Phase Measurements at the Theoretical Limit

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Institute for Quantum Computing

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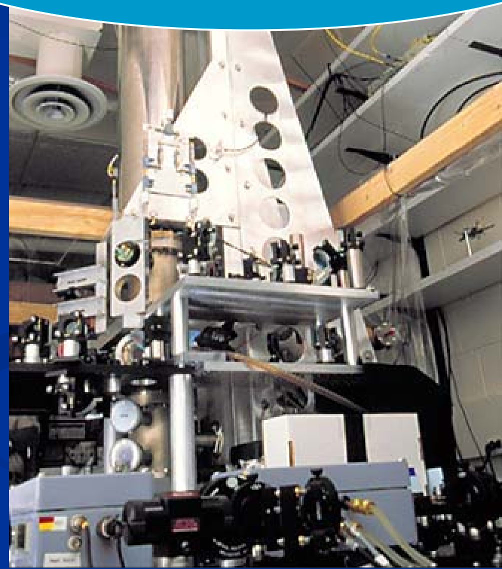
Applications of phase measurement

Distance measurement

Frequency and time measurement



Communication

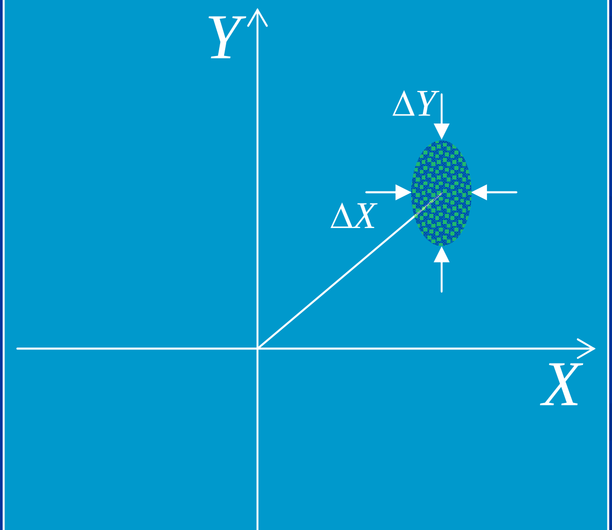
The Heisenberg Uncertainty Principle

position & momentum

$$\Delta p \Delta x \geq \frac{\hbar}{2}$$

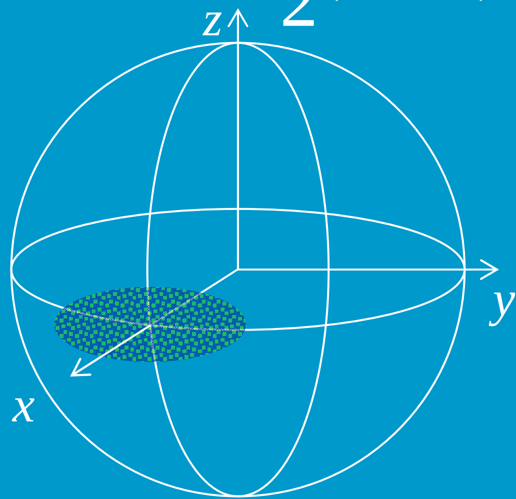
quadratures

$$\Delta X \, \Delta Y \geq 1$$



spin

$$\Delta J_y \Delta J_z \geq \frac{1}{2} \left| \langle J_x \rangle \right|$$



The Heisenberg limit vs the standard quantum limit

The Standard Quantum Limit

- If the two uncertainties are equal.
- Uncertainty scaling

$$\Delta\phi \propto 1/\sqrt{N}$$

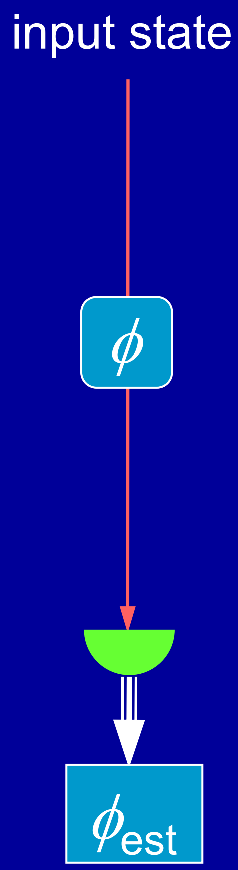
The Heisenberg Limit

- If one uncertainty is reduced as much as possible.
- Uncertainty scaling

$$\Delta\phi \propto 1/N$$

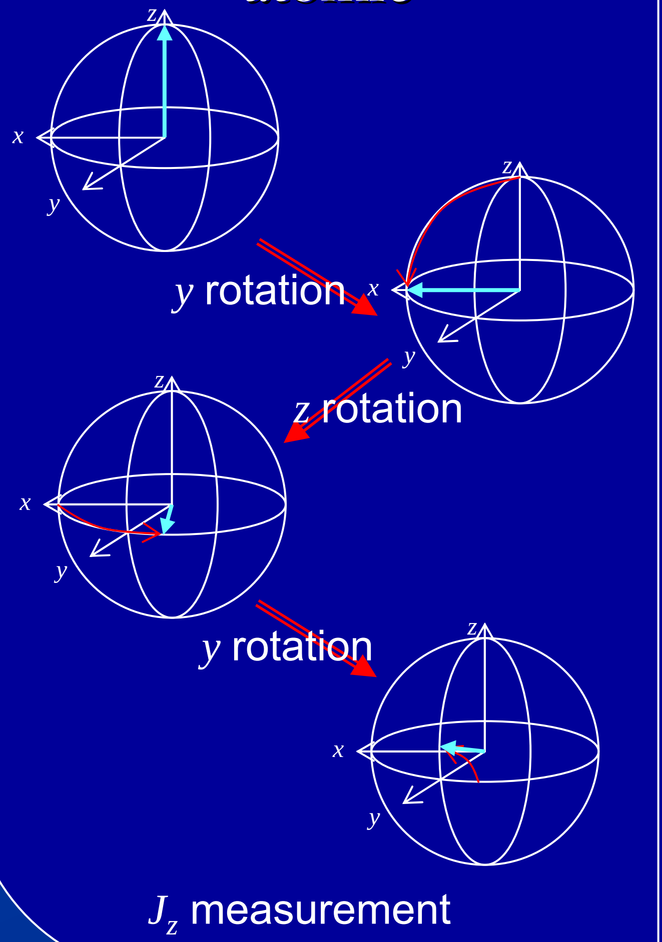
Types of phase measurement

Single-mode phase

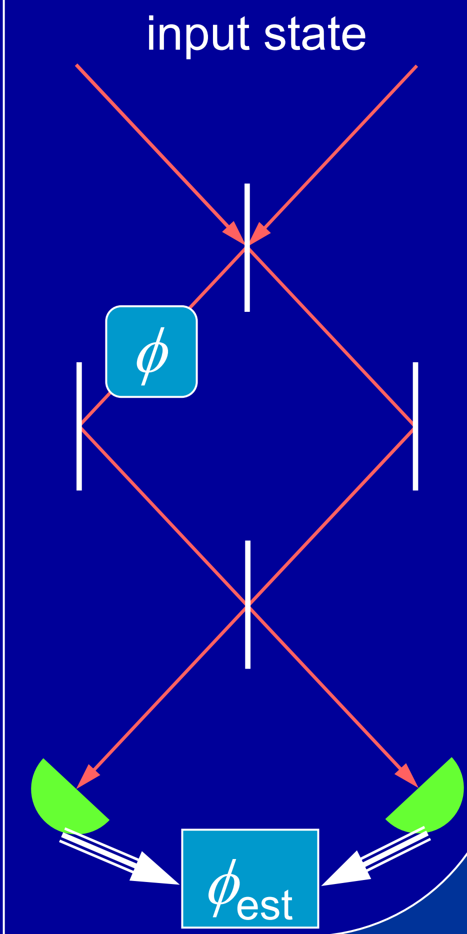


Interferometric measurement

atomic

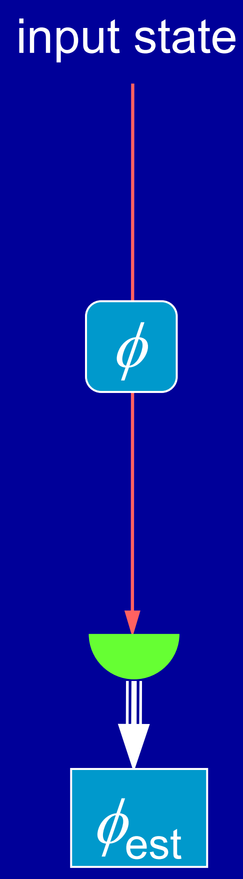


optical



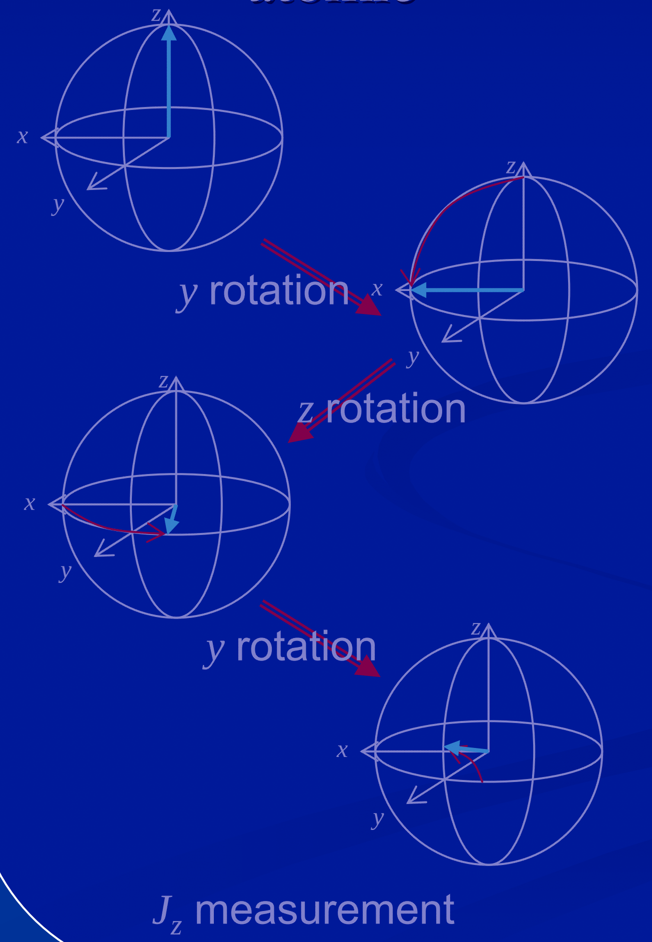
Types of phase measurement

Single-mode phase

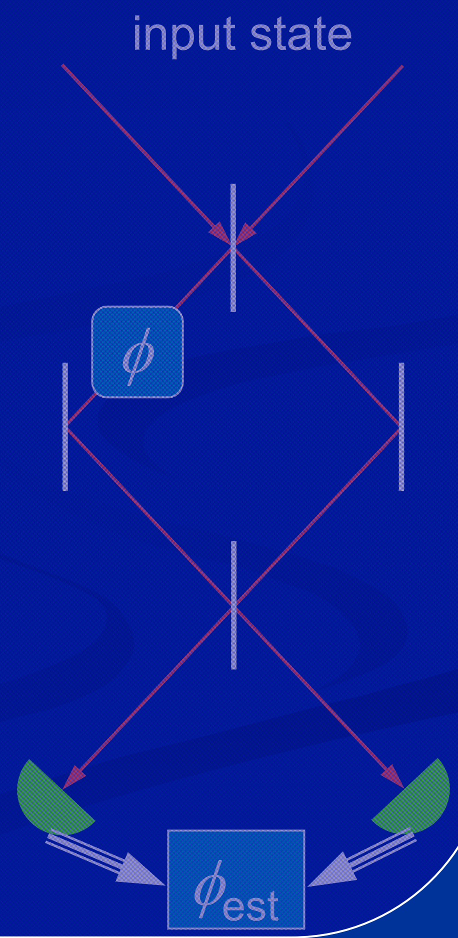


Interferometric measurement

atomic

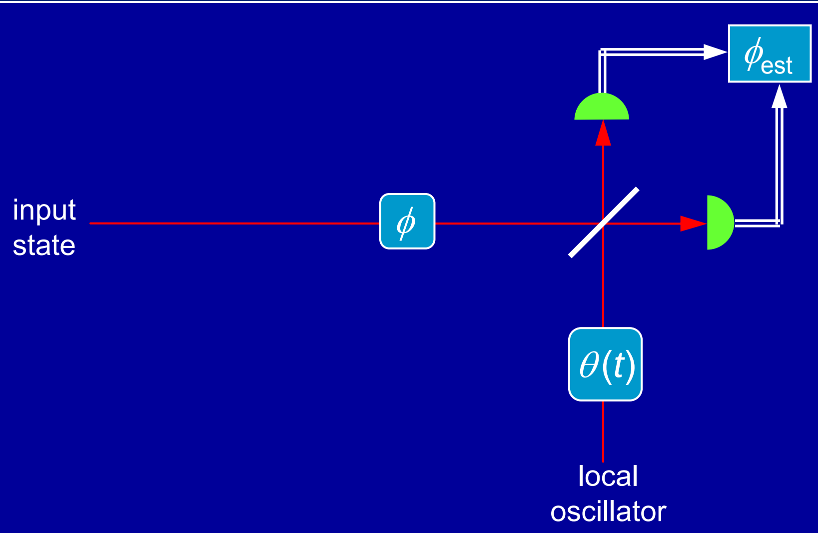


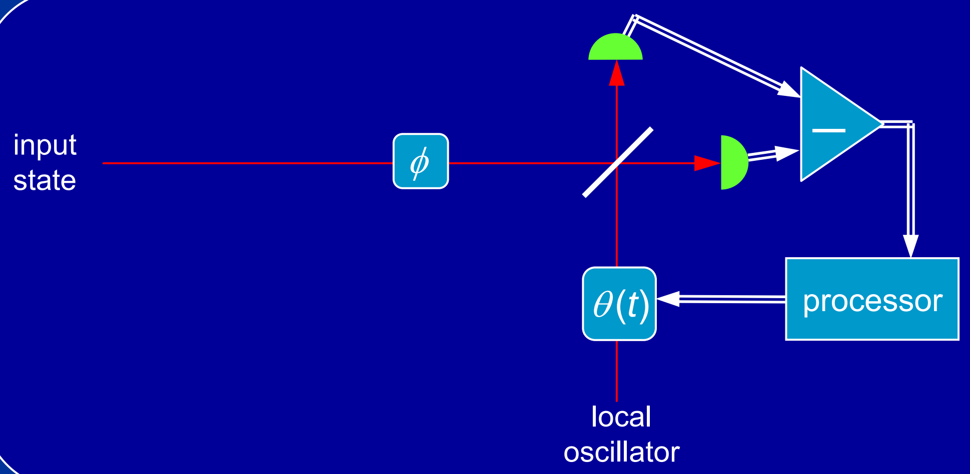
optical



Single-mode measurements

- Signal is beam is mixed with strong "local oscillator".
- Heterodyne – linear variation of θ.
- Homodyne – θ close to ϕ.

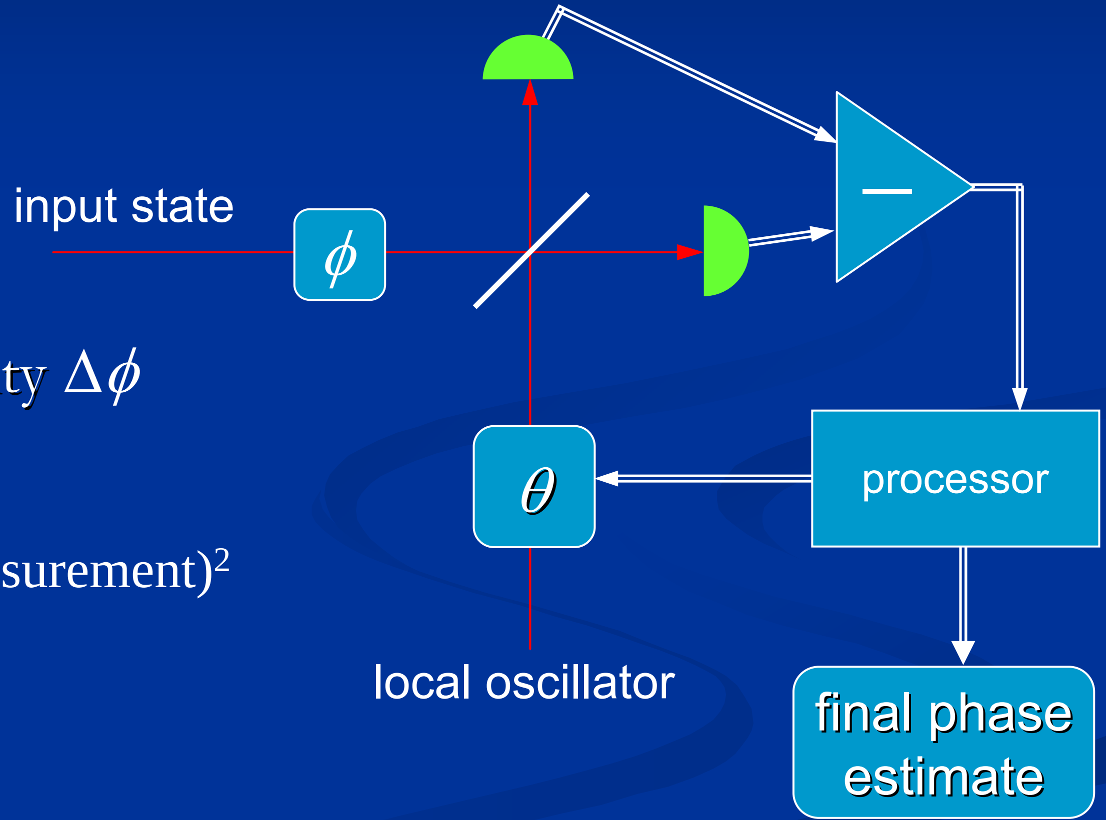




- Use an estimate of the phase to approximate a homodyne measurement.

Adaptive phase measurement

Task: measure an arbitrary phase



Total phase uncertainty $\Delta\phi$

$$\Delta\phi^2 = (\text{intrinsic uncertainty})^2 + (\text{uncertainty due to measurement})^2$$

H. M. Wiseman, Phys. Rev. A **56**, 944 (1997).
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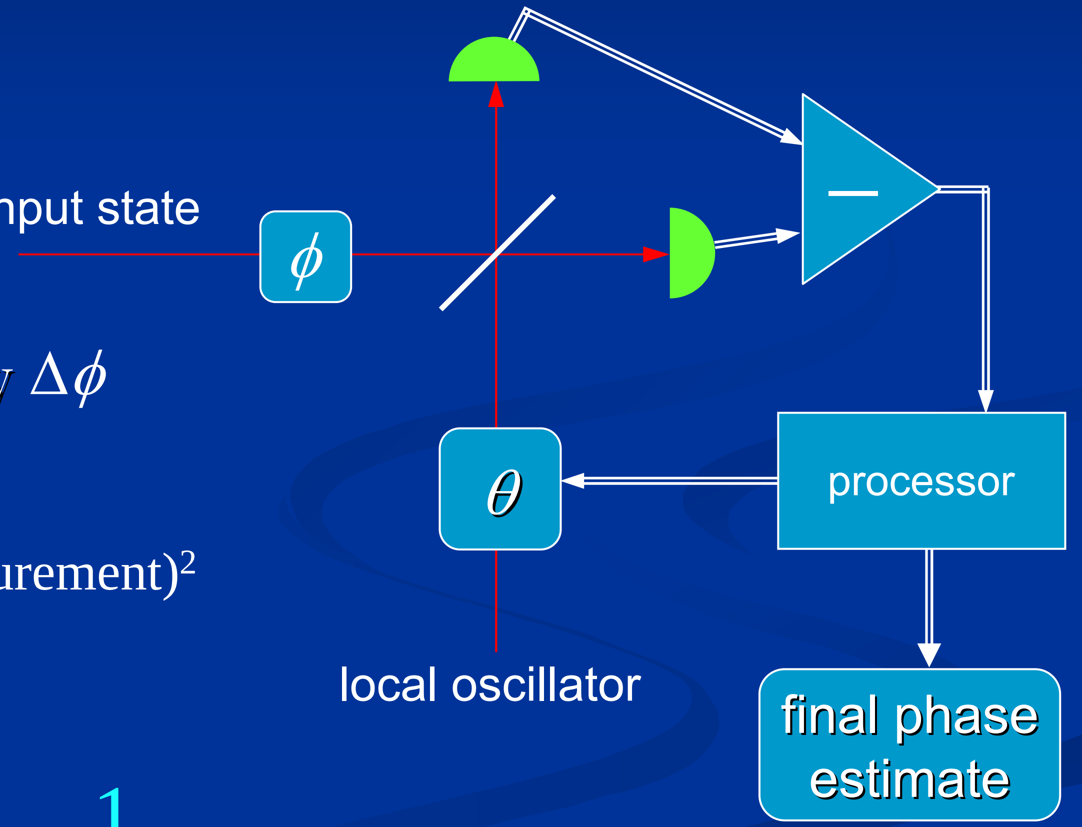
Adaptive phase measurement

Task: measure an arbitrary phase



Total phase uncertainty $\Delta\phi$

$\Delta\phi^2$ = (intrinsic uncertainty)^2
+ (uncertainty due to measurement)^2

heterodyne measurements $\propto \dfrac{1}{\sqrt{N}}$

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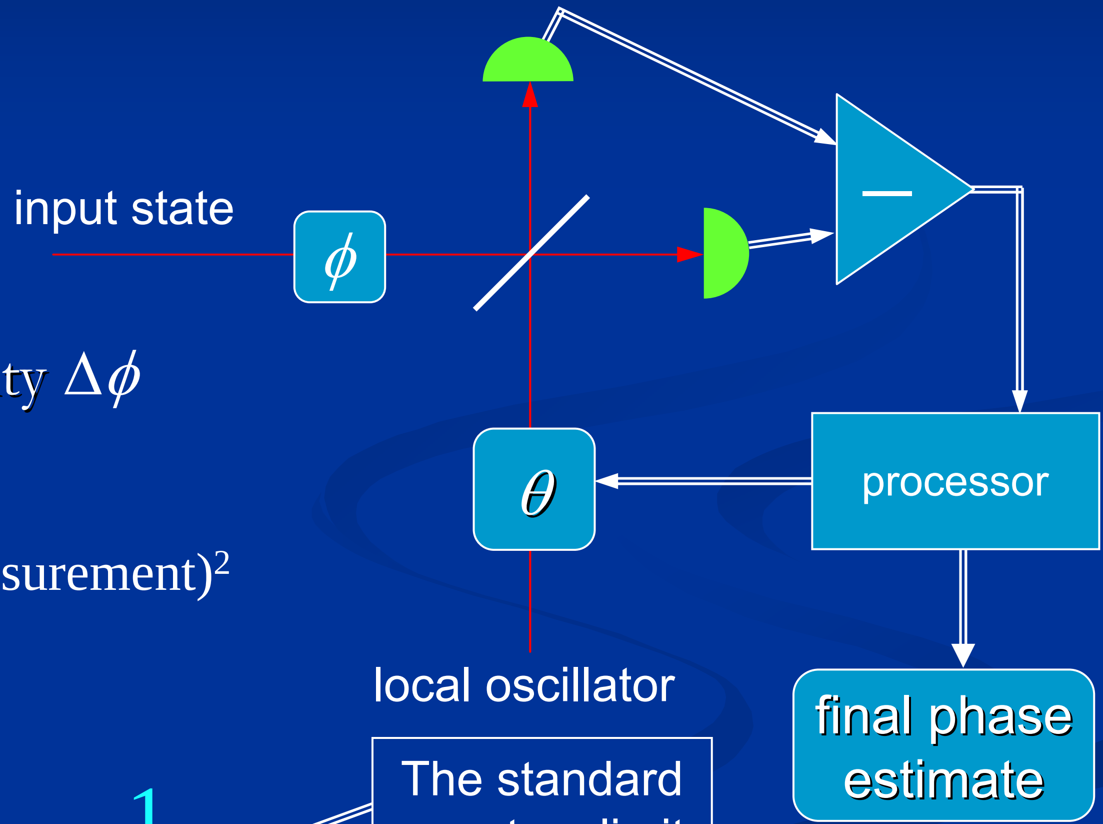
Adaptive phase measurement

Task: measure an arbitrary phase



Total phase uncertainty $\Delta\phi$

$$\Delta\phi^2 = (\text{intrinsic uncertainty})^2 + (\text{uncertainty due to measurement})^2$$

heterodyne measurements $\propto \dfrac{1}{\sqrt{N}}$ ← The standard quantum limit

H. M. Wiseman, Phys. Rev. A **56**, 944 (1997).
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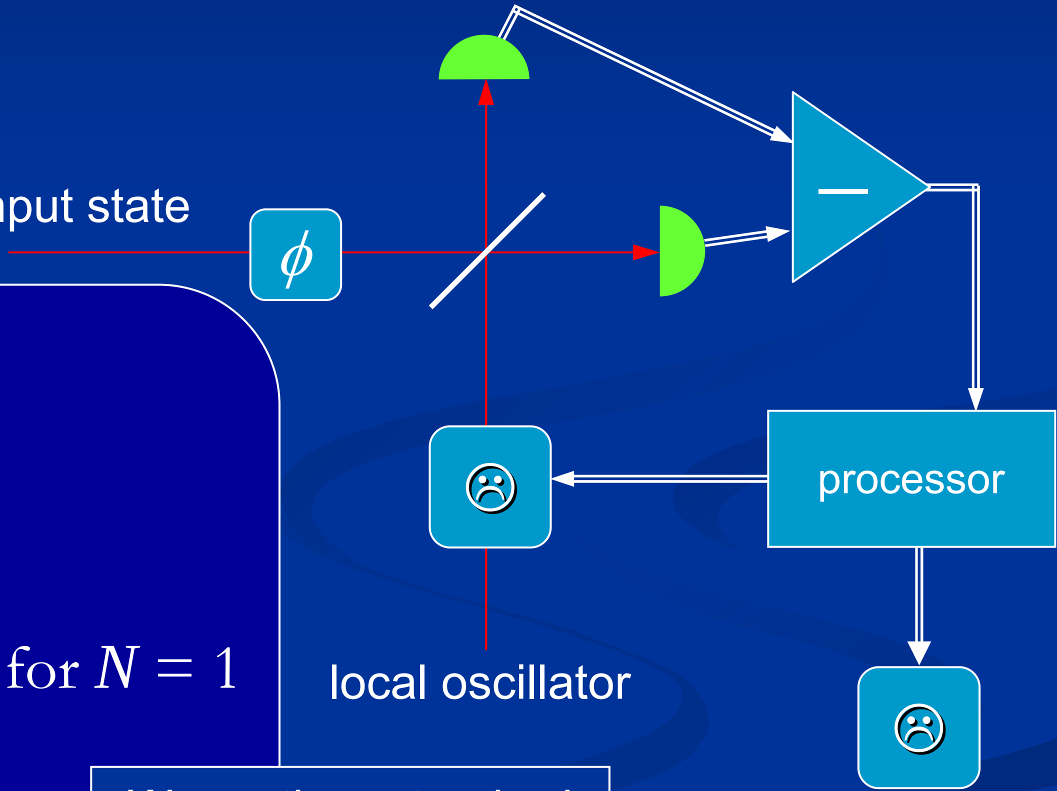
Wiseman Mark I

Task: measure an arbitrary phase

☺ ≡ best phase estimate

☹ ≡ poor phase estimate



Mark I

feedback phase estimate: ☹
final phase estimate: ☹

⇒ ideal phase measurement for $N = 1$
⇒ for $N >> 1$

$$\Delta\phi \propto \frac{1}{N^{1/4}}$$

Worse than standard quantum limit!

