Unitary Process Discrimination with Error Margin

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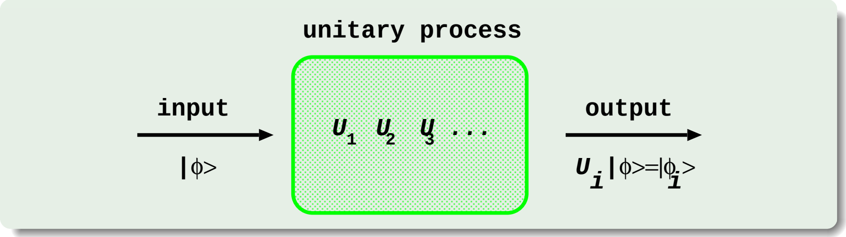
Unitary process discrimination with error margin



- State discrimination $\{|\,\phi_i\,\rangle \equiv U_i|\,\phi\,\rangle\}$
- Inconclusive result ("I don't know") is allowed
- Maximize the success probability P_{success}
- Margin on error probability $P_{\mathrm{error}} < m$

  $m = 1$: minimum-error discrimination

  $m = 0$: unambiguous discrimination
- Two solvable cases:
  - $\{U_1, U_2\}$ (two unitary processes)
  - $\{T_g\}_{g\in G}$ (T_g is a projective representation of a finite group G)

Two unitary processes $\{U_1, U_2\}$

First, fix input $|\,\phi\,\rangle$

Two-state discriminaiton

- $\rho_1 = |\,\phi_1\,\rangle\langle\,\phi_1\,|, \quad |\,\phi_1\,\rangle \equiv U_1|\,\phi\,\rangle$
  $\rho_2 = |\,\phi_2\,\rangle\langle\,\phi_2\,|, \quad |\,\phi_2\,\rangle \equiv U_2|\,\phi\,\rangle$
- We assume the same occurrence probabilities

POVM $\{E_\mu\}_{\mu=1,2,3}$

E_1 for ρ_1, E_2 for ρ_2, $E_3 \equiv E_?$ for "I don't know",

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Joint probabilities

The state is ρ_a and measurement outcome is μ:
$P_{\rho_a,E_\mu} = \mathrm{tr}\,[E_\mu \rho_a]$

Strong error-margin conditions

Success probability to be optimized

$$p_\circ = \frac{1}{2}\left(P_{\rho_1,E_1} + P_{\rho_2,E_2}\right)$$

Margin m on conditional error probabilities

$$P_{\rho_2|E_1} = \frac{\mathrm{tr}\left[E_1\rho_2\right]}{\mathrm{tr}\left[E_1\rho_1\right] + \mathrm{tr}\left[E_1\rho_2\right]} \leq m$$

$$P_{\rho_1|E_2} = \frac{\mathrm{tr}\left[E_2\rho_1\right]}{\mathrm{tr}\left[E_2\rho_1\right] + \mathrm{tr}\left[E_2\rho_2\right]} \leq m$$

$m = 1 \iff$ **Minimum error discrimination**

$$p_\circ = \frac{1}{2}\left(1 + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2}\right)$$

$m = 0 \iff$ **Unambiguous discrimination**

$$p_\circ = 1 - |\langle \phi_1 | \phi_2 \rangle|$$

Optimization problem

Optimization

maximize:

$$p_\circ = \frac{1}{2}\Big(\mathrm{tr}\,[E_1\rho_1] + \mathrm{tr}\,[E_2\rho_2]\Big),$$

subject to:

$$E_1 \geq 0, \quad E_2 \geq 0,$$

$$E_1 + E_2 \leq 1,$$

$$\mathrm{tr}\,[E_1\rho_2] \leq m\Big(\mathrm{tr}\,[E_1\rho_1] + \mathrm{tr}\,[E_1\rho_2]\Big),$$

$$\mathrm{tr}\,[E_2\rho_1] \leq m\Big(\mathrm{tr}\,[E_2\rho_1] + \mathrm{tr}\,[E_2\rho_2]\Big).$$

