we perform the map on a system which is entangled in the state $|\psi\rangle\rangle \in \mathcal{H} \otimes \mathcal{H}$ with an identical system; i.e.,

$$|\psi\rangle\rangle \rightarrow |\phi\rangle\rangle = \frac{A \otimes I|\psi\rangle\rangle}{\|A\psi\|_{HS}} . \tag{6}$$

With the double ket we denote bipartite vectors $|\psi\rangle\rangle \in \mathcal{H} \otimes \mathcal{H}$, which, keeping the basis $\{|j\rangle\}$ as fixed, are in one-to-one correspondence with matrices as follows:

$$|\psi\rangle\rangle = \sum_{ij} \psi_{ij}|i\rangle \otimes |j\rangle . \tag{7}$$

$$A_{ij} = \kappa\langle E_{ij}(\psi)\rangle , \tag{10}$$

where the operator $E_{ij}(\psi)$ is given by

$$E_{ij}(\psi) = |i_0\rangle\langle i| \otimes |j_0\rangle\langle\psi^{-1*}(j)| , \tag{11}$$

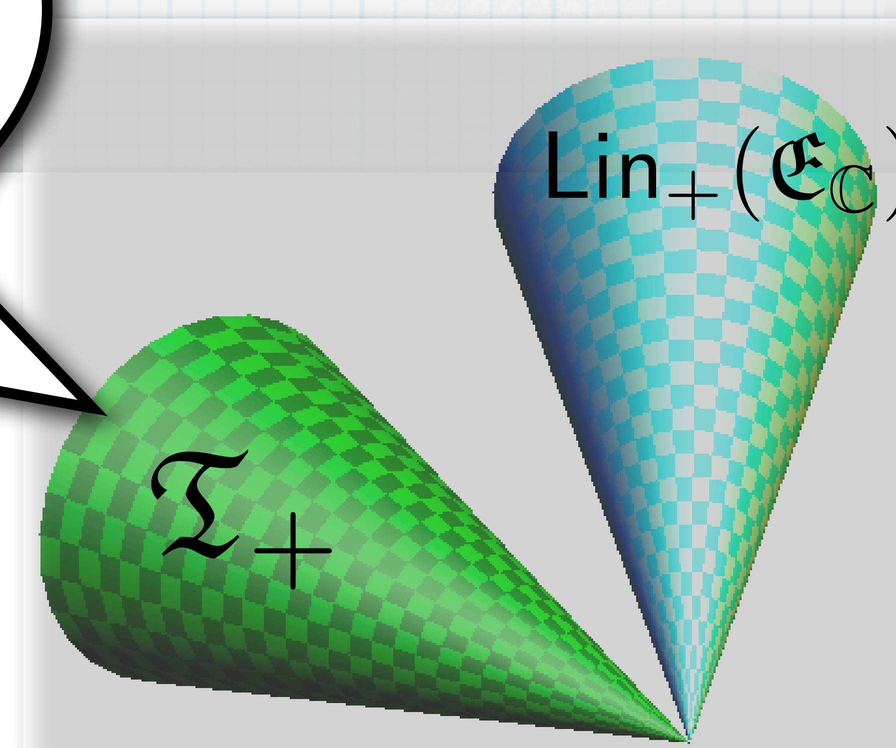
and the proportionality constant is given by

$$\kappa = e^{i\theta}\sqrt{\frac{p_A(\psi)}{\langle|i_0, j_0\rangle\rangle\langle\langle i_0, j_0|\rangle}} . \tag{12}$$

Since A_{ij} is written only in terms of output ensemble averages, it can be estimated through quantum tomography. Quantum tomography [10,11] is a method to estimate the ensemble average $\langle H\rangle$ of any arbitrary operator H on $\mathcal{H}$ by using only measurement outcomes of a *quorum* of observables $\{O(l)\}$. A *quorum* is just a set of operators $\{O(l)\}$ which are observable (i.e., have orthonormal resolution) and span the linear space of operators on $\mathcal{H}$. This means that any operator H can be expanded as $H = \sum_l \mathrm{Tr}[Q^\dagger(l)H]O(l)$, where $\{Q(l)\}$ and $\{O(l)\}$ form a biorthogonal set such that $\mathrm{Tr}[Q^\dagger(i)O(j)] = \delta_{ij}$. Hence, the tomographic estimation of the ensemble average $\langle H\rangle$ is obtained as the double average—over both the ensemble and the quorum—of the unbiased

Big problem:
howto introduce
composition of effects?

