OPTIMAL QUANTUM LEARNING AND MULTIROUND REFERENCE FRAME ALIGNMENT

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OUTLINE

- Optimal quantum learning of a unitary transformation from finite examples (arXiv:0903.0543v1)

- Optimal correction of an unknown rotation
  (a little variation on the theme of quantum learning)

- Multi-round and adaptive alignment of reference frames
  equivalence of backward communication with forward communication of charge-conjugate particles

OPTIMAL QUANTUM LEARNING:

WHAT IS IT ABOUT

LEARNING AN UNKNOWN FUNCTION

Problem: a black box computes an unknown function y = f(x)

We can evaluate f on a finite set of points $x_1, \ldots, x_N$

getting outcomes $y_1, \ldots, y_N$

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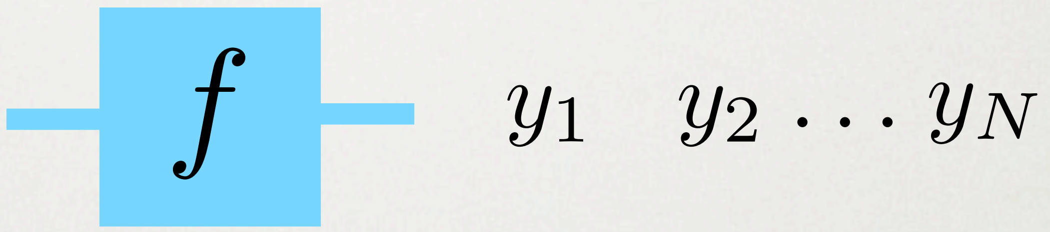

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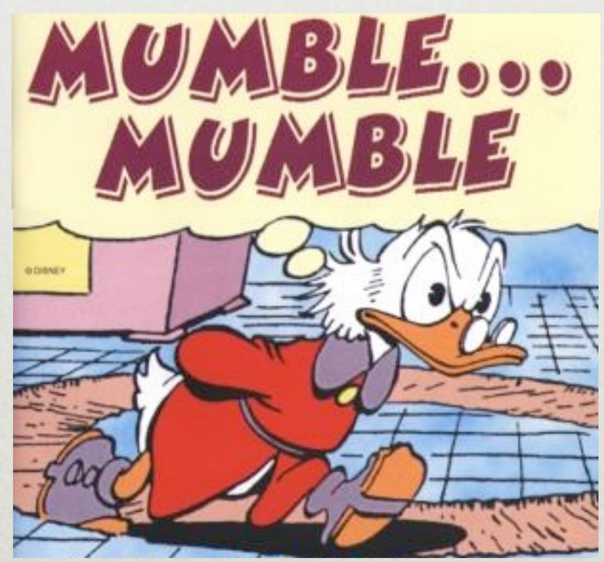

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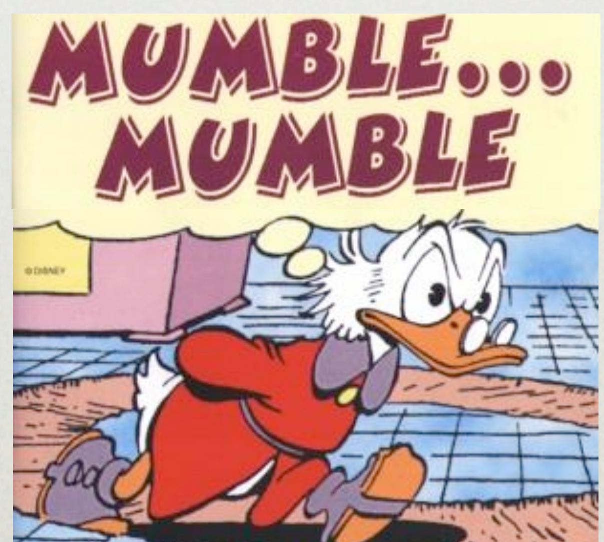
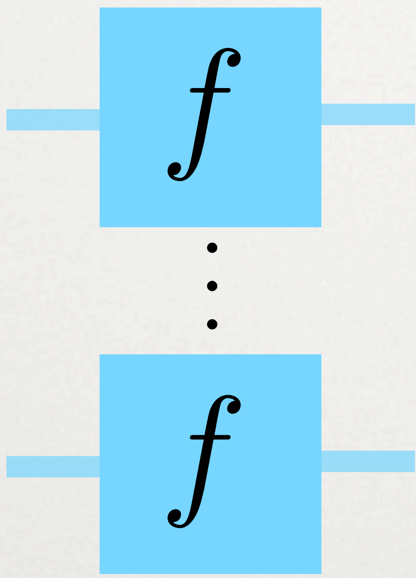


$$f(x) = \ ?$$

In classical computer science, statistical learning provides several efficient solutions for this problem

CLASSICAL NETWORKS FOR LEARNING

Comparing x with f(x) for N times is not the only possibility: this just corresponds to the parallel configuration



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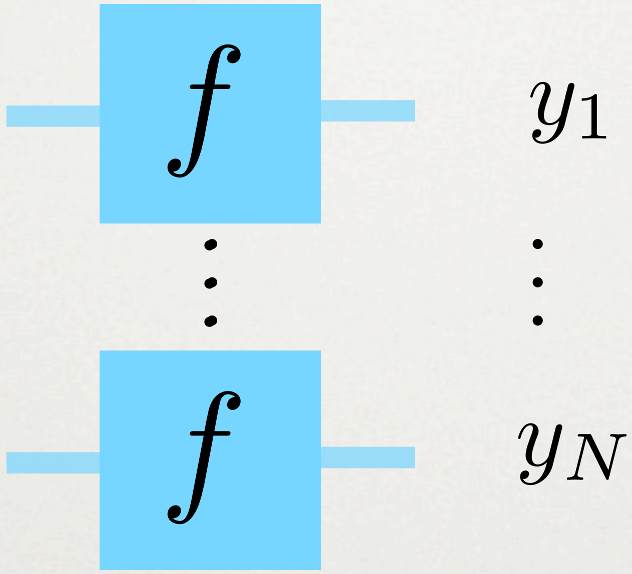

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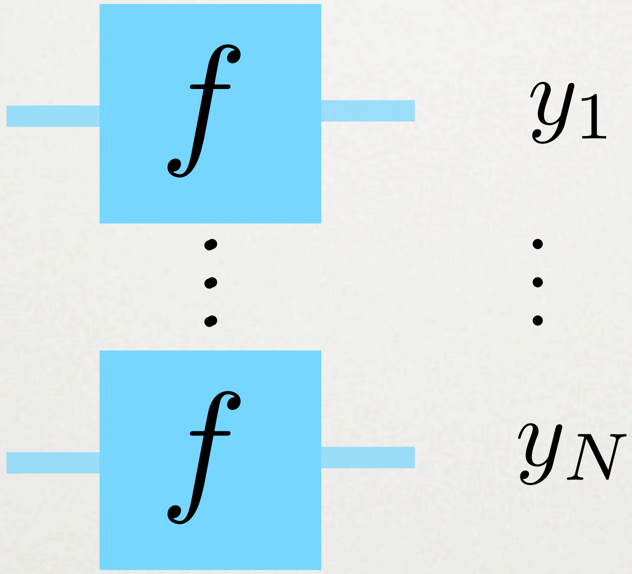
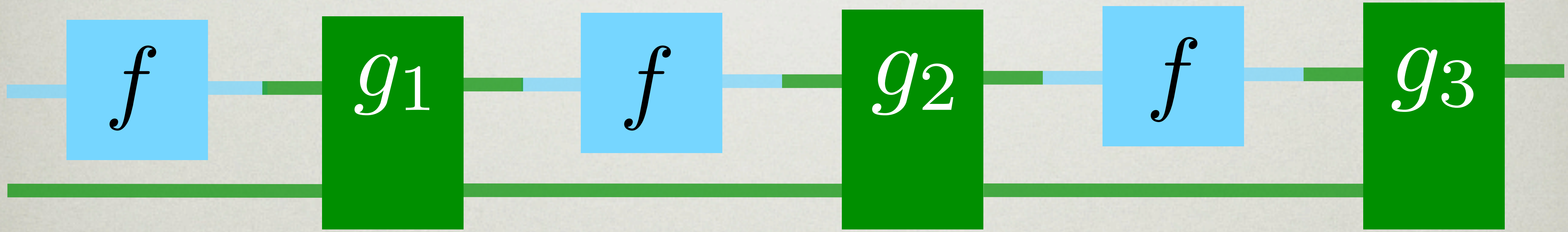


To learn better, one could use a sequential network:



where $g_1, g_2, \dots, g_N$ are known functions

OPTIMIZATION PROBLEM

Find the optimal strategy to learn an unknown function $f \in \mathcal{F}_0$

This means:

- find the best network $\rightarrow$ $F = g_N \circ f \circ \cdots \circ g_2 \circ f \circ g_1 \circ f$
- find the best input X $\rightarrow$ $Y = F(X)$
- for outcome Y, find the optimal guess $\hat{f}$

$$Y \rightarrow \hat{f}$$

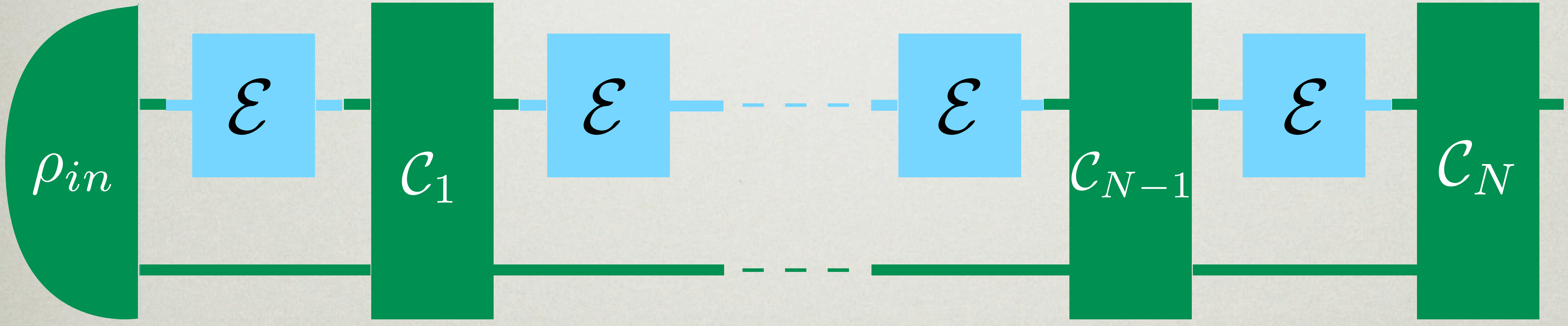
- Difference with estimation of the function f

Estimation corresponds to the special case $\hat{f} \in \mathcal{F}_0$

In general, the optimal guess does not have to be in $\mathcal{F}_0$.