Equal occurrence probabilities
Semidefinite programming

Bloch vector representation

In the 2-dim subspace $= < |\,\phi_1\,\rangle, |\,\phi_2\,\rangle >$

Bloch vector representation

$$\rho_a = \frac{1 + \mathbf{n}_a \cdot \boldsymbol{\sigma}}{2}, \quad E_\mu = \alpha_\mu + \boldsymbol{\beta}_\mu \cdot \boldsymbol{\sigma}$$

Optimization in terms of parameters $\{\alpha_\mu, \boldsymbol{\beta}_\mu\}$

maximize:

$$p_\circ = \frac{1}{2}\Big(\alpha_1 + \boldsymbol{\beta}_1 \cdot \mathbf{n}_1 + \alpha_2 + \boldsymbol{\beta}_2 \cdot \mathbf{n}_2\Big),$$

subject to:

$$\alpha_1 \geq |\boldsymbol{\beta}_1|, \quad \alpha_2 \geq |\boldsymbol{\beta}_2|,$$

$$\alpha_1 + \alpha_2 + |\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2| \leq 1,$$

$$\alpha_1 + \boldsymbol{\beta}_1 \cdot \mathbf{n}_2 \leq m\Big(2\alpha_1 + \boldsymbol{\beta}_1 \cdot (\mathbf{n}_1 + \mathbf{n}_2)\Big),$$

$$\alpha_2 + \boldsymbol{\beta}_2 \cdot \mathbf{n}_1 \leq m\Big(2\alpha_2 + \boldsymbol{\beta}_2 \cdot (\mathbf{n}_1 + \mathbf{n}_2)\Big).$$

Optimal success probability

$$p_\circ = \begin{cases} A_m\Bigl(1 - |\langle \phi_1 | \phi_2 \rangle|\Bigr), & (0 \le m \le m_c), \\ \frac{1}{2}\left(1 + \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2}\right), & (m_c \le m \le 1), \end{cases}$$

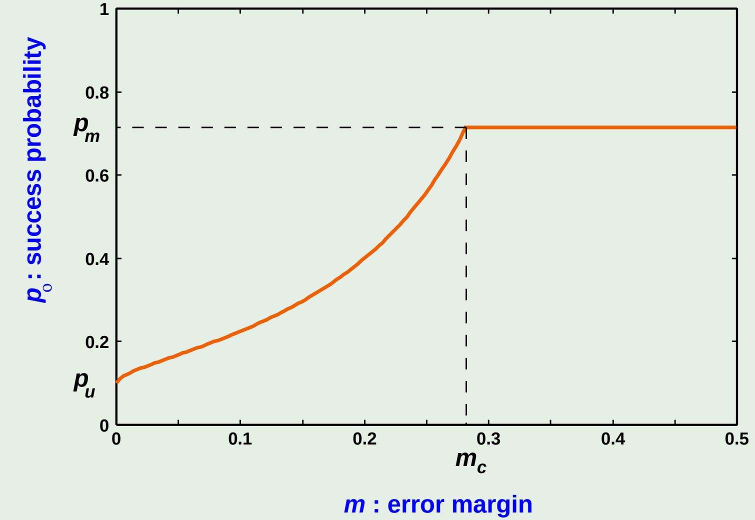
where

$$m_c = \frac{1}{2}\left(1 - \sqrt{1 - |\langle \phi_1 | \phi_2 \rangle|^2}\right),$$

and A_m is an increasing function of error margin and defined to be

$$A_m = \frac{1-m}{(1-2m)^2}\left(1 + 2\sqrt{m(1-m)}\right).$$

Success probability $p_\circ(m)$



p_m : minimum error discrimination

p_u : unambiguous discrimination

Strong and weak error-margin conditions I

Strong conditions

$P_{\rho_2|E_1} \leq m, \quad P_{\rho_1|E_2} \leq m$

$$
\begin{aligned}
p_\circ &= A_m\Big(1 - |\langle\, \phi_1 \,|\, \phi_2 \,\rangle|\Big), \\
&\qquad\qquad\qquad (0 \leq m \leq m_c) \\
A_m &\equiv \frac{1-m}{(1-2m)^2}\left(1 + 2\sqrt{m(1-m)}\right)
\end{aligned}
$$