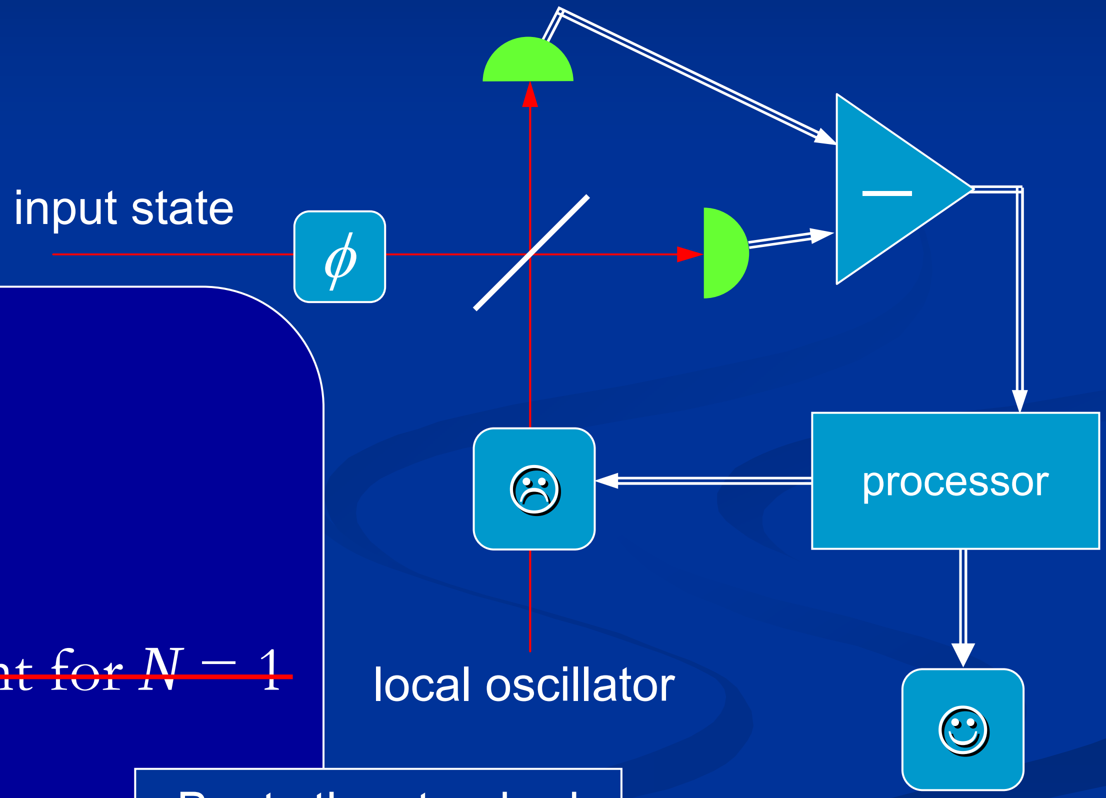
H. M. Wiseman, Phys. Rev. A **56**, 944 (1997).
H. M. Wiseman, Phys. Rev. A **57**, 2169 (1998).

Wiseman Mark II

Task: measure an arbitrary phase

☺ ≡ best phase estimate

☹ ≡ poor phase estimate



Mark II

feedback phase estimate: ☹
final phase estimate: ☺

⇒ ~~ideal phase measurement for $N = 1$~~
⇒ for $N >> 1$

$$\Delta\phi \propto \frac{1}{N^{3/4}}$$

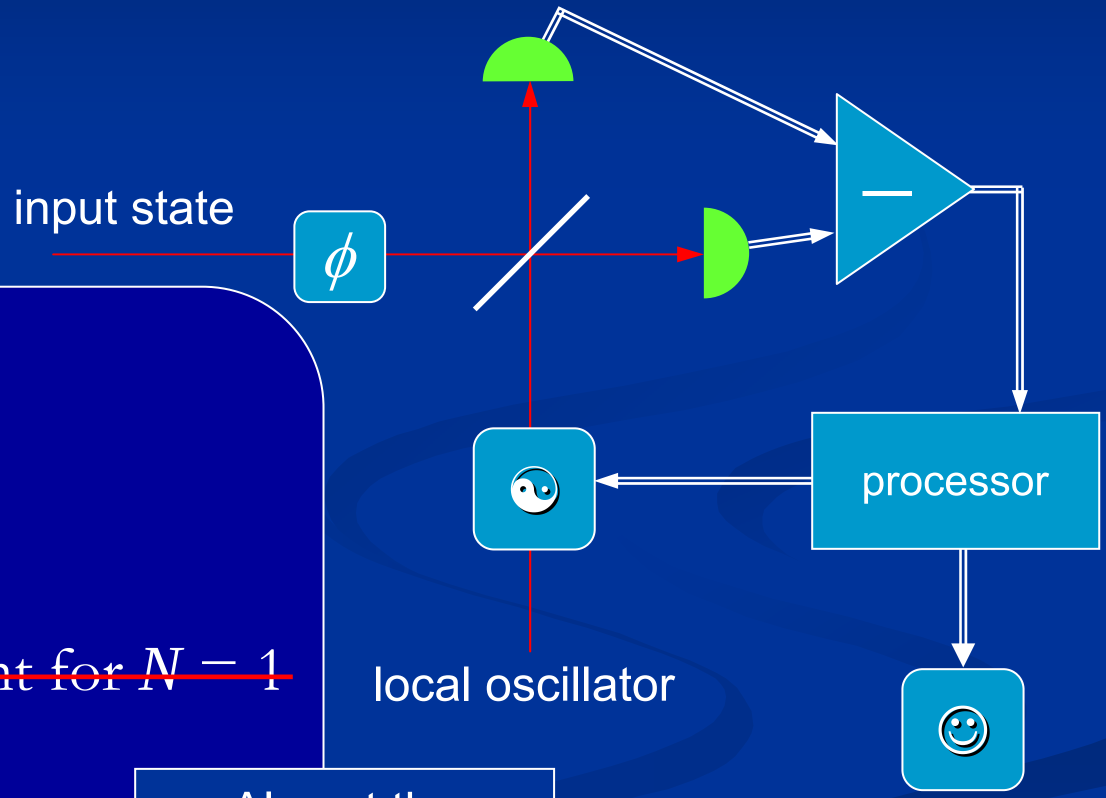
Beats the standard quantum limit

H. M. Wiseman, Phys. Rev. A **56**, 944 (1997).
H. M. Wiseman, Phys. Rev. A **57**, 2169 (1998).

Optimal adaptive

Task: measure an arbitrary phase

☺ ≡ best phase estimate
☹ ≡ poor phase estimate
☯ ≡ intermediate



Mark II

feedback phase estimate: ☯
final phase estimate: ☺

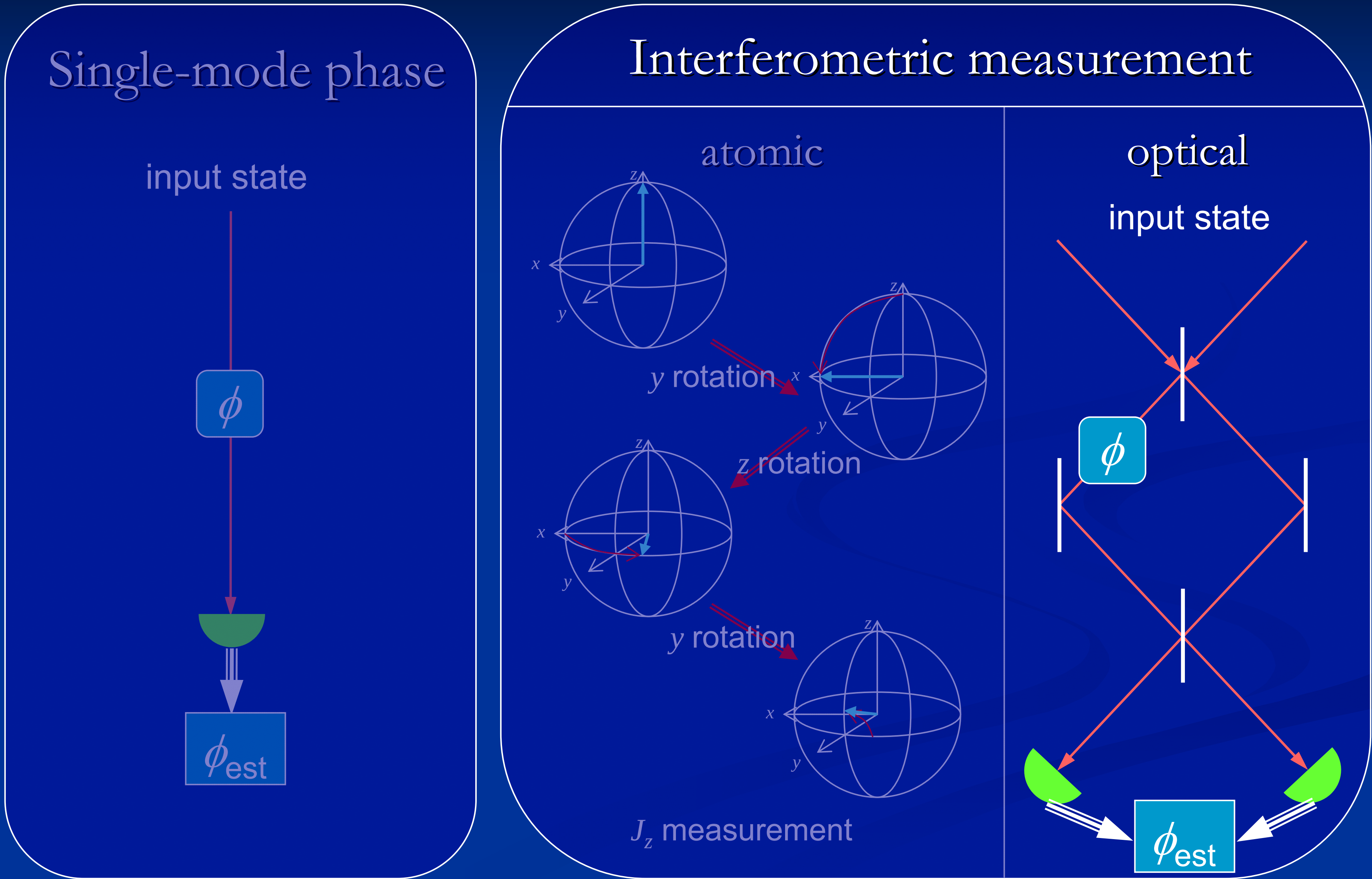
⇒ ~~ideal phase measurement for $N = 1$~~
⇒ for $N >> 1$

$$\Delta\phi \propto \frac{\sqrt{\ln N}}{N}$$

Almost the Heisenberg limit

D. W. Berry and H. M. Wiseman,
Phys. Rev. A **63**, 013813 (2001).

Types of phase measurement



Optical interferometry

- Theoretical limit
- Squeezed states[1]
- NOON states[2]
- Theoretical-limit adaptive measurements[3]
- Theoretical-limit **non**adaptive measurements[4]
- Hybrid measurements[4]

[1] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
[2] B. C. Sanders, Phys. Rev. A **40**, 2417 (1989).
[3] B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature 450, 393 (2007).
[4] B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, e-print: 0809.3308 (2008).

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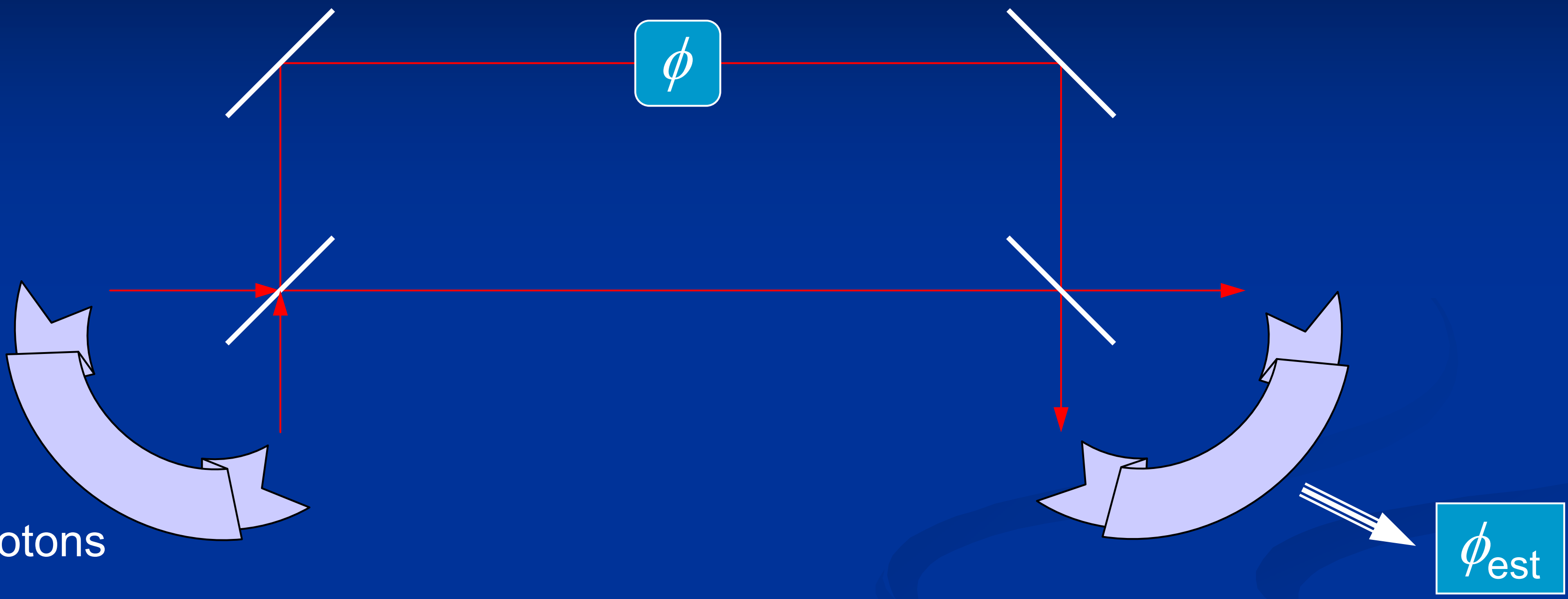
Optimal measurements



N photons

optimal two-mode entangled input state[1,2]

optimal two-mode joint measurement[3]

$$\Delta\phi \approx \pi / N$$

The theoretical limit

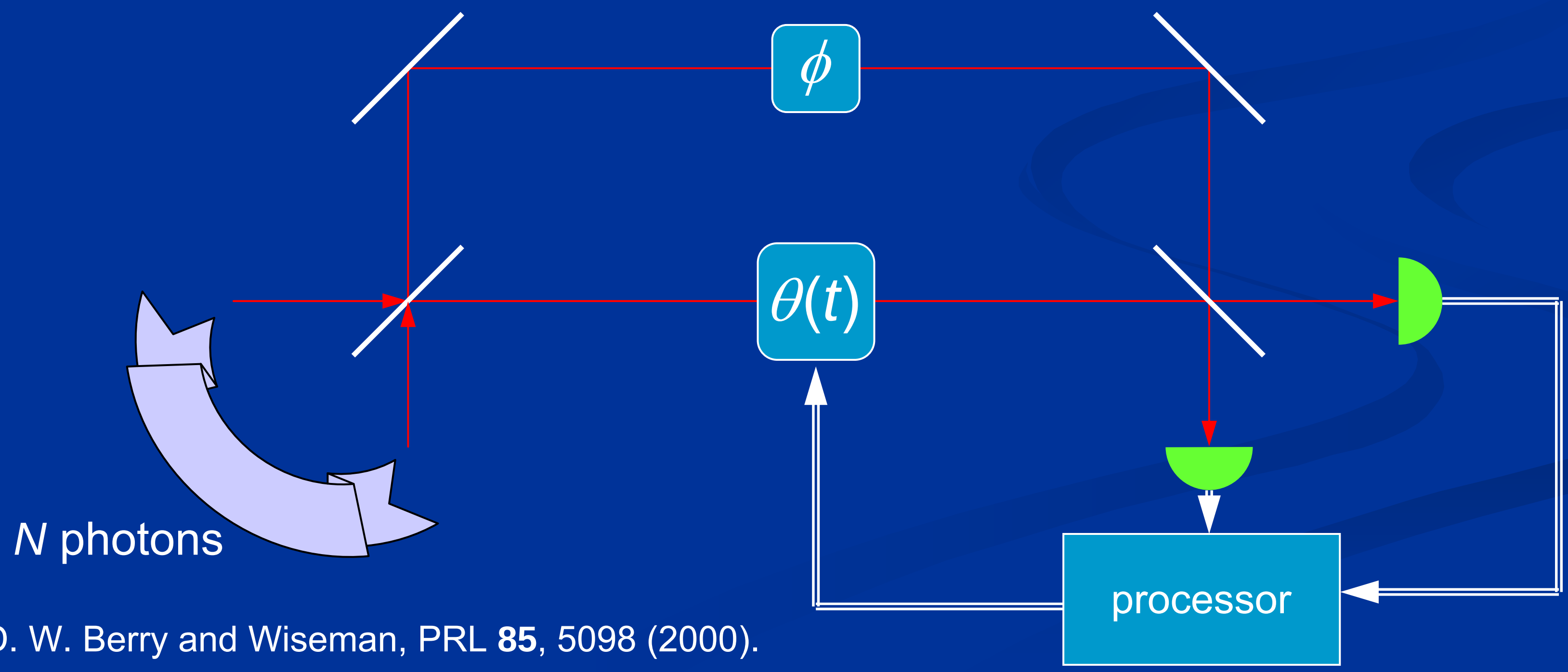
[1] A. Luis and J. Peřina, Phys. Rev. A 54, 4564 (1996).

[2] D. W. Berry and H. M. Wiseman, PRL **85**, 5098 (2000).

[3] B. C. Sanders and G. J. Milburn, PRL **75**, 2944 (1995).

How to perform the measurement?

- $\theta(t)$ is adjusted to minimise the expected variance after the next detection.
- Gives uncertainty $\Delta\phi \sim 1/N$



D. W. Berry and Wiseman, PRL **85**, 5098 (2000).

How to create the input state?

Two problems:

1. The state needs to be a special coherent superposition of the form

$$\sum_{n=0}^{N} \psi_n |n\rangle |N-n\rangle$$

There is no known way of producing such a state.

2. The input mode needs to be very long so that $\theta(t)$ can be adjusted between detections.

Optical interferometry

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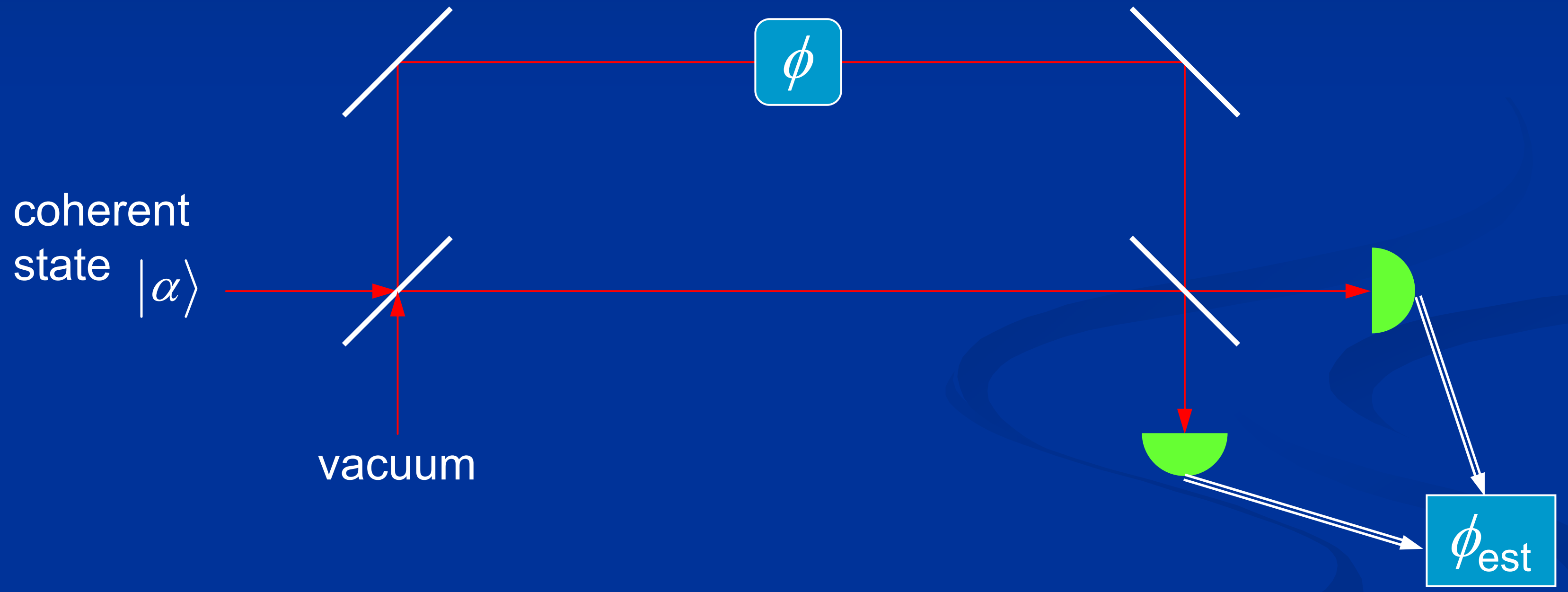
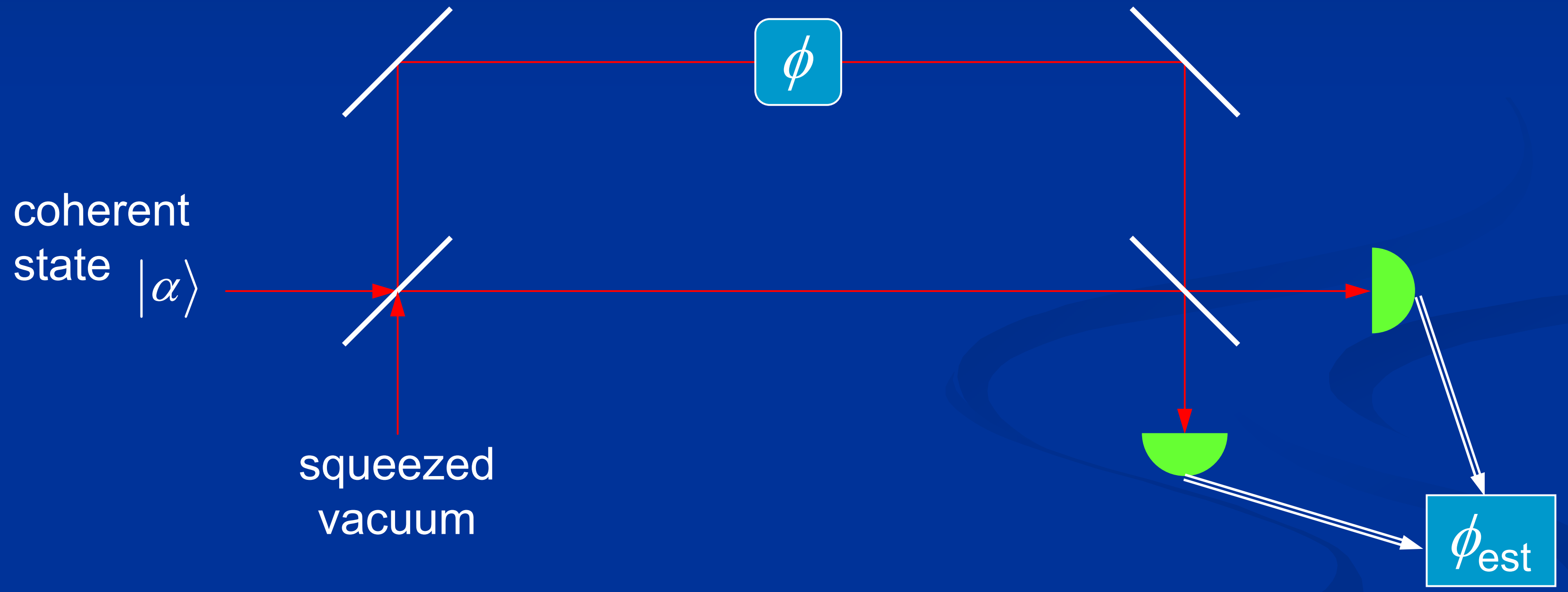
Mach-Zehnder interferometer with coherent states



without squeezing:

$$\Delta\phi \approx 1/\sqrt{N}$$ ⟸ The standard quantum limit

Mach-Zehnder interferometer with squeezed states



without squeezing:

$$\Delta\phi \approx 1/\sqrt{N}$$

with squeezing:

$$\Delta\phi \approx e^{-r}/\sqrt{N}$$

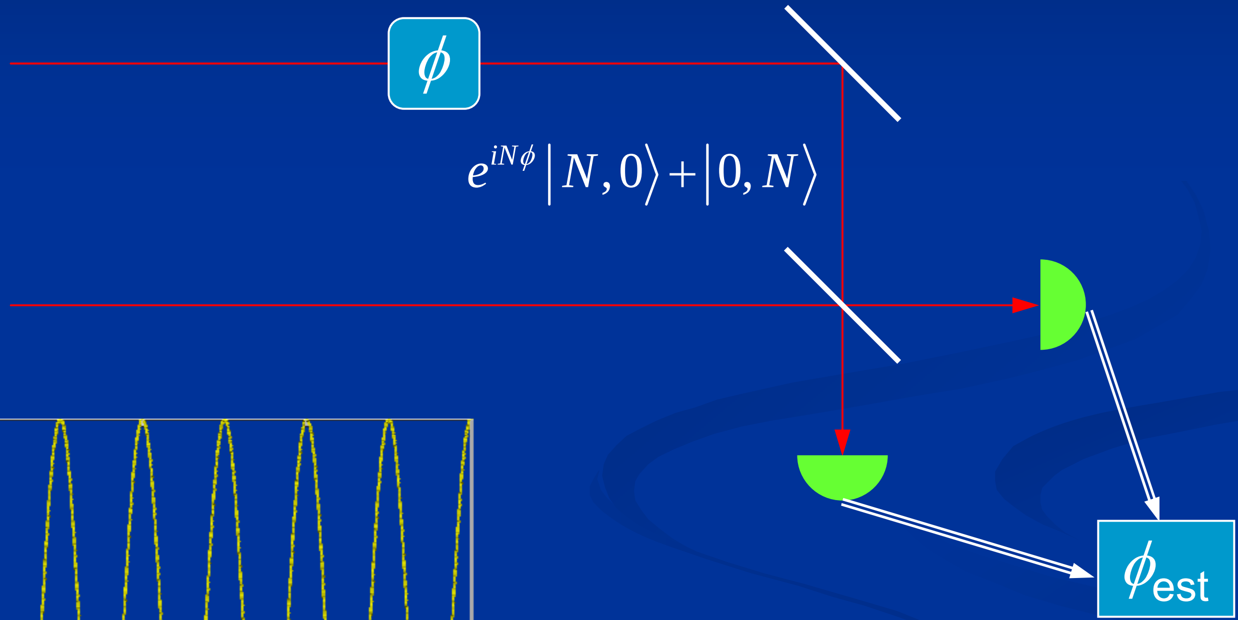
⟸ Beats the standard quantum limit

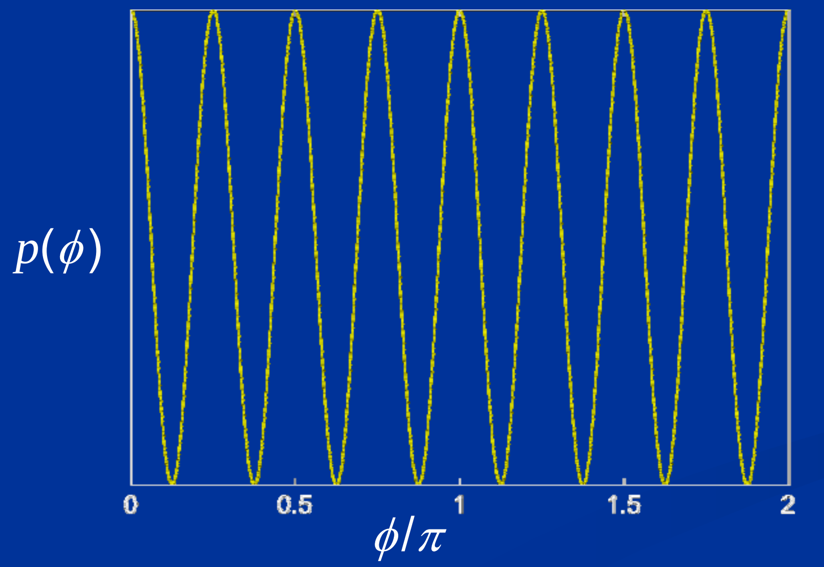
C. M. Caves, Phys. Rev. D **23**, 1693 (1981).