FROM CLASSICAL TO QUANTUM LEARNING

- Unknown function f $\longrightarrow$ unknown quantum channel $\mathcal{E}$
- Classical network $\longrightarrow$ quantum network
- Input X $\longrightarrow$ quantum state ρ_{in}
- Output Y $\longrightarrow$ quantum state ρ_{out}



GUESSING A CHANNEL FROM A STATE

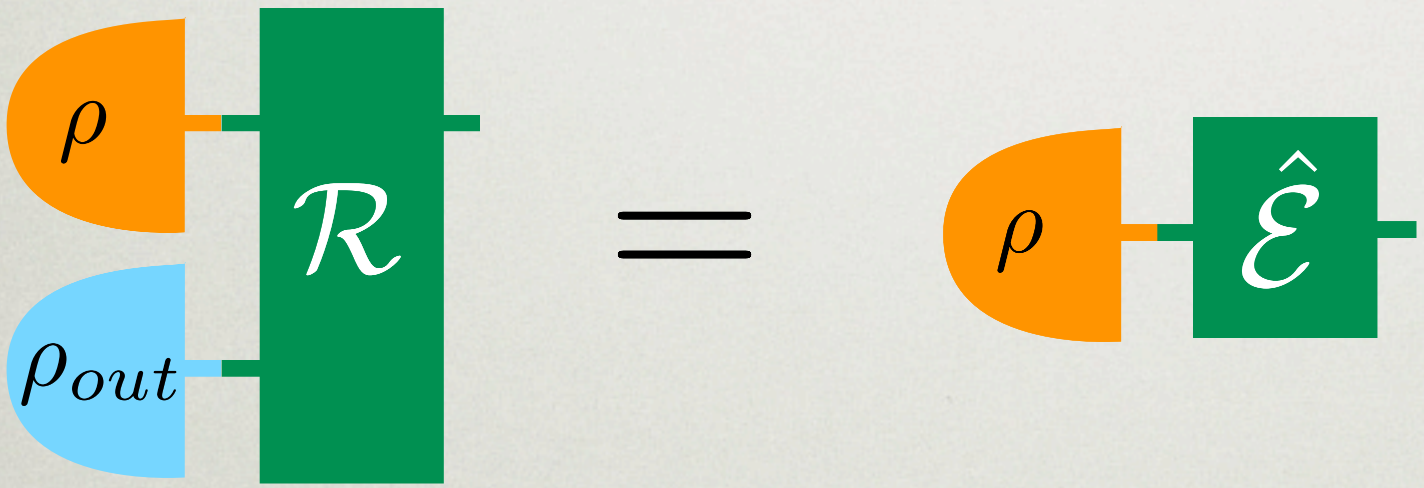
- Classical guess $\longrightarrow$ Quantum "guess"

$Y \to \hat{f}$ $\qquad$ $\rho_{out} \to \hat{\mathcal{E}}$

Physical implementation of the quantum guess:

retrieving channel $\mathcal{R}$

It retrieves the unknown transformation from the output state ρ_{out} and performs it on a new state ρ



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- Classical guess

  $Y \to \hat{f}$

$\longrightarrow$

Quantum "guess"

$\rho_{out} \to \hat{\mathcal{E}}$

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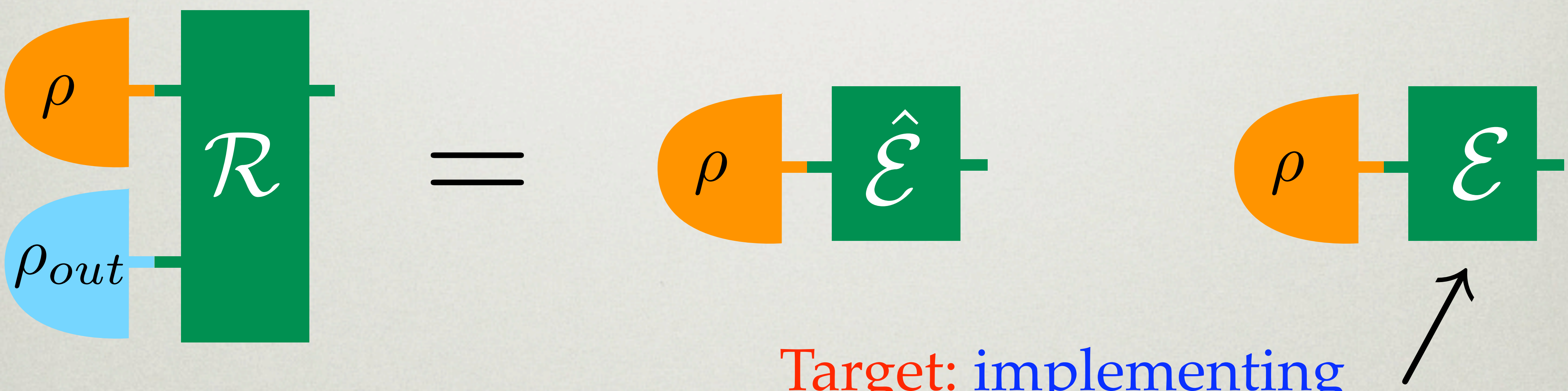


Target: implementing the unknown channel with maximum fidelity

OPTIMAL QUANTUM LEARNING

Find the optimal strategy to learn an unknown channel $\mathcal{E} \in \mathsf{E}_0$

This means:

- find the best network $\rightarrow \quad \mathcal{N} = \mathcal{C}_N \circ \mathcal{E} \circ \cdots \circ \mathcal{C}_2 \circ \mathcal{E} \circ \mathcal{C}_1 \circ \mathcal{E}$
- find the best input $\rho_{in} \rightarrow \quad \rho_{out} = \mathcal{N}(\rho_{in})$
- find the optimal retrieving channel $\mathcal{R}$

$$\rightarrow \quad \hat{\mathcal{E}}(\rho) = \mathcal{R}(\rho \otimes \rho_{out})$$

Figure of merit: input-output fidelity

$$F(\mathcal{E}, \hat{\mathcal{E}}) = \int \mathrm{d}\varphi \;\; F(\mathcal{E}(\varphi), \hat{\mathcal{E}}(\varphi))$$

$$F(\rho, \sigma) = \mathrm{Tr}\left[(\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}})^{\frac{1}{2}} \right]$$

“MEASURE-AND-PREPARE” SCHEMES

- Particular scheme to retrieve the unknown transformation:

-perform a measurement on the output state,
-for outcome Y perform channel $\hat{\mathcal{E}}_Y$

In this case, the retrieving channel is:

$$\mathcal{R}_{meas}(\rho \otimes \rho_{out}) = \sum_Y \mathrm{Tr}[P_Y \rho_{out}] \, \hat{\mathcal{E}}_Y(\rho)$$

- Particular measure-and-prepare scheme:

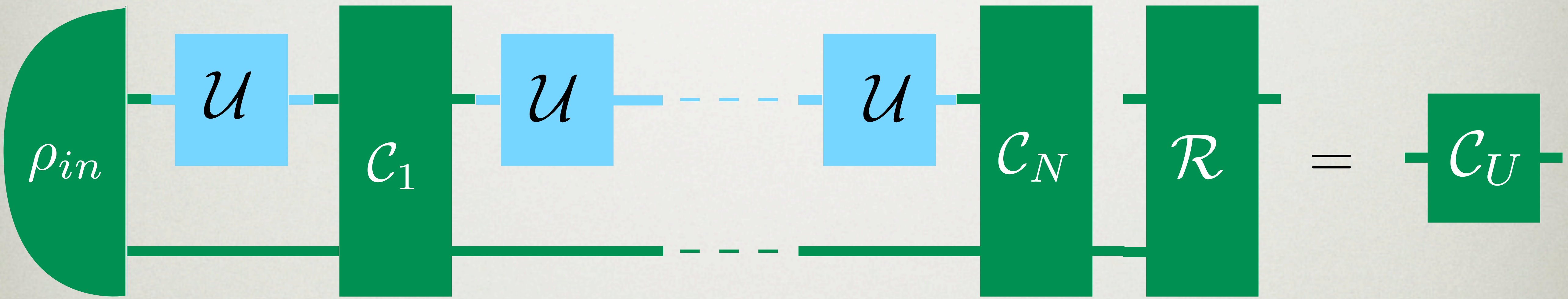
estimation of the channel $\mathcal{E} \in \mathsf{E}_0$

In this case, one has $\hat{\mathcal{E}}_Y \in \mathsf{E}_0$

Estimation $\in$ {measure-and-prepare schemes} $\subset$ {retrieving channels}

LEARNING AN UNKNOWN UNITARY

Consider the case where the set of channels E_0 is a group of unitary transformations.



Assuming a uniform prior for the unknown unitaries, we have the average fidelity

$$F = \int \mathrm{d}U \quad F(\mathcal{U}, \mathcal{C}_U)$$

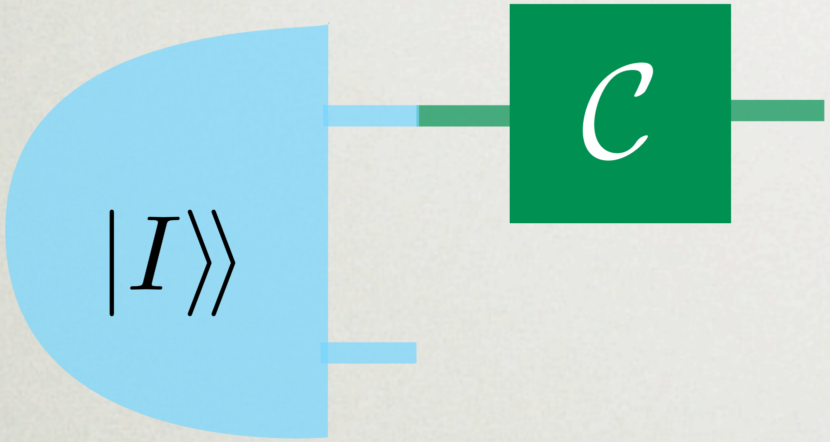
HOW TO OPTIMIZE A QUANTUM NETWORK:

QUANTUM COMBS

CHOI-JAMIOLKOWSKI OPERATORS

Convenient representation of linear maps:
Choi-Jamiolkowski-Belavkin-Staszewski operator (CJBS)

$$C = (\mathcal{C} \otimes \mathcal{I})(|I\rangle\rangle\langle\langle I|) \qquad |I\rangle\rangle = \sum_n |n\rangle|n\rangle$$



For a unitary channel:

$$(\mathcal{U} \otimes \mathcal{I})(|I\rangle\rangle\langle\langle I|) = |U\rangle\rangle\langle\langle U| \qquad |U\rangle\rangle = (U \otimes I)|I\rangle\rangle$$

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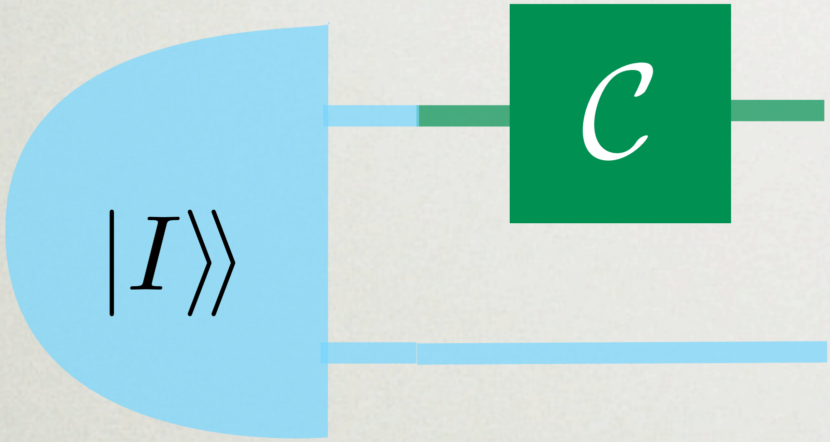

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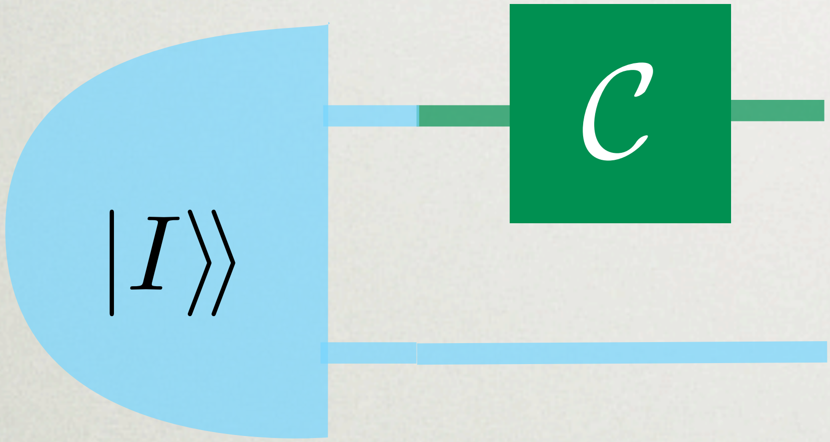