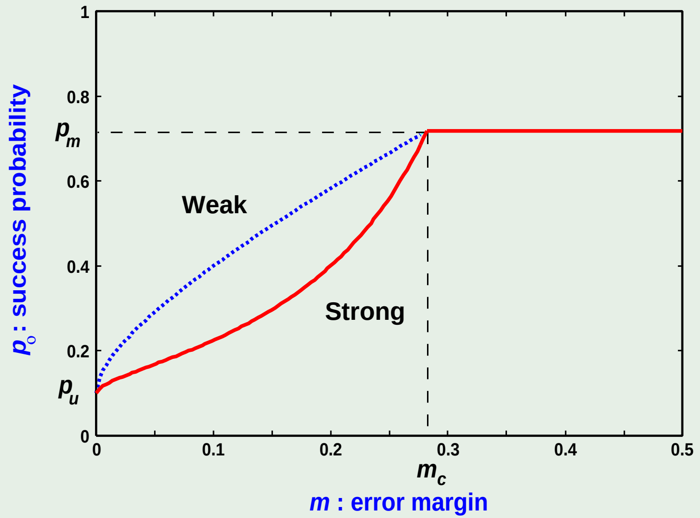
Weak conditions

$p_\times = P_{E_1,\rho_2} + P_{E_2,\rho_1} \leq m$

$$
\begin{aligned}
p_\circ &= \left(\sqrt{m} + \sqrt{1 - |\langle\, \phi_1 \,|\, \phi_2 \,\rangle|}\right)^2, \\
&\qquad\qquad\qquad (0 \leq m \leq m_c)
\end{aligned}
$$

POVM E_1, E_2, E_3 : rank 0 or 1

Strong and weak error-margin conditions II

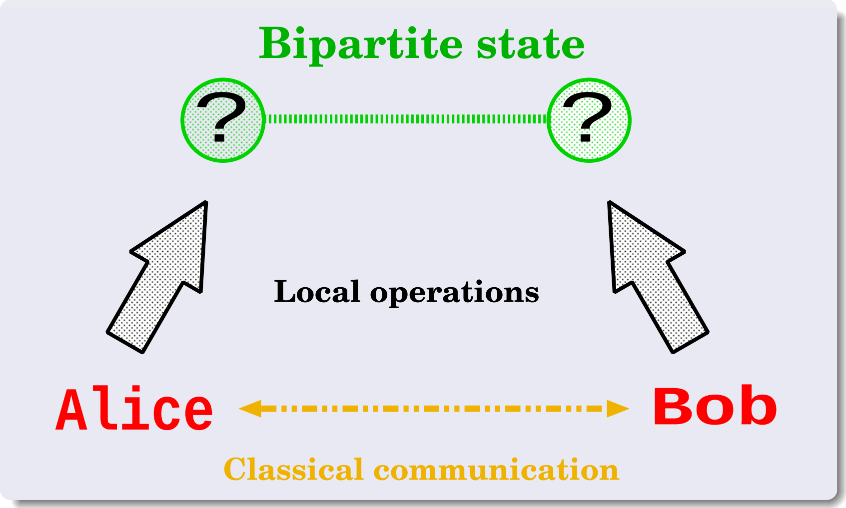


p_m : **minimum error discrimination**

p_u : **unambiguous discrimination**

Optimal discrimination by LOCC

Local Operations and Classical Communication (LOCC)



Two Orthogonal pure states

Two orthogonal pure states can be perfectly discriminated by LOCC:(Local Operations and Classical Communication)
(Walgate *et al.*, 2000)

Example：

$$
\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle + |-\rangle|-\rangle) \\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle) = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle)
\end{aligned}
$$

$$\text{where, } |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

In general, if $\langle \phi_1 | \phi_2 \rangle = 0$

$$
\begin{cases} |\phi_1\rangle = \sum_i |i\rangle|\xi_i\rangle \\ |\phi_2\rangle = \sum_i |i\rangle|\eta_i\rangle \end{cases}, \text{ where} \begin{cases} |i\rangle \ : \ \text{Orthonormal basis} \\ \langle \xi_i | \eta_i \rangle = 0 \end{cases}
$$

Local discrimination of nonorthogonal states

Local discrimination

Two non-orthogonal pure states (generally entangled) can be optimally discriminated by LOCC.