Mach-Zehnder interferometer with NOON states

input state

$$|N,0\rangle + |0,N\rangle$$





$$\Delta\phi \approx 1/N$$

B. C. Sanders, Phys. Rev. A **40**, 2417 (1989).

Optical interferometry

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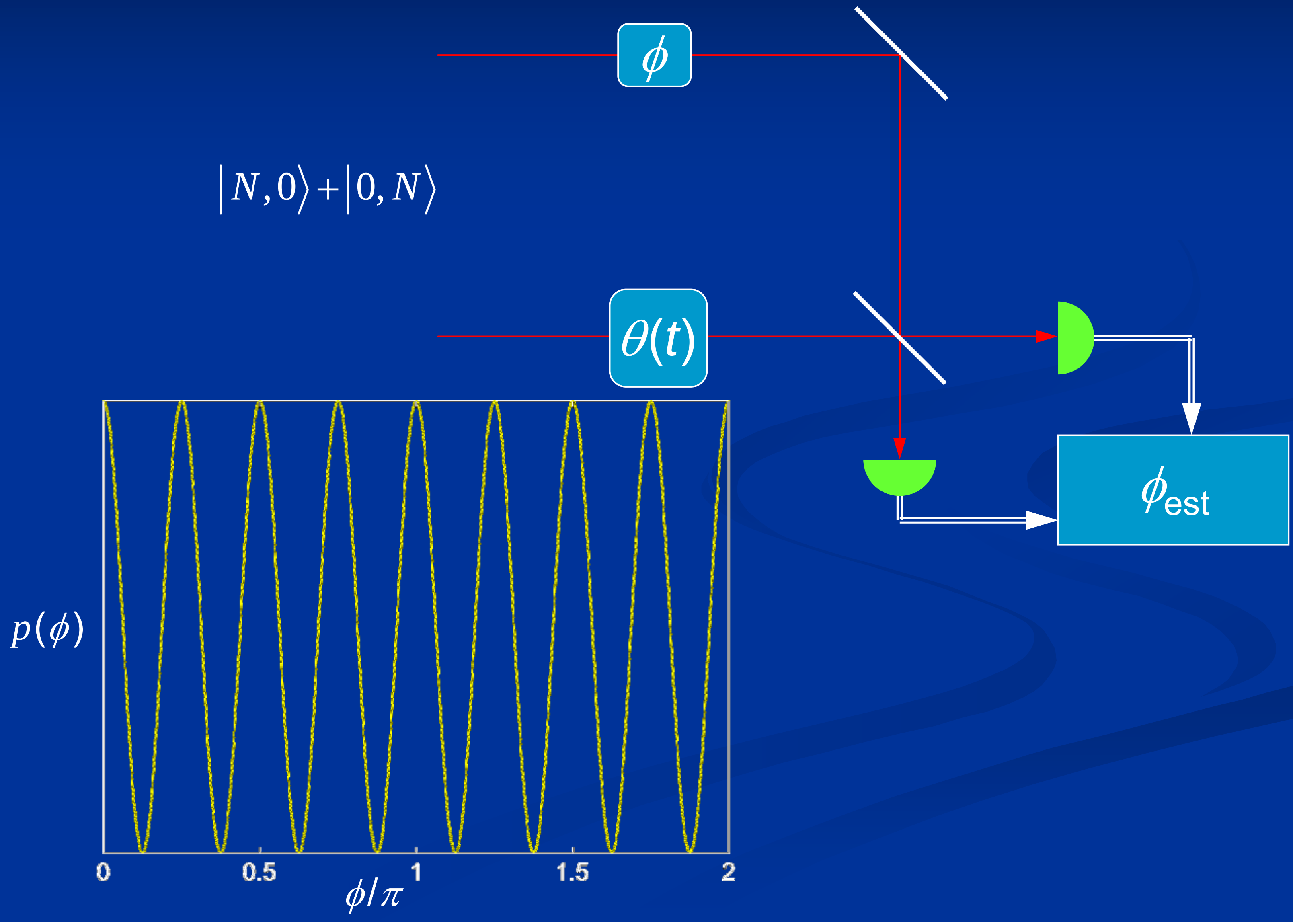
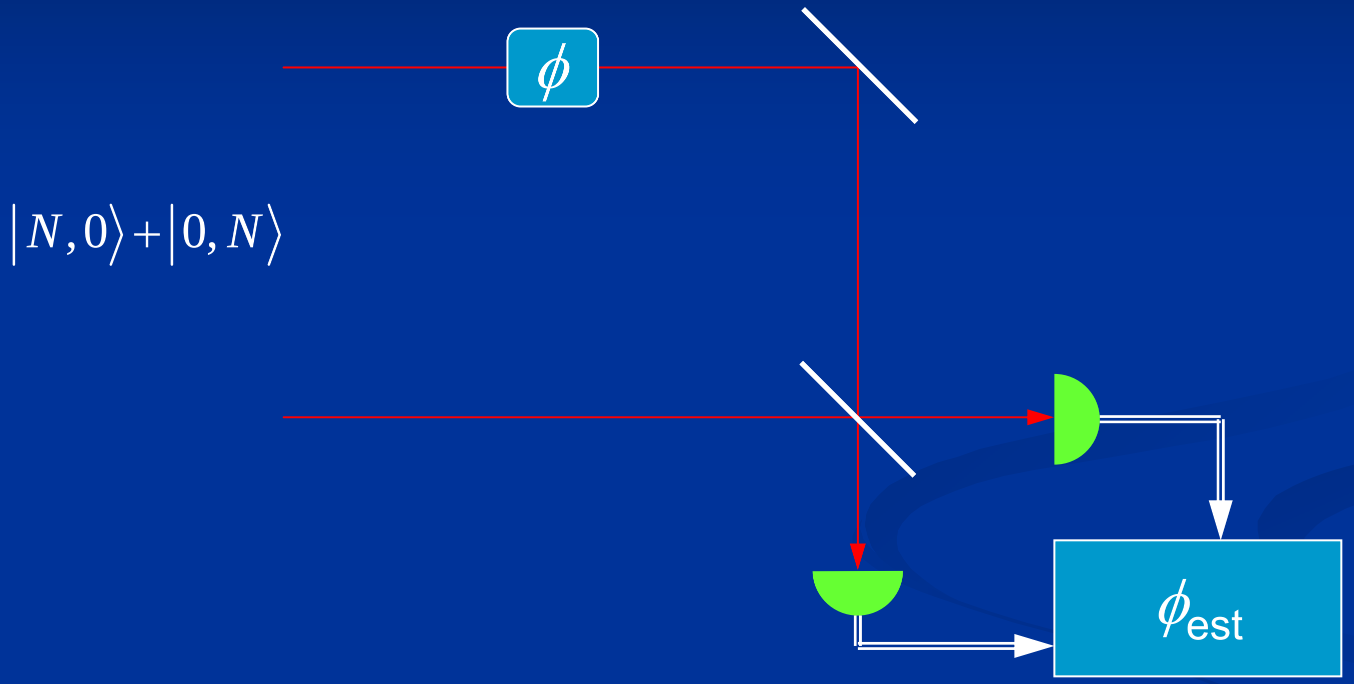
Eliminating the fringes

$$|N,0\rangle + |0,N\rangle$$



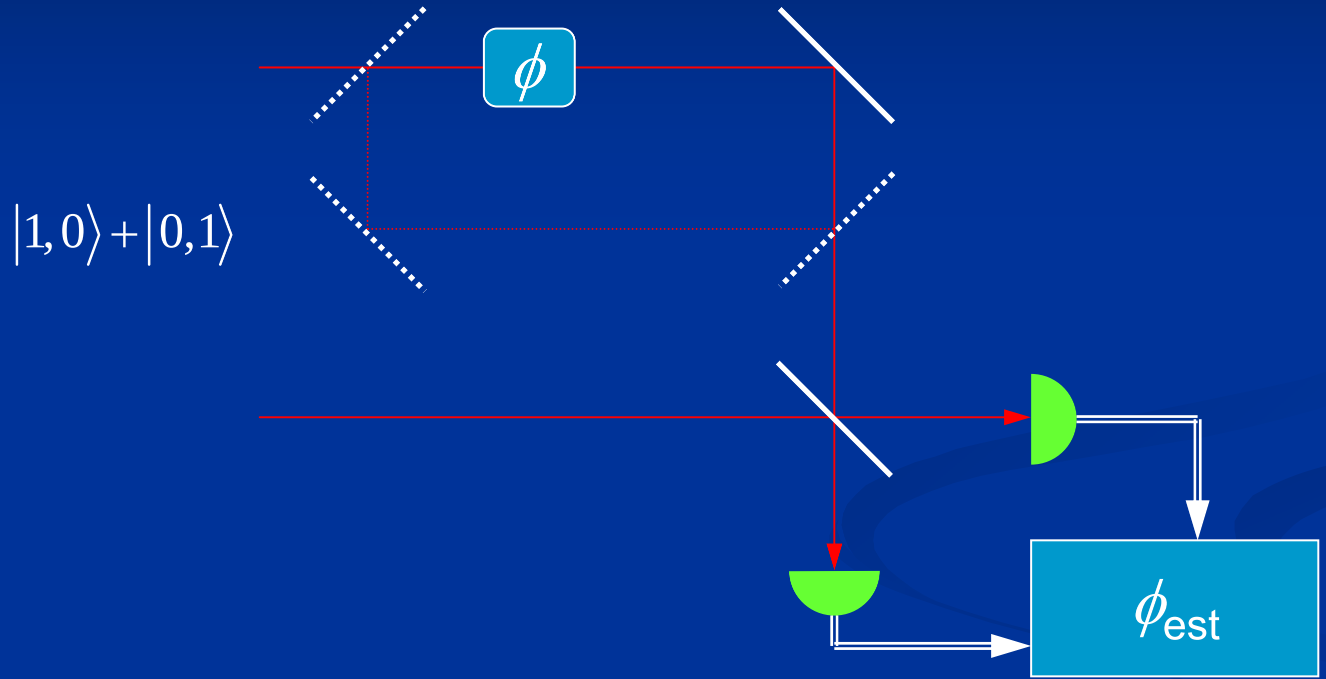
Equivalence of NOON states and multiple passes



Photons detected at times $t_1, t_2, \ldots t_N$.

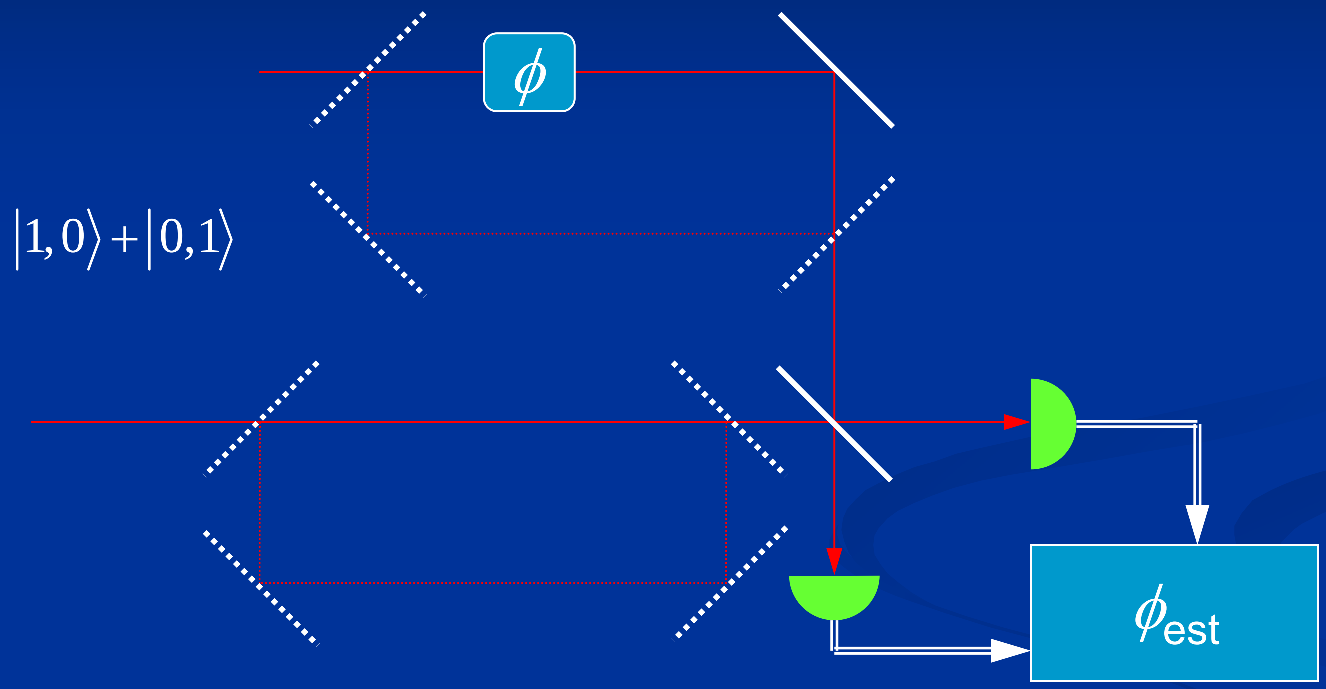
$\Rightarrow$ Passed through phase shift at times
$t_1 - \Delta t, t_2 - \Delta t, \ldots t_N - \Delta t$.

Equivalence of NOON states and multiple passes



Electro-optic switches pass single photon through phase shift at times $t_1 - \Delta t$, $t_2 - \Delta t$, … $t_N - \Delta t$.

Equivalence of NOON states and multiple passes



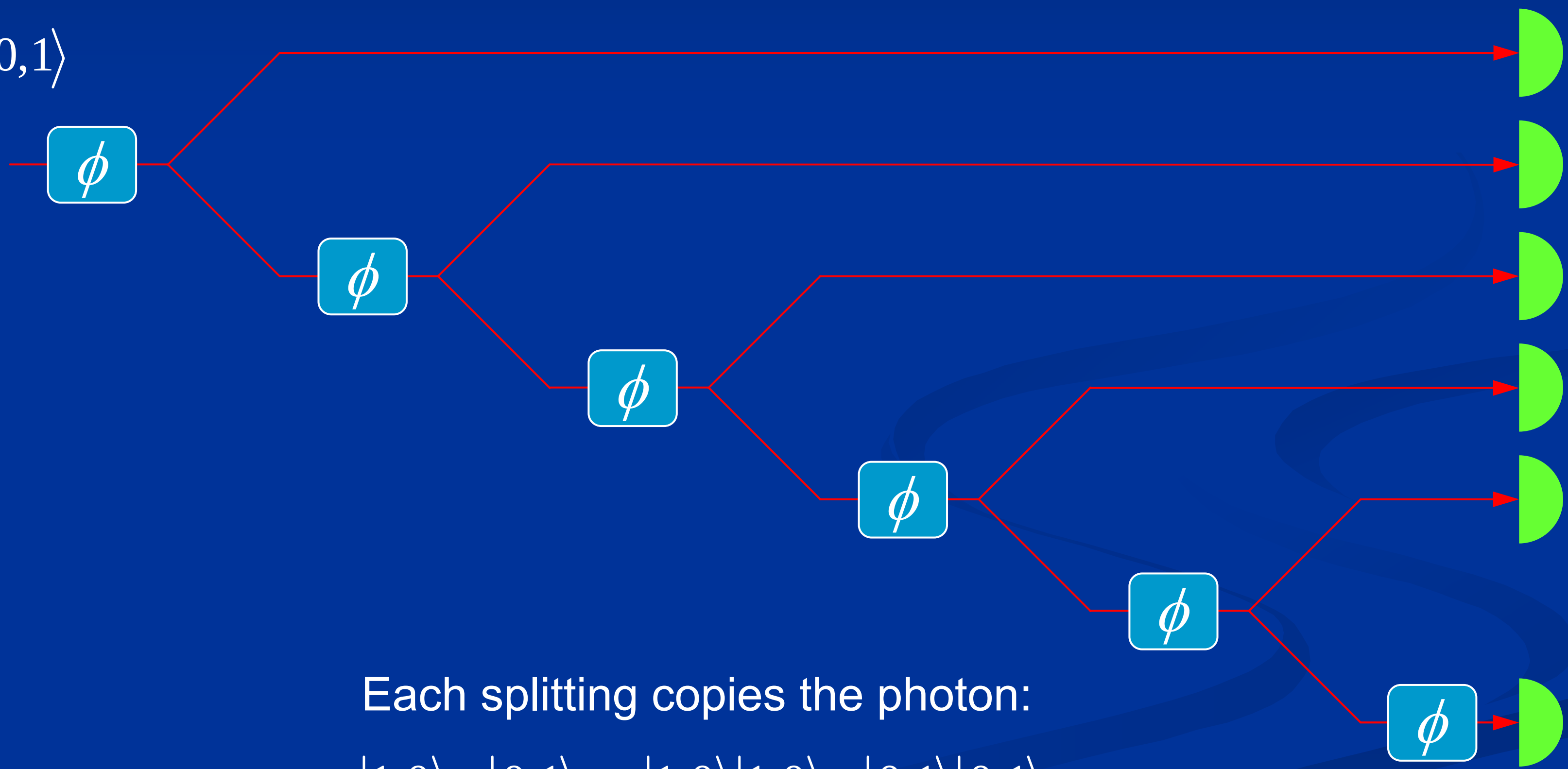
Electro-optic switches pass single photon through phase shift at times $t_1 - \Delta t$, $t_2 - \Delta t$, … $t_N - \Delta t$.

Equivalence of NOON states and multiple passes

$|1,0\rangle + |0,1\rangle$

Equivalence of NOON states and multiple passes

$|1,0\rangle + |0,1\rangle$



Each splitting copies the photon:

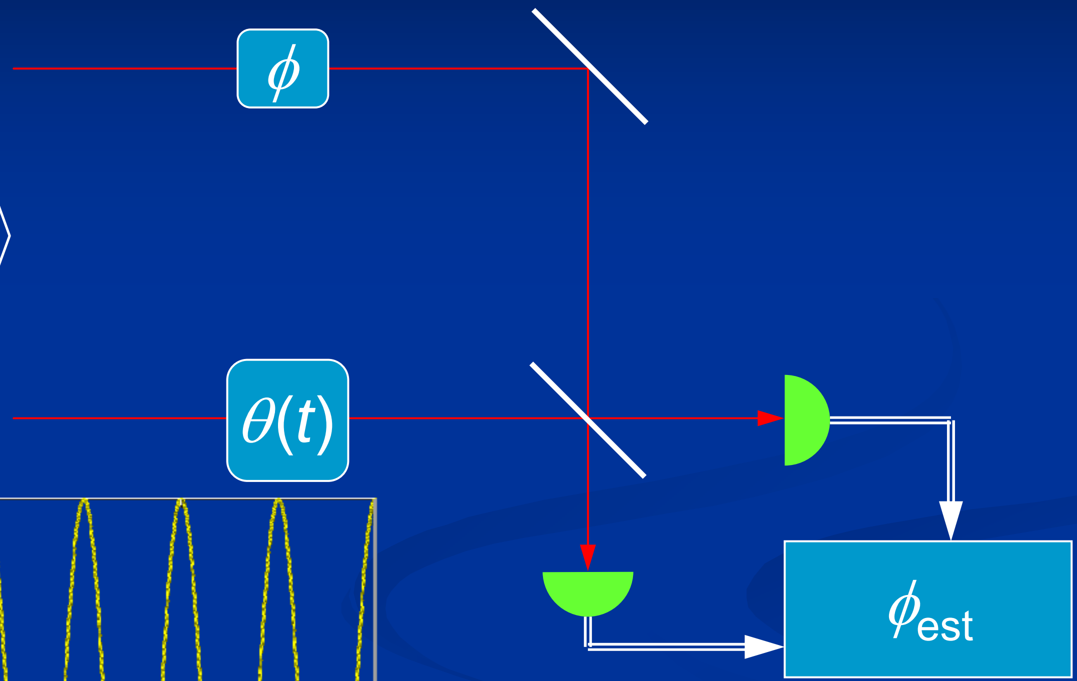
$$|1,0\rangle + |0,1\rangle \mapsto |1,0\rangle|1,0\rangle + |0,1\rangle|0,1\rangle$$

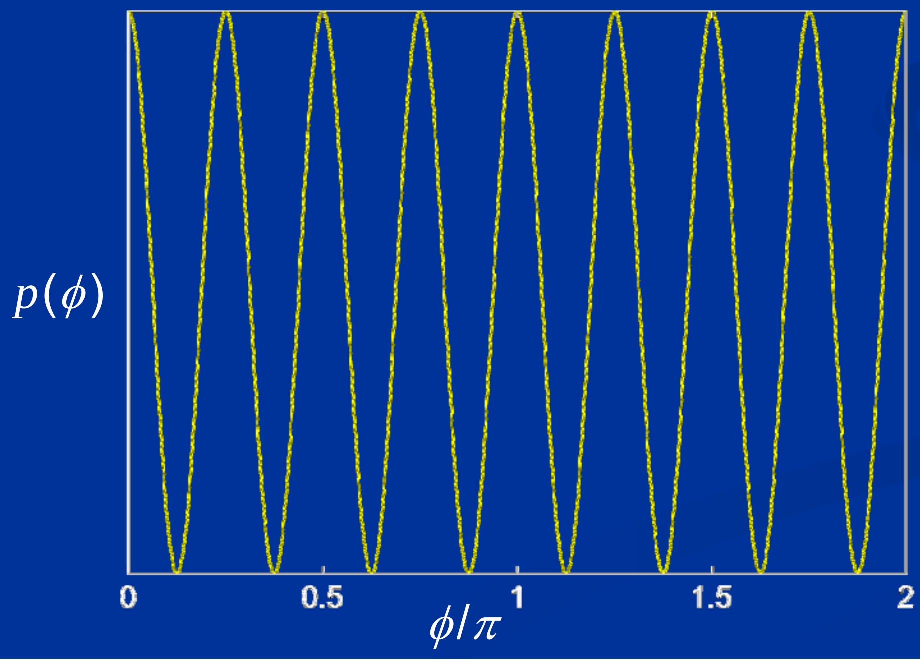
Equivalence of NOON states and multiple passes

Copy the photons at the beginning to get the NOON state.