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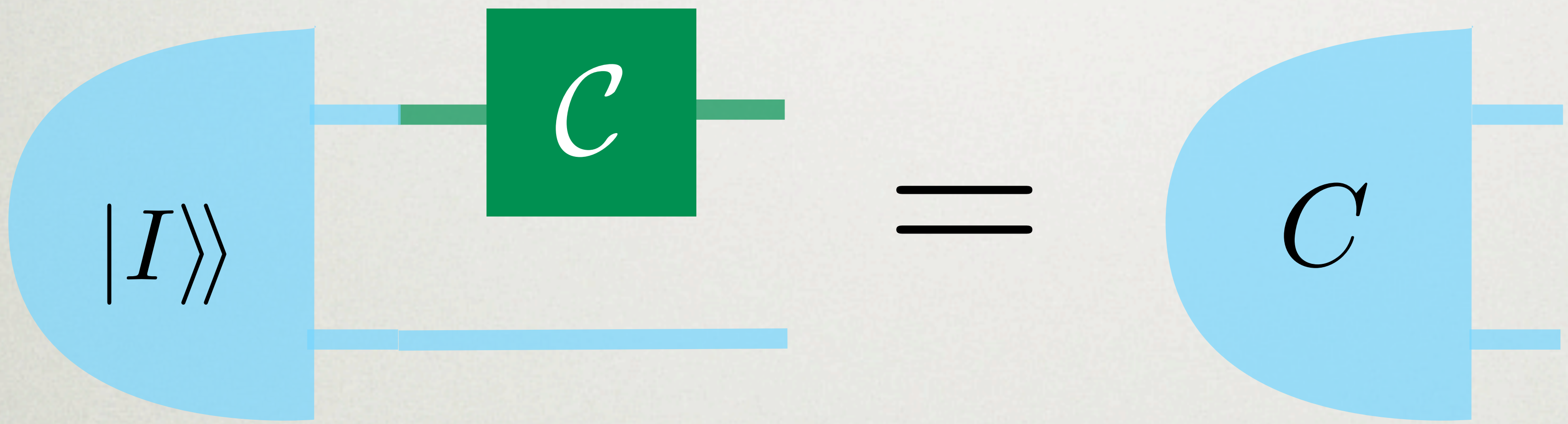

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LINK PRODUCT

Convenient representation of composition of linear maps: link product

$$\mathcal{F} \circ \mathcal{E} \Longleftrightarrow F_{cb} * E_{ba} := \mathrm{Tr}_b[(F_{cb} \otimes I_a)(I_c \otimes E_{ba}^{\tau_b})]$$

$$F_{cb} * E_{ba} = E_{ba} * F_{cb} \qquad \text{up to permutation of Hilbert spaces}$$

GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 060401 (2008)

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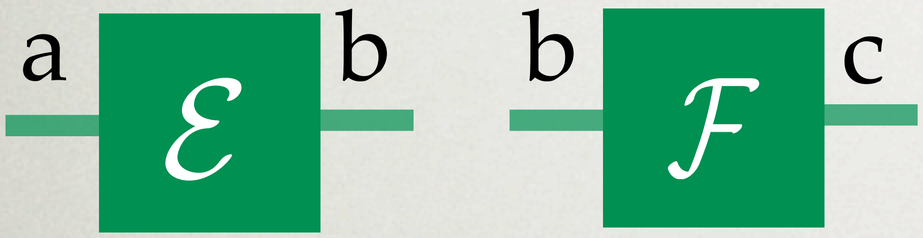

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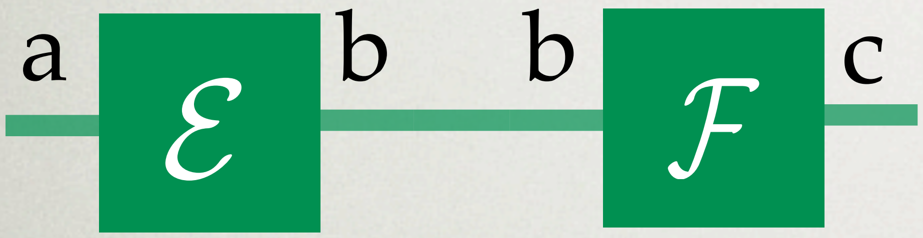

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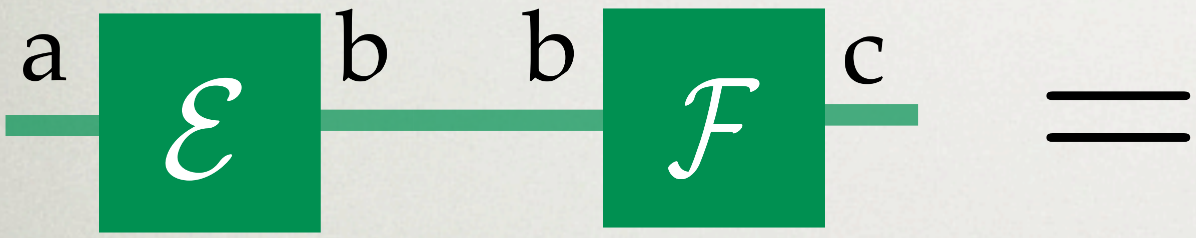

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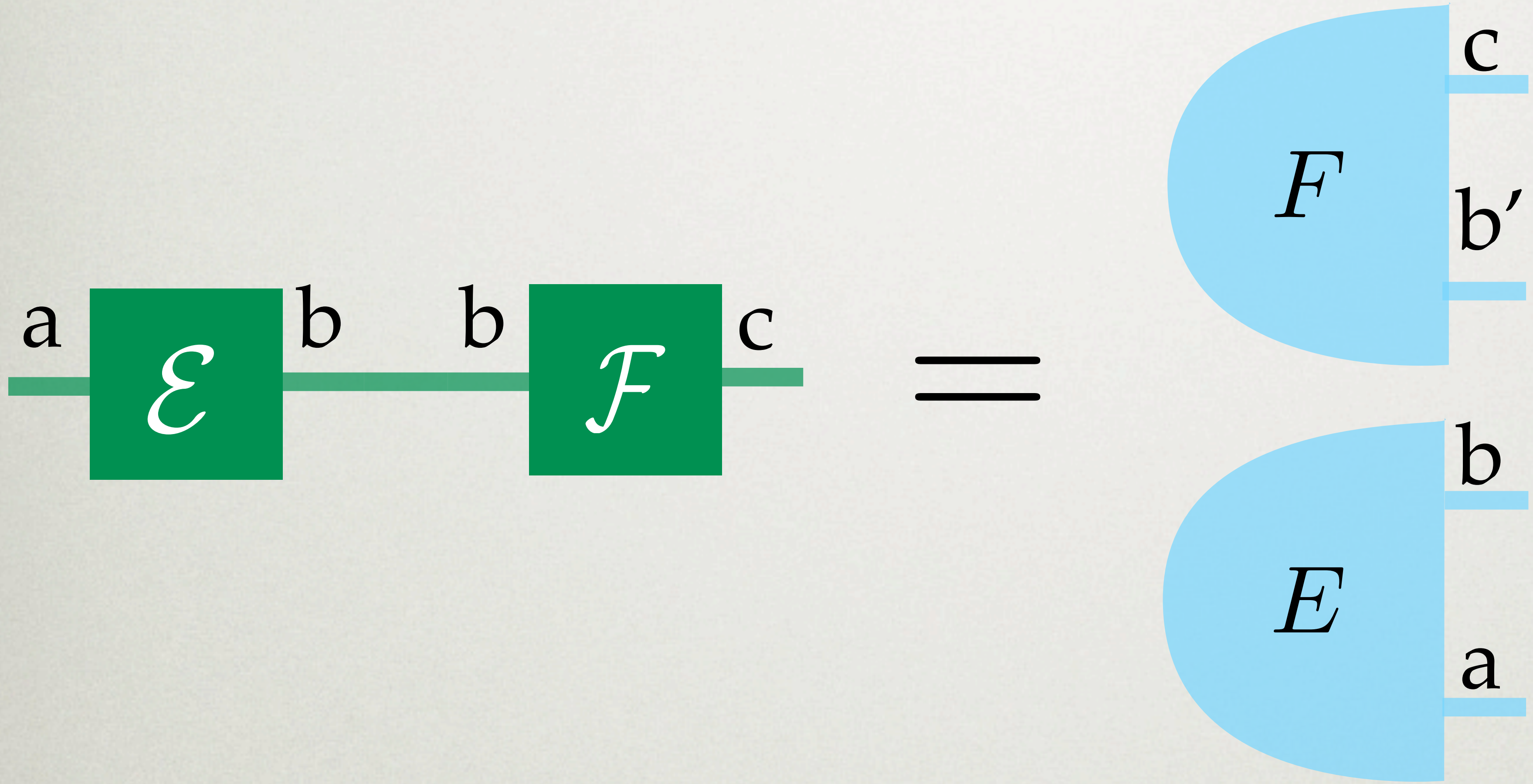

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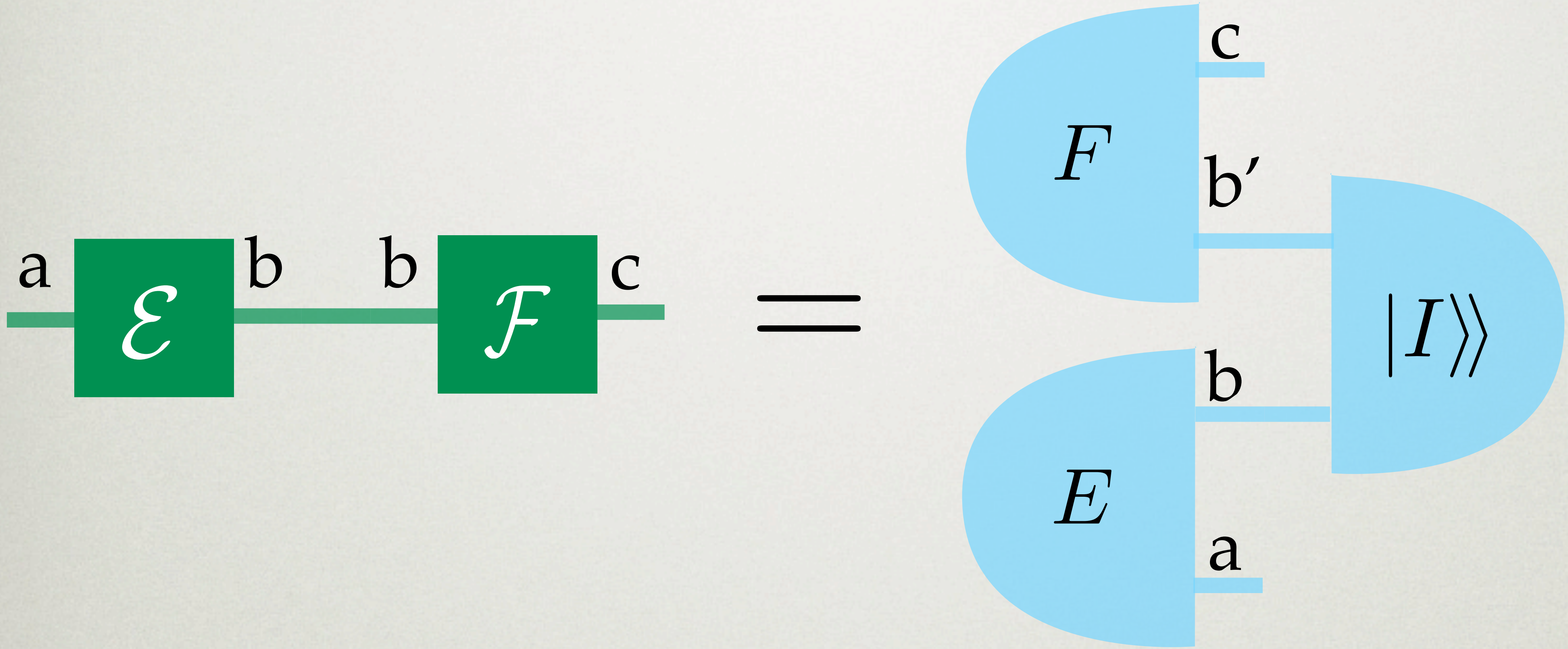

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KNOWN FORMULAS IN TERMS OF LINK PRODUCT

- Tensor product of states:

$$\rho_a \otimes \sigma_b = \rho_a * \sigma_b$$

- Born statistical formula:

$$\mathrm{Tr}[\rho P] = \rho_a * P_a^{\tau}$$

- Transformation of states:

$$\mathcal{E}(\rho) = E_{out,in} * \rho_{in}$$

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States and transformations are treated on an equal footing.

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Is this a state
or a transformation?