Discrimination with minimum error:

Virmani *et al*. (2001)

Error margin : $m_c \leq m \leq 1, \quad m_c = \frac{1}{2}\left(1 - |\langle \phi_1 | \phi_2 \rangle|^2\right)$

Unambiguous discrimination:

Chen *et al*. (2001,2002), Ji *et al*. (2005)

Error margin : $m = 0$

We can show

For any error margin, two pure states can be optimally discriminated by LOCC.

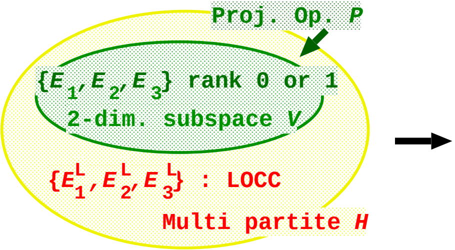
Three-element POVM of rank 0 or 1

Optimal POVM of discrimination with error margin

$\{E_1, E_2, E_3\}$: each element is of rank 0 or 1

Theorem

Let V be a two-dimensional subspace of a multipartite tensor-product space H, and P be the projector onto the subspace V. Then, for any three-element POVM $\{E_1, E_2, E_3\}$ of V with every element being of rank 0 or 1, there exists a one-way LOCC POVM $\{E_1^{\mathrm{L}}, E_2^{\mathrm{L}}, E_3^{\mathrm{L}}\}$ of H such that $E_\mu = PE_\mu^{\mathrm{L}}P \ (\mu = 1, 2, 3)$.



$$E_\mu = P\, E_\mu^{\mathrm{L}} P$$

Unitary process $\{U_1, U_2\}$ discrimination I

Discrimination between states $\{|\phi_1\rangle, |\phi_2\rangle\}$

$$P_{\max}(m, |\phi_1\rangle, |\phi_2\rangle) = f(m, |\langle\phi_1|\phi_2\rangle|)$$

$$f(m,s) = \begin{cases} \left(\sqrt{m} + \sqrt{1-s}\right)^2, & 0 \le m < \frac{1}{2}\left(1 - \sqrt{1-s^2}\right) \\ \frac{1}{2}\left(1 + \sqrt{1-s^2}\right), & \frac{1}{2}\left(1 - \sqrt{1-s^2}\right) \le m \le 1 \end{cases}$$

- $f(m,s)$ is decreasing with respect to s

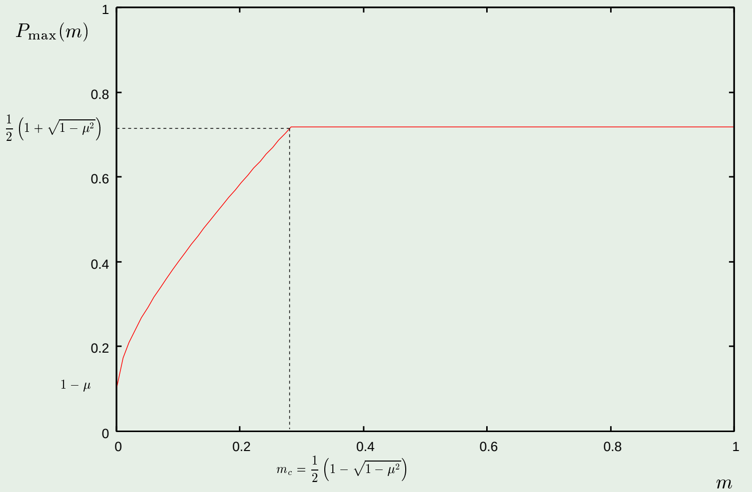
Discrimination between processes $\{U_1, U_2\}$

$$P_{\max}^{\mathrm{pure}}(m) = \max_{|\phi\rangle} P_{\max}(m, |\phi_1\rangle, |\phi_2\rangle) = f(m, \mu)$$

$$\mu \equiv \min_{|\phi\rangle} \left|\langle\phi|U_1^{+} U_2|\phi\rangle\right|$$