Eliminating the fringes

$$|N,0\rangle + |0,N\rangle$$





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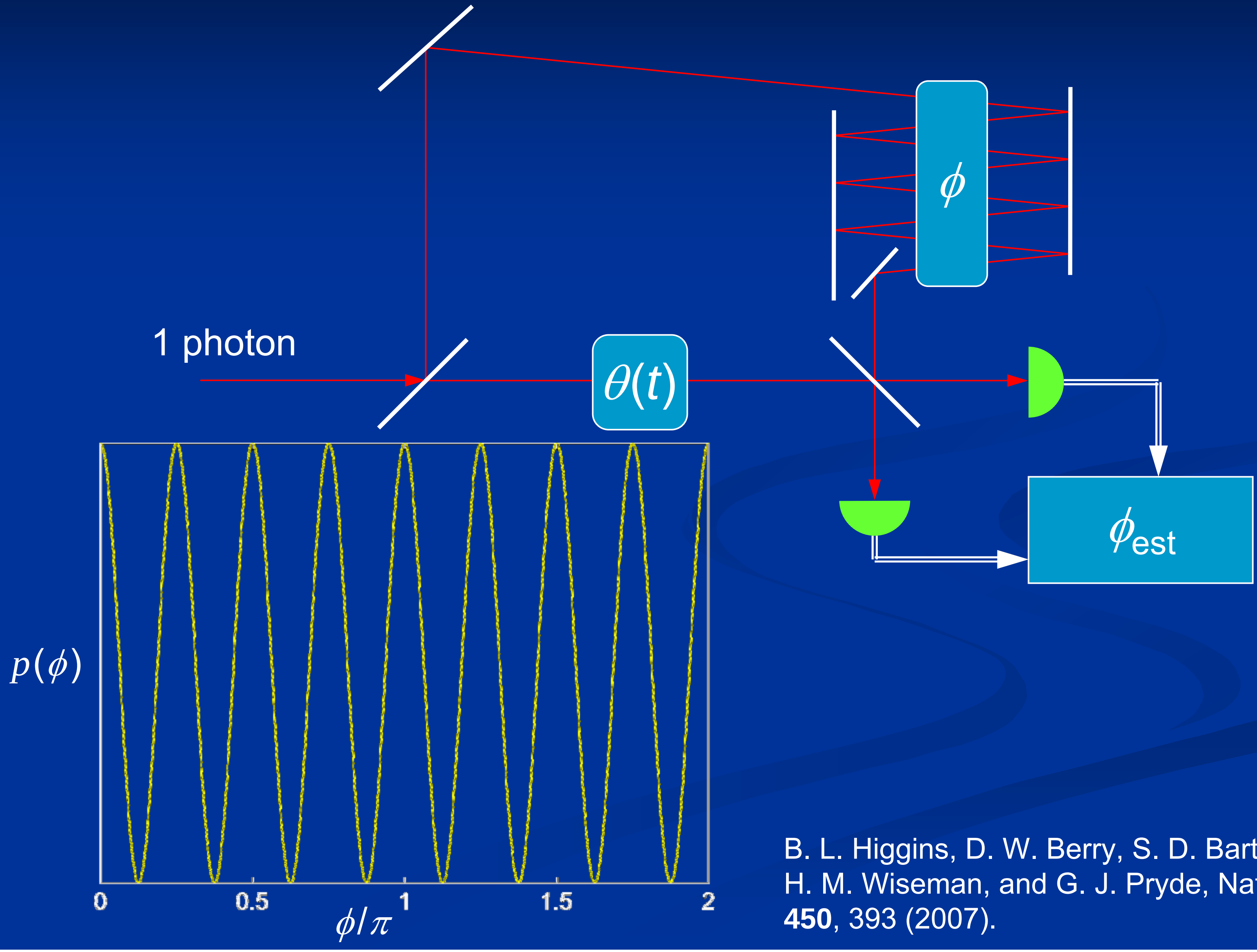
Eliminating the fringes



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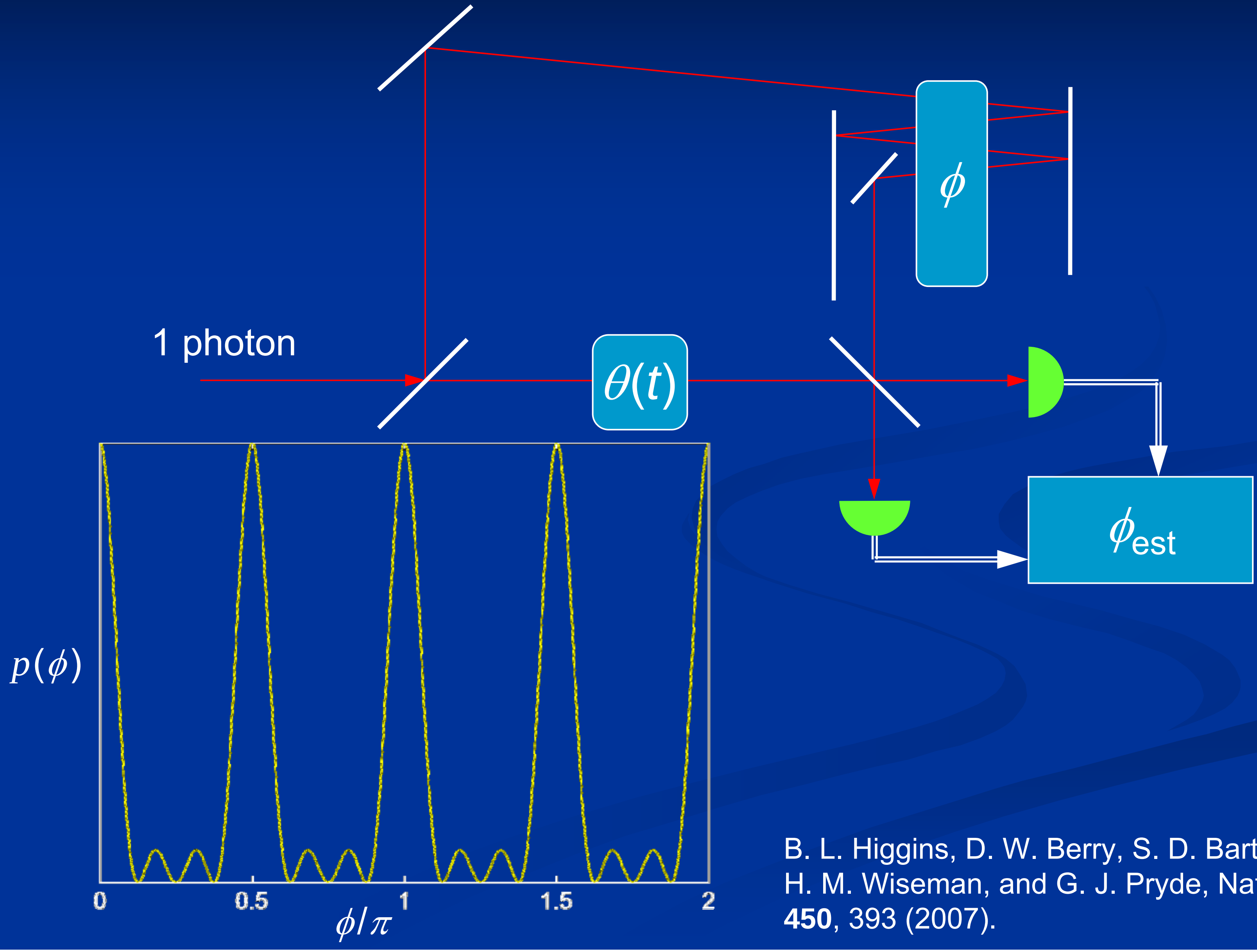

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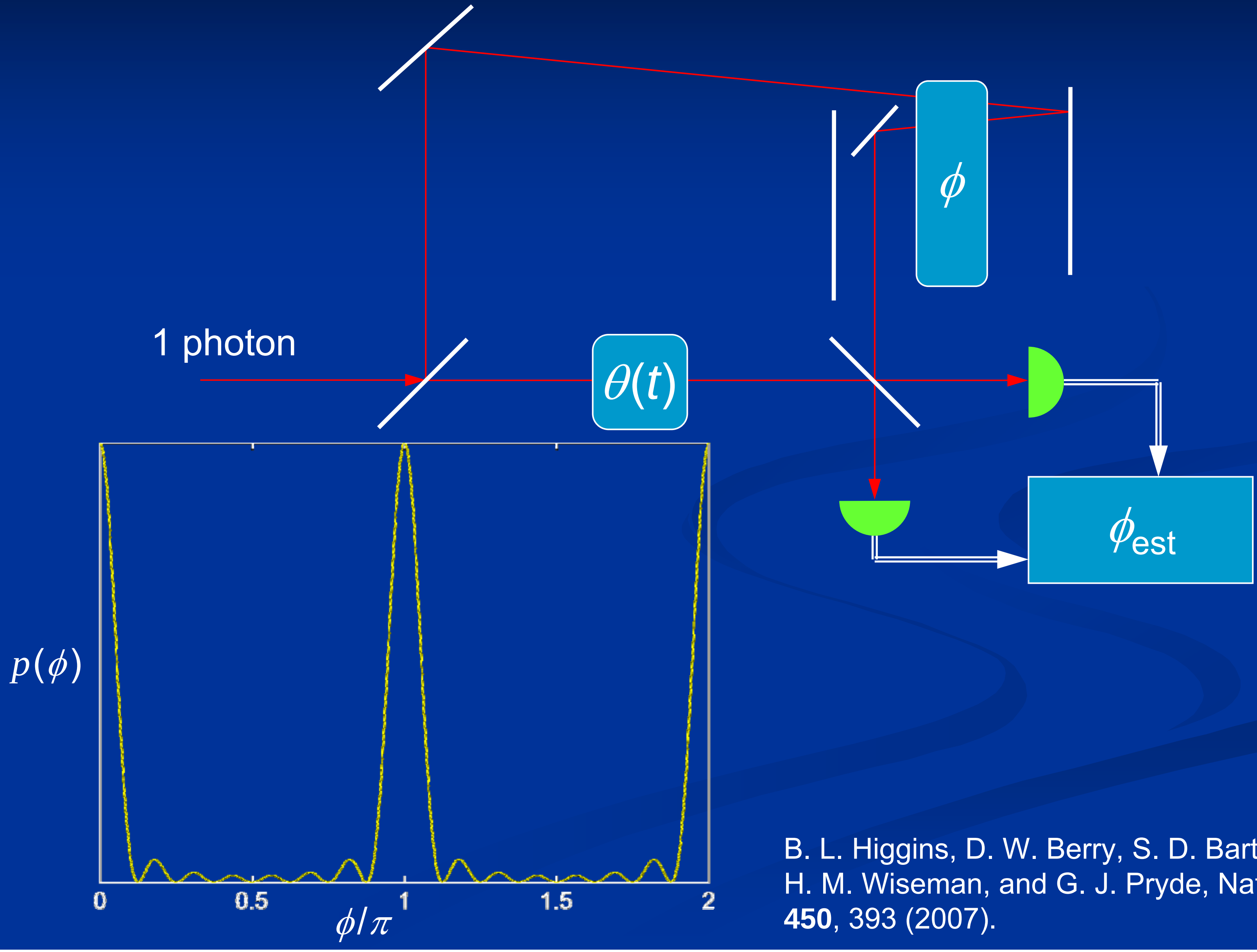

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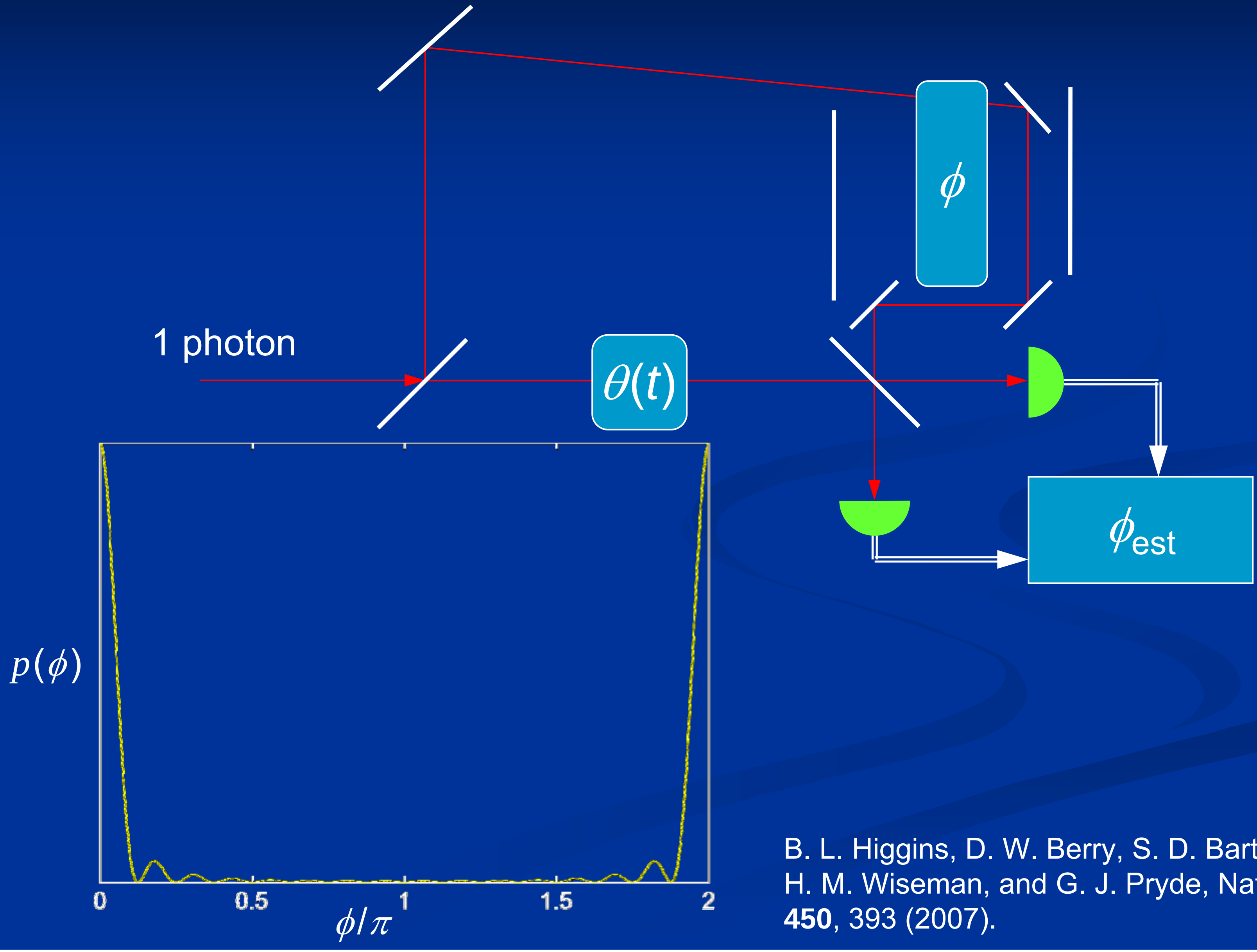


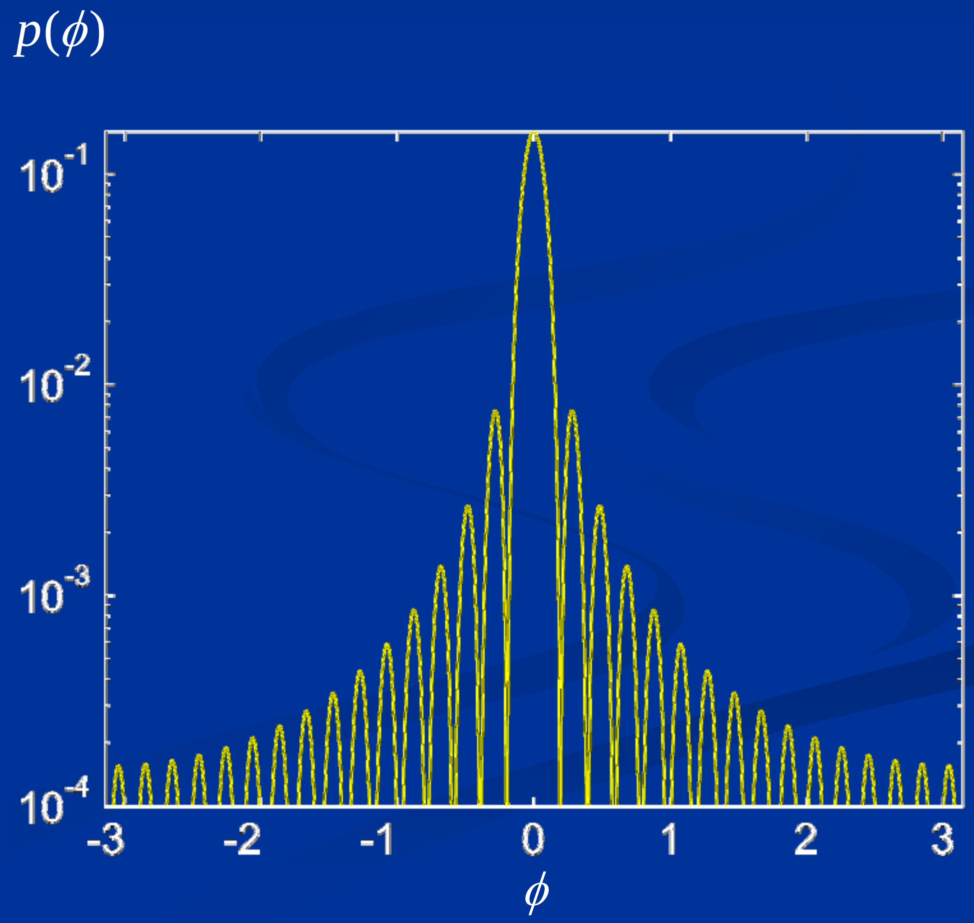
B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature **450**, 393 (2007).

The uncertainty

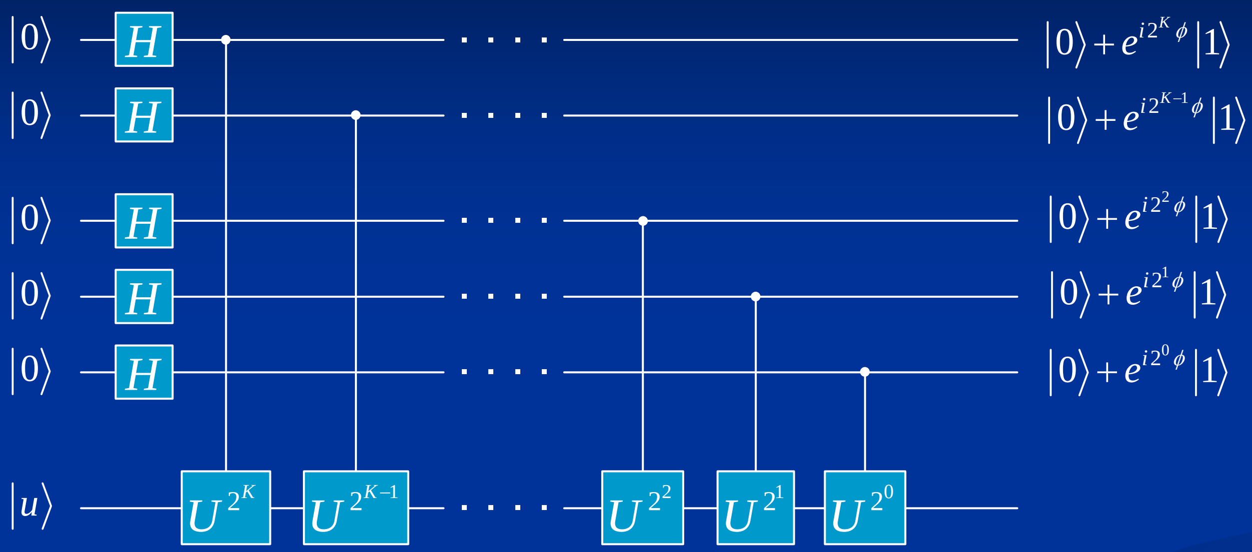
- The uncertainty is

$$\Delta\phi \approx \sqrt{2/N}$$

- This does not beat the SQL!
- The distribution has fat tails.



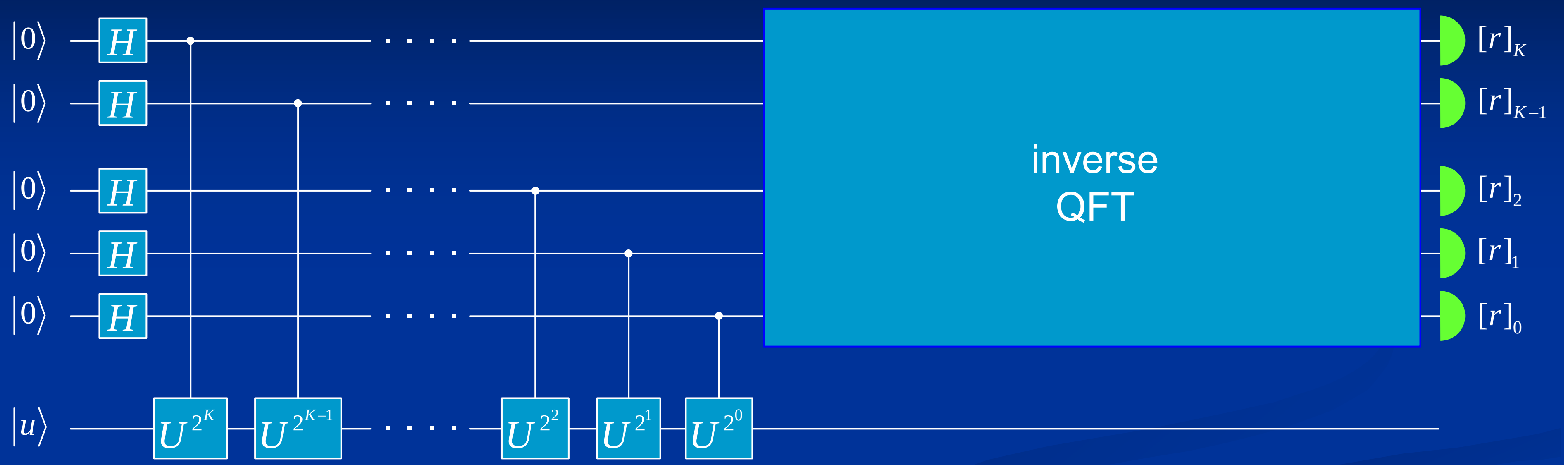
Inverse quantum Fourier transform



- The phase shifts are obtained from unitary U satisfying

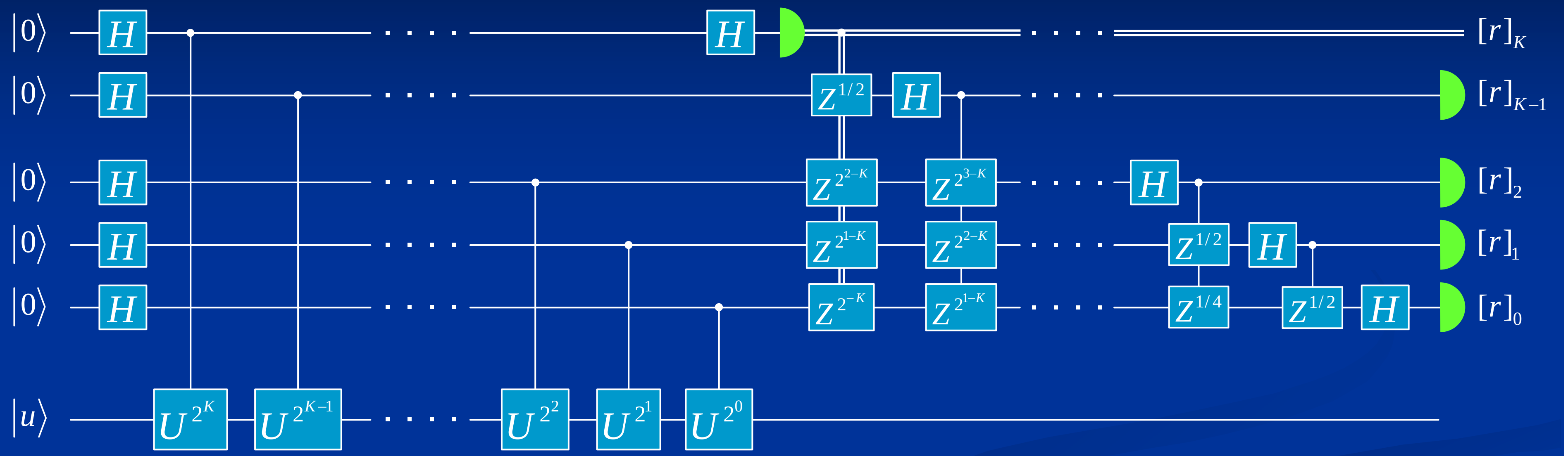
$$U\left|u\right\rangle = e^{i\phi}\left|u\right\rangle$$

Inverse quantum Fourier transform



- Provided ϕ is of the form $\phi = \pi r / 2^K$, the inverse quantum Fourier transform gives the bits of r at the output.