- Transformation of states:

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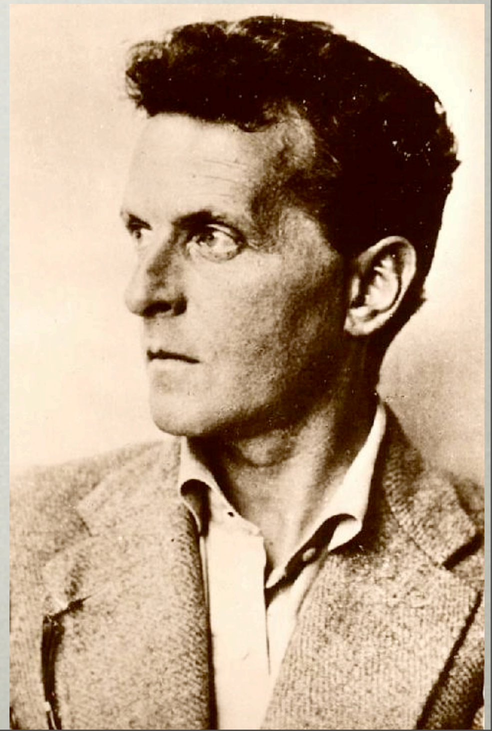

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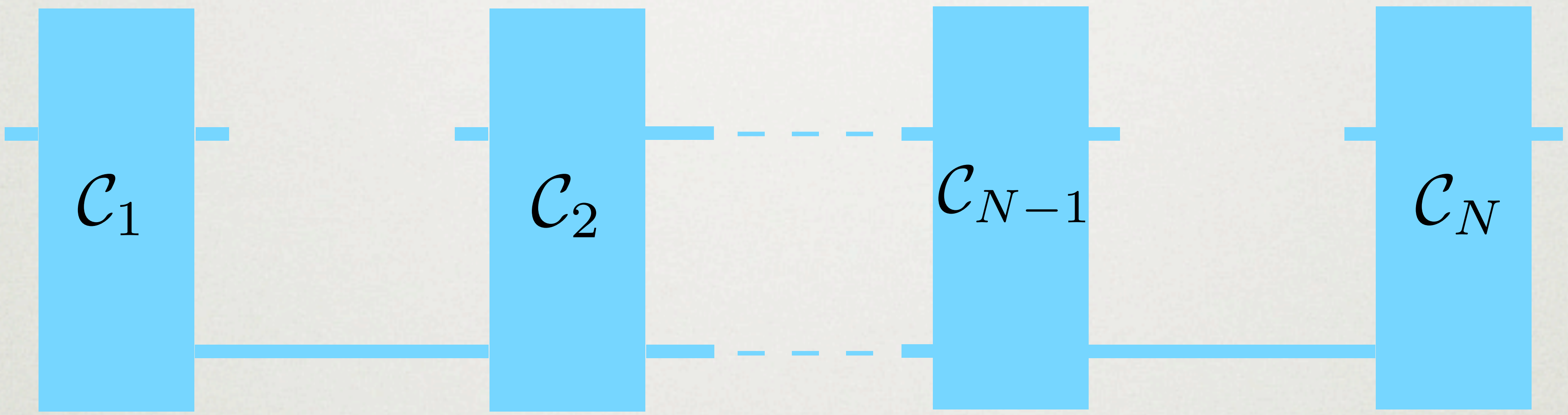
QUANTUM COMBS

Quantum comb = sequential networks of quantum operations

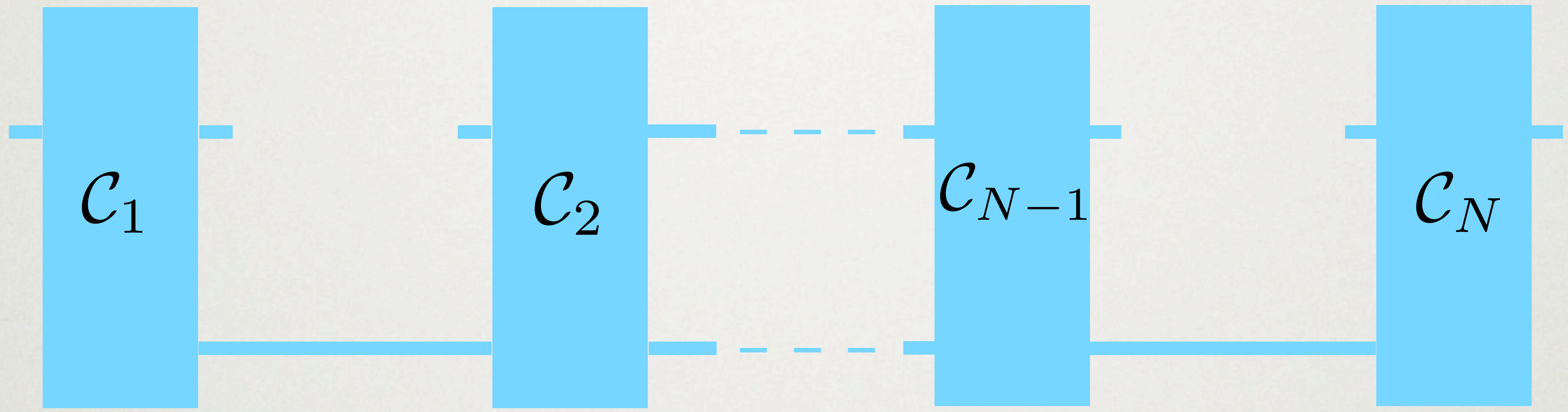


The quantum comb is represented by the Choi operator

$$S^{(N)} = C_N * \cdots * C_2 * C_1$$

NORMALIZATION OF COMBS

- Deterministic comb = network of channels



Recursive normalization of deterministic combs:

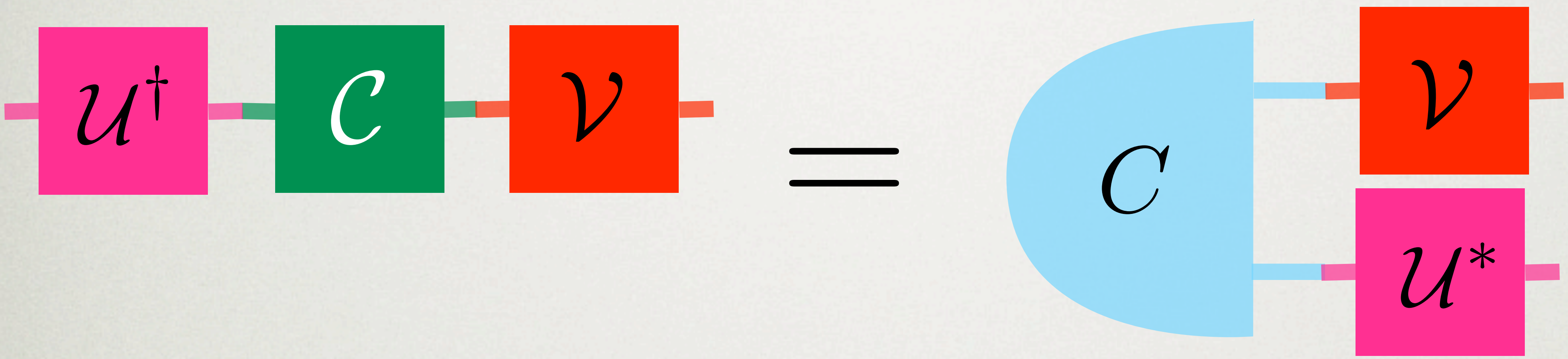
$$\mathrm{Tr}_{2N-1}[S^{(N)}] = I_{2N-2} \otimes S^{(N-1)}$$

Optimize a network = optimize a positive operator under this constraint

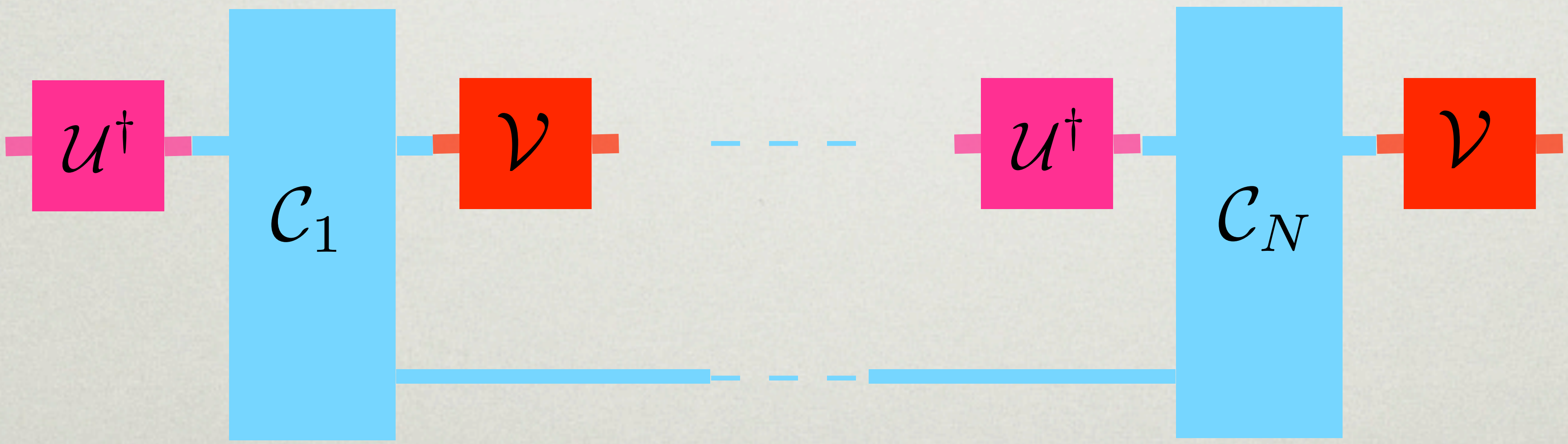
GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 060401 (2008)

ROTATION OF COMBS

- Rotation of input/output of a channel = rotation of the Choi operator



- Rotation of inputs/outputs of a network = rotation of the comb



$$S^{(N)} \longmapsto (V \otimes U^*)^{\otimes N} S^{(N)} (V^\dagger \otimes U^\tau)^{\otimes N}$$

OPTIMIZATION OF LEARNING

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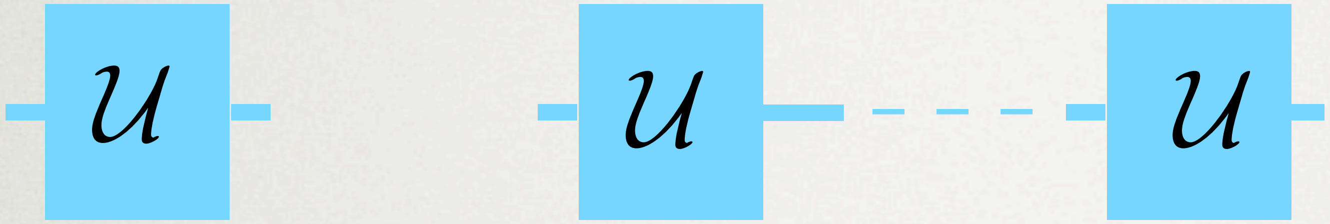

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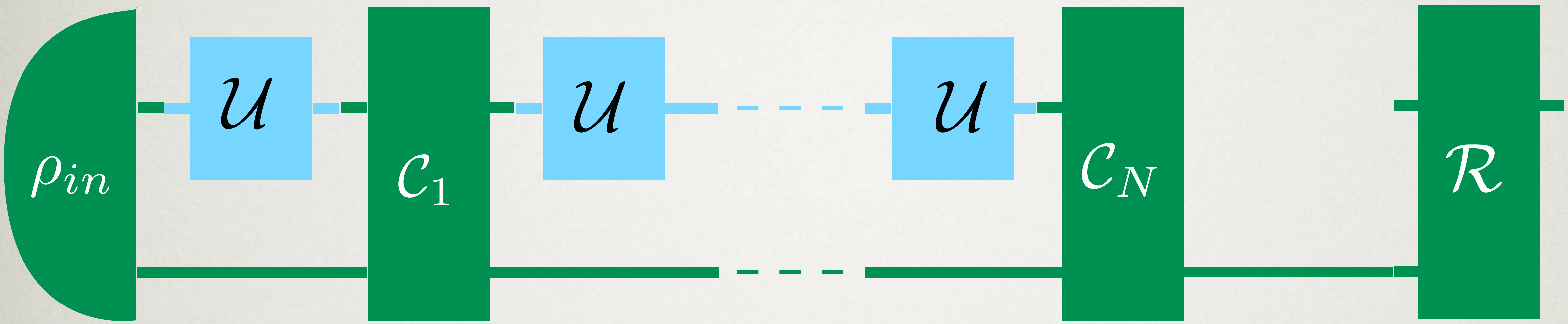