Remark: The optimal success probability is attained by a pure-state input, since $P_{\max}^{\mathrm{pure}}(m)$ is concave with respect to m

Discrimination between processes $\{U_1, U_2\}$ II



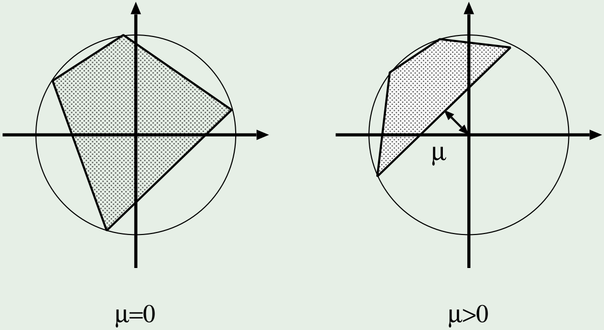
m **: error margin**

$\mu \equiv \min_{|\phi\rangle} \left|\langle \phi | U_1^{\dagger} U_2 | \phi \rangle\right|$

Minimum fidelity μ

$$\mu \equiv \min_{|\phi\rangle} \left| \langle \phi | U_1^{+} U_2 | \phi \rangle \right| = \min_{q_a \geq 0, \sum_a q_a = 1} \left| \sum_{a=1}^{d} q_a e^{i\theta_a} \right|$$

where $\{e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d}\}$ are eigenvalues of $U_1^{+} U_2$



Unitary processes $\{T_g\}_{g\in G}$: T_g is a unitary projective repersentation of a finite group G

Unitary projective representation

$$T_g T_h = c_{g,h} T_{gh} \ \ (T_g^{+} = T_g^{-1},\ |c_{g,h}| = 1,\ g,h \in G)$$

where $\{c_{g,h}\}$ is a factor set.

State discrimination

$$\left\{ P_{\phi_g} = \frac{1}{|G|},\ |\,\phi_g\,\rangle = T_g|\,\phi\,\rangle \right\}_{g\in G}$$

where $|G|$ is the order of G.

Covariant measurement

Covariant POVM

Optimal POVM $\{E_g, E_?\}$ can be assumed covariant:

$$T_g E_? T_g^+ = E_?, \quad T_g E_h T_g^+ = E_{gh}$$

Optimization with covariant POVM

maximize: $P_\circ = \langle \phi | E_1 | \phi \rangle$

subject to: $E_1 \geq 0, \ \sum_{g \in G} T_g E_1 T_g^+ \leq 1$

weak error-margin condition

$$P_\times = \sum_{h(\neq 1)} \langle \phi | T_h^+ E_1 T_h | \phi \rangle \leq m$$

If $\{T_g\}$ is irreducible (I)

By Schur's lemma

$$\sum_{g\in G} T_g E_1 T_g^{+} = \frac{|G|}{d}\mathrm{tr}\,[E_1]\cdot \mathbf{1} \quad (d = \text{dimension})$$

Completeness of POVM : $\sum_{g\in G} T_g E_1 T_g^{+} \leq \mathbf{1}$

$$P_\circ = \mathrm{tr}\,[E_1\rho] \leq \mathrm{tr}\,[E_1] \leq \frac{d}{|G|}$$

Error-margin condition : $\sum_{g(\neq 1)} \mathrm{tr}\left[T_g^{+} E_1 T_g \rho\right] \leq m$

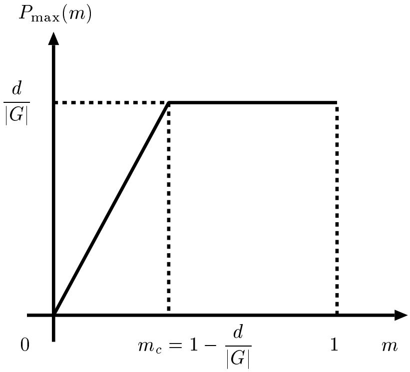
$$P_\circ = \mathrm{tr}\,[E_1\rho] \leq \mathrm{tr}\,[E_1] \leq \frac{m}{\frac{|G|}{d} - 1}$$