Inverse quantum Fourier transform



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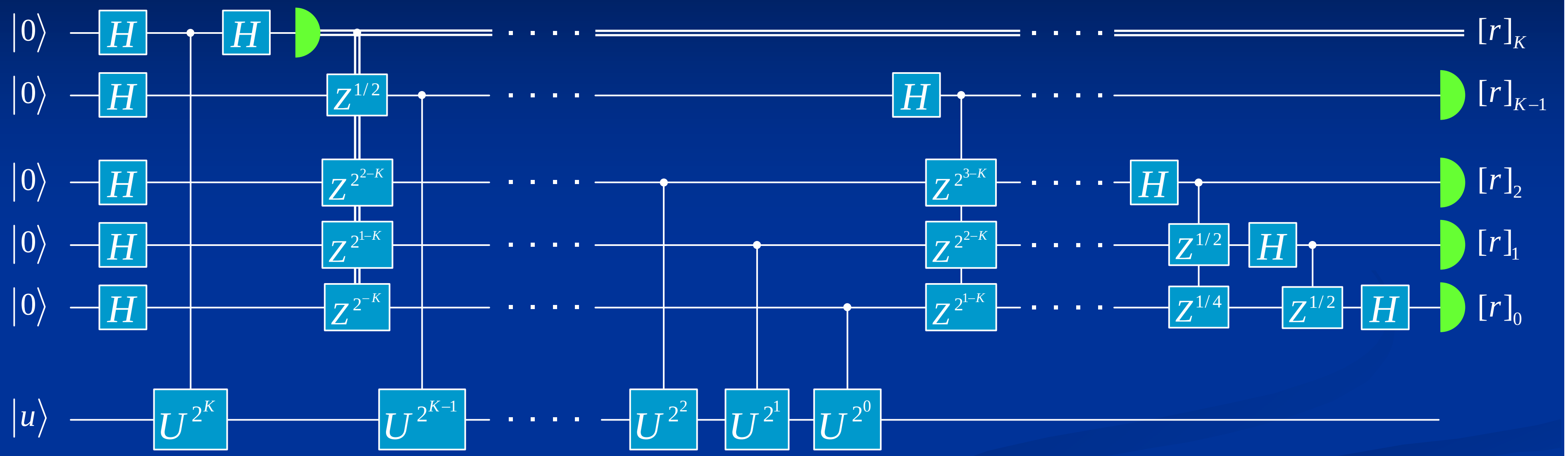

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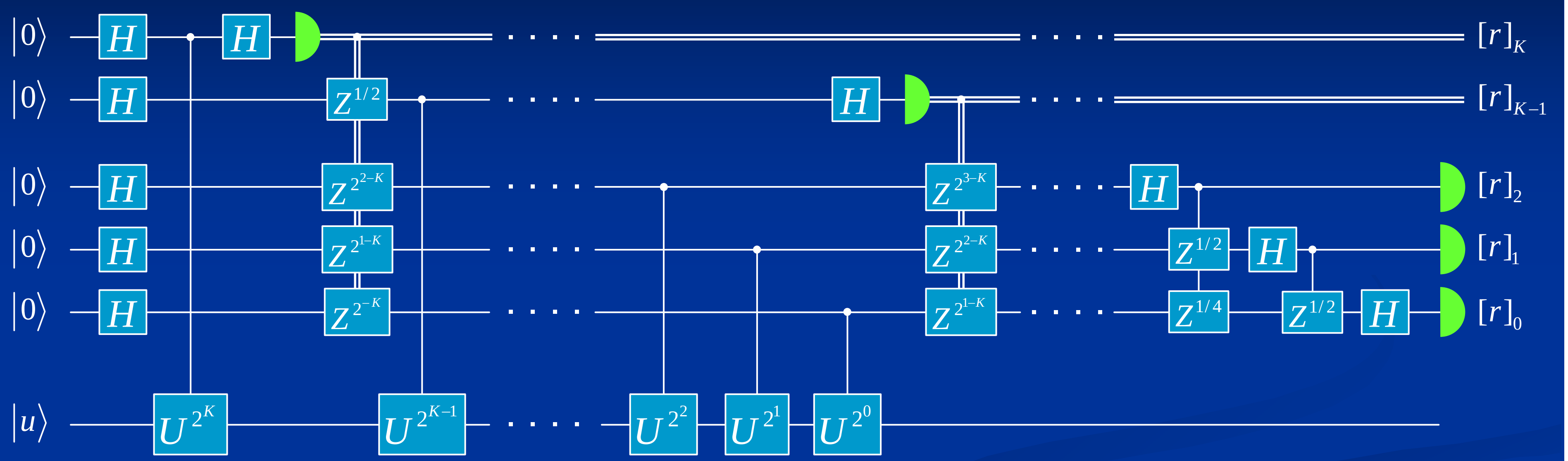
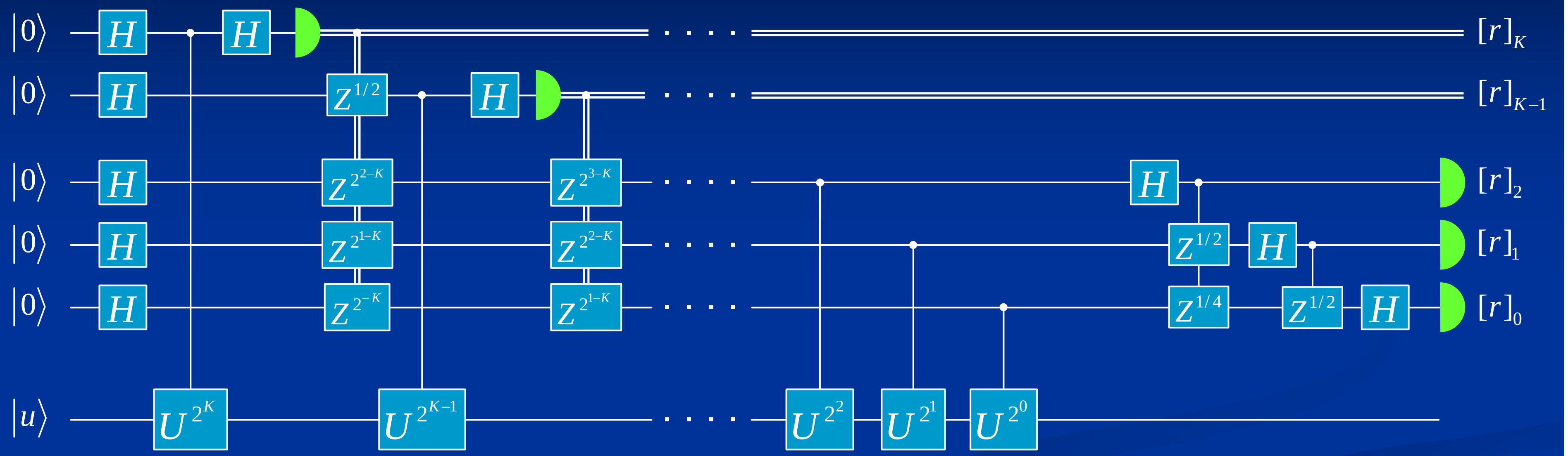


Inverse quantum Fourier transform



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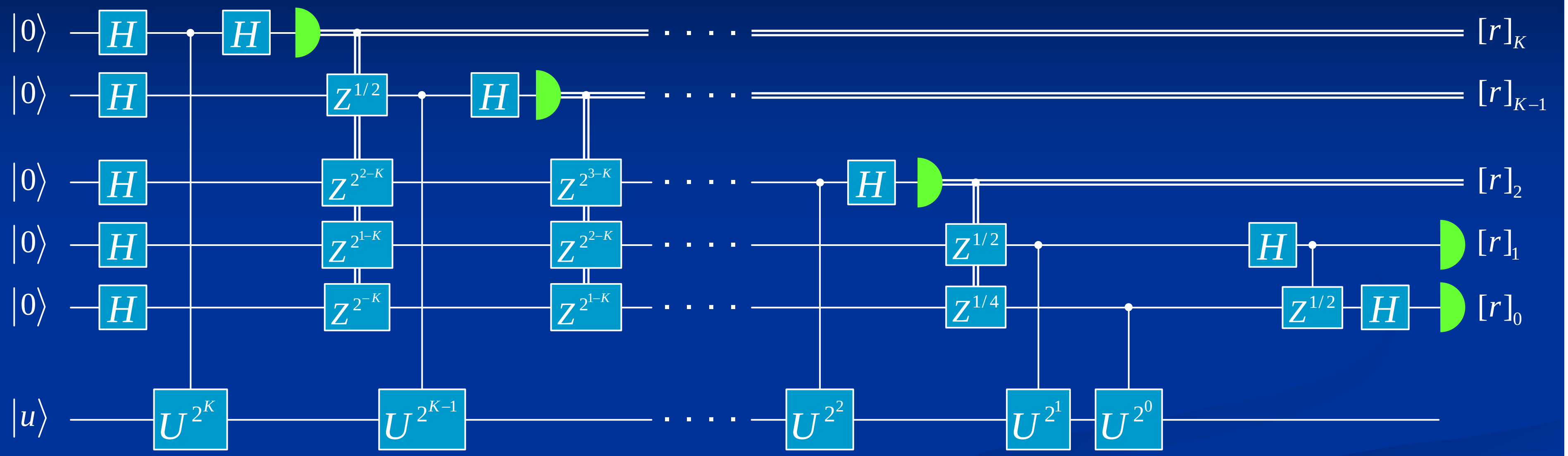
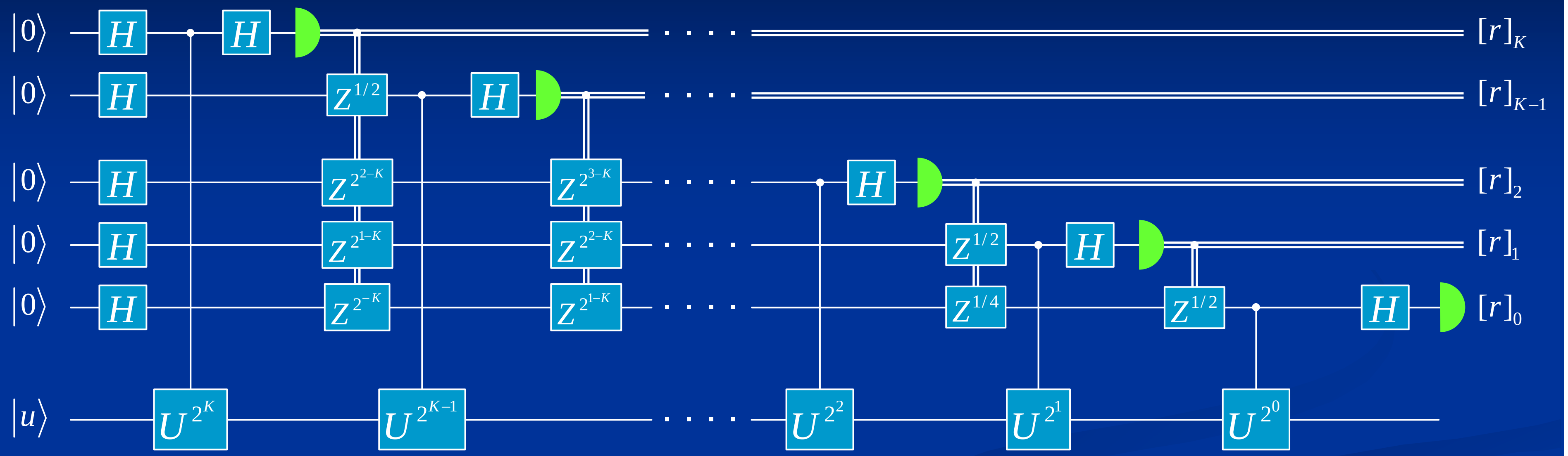
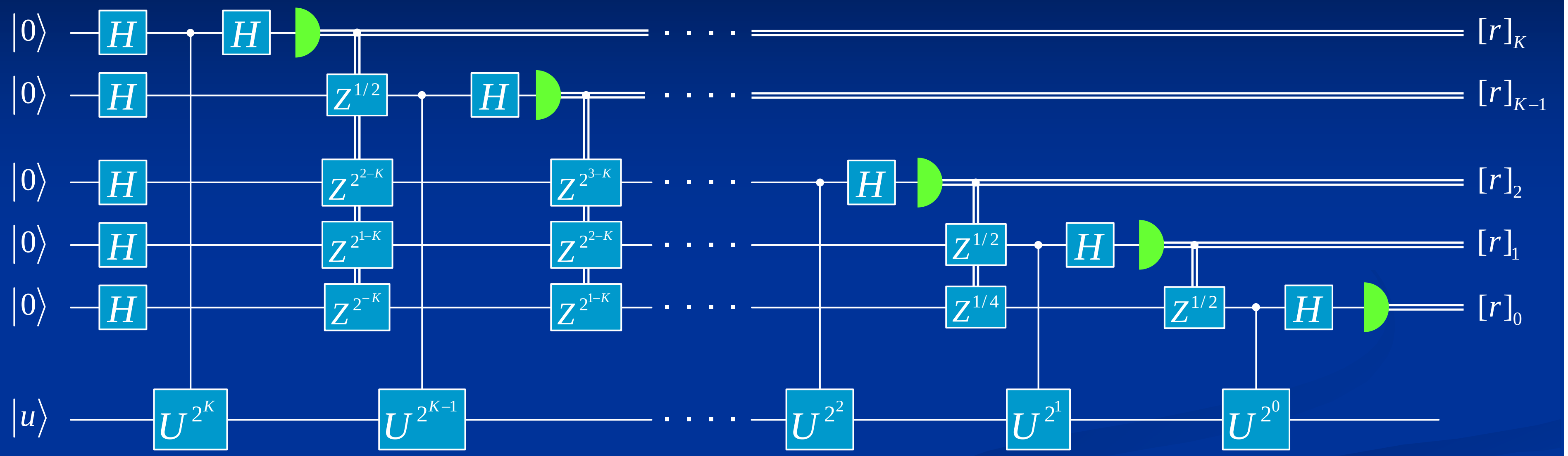


Inverse quantum Fourier transform



Inverse quantum Fourier transform



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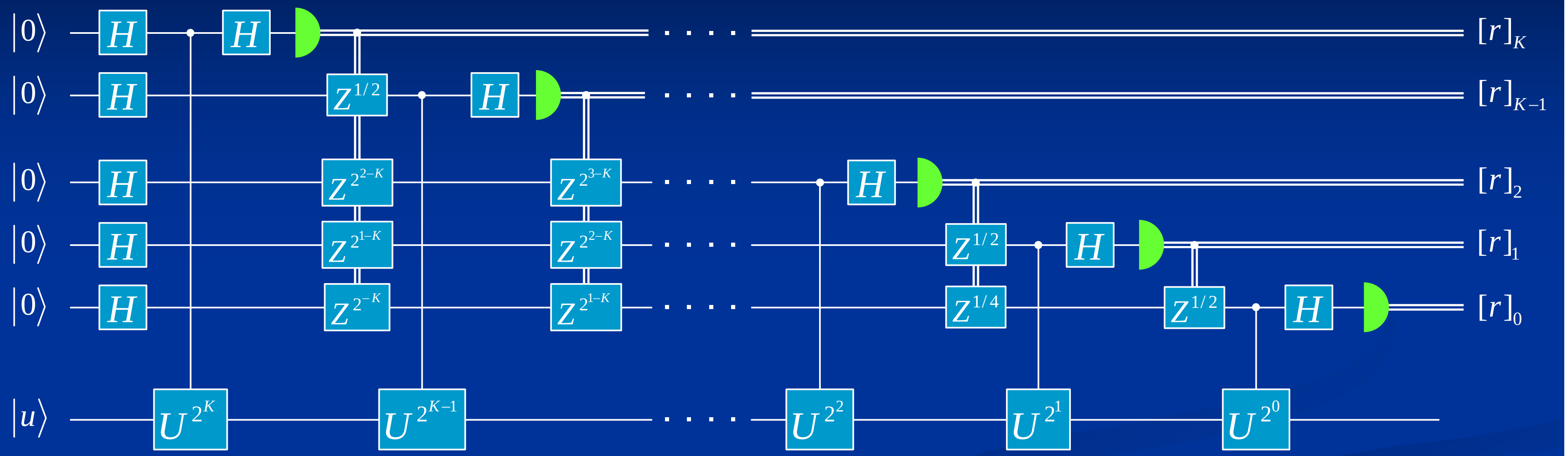
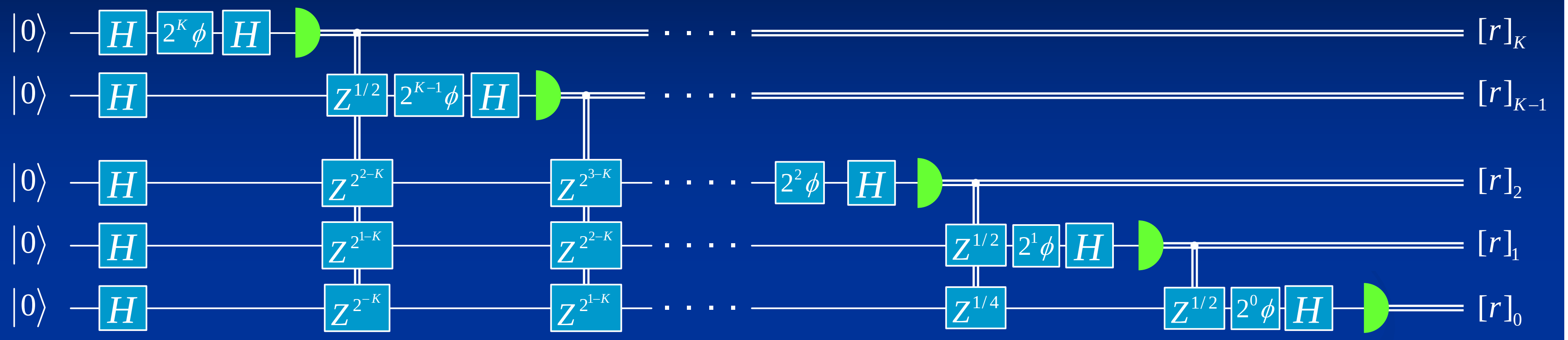


Inverse quantum Fourier transform



1. The qubits are dual-rail single photons.
2. The Hadamard is a beam splitter.
3. The controlled unitaries are the unknown phase in the interferometer.
4. The controlled phase operations are feedback to the phase $\theta(t)$.
5. The operations may be performed in sequence to reuse the same interferometer.

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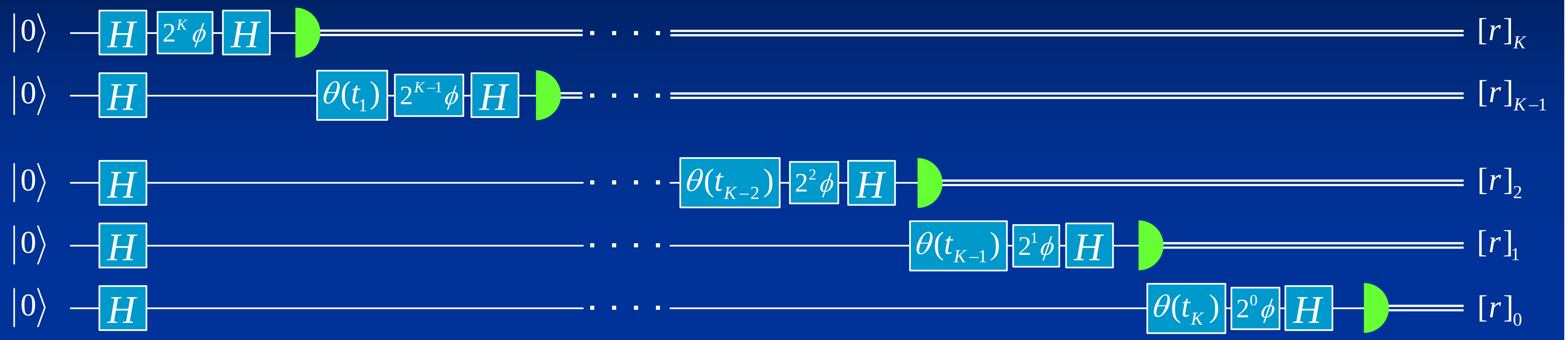

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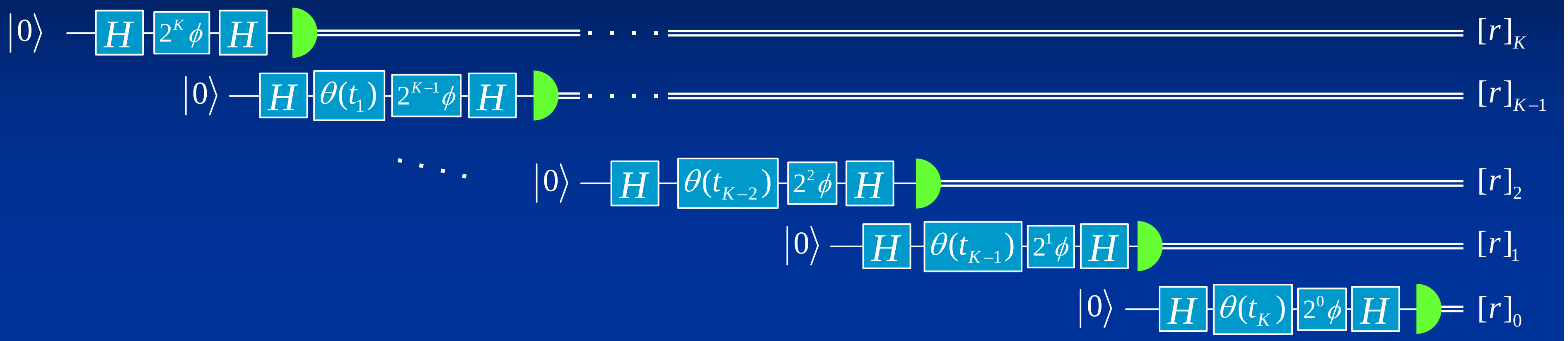

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The equivalent state

- The sequence of different numbers of passes is equivalent to a tensor product of NOON states:

$$\left(\left|2^K,0\right\rangle+\left|0,2^K\right\rangle\right)\otimes\ldots\otimes\left(\left|2^1,0\right\rangle+\left|0,2^1\right\rangle\right)\otimes\left(\left|1,0\right\rangle+\left|0,1\right\rangle\right)$$

- This is equivalent to

$$\sum_{n=0}^{N}\left|n,N-n\right\rangle$$

for $N = 2^{K+1}-1$.

How to create the input state?

Two problems:

1. Using multiple passes of single photons we obtain an effective state of the form

$$\sum_{n=0}^{N} \psi_n |n\rangle |N-n\rangle$$

even though the actual state is just single photons.

2. The input mode does not need to be long – we can send photons through one at a time.

What do we need for theoretical-limit scaling?

- The squared error is approximately (for real ψ_n)

$$\Delta\phi^2 \approx \sum_{n=-1}^{N} \left(\psi_n - \psi_{n+1}\right)^2$$

where we add the dummy state coefficients $\psi_{-1} = \psi_{N+1} = 0$.

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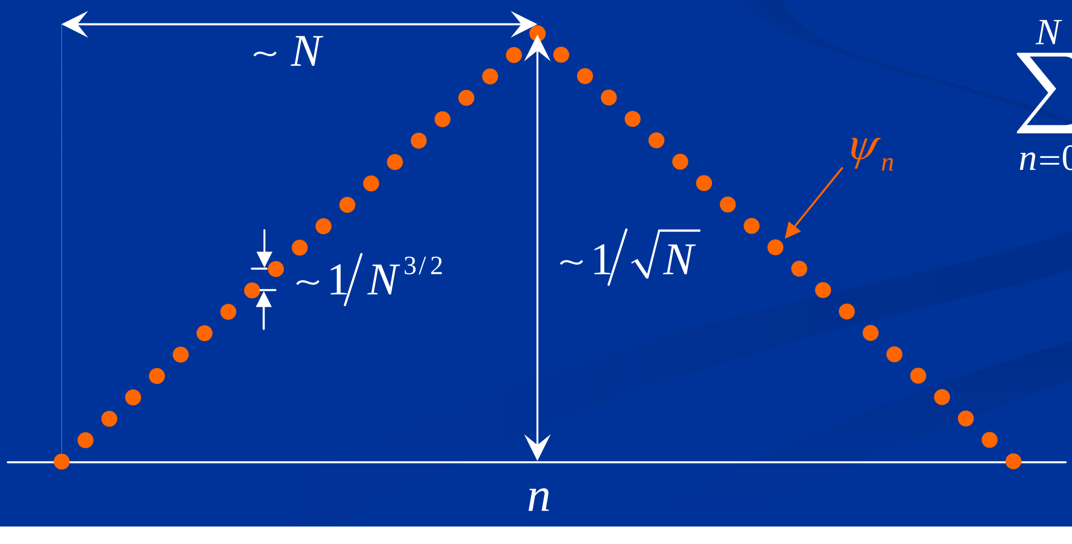
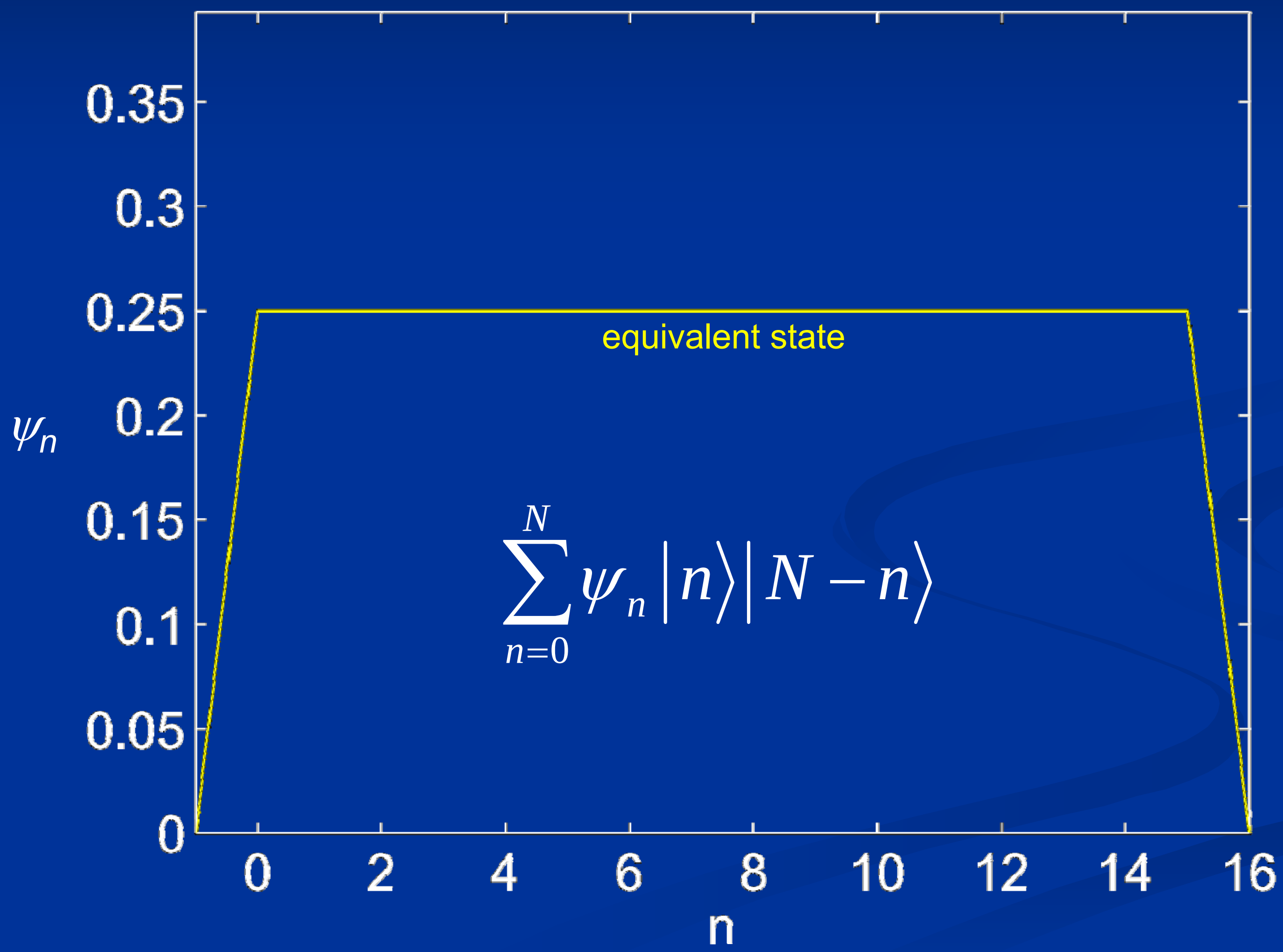
where we add the dummy state coefficients $\psi_{-1} = \psi_{N+1} = 0$.
- For scaling at the theoretical limit we need $\psi_{n+1} - \psi_n \propto 1/N^{3/2}$.
- The state coefficients just need to increase then decrease in a gradual way.