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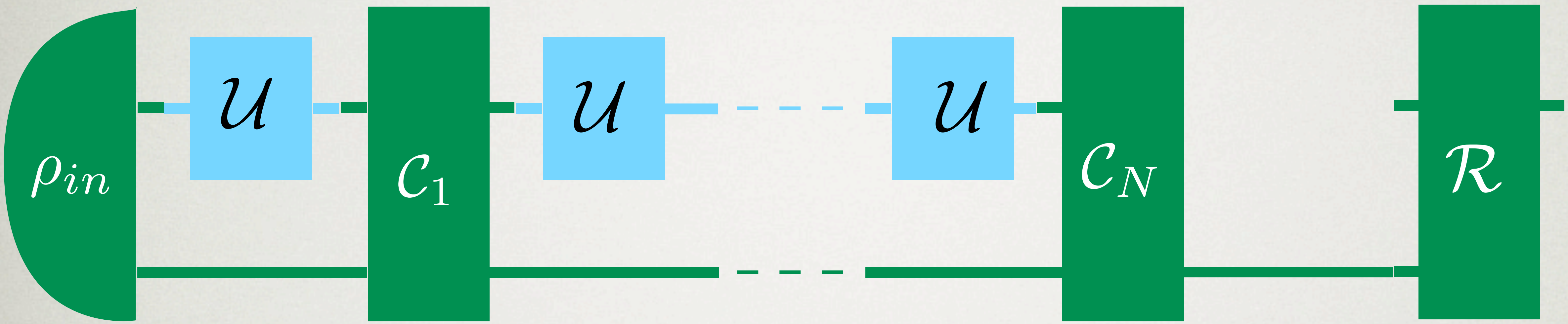


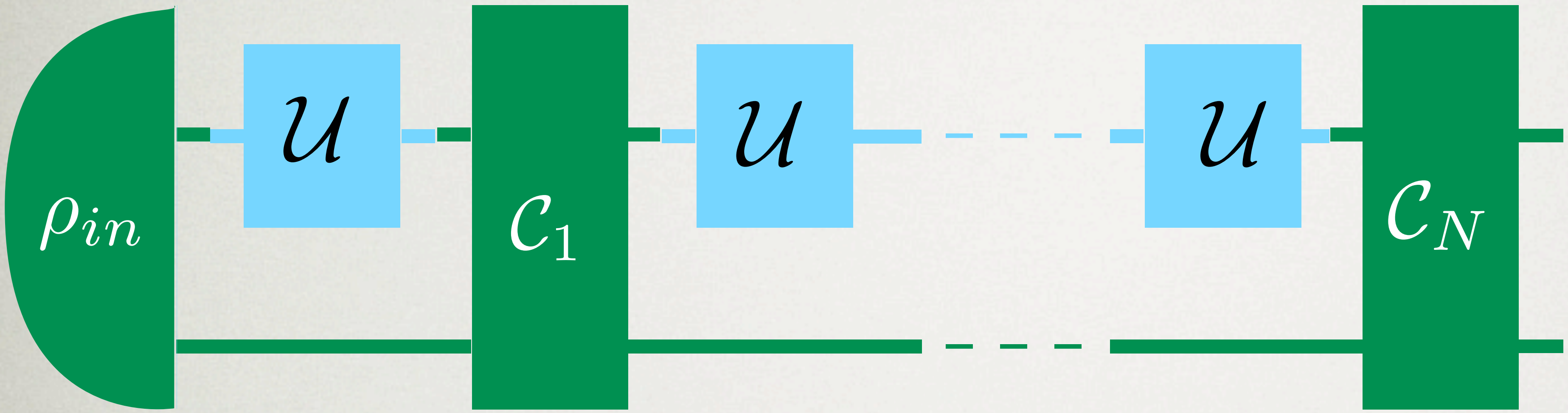
Comb of the learning network: $L = R * C_N * \cdots * C_2 * C_1 * \rho_{in}$

Fidelity: $$F = \frac{1}{d^2} \int \mathrm{d}U \;\; \langle\langle U| \langle\langle U^*|^{\otimes N}| \; L \; |U\rangle\rangle |U^*\rangle\rangle^{\otimes N}$$

We can always optimize over covariant combs:

$$[L, U \otimes V^* \otimes U^{*\otimes N} \otimes V^{\otimes N}] = 0 \qquad \forall U, V$$

OPTIMALITY OF PARALLEL STRATEGIES



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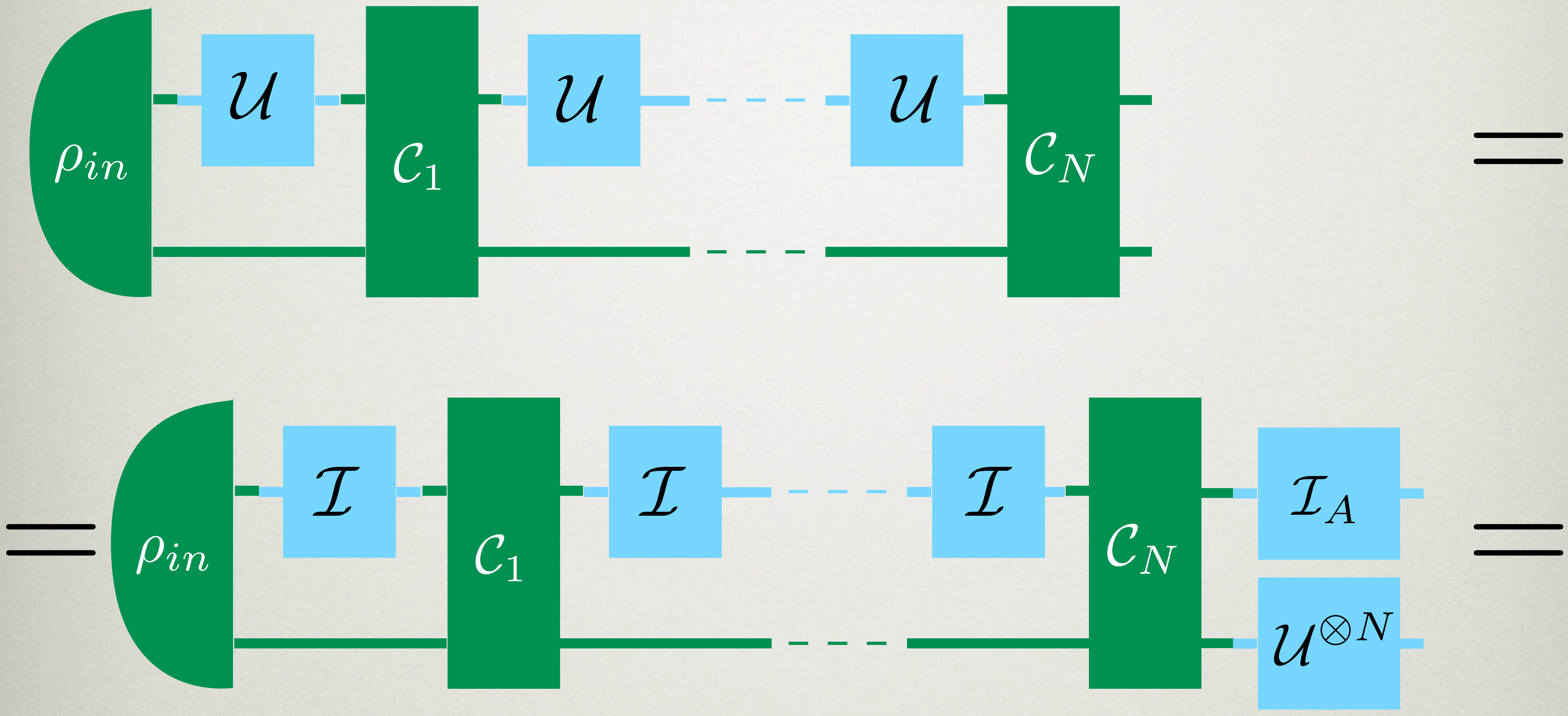

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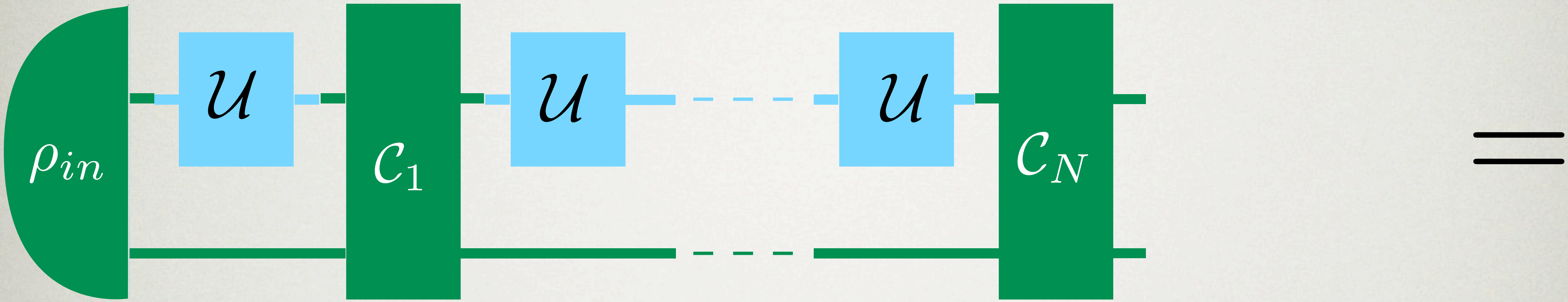

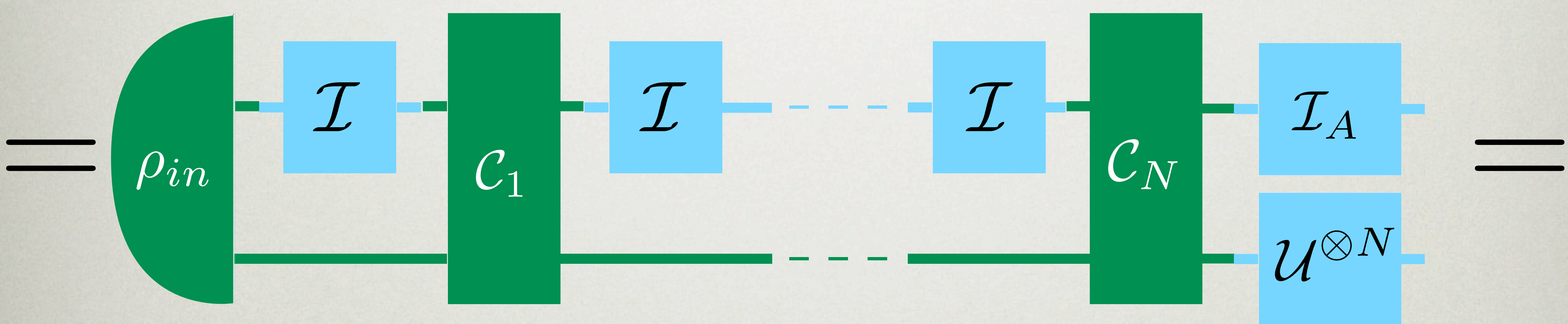

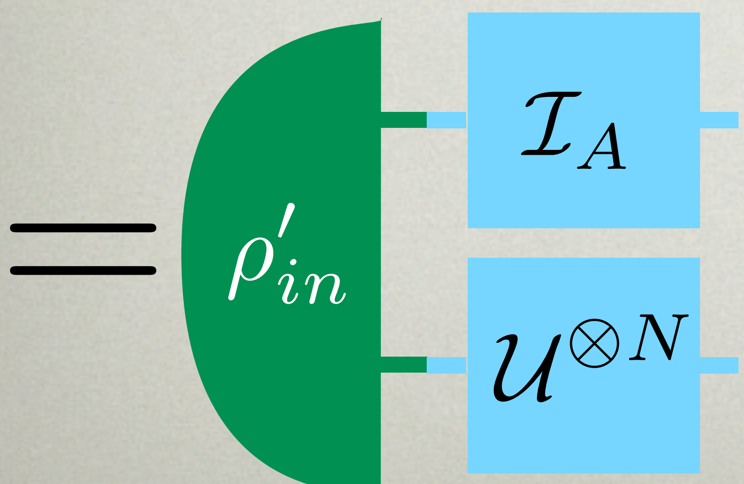

Any covariant network is equivalent to a parallel scheme with ancilla!