Math. short-circuit:
use Choi-Jamiolkowski
isomorphism

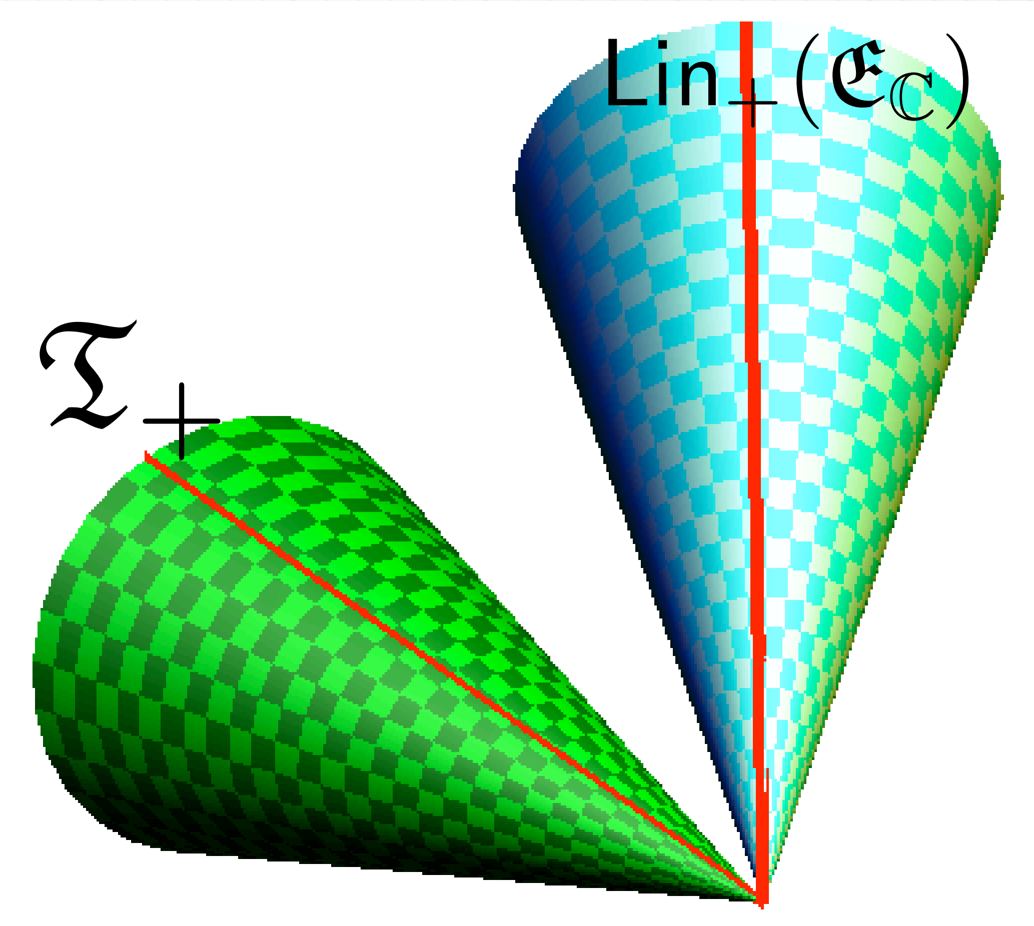


$$\mathbf{I} : \mathfrak{T}_+ \simeq \mathrm{Lin}_+(\mathfrak{E}_{\mathbb{C}})$$

CJ Isomorphism ⇒ composition of effects

Effects are identified with "atomic" events
(apart from a phase) i.e. events that cannot be written as sum of other events

AE (Atomicity of evolution):
the composition of "atomic" events is atomic



One can prove that the phase (two-cocycle) is trivial.

Introduce the generalized transformation via the polar identity:

$$\mathscr{T}_{a,b} := \frac{1}{4}\sum_{k=0}^{3} i^k \mathscr{T}_{a+i^k b}$$

compositon of effects as: $ab = e \circ \mathscr{T}_{e,a} \circ \mathscr{T}_{e,b}$

QUit quantum information theory group

SUMMARY