If $\{T_g\}$ is irreducible (II)

Maximal success probability

$$P_{\max}(m) = \begin{cases} \frac{m}{\frac{|G|}{d} - 1}, & (0 \le m \le m_c) \\ \frac{d}{|G|}, & (m_c \le m \le 1) \end{cases}$$

$$m_c = 1 - \frac{d}{|G|}$$



General case

For any $E_1 \geq 0$

$$\kappa \sum_{g\in G} T_g E_1 T_g^{+} \geq E_1, \quad \kappa \equiv \sum_r \frac{\min(m_r, d_r) d_r}{|G|}$$

$|G| =$ order of G, $r =$ irreducible representation

$d_r =$ dimension of r, $m_r =$ multiplicity of r

(Proof: orthogonality of representation matrices and Schwarz inequality)

Note: $\sum_r \frac{d_r d_r}{|G|} = 1$ **(Plancherel measure)**

Completeness of POVM : $\sum_{g\in G} T_g E_1 T_g^{+} \leq \mathbf{1}$

$$P_\circ = \mathrm{tr}\,[E_1\rho] \leq \mathrm{tr}\,[E_1] \leq \kappa$$

Error-margin condition : $\sum_{g(\neq 1)} \mathrm{tr}\left[T_g^{+} E_1 T_g \rho\right] \leq m$

$$P_\circ = \mathrm{tr}\,[E_1\rho] \leq \mathrm{tr}\,[E_1] \leq \frac{\kappa}{1-\kappa} m$$

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Maximal success probability

$$P_{\circ}^{\max}(m) = \begin{cases} \frac{\kappa}{1-\kappa} m, & (0 \leq m \leq m_c) \\ \kappa, & (m_c \leq m \leq 1) \end{cases}$$

$$m_c = 1 - \kappa$$

$$\kappa = \sum_r \frac{\min(m_r, d_r) d_r}{|G|}$$

Note : With a sufficient large ancilla,

$$\kappa \to \sum_{r(m_r \geq 1)} \frac{d_r^2}{|G|} \to \sum_r \frac{d_r^2}{|G|} = 1 \text{ (Plancherel measure)}$$

Optimal input and POVM

$$|\,\phi\,\rangle = \frac{1}{\sqrt{\kappa}} \sum_r \sum_{a=1}^{\min(m_r, d_r)} \sqrt{\frac{d_r}{|G|}} |\, r, a, a\,\rangle$$

$$E_1 = P_{\circ}^{\max}(m) |\,\phi\,\rangle\langle\,\phi\,|$$

Example I : Super dense coding in d dimension

- Define unitaries T_{mn} on C^d

$$
\begin{aligned}
T_{mn} &= X^m Z^n \quad (m, n = 0, 1, \ldots, d-1) \\
X &= \sum_{a=0}^{d-1} |\, a \,\rangle\langle\, a+1 \,|, \quad (\sim \sigma_x) \\
Z &= \sum_{a=0}^{d-1} e^{i\frac{2\pi}{d}a} |\, a \,\rangle\langle\, a \,|, \quad (\sim \sigma_z) \\
& XZ = e^{i\frac{2\pi}{d}} ZX
\end{aligned}
$$

- $\{T_{mn}\}$ is an irreducible projective representation of $G = Z_d \times Z_d$

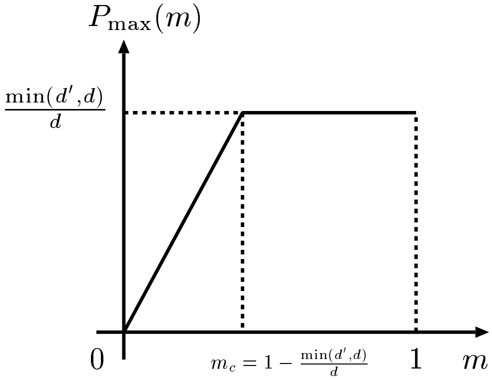
$$
T_{mn} T_{m'n'} = e^{-i\frac{2\pi}{d}nm'} T_{m+m', n+n'}
$$