Learning can be parallelized, in the same way as estimation (cf previous talk)

OPTIMAL INPUT STATES

Decomposing the unitaries as $$U^{\otimes N} \otimes I_A = \bigoplus_J (U_J \otimes I_{m_J})$$

one can prove that the optimal input states have the form

$$|\psi\rangle = \bigoplus_J a_J \frac{|I_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

where $|I_J\rangle\rangle \in \mathcal{H}_J^{\otimes 2}$ is a maximally entangled state

This is the same form of the optimal states for
estimation of the unknown unitary U with N copies

GC, G M D'Ariano, and M F Sacchi, Phys. Rev. A 72, 043448 (2005).

OPTIMAL RETRIEVING CHANNEL

Theorem: for any group of unitaries,
for an input state of the optimal form

$$|\psi\rangle = \bigoplus_J a_J \frac{|I_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

the optimal retrieving channel to extract U from the states

$$(U^{\otimes N} \otimes I_A)|\psi\rangle = \bigoplus_J a_J \frac{|U_J\rangle\rangle}{\sqrt{d_J}} \qquad a_J \geq 0$$

is achieved by a “measure-and-prepare” scheme.
Precisely, it is achieved by estimation of the unknown unitary U:
for outcome $\hat{U}$, just perform the unitary $\hat{U}$

For the optimal POVM, see
GC, G M D’Ariano, and M F Sacchi, Phys. Rev. A 72, 043448 (2005).

QUANTUM MEMORY DOES NOT IMPROVE LEARNING

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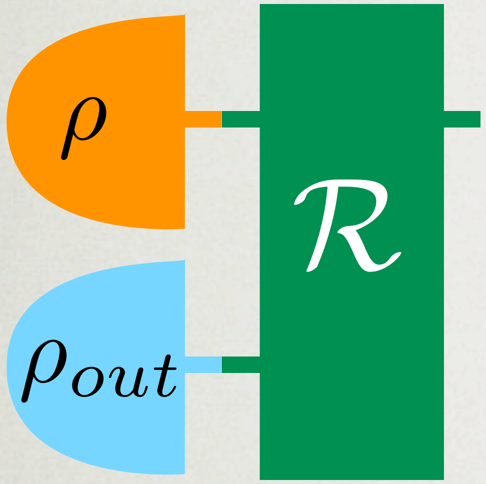

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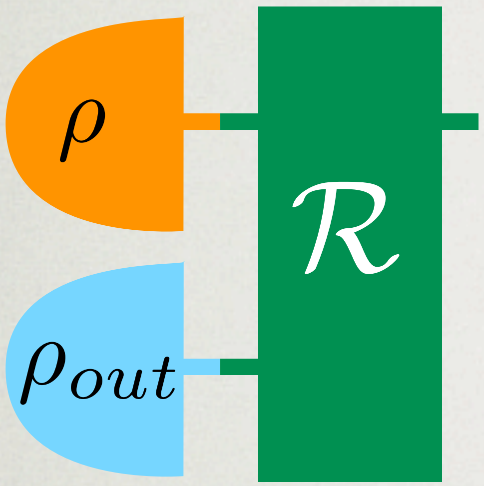

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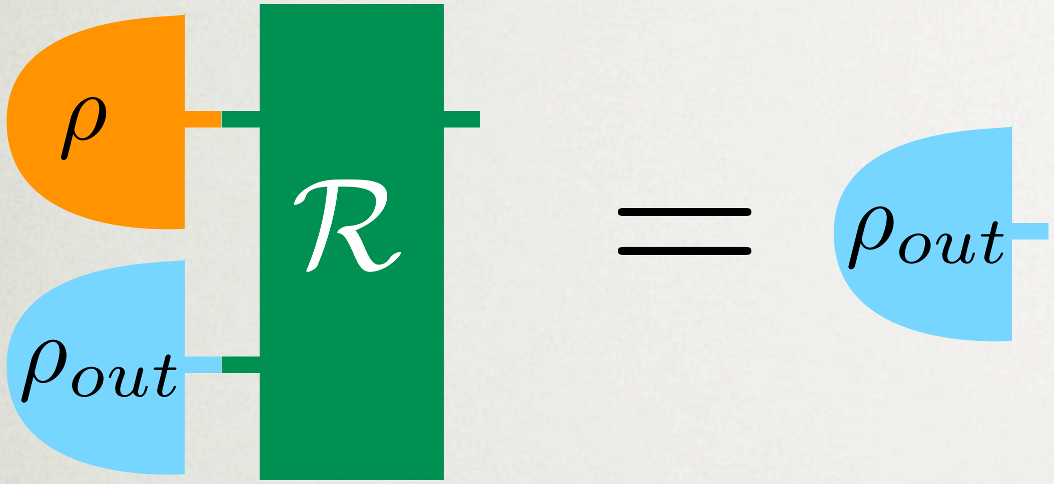

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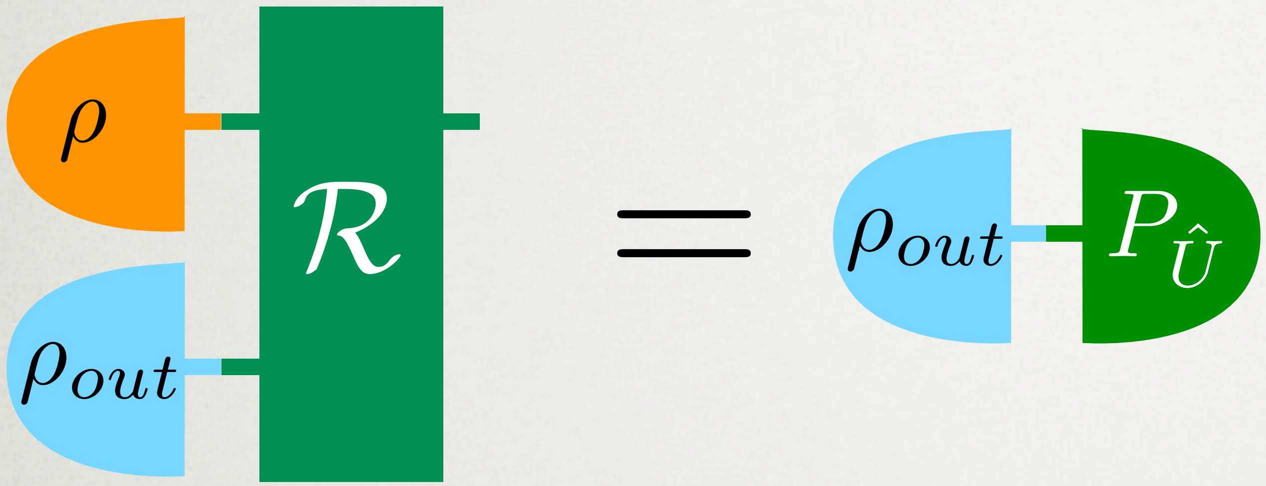

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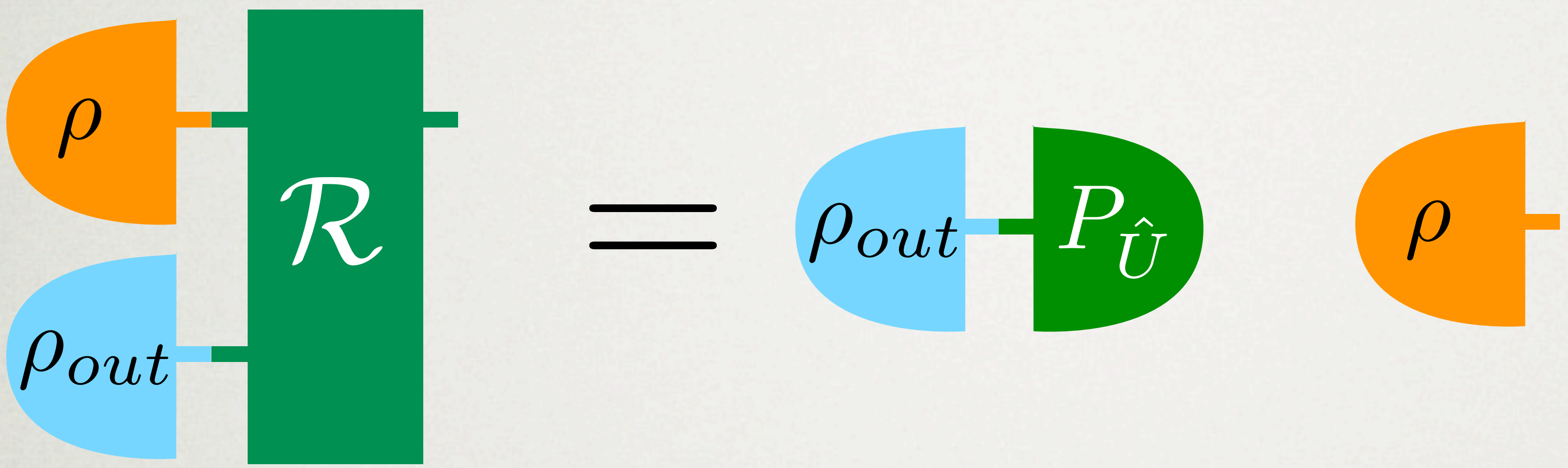

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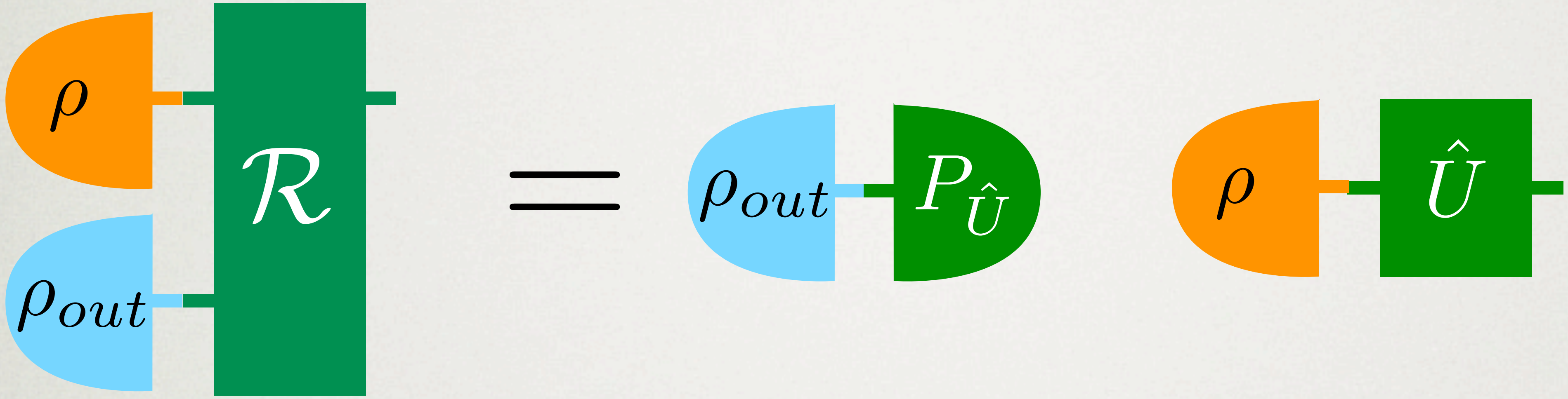

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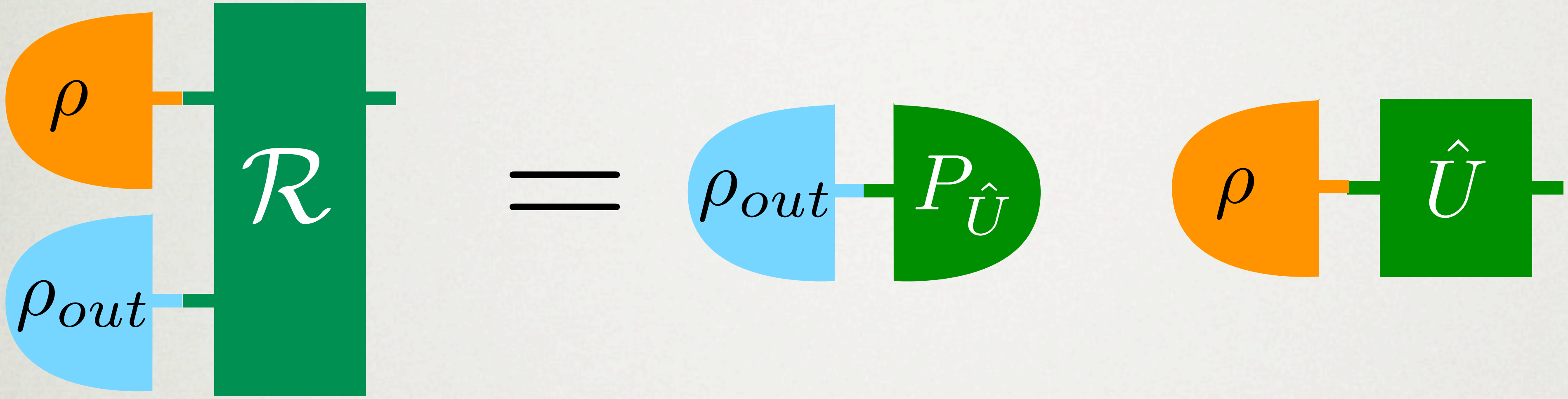

Optimal retrieving is "measure-and-prepare":
no need of waiting for the input state ρ

We can measure immediately after having applied U, and store the outcome $\hat{U}$ in a classical memory.