$$\sum_{n=0}^{N} \psi_n |n\rangle |N-n\rangle$$

The equivalent state



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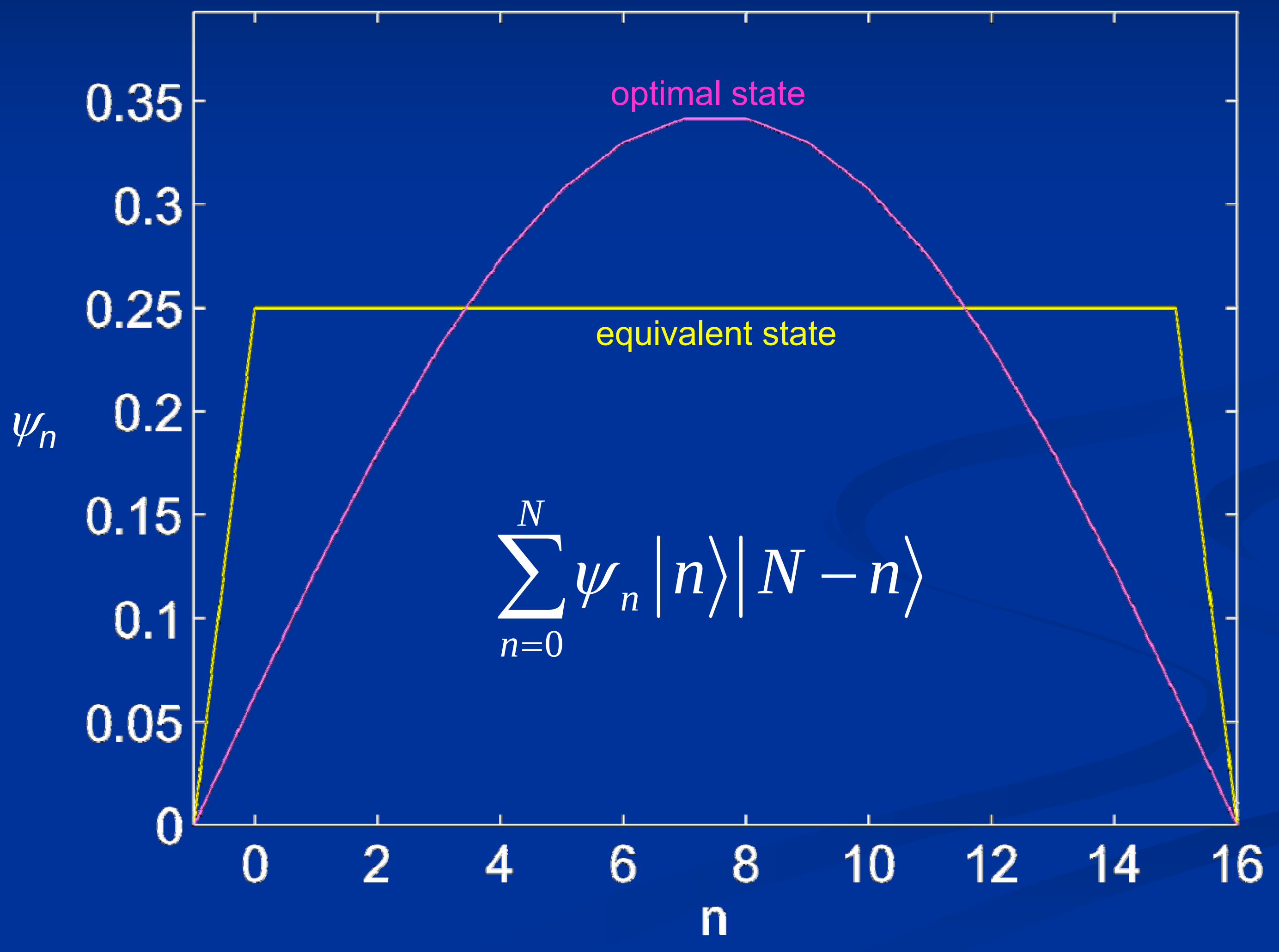
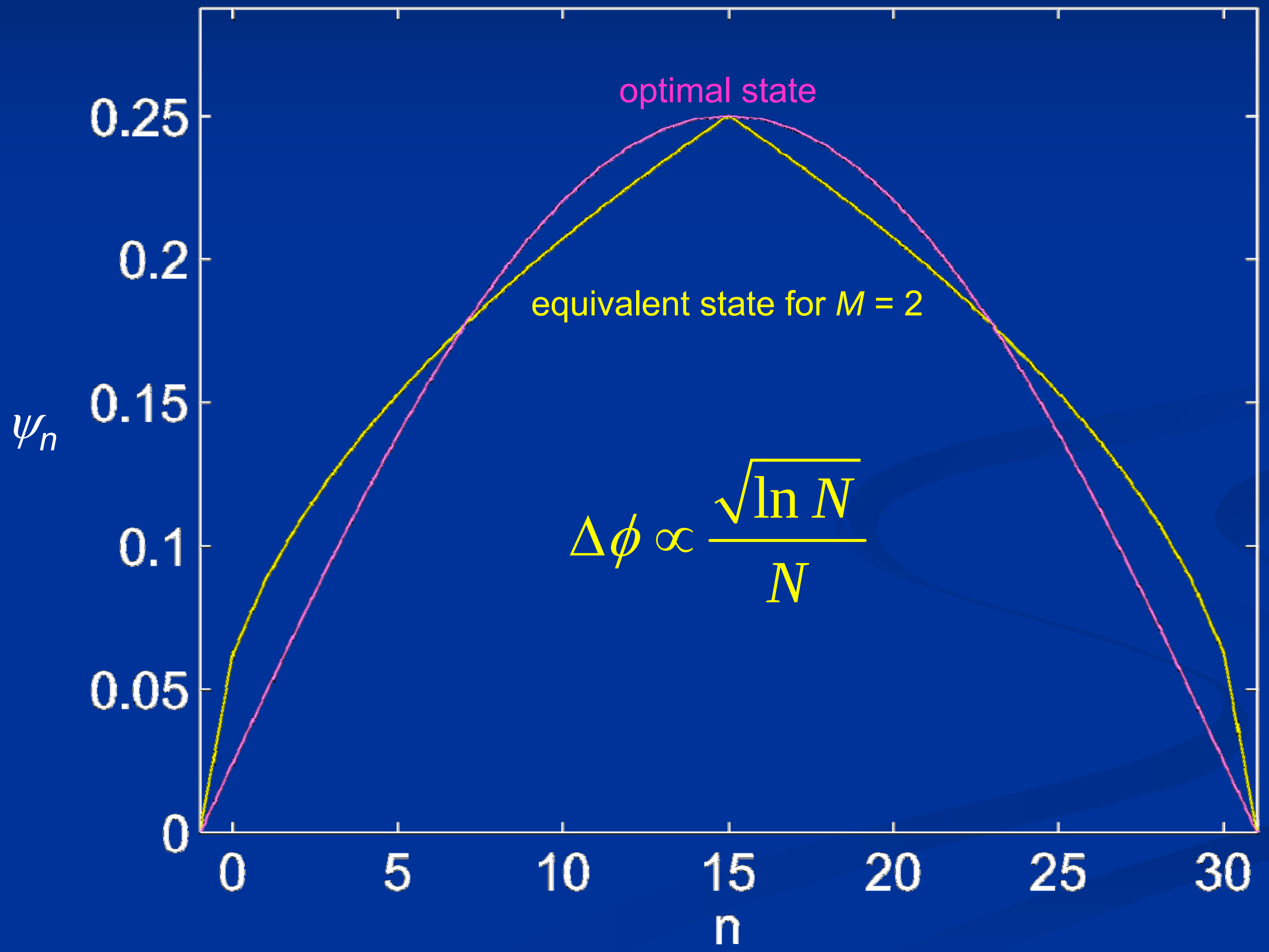
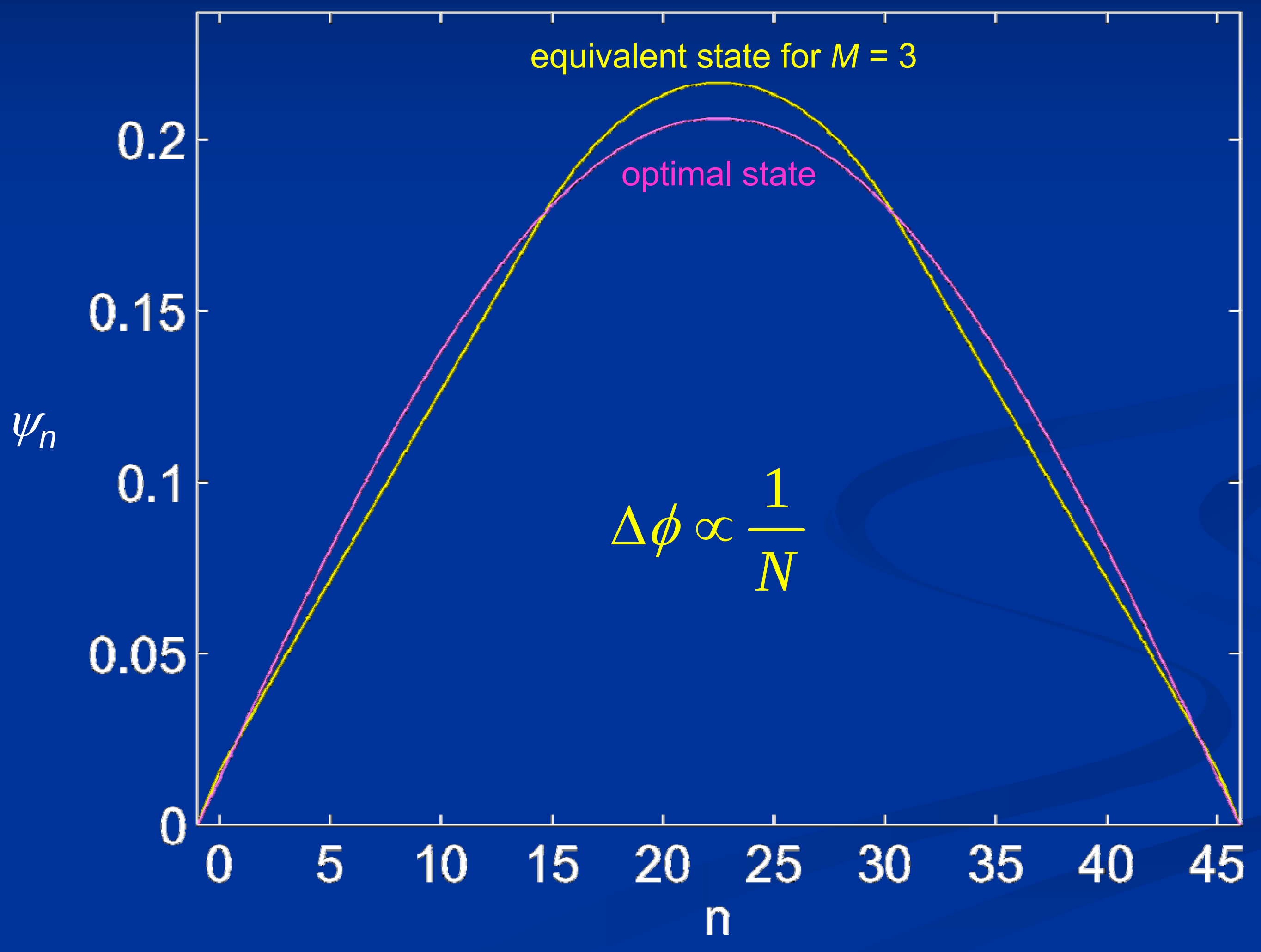


The equivalent state



The equivalent state



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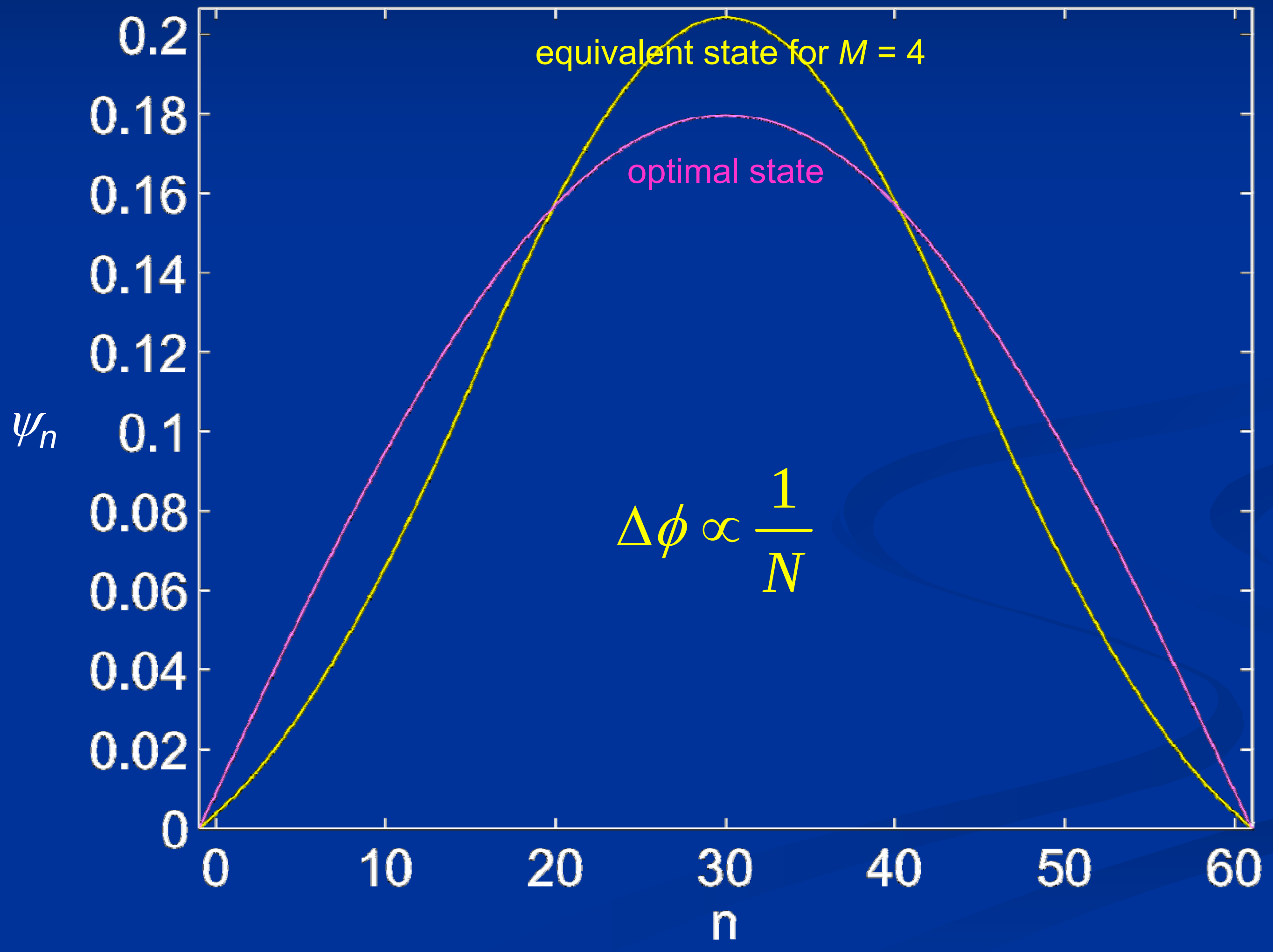
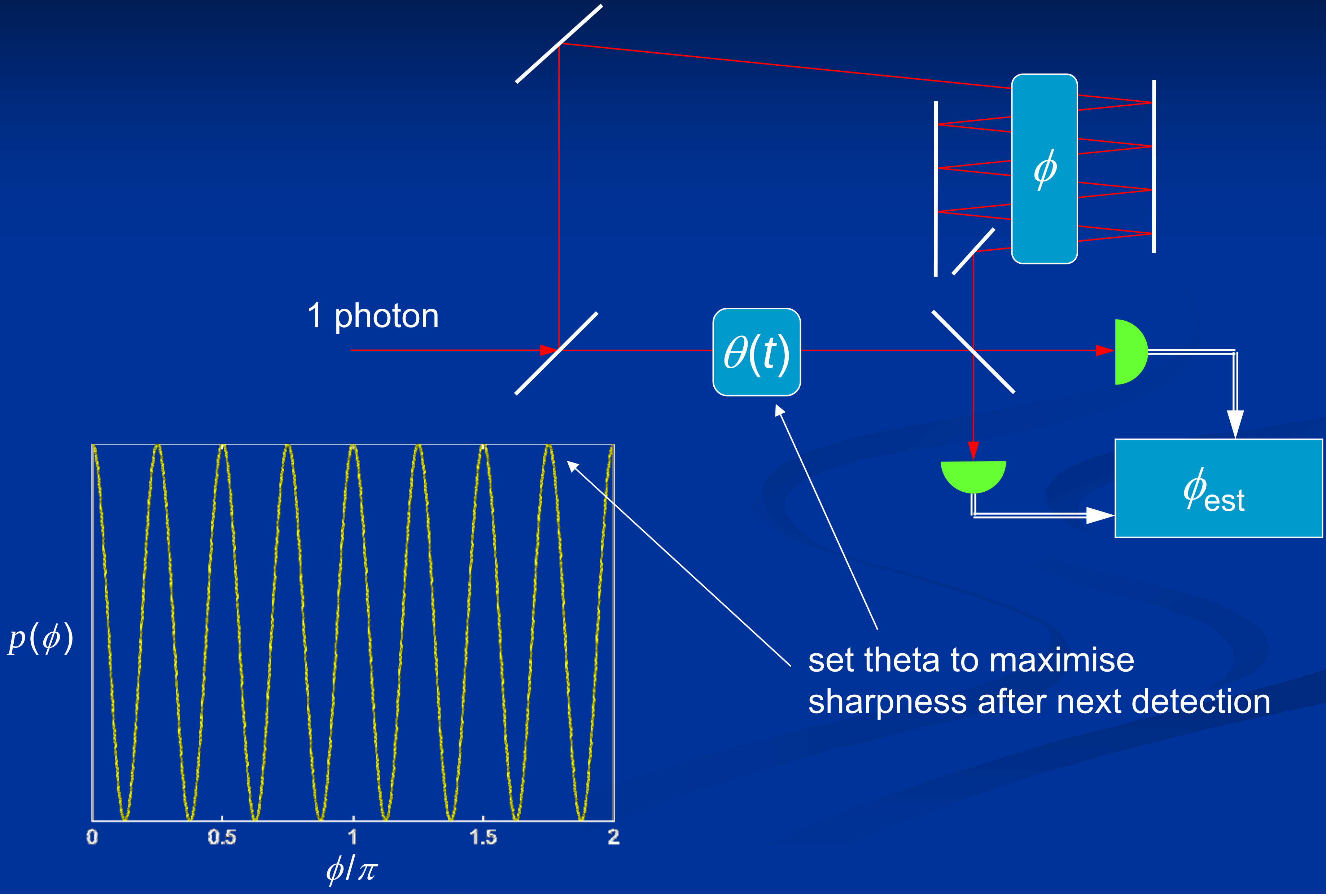
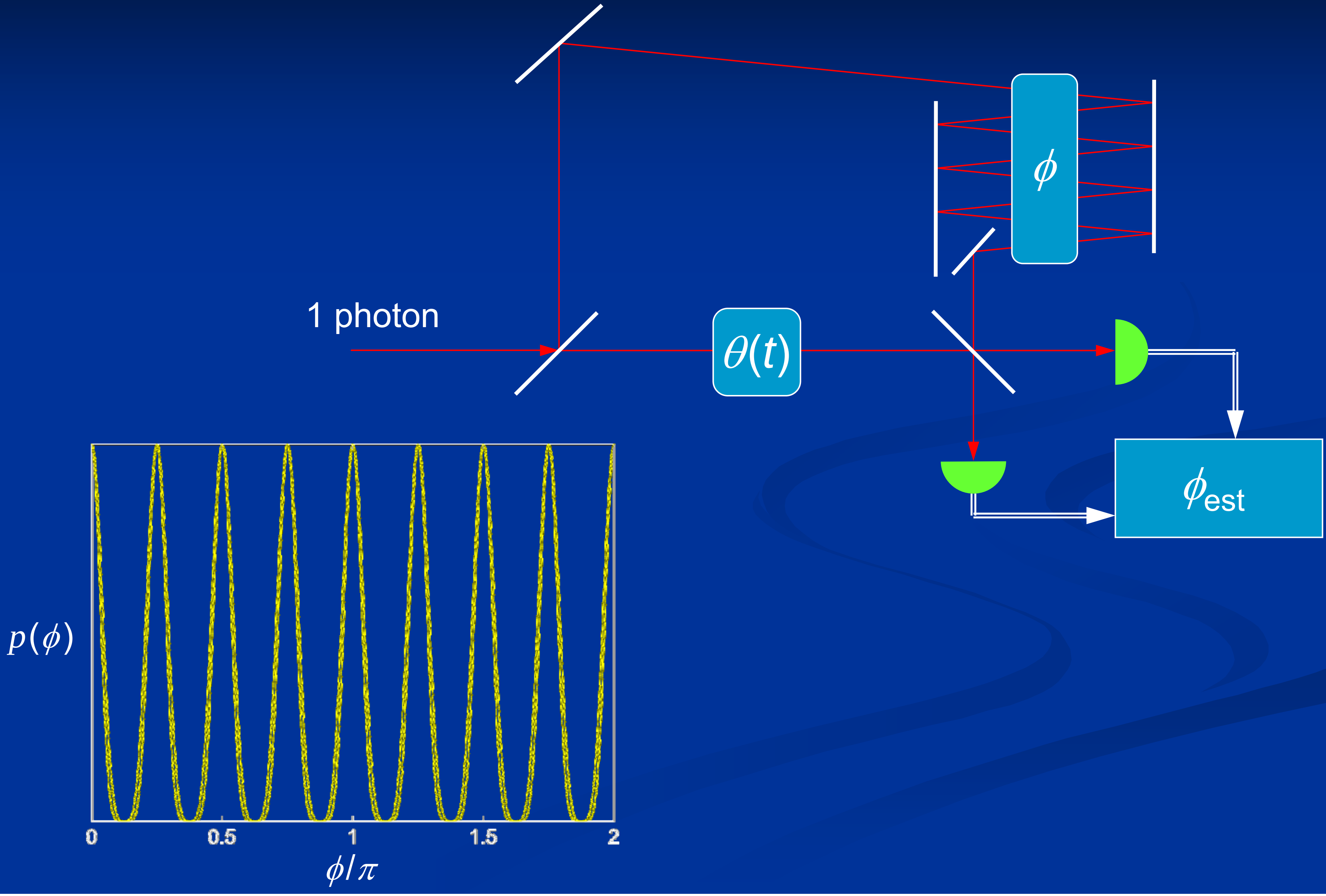


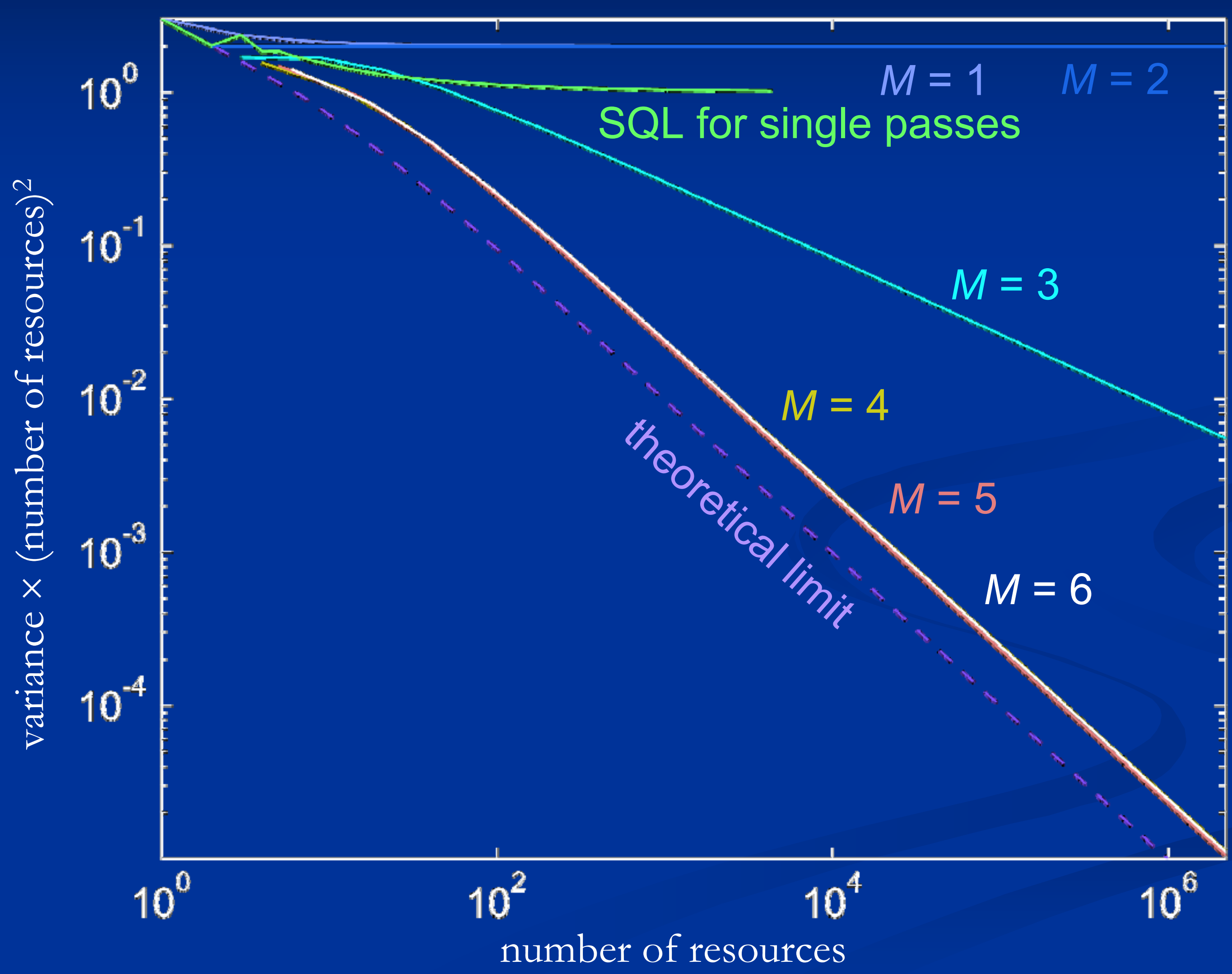
What about the feedback?