What's more, once we have measured, we can make as many copies as we want.
On the contrary, a quantum memory would be degraded every time we access it.

STABILITY AND INSTABILITY OF OUR RESULT

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Our result is stable under the following variations:

- learning from N to M copies with global fidelity: target $U^{\otimes M}$ (optimality for single-copy fidelity is trivial)

- N non-identical input unitaries and/or non-identical target unitaries

- perform the inverse of U: target $U^{\dagger}$

- any combination of the above things

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- any combination of the above things

Our result is not stable under the following variations:

- learning general channels
- learning unitaries that do not form a group
- learning with restrictions on the available input states (entanglement)

ERROR CORRECTION WITH CORRELATED NOISE

Consider the following correlated error model:

$$\mathcal{D}_N(\rho) = \int_G \mathrm{d}g \; U_g^{\otimes N} \; \rho \; U_g^{\dagger \otimes N}$$

Possible coding strategy:

- use k particles to detect the unitary error
- use the remaining (N-k) particles to carry the message

$$\mathcal{D}_N(|e\rangle\langle e|^{(k)} \otimes \rho^{(N-k)})$$

Problem: find the best decoding to maximize the fidelity between

$$\mathcal{R} \circ \mathcal{D}_N(|e\rangle\langle e|^{(k)} \otimes \rho^{(N-k)}) \quad \text{and} \quad \rho^{(N-k)}$$

OPTIMAL CORRECTION SCHEME

The correction problem is equivalent to learning $U^{\dagger\otimes(N-k)}$ from k examples of U .

We know that the optimal scheme is just estimation and preparation
In particular, the optimal states for error correction are the optimal states for estimation.

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The optimality of measure-and-prepare retrieving has been also observed for k =1, and for a maximum likelihood input state

$$|\psi\rangle \propto \bigoplus_J |I_J\rangle\rangle \qquad (a_J \propto \sqrt{d_J} \; in \; the \; optimal \; form)$$

For SU(2) and U(1) the state assumed in arXiv:0812.5040 allows

$$p_{succ} = 1 - \frac{\alpha}{N}$$

PRO AND CONTRA

The max-likelihood state is not optimal for the fidelity

The optimal state is $|\psi\rangle \propto \sum_{n=0}^{N} \sin\left(\frac{n\pi}{N}\right) |n\rangle \qquad for\ U(1)$

$$|\psi\rangle \propto \bigoplus_{j=0}^{N/2} \sin\left(\frac{2j\pi}{N}\right) \frac{|I_J\rangle\rangle}{\sqrt{2j+1}} \qquad for\ SU(2)$$

and gives fidelity $F_{opt} = 1 - \frac{\beta}{N^2}$

The max-likelihood state gives $F = 1 - \frac{\gamma}{N}$

On the other hand,
the optimal state for fidelity does not allow probabilistically perfect error correction

OPTIMAL MULTIROUND PROTOCOLS FOR REFERENCE FRAME ALIGNMENT

QUANTUM GYROSCOPES

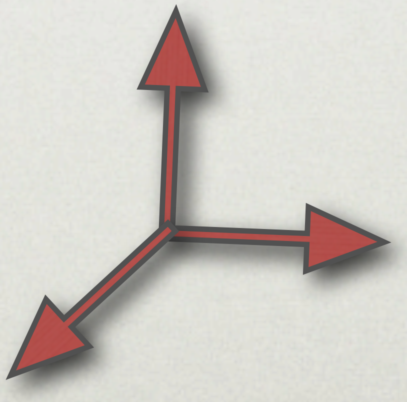
Spin $\frac{1}{2}$ particle, rotation $g \in \mathbb{SO}(3)$ $g = (\mathbf{n}, \varphi)$

State change: $U_g = e^{i\varphi\mathbf{n}\cdot\sigma} = \cos(\varphi/2) + i\sin(\varphi/2)\mathbf{n}\cdot\sigma$

encodes a spatial direction:

N qubits: $|A\rangle \in \mathcal{H}^{\otimes N}$ $|A_g\rangle = U_g^{\otimes N}|A\rangle$

encode a Cartesian frame:



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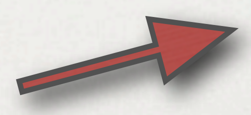

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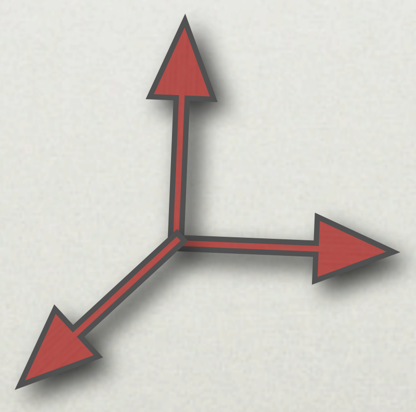

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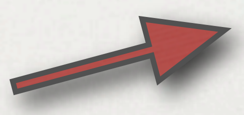

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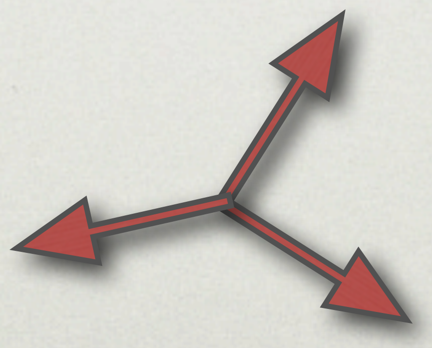
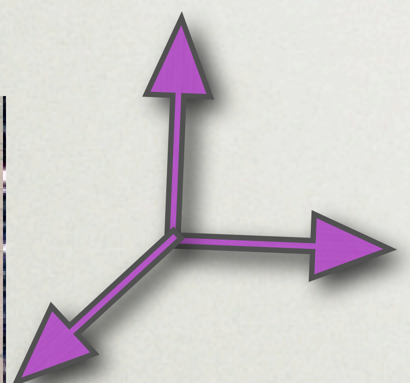


ALIGNING AXES WITH QUANTUM GYROSCOPES

Suppose Alice and Bob have different Cartesian frames (different axes): a state that is $|A\rangle$ for Alice is $U_g|A\rangle$ for Bob.
However, using quantum communication they can try to establish a shared reference frame:

Alice



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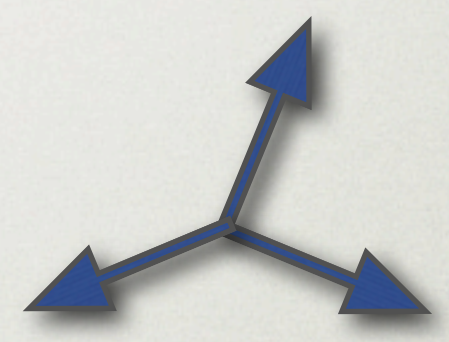

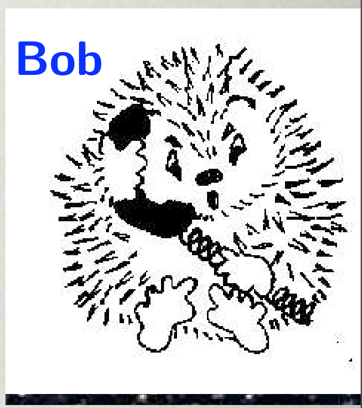

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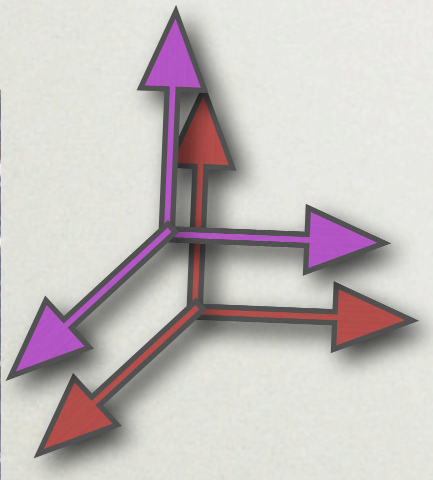

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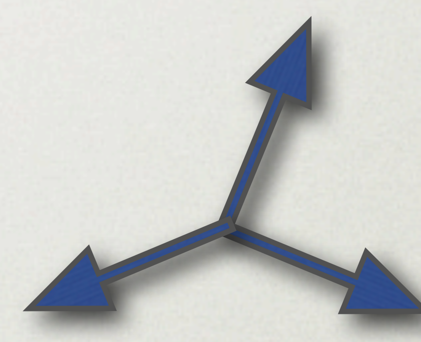

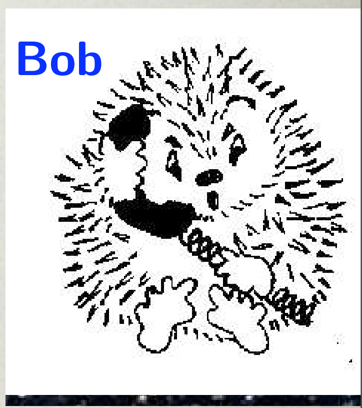

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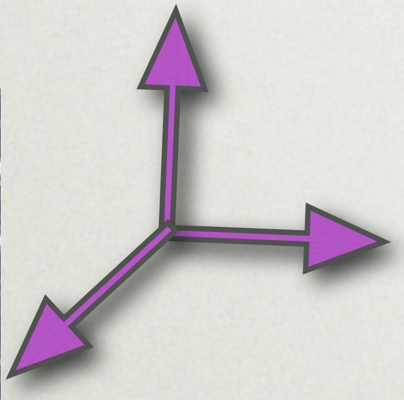

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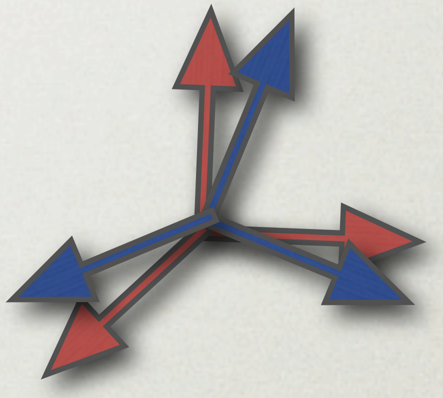

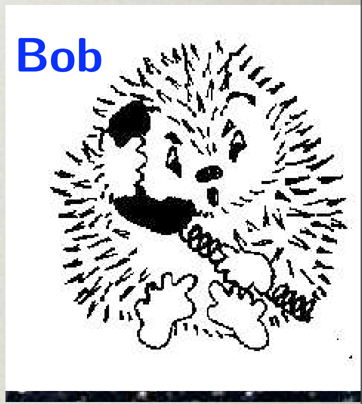

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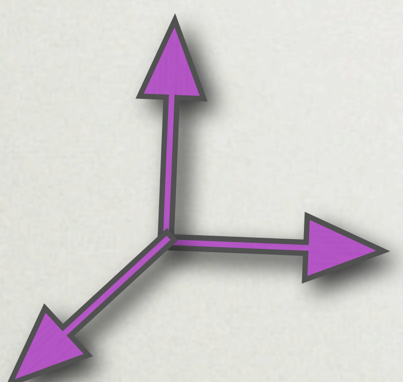

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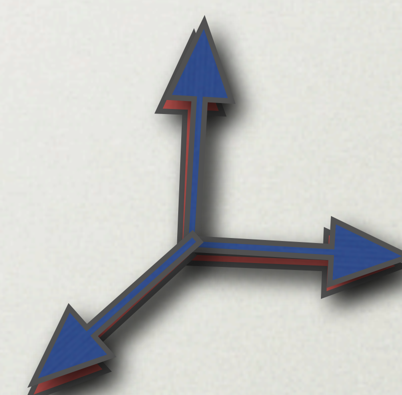

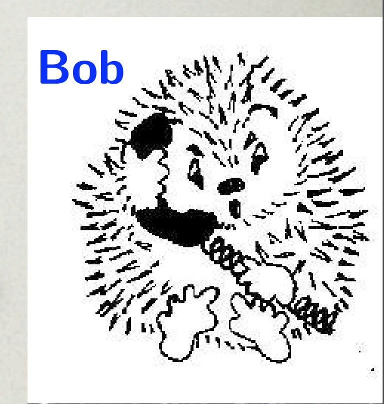

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ULTIMATE PRECISION LIMITS FOR N PARTICLES

- For a quantum gyroscope made of N identical spin 1/2 particles:

$$\langle c \rangle \approx \sum_{i=x,y,z} \Delta\theta_i^2 = 3\Delta\theta_x^2 \approx \frac{2\pi^2}{N^2}$$

GC, D'Ariano, Perinotti, Sacchi, PRL 93, 180503 (2004)
Bagan, Baig, Muñoz-Tapia, PRA 70, 030301 (2004)
Hayashi, PLA 354, 183 (2006)

However, this result is provenly the optimal one only if we assume that Alice sends all particles in a single shot.