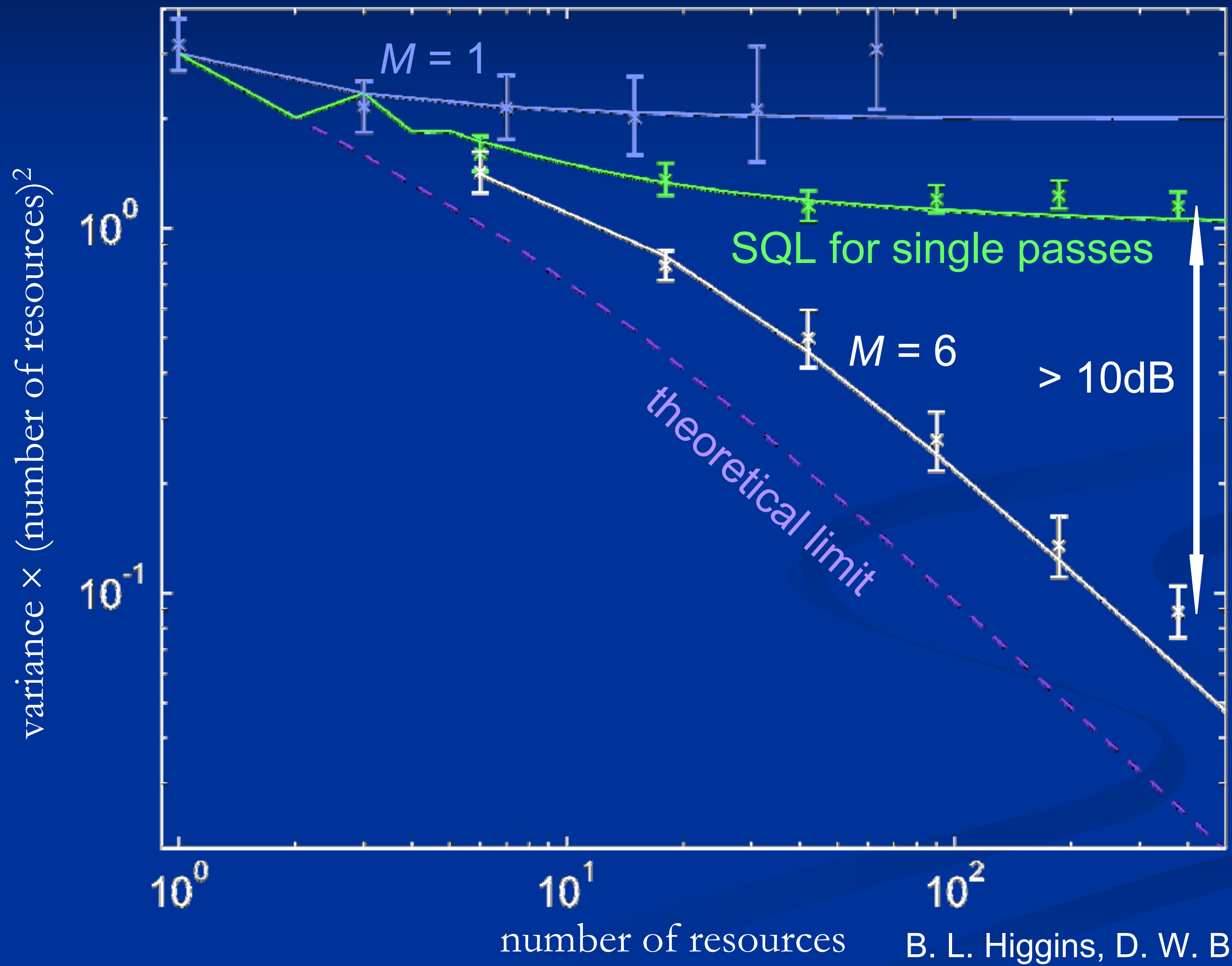
What about the feedback?



Predicted variances



Experimental results



B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature **450**, 393 (2007).

Optical interferometry

- Theoretical limit
- Squeezed states[1]
- NOON states[2]
- Theoretical-limit adaptive measurements[3]
- Theoretical-limit **non**adaptive measurements[4]
- Hybrid measurements[4]

[1] C. M. Caves, Phys. Rev. D **23**, 1693 (1981).
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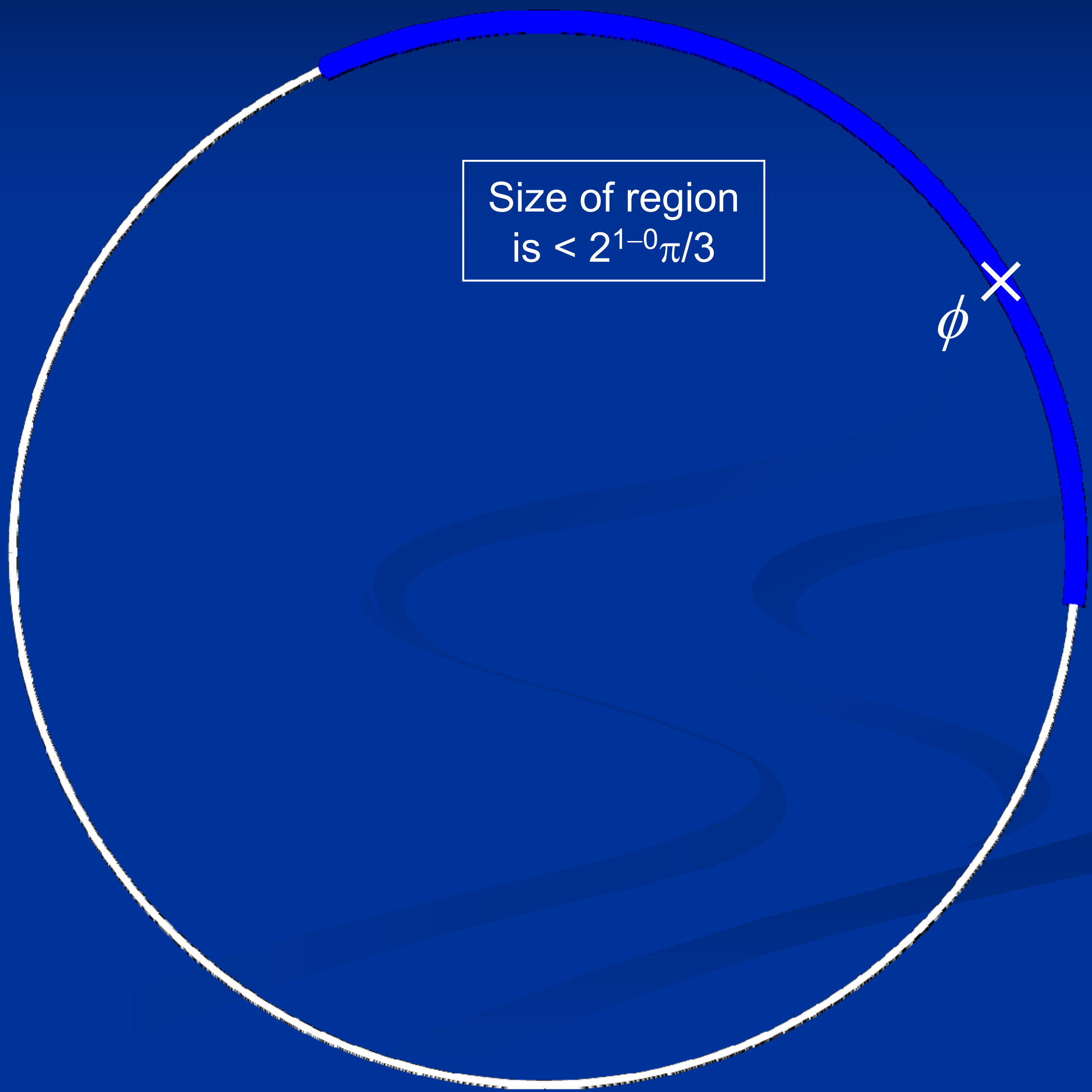
[4] B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, e-print: 0809.3308 (2008).

Nonadaptive measurements

Nonadaptive measurements

0. Perform enough measurements with $2^0 = 1$ pass to ensure that the system phase is in the blue region with high probability.



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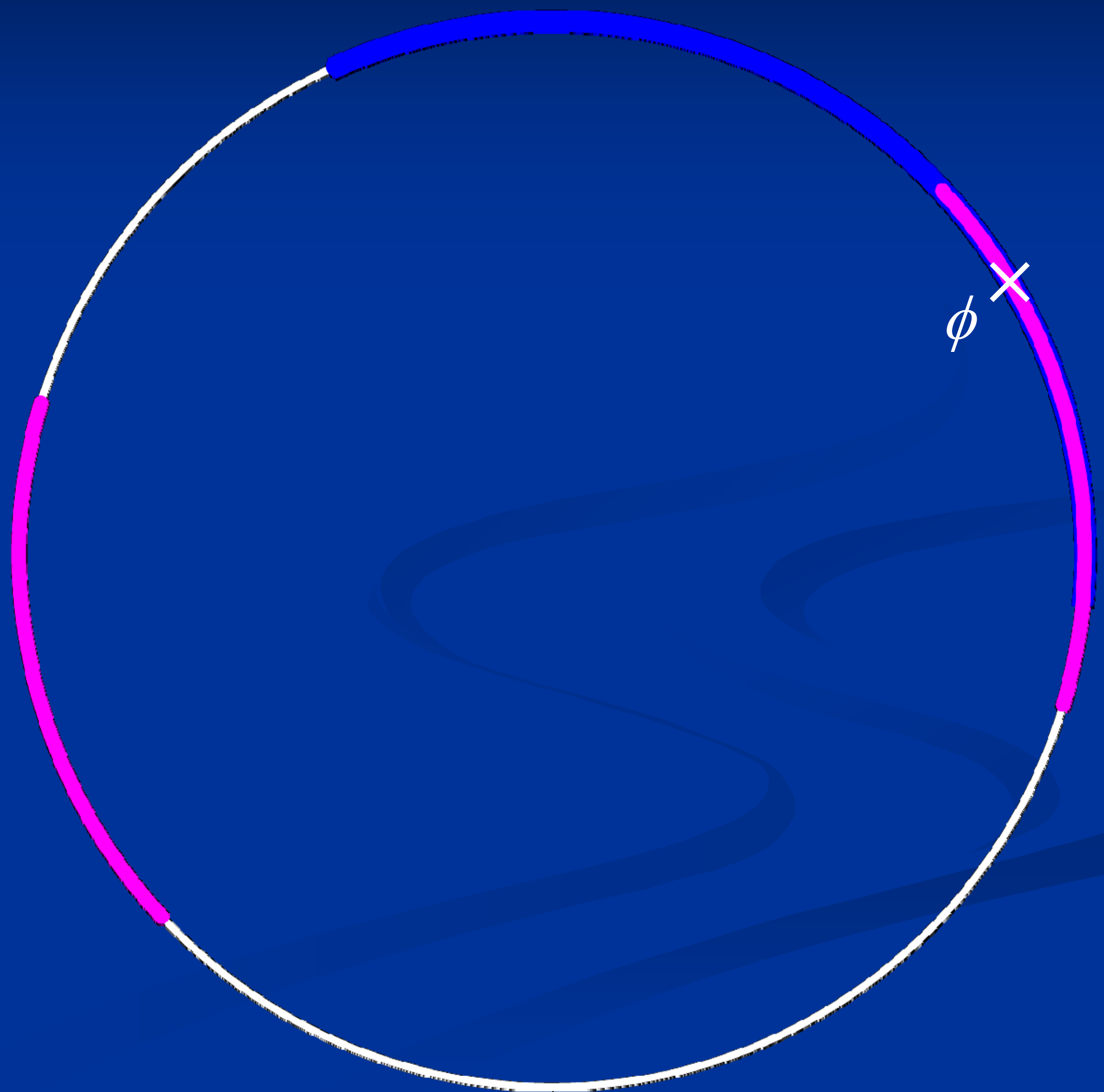

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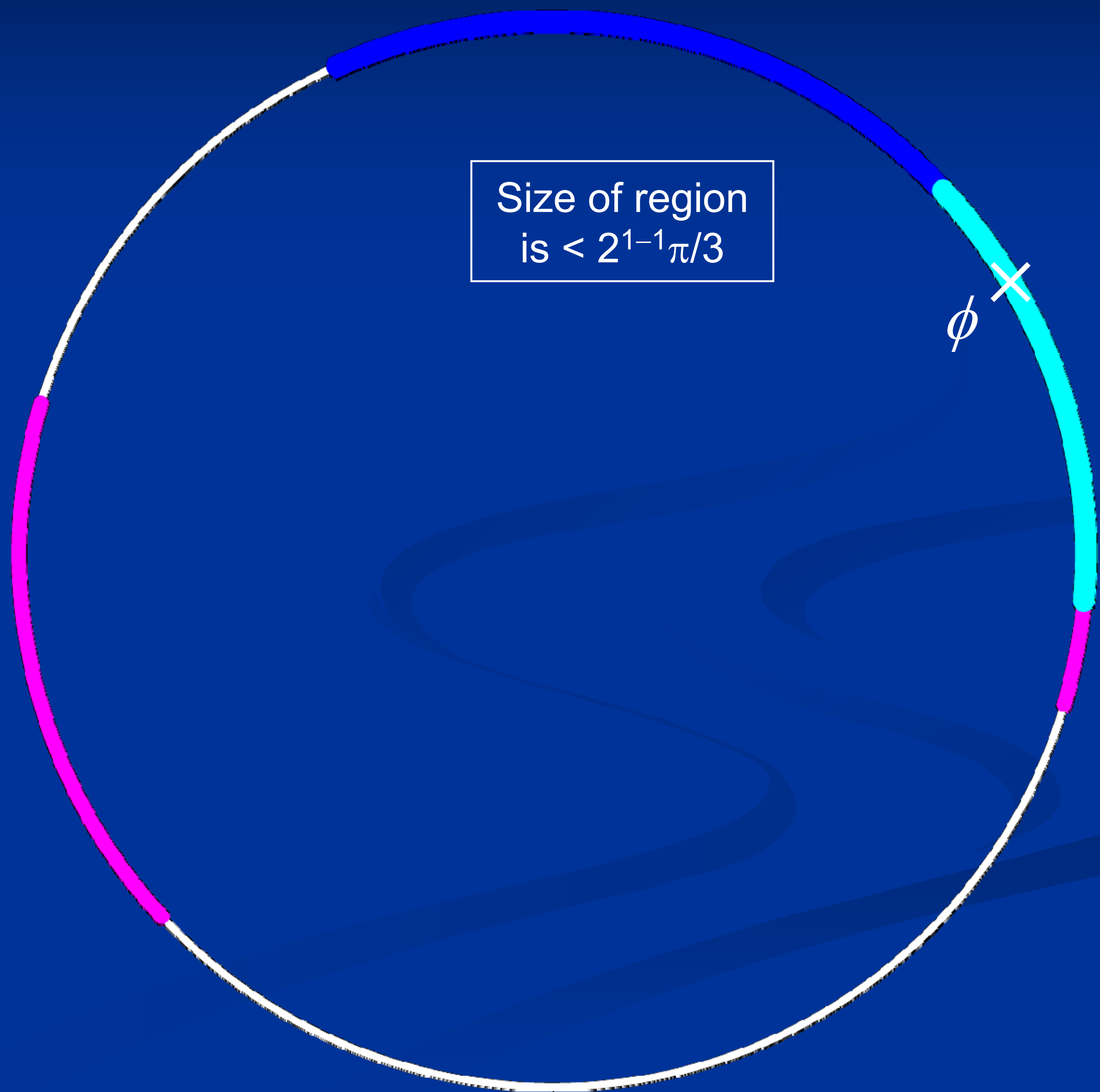

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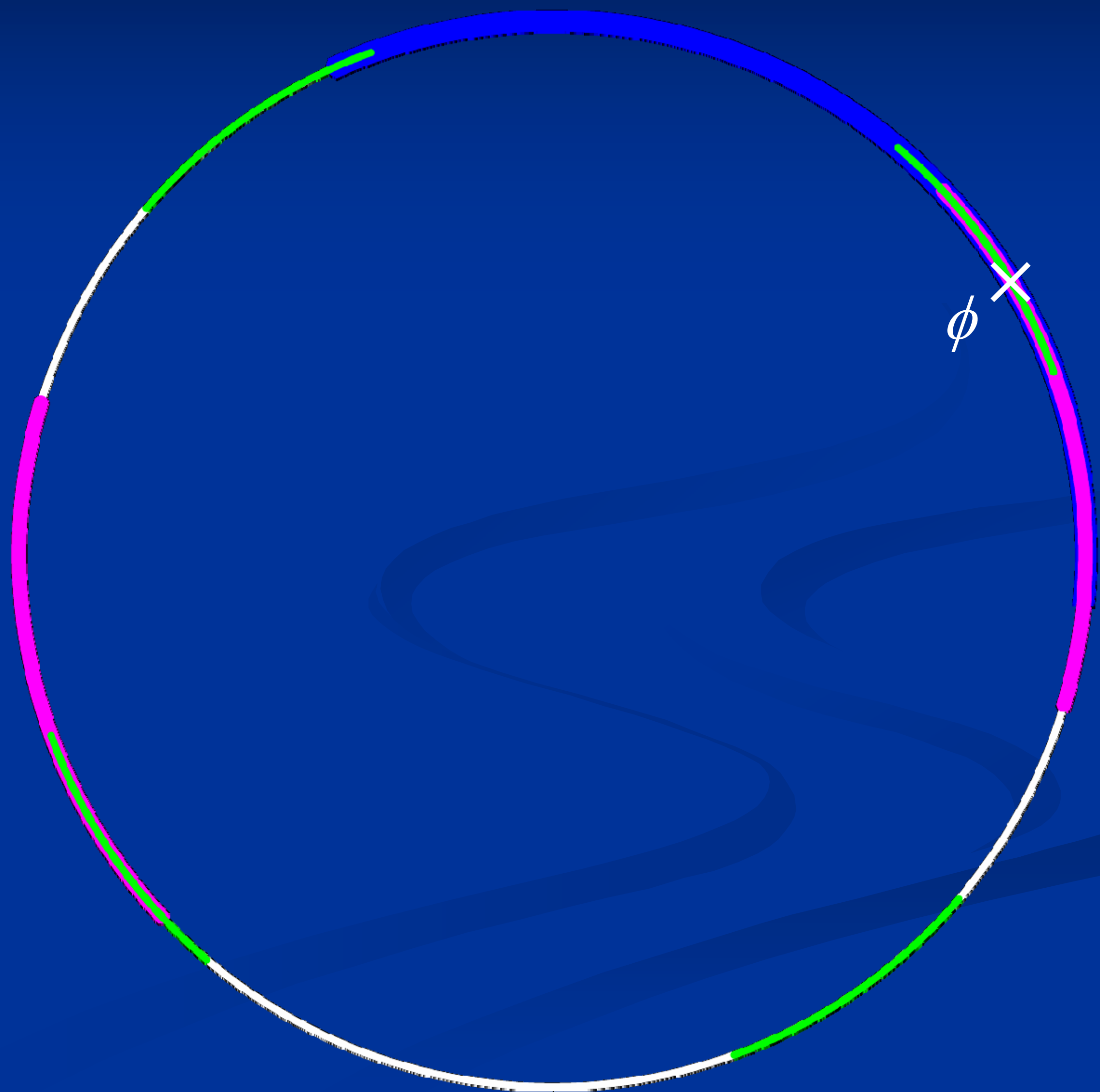
2. Perform enough measurements with 2^2 passes to ensure that the system phase is in one of the four green regions with high probability.



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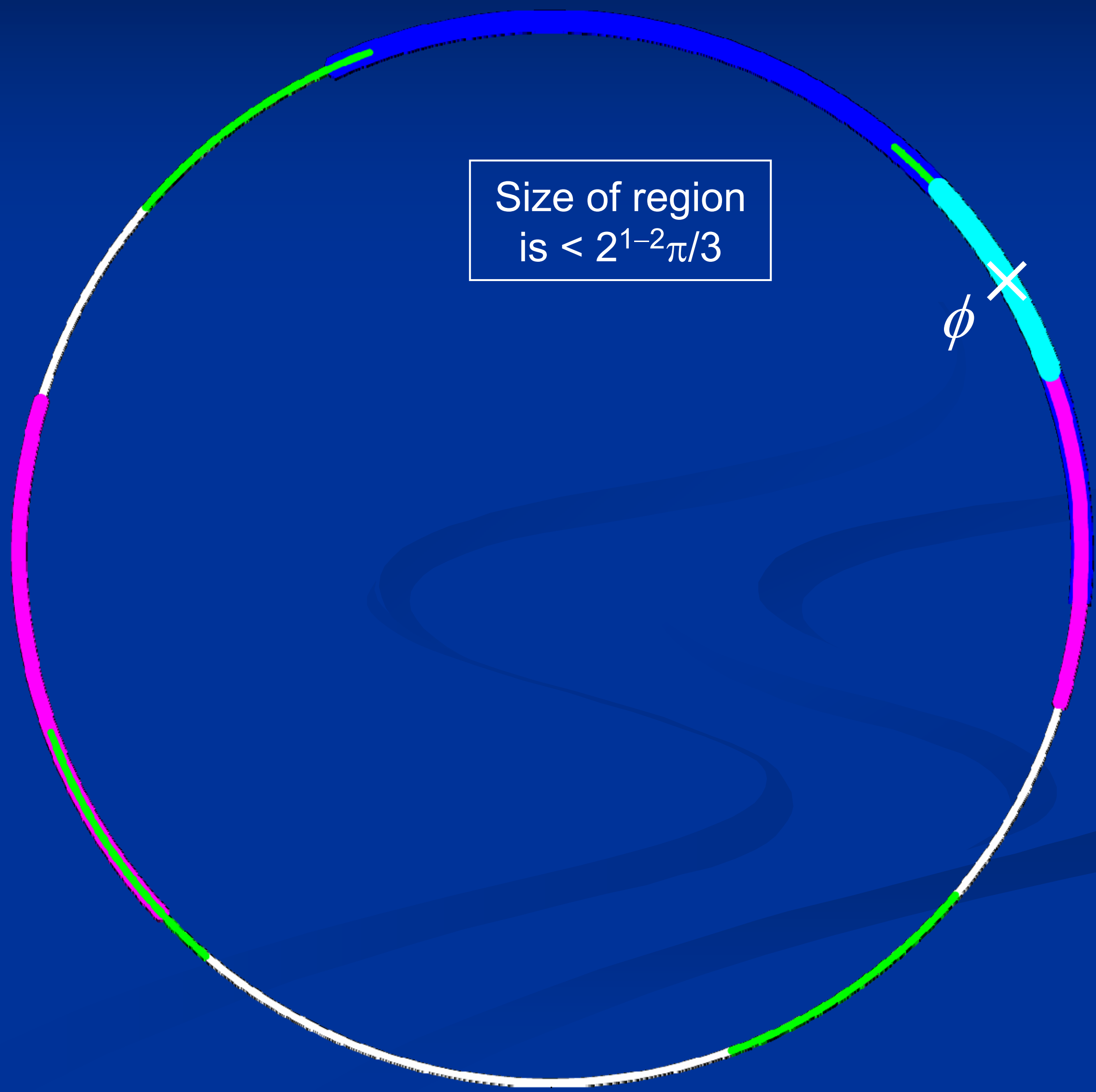

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   ⋮

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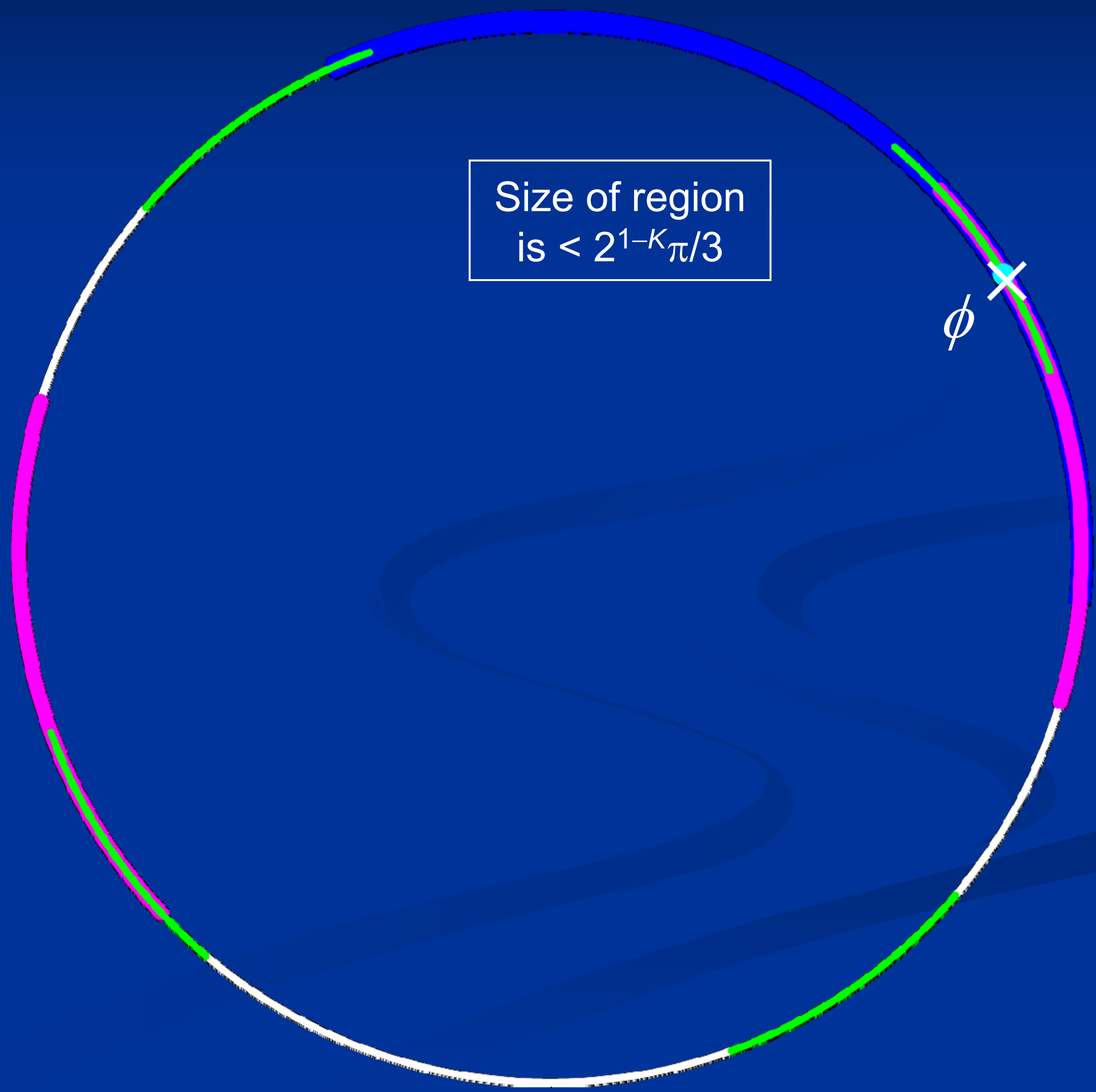

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K. Perform enough measurements with 2^K passes to ensure that the system phase is in one of 2^K regions with high probability.