In other words, this result is about protocols with a single-round of forward quantum communication.

What about multi-round protocols?

MULTI-ROUND ALIGNMENT PROTOCOLS

- For a quantum gyroscope made of N identical spin 1/2 particles:

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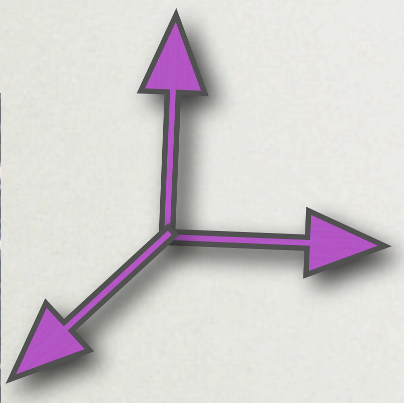

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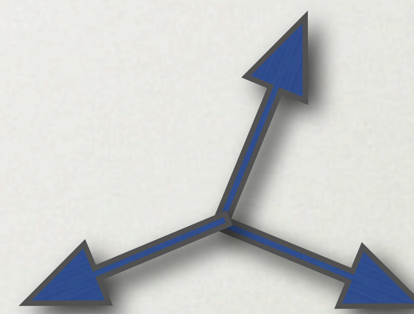

Allow

- unlimited amount of classical communication
- k rounds of quantum communication, in which batches of spin 1/2 particles are sent.

Then find the best way of estimating the mismatch of alignment.

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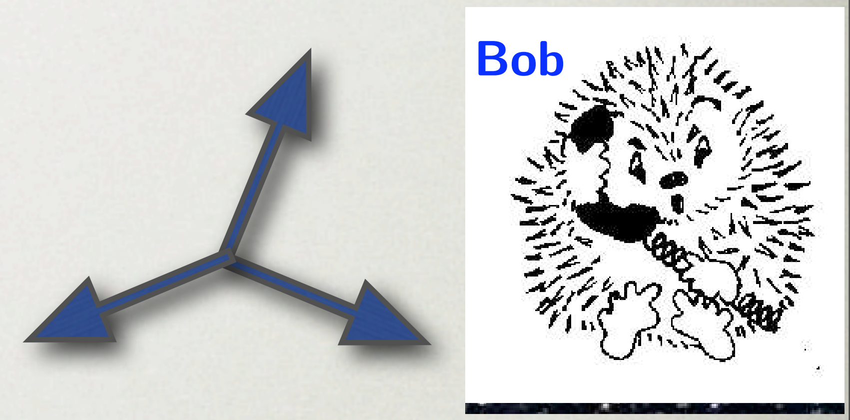

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QUANTUM COMB FORMULATION

Alice's moves, in her description, are given by comb S

In Bob's description:

$$S_g = (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C) S (U_g^{\dagger\otimes N_{A\to B}} \otimes U_g^{\tau *\otimes N_{B\to A}} \otimes I_C)$$

Bob's estimation strategy: tester $T_{\hat{g}}$

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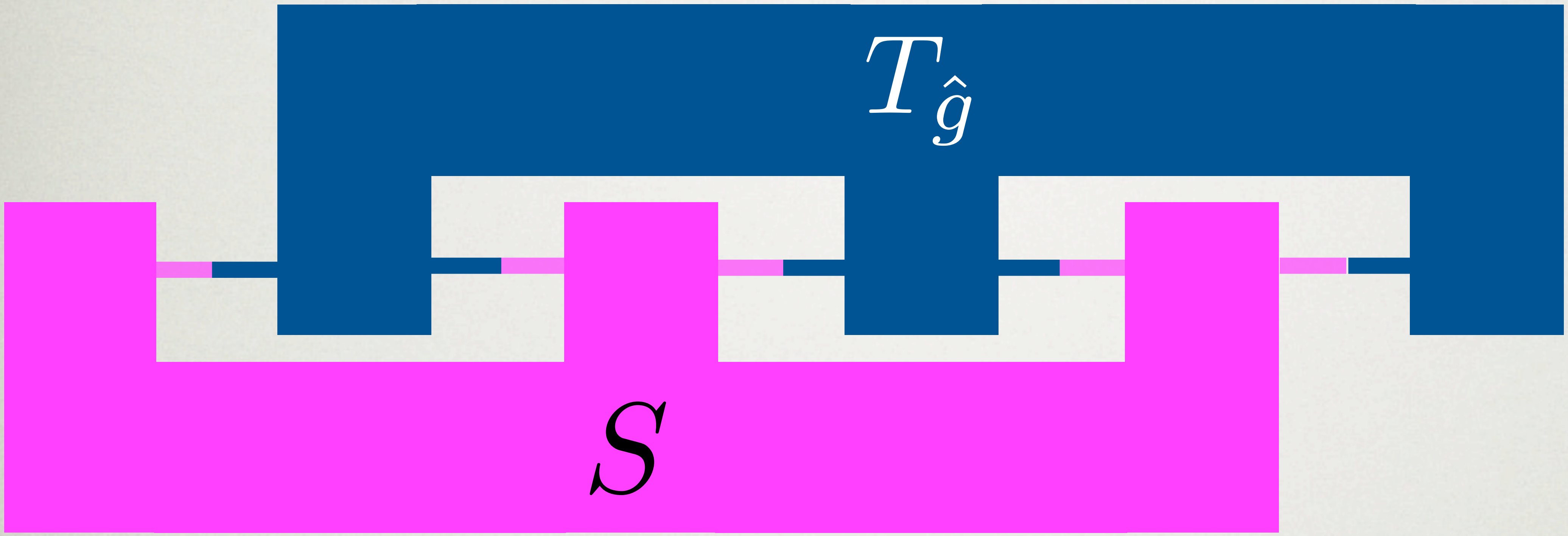

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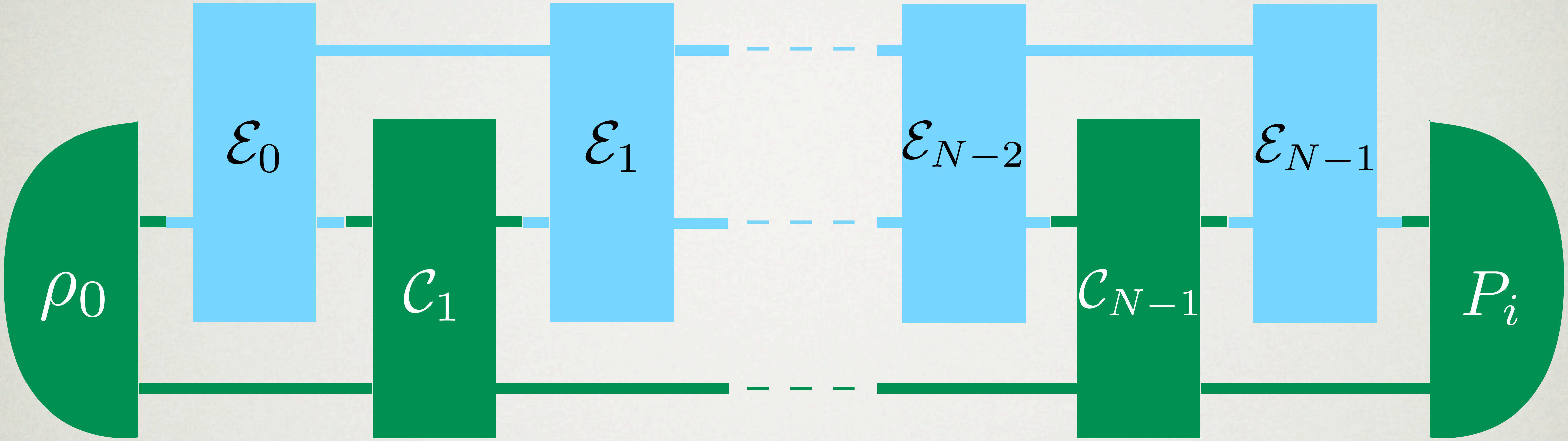
QUANTUM TESTERS



Quantum tester = network beginning with a state preparation and ending with a measurement
= collection of positive operators with suitable normalization.

$$\{T_i\} \quad T_i \geq 0 \quad \sum_i T_i = \textit{deterministic comb}$$

Born rule for quantum networks: $p_i = S * T_i = \mathrm{Tr}[S\ T_i^{\tau}]$

OPTIMALITY OF COVARIANT TESTERS

$$\{S_g = W_g S_0 W_g^\dagger \mid g \in G\}$$ invariant family of quantum combs with uniform prior dg

$$c(\hat{g}, g)$$ left-invariant cost function

$$c(k\hat{g}, kg) = c(\hat{g}, g) \qquad \forall k \in G$$

The optimal tester for

- minimizing the average cost $$\langle c \rangle = \int \mathrm{d}g \int \mathrm{d}\hat{g}\ c(\hat{g}, g)\ p(\hat{g}|g)$$
- minimizing the worst-case cost $$c_{wc} = \max_g \int \mathrm{d}\hat{g}\ c(\hat{g}, g)\ p(\hat{g}|g)$$

is covariant $$T_{\hat{g}} = \left(W_{\hat{g}}\ T_0\ W_{\hat{g}}^\dagger\right)^\tau$$

and $$\langle c \rangle^{opt} = c_{wc}^{opt}$$

GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 180501 (2008)

DECOMPOSITION OF QUANTUM TESTERS

Theorem

Any tester can be split into two parts

- a deterministic supermap transforming quantum combs into states

$$\mathcal{T}(S) = T^{\frac{1}{2}} \; S \; T^{\frac{1}{2}} \qquad T = \sum_i T_i^{\tau}$$

- an ordinary quantum measurement $\{P_i\}$ on the output states

$$p_i = S * T_i = \mathcal{T}(S) * P_i = \mathrm{Tr}[\mathcal{T}(S) P_i^{\tau}]$$

OPTIMALITY PROOF FOR ONE-WAY STRATEGIES

Decomposition of the tester: measurement on the quantum state

$$\mathcal{T}(S) = T^{\frac{1}{2}}\ S\ T^{\frac{1}{2}} \qquad T = \int \mathrm{d}\hat{g}\ T_{\hat{g}}^{\tau}$$

where

$$T_{\hat{g}} = (W_g T_0 W_g^{\dagger})^{\tau} \qquad W_g = U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C$$

Since $\quad [T, W_g] = 0 \qquad \forall g \in G$

the output state is of the form

$$\rho_g = (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C)\ \rho_0\ (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C)^{\dagger}$$

But a state like this can be obtained in a single round!

OPTIMALITY PROOF FOR ONE-WAY STRATEGIES

Theorem:

For any multi-round protocol, there is a protocol with a single round of forward quantum communication from Alice to Bob, using

- $N_{A\to B}$ particles and
- $N_{B\to A}$ charge-conjugate particles

that achieves the same average (or worst case) cost.

G C, G M D'Ariano, and P Perinotti, Proc. QCMC 2008 (arXiv:0812.3922)

In particular,

- for quantum clocks G = U(1)
- for quantum gyroscopes G = SU (2)

the only thing that matters is the total number of transmitted particles

$$N_{tot} = N_{A\to B} + N_{B\to A}$$

CONCLUSIONS

- the optimal learning of a group transformation is “measure-and-prepare”

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- the proper way to solve these problem is the formalism of quantum combs and testers.