- At stage k, if the system phase is not in the region, then the maximum error is $\propto 2^{-k}$.
- More measurements are needed for small k to ensure that the contribution to the variance is not large.
- The resource cost of additional measurements is less for small k.
- The best results are obtained if M decreases linearly with k.

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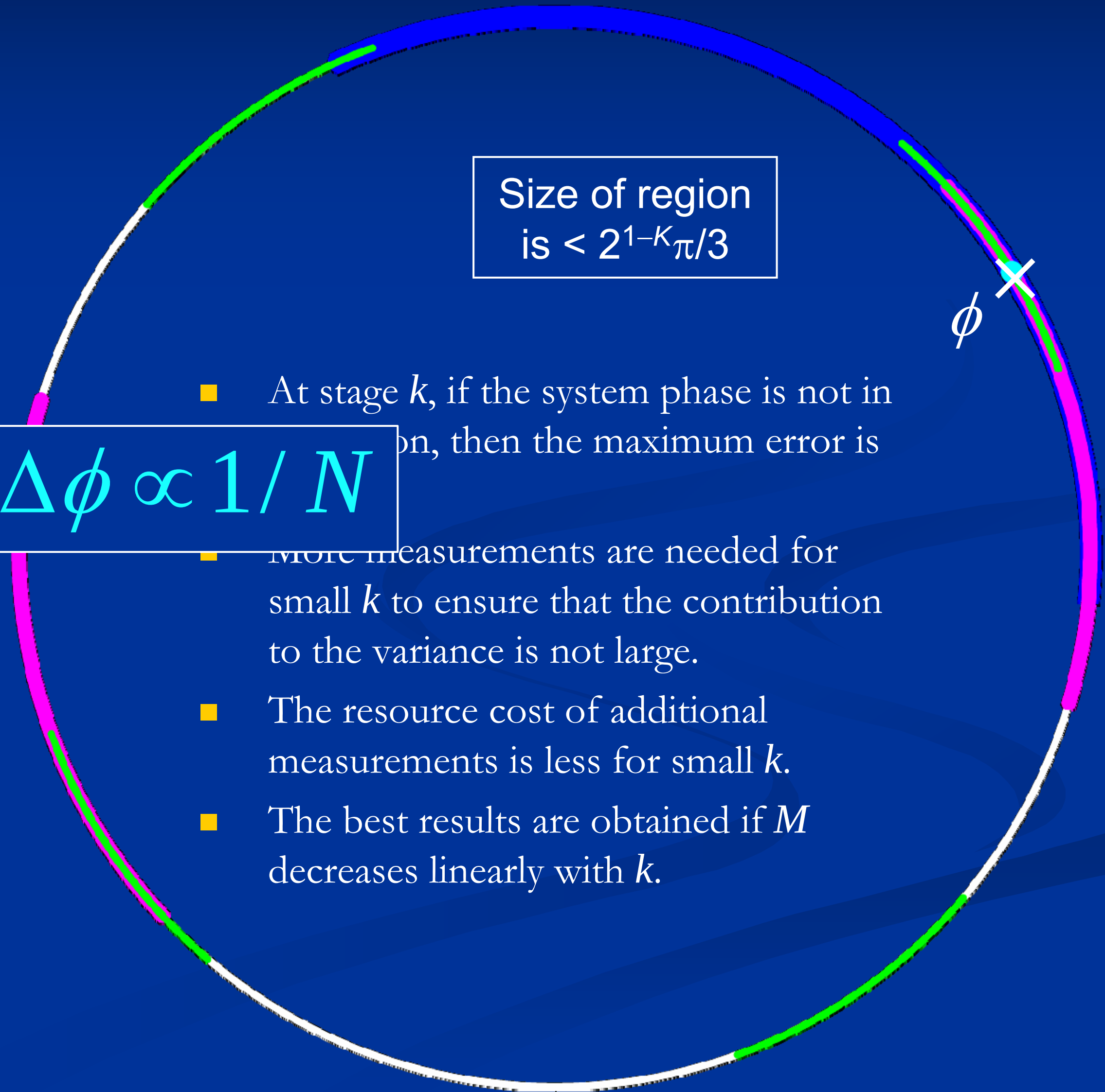

$$\Delta\phi \propto 1/N$$

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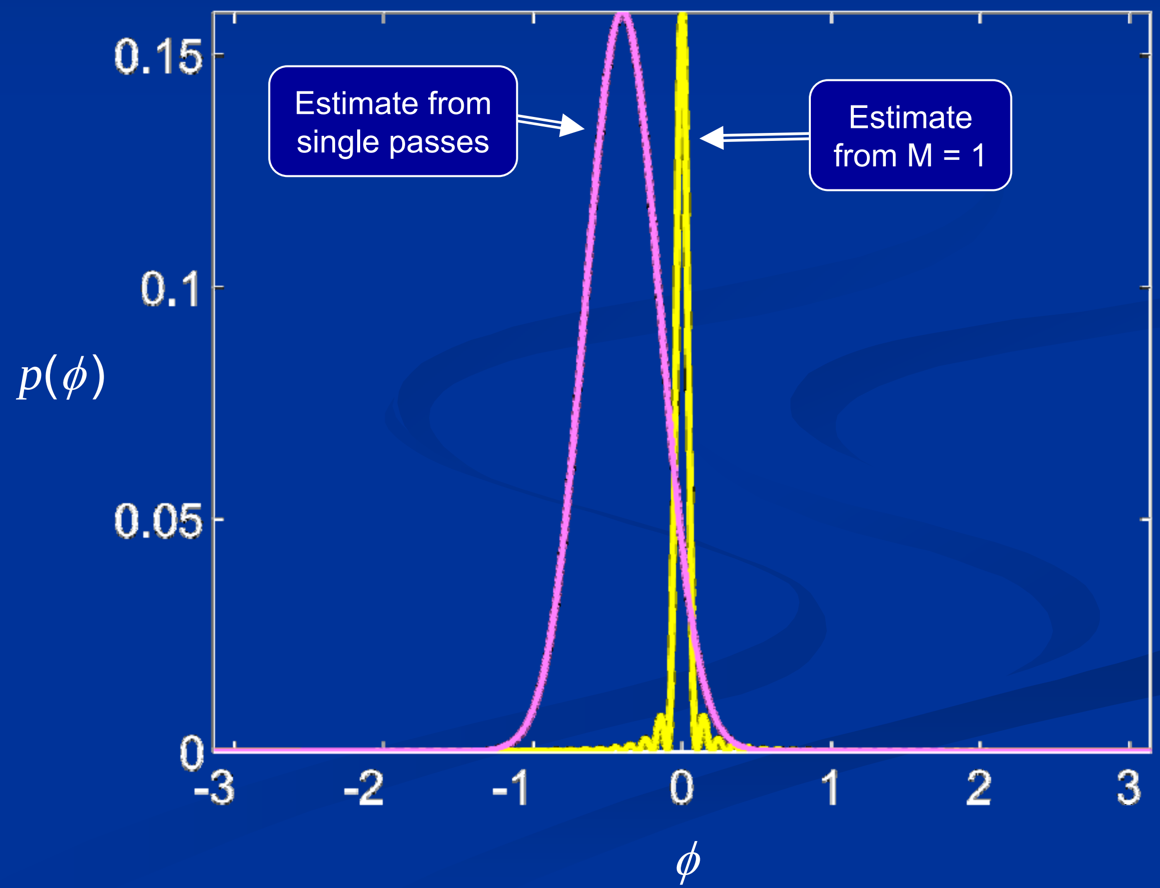
Hybrid measurements

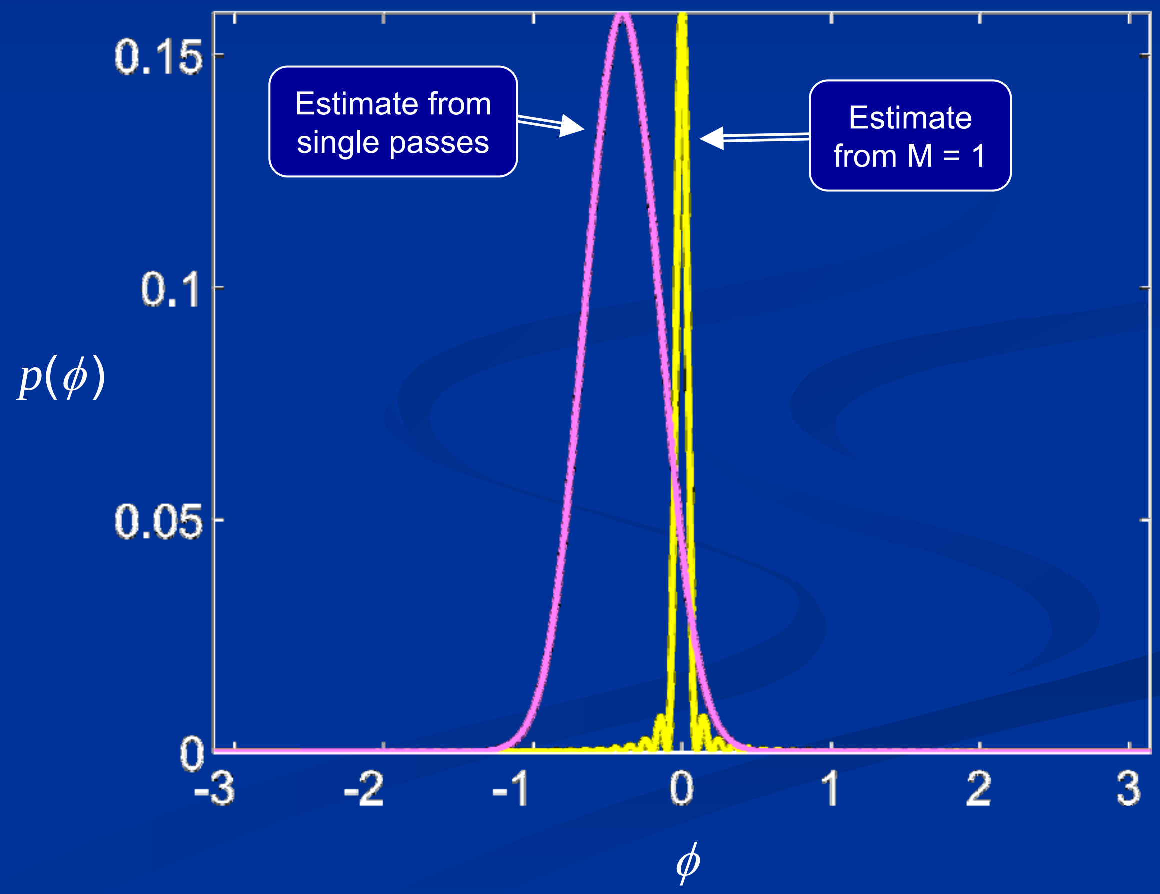
- Supplement the $M = 1$ measurement with additional measurements with single passes.



Hybrid measurements

- Supplement the $M = 1$ measurement with additional measurements with single passes.

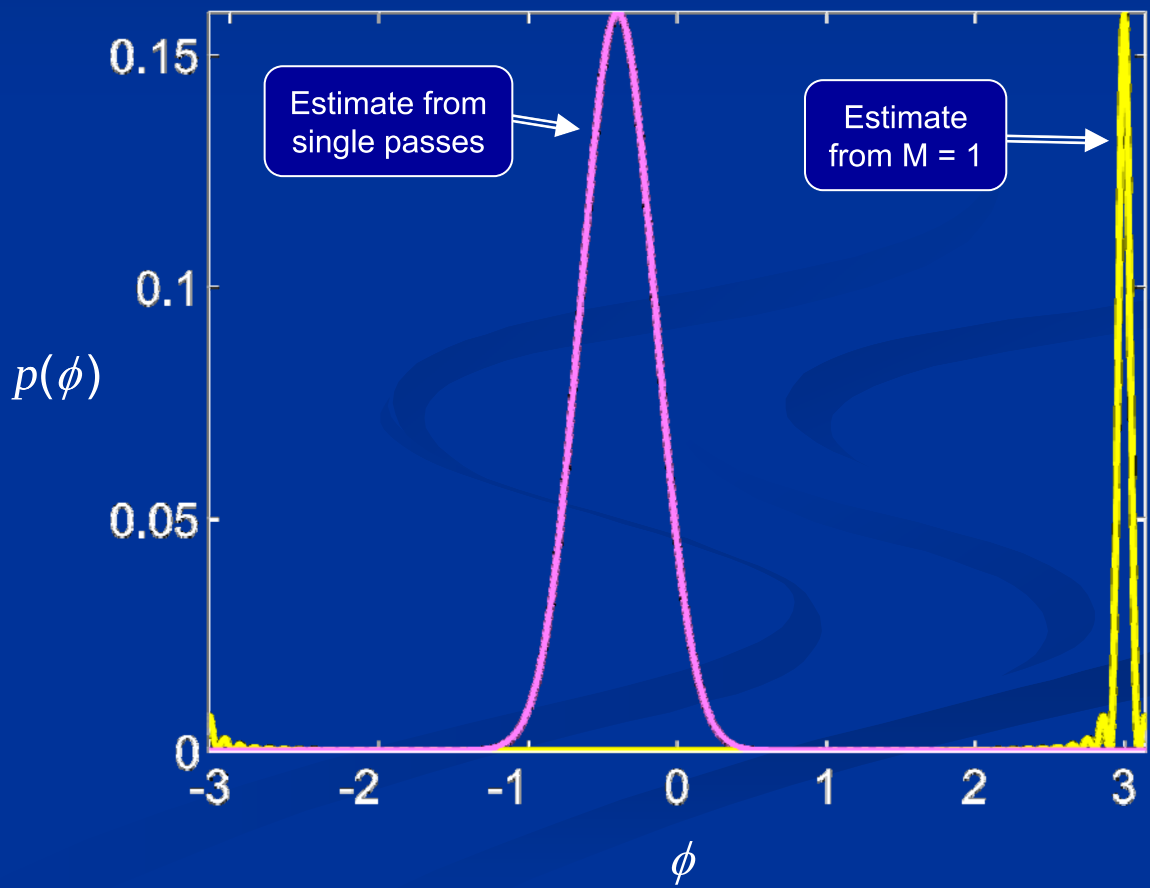
- If estimates agree, use the $M = 1$ estimate.



Hybrid measurements

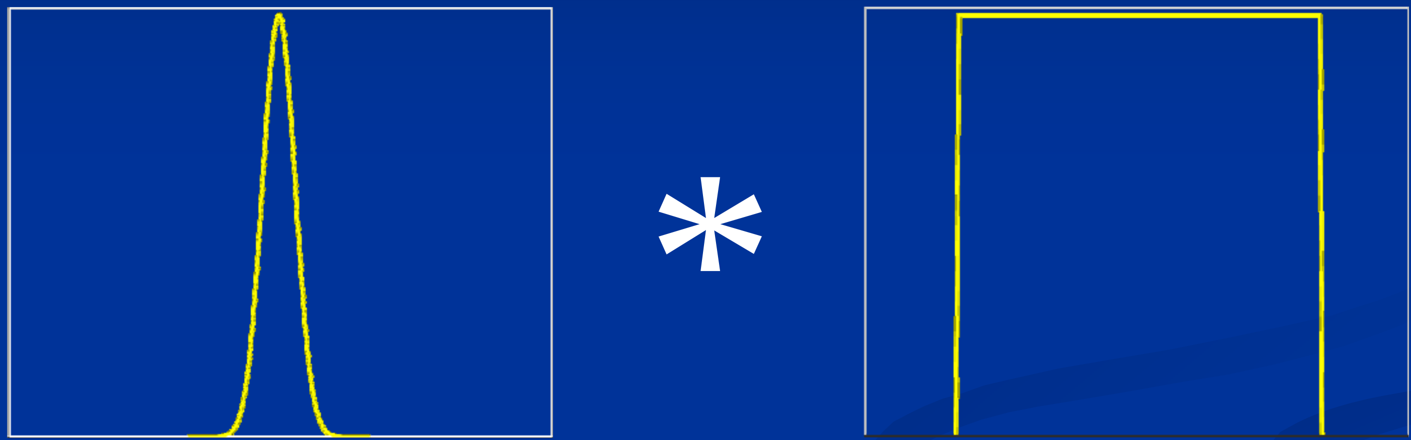
- Supplement the $M = 1$ measurement with additional measurements with single passes.
- If estimates agree, use the $M = 1$ estimate.
- If the estimates differ, use estimate from single photons.
- This yields error

$$\Delta\phi \propto 1/N^{3/4}$$

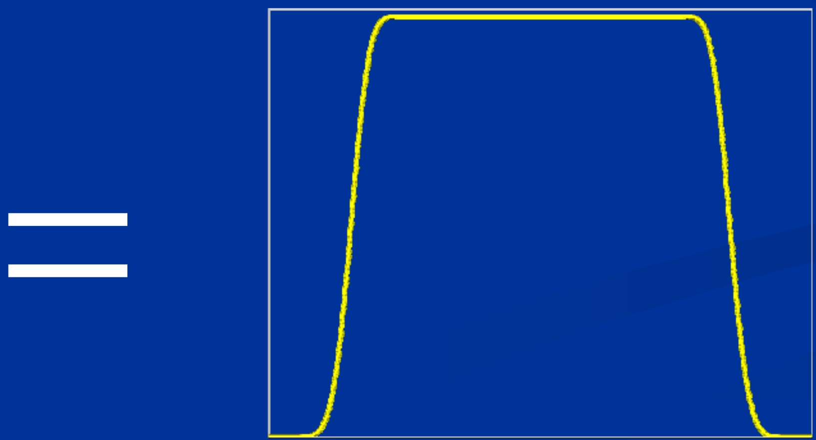


Hybrid measurements

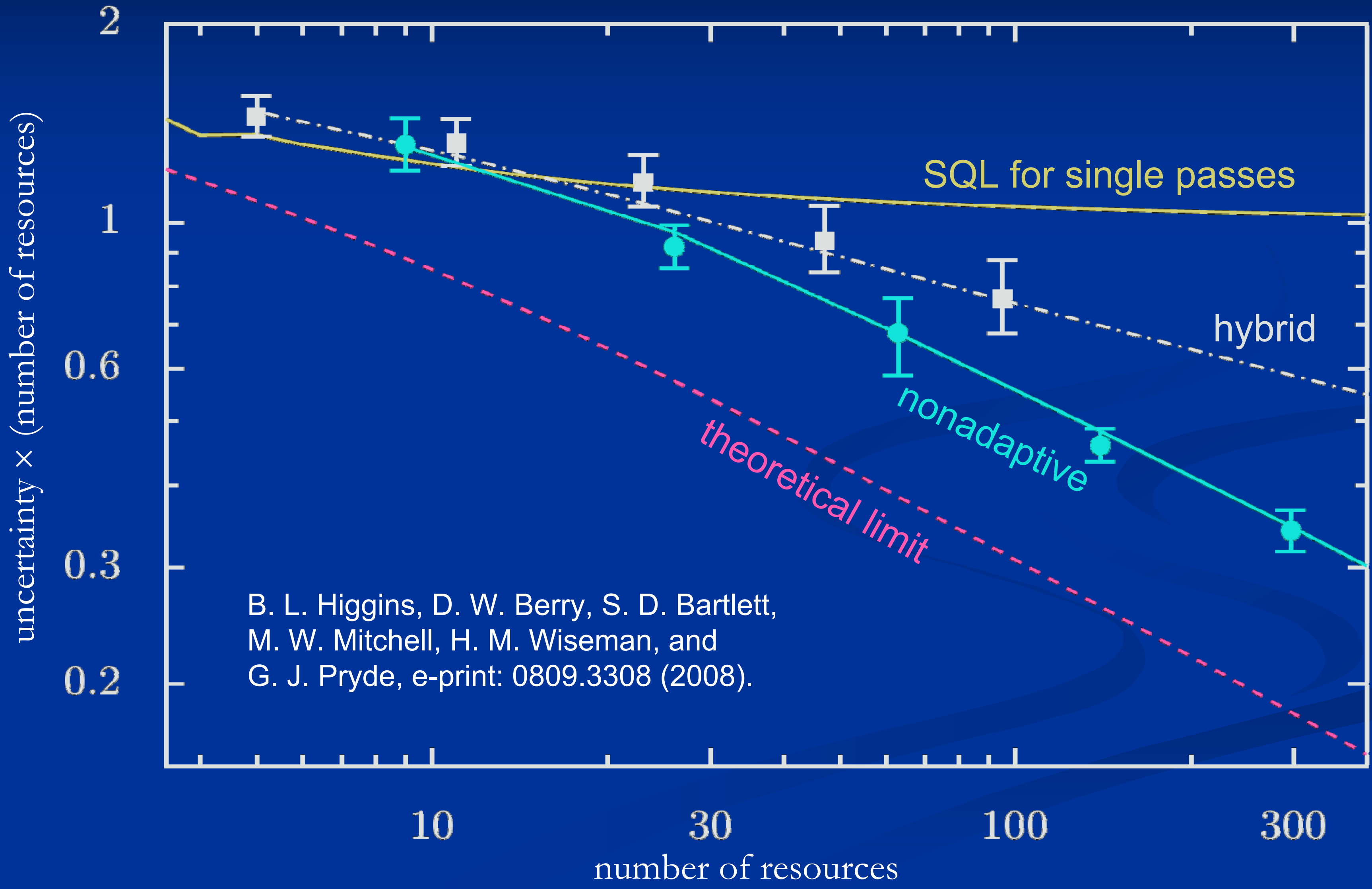
- The equivalent state is the (approximate) Gaussian from single photon measurements convoluted with the flat distribution from the $M = 1$ measurement:



- The resulting equivalent state still has a region where the state coefficients rise sharply:



Hybrid measurements

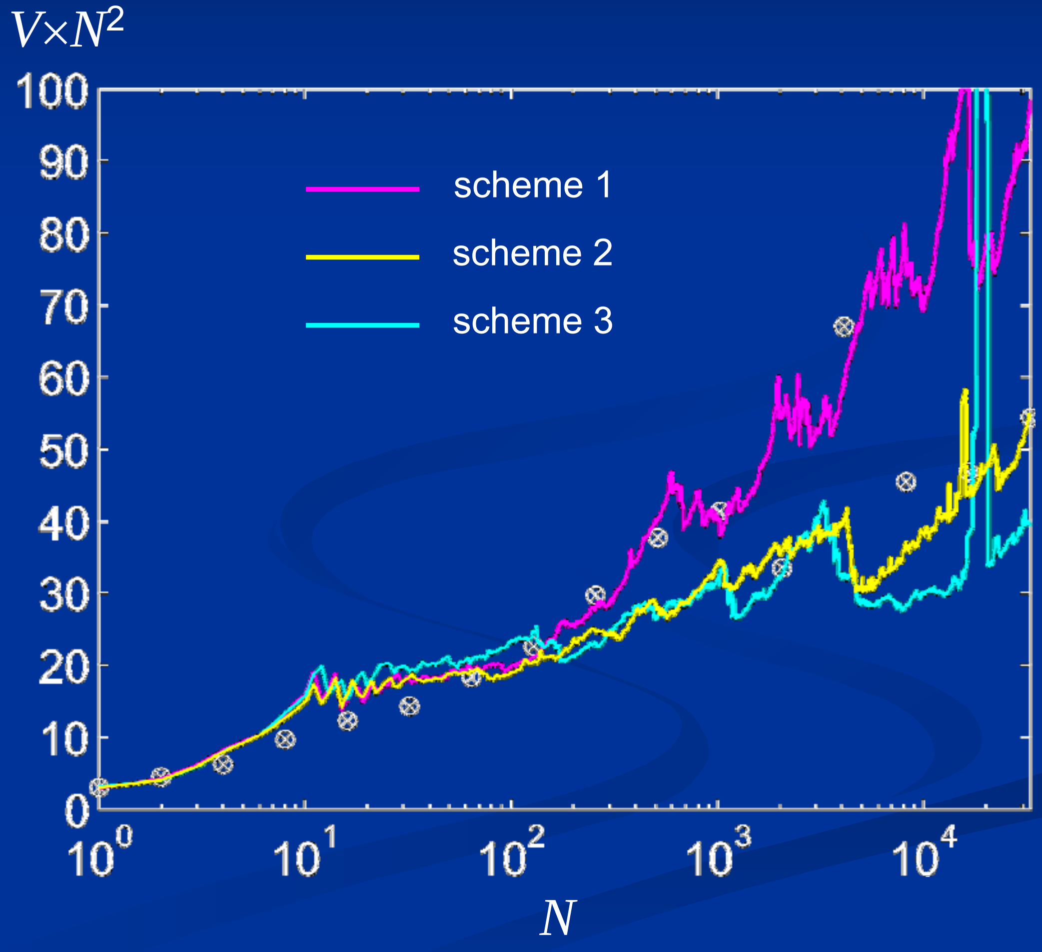


Adapting the number of passes

- As well as adapting a feedback phase, the number of passes can be adapted.

$$\Delta\phi \sim \frac{\ln N}{N}$$

Almost the theoretical limit



Summary

Single mode phase

- Feedback is needed to beat the standard quantum limit.
- The best feedback is not the best phase estimate.

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Single mode phase

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Interferometry

- Special states give improved accuracy, but have problem with ambiguity.
- Using multiple measurements gives true scaling at the theoretical limit.
- This may be achieved even without adaptive measurements!

Further Reading

- Optimal single-mode phase measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. A **63**, 013813 (2001).

- Continuous phase measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. A **73**, 063824 (2006).

- Adaptive interferometric measurements:

  D. W. Berry and H. M. Wiseman, Phys. Rev. Lett. **85**, 5098 (2000).

- Theoretical-limit interferometry:

  B. L. Higgins, D. W. Berry, S. D. Bartlett, H. M. Wiseman, and G. J. Pryde, Nature **450**, 393 (2007).

- Nonadaptive theoretical-limit interferometry:

  B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, e-print 0809.3308 (2008).