- For $C^d \otimes C^{d'}$(ancilla), $d_r = d, m_r = d', |G| = d^2$

Maximum success probability of $\{T_{nm}\}$

$$P_{\circ}^{\max}(m) = \begin{cases} \frac{\tilde{d}}{d-\tilde{d}} m, & (0 \leq m \leq m_c) \\ \frac{\tilde{d}}{d}, & (m_c \leq m \leq 1) \end{cases}$$

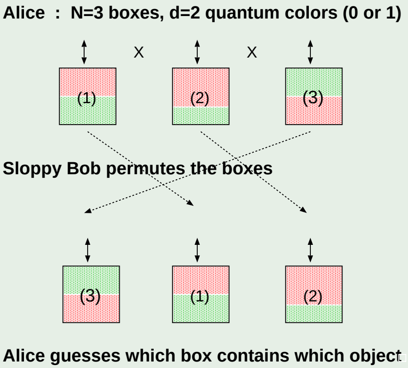
$$\tilde{d} = \min(d', d), \quad m_c = 1 - \frac{\tilde{d}}{d}$$



Example II : Color coding (symmetric group S_N)

- Consider $(C^d)^{\otimes N}$
- Permutation of N subsystems : T_σ $(\sigma \in S_N)$ on $(C^d)^{\otimes N}$
- $\{T_\sigma\}_{\sigma \in S_N}$ is a representation of S_N

Korff and Kempe (PRL 2005)
A. Hayashi, T. Hashimoto, and M. Horibe (PRA 2005)



N : number of boxes d : number of colors

- $N = 3, d = 2$

$$P_\circ^{\text{classical}} = \frac{1}{2}, \quad P_\circ = P_\circ^{\text{ancilla}} = \frac{5}{6}$$

- $N = 4, d = 2$

$$P_\circ^{\text{classical}} = ?, \quad P_\circ = \frac{13}{24}, \quad P_\circ^{\text{ancilla}} = \frac{14}{24}$$

When $N \to \infty$

$$P_\circ \to 1 \qquad \text{if } d \sim \frac{N}{e} \qquad \text{(Korff and Kempe)}$$

$$P_\circ^{\text{ancilla}} \to 1 \qquad \text{if } d \sim 2\sqrt{N} \qquad \text{(HHH)}$$

Strong error-margin condition

Error-margin condition

$$P_{\times|E_1} \equiv \frac{\sum_{g(\neq 1)} \mathrm{tr}\left[\rho T_g E_1 T_g^{+}\right]}{\sum_g \mathrm{tr}\left[\rho T_g E_1 T_g^{+}\right]} \leq m$$

$$\mathrm{tr}\left[\rho E_1\right] \leq \kappa \mathrm{tr}\left[\rho \sum_g T_g E_1 T_g^{+}\right] \leq \frac{\kappa}{1-m} \mathrm{tr}\left[\rho E_1\right]$$

Maximum success probability

$$P_{\circ}^{\max}(m) = \begin{cases} 0, & (0 \leq m < m_c) \\ \kappa, & m_c \leq m \leq 1 \end{cases}$$

$$m_c = 1-\kappa, \quad \kappa = \sum_r \frac{\min(m_r, d_r) d_r}{|G|}$$

Summary

- Unitary process discrimination with error margin
- Two-unitary case, $\{U_1, U_2\}$, solved
- Group representation case, $\{T_g\}_{g\in G}$, solved
  - Unambiguous discrimination $(m = 0)$: $P_\circ^{\max}(0) = 0$ or 1
  - Weak error margin : $P_\circ^{\max}(m)$ is linear in m for $m \le m_c$
  - Strong error margin : $P_\circ^{\max}(m) = 0$ for $m \le m_c$
- Many applications (super dense coding, color coding, $\cdots$)
  - Ancilla
  - Entangled input state for multiple uses of the process

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Appendix 1 (The strong condition is stronger than the weak condition)

$$\begin{aligned} P_{\times} &= P_{E_1,\rho_2} + P_{E_2,\rho_1} \\ &= P_{\rho_2|E_1} P_{E_1} + P_{\rho_1|E_2} P_{E_2} \\ &\leq m(P_{E_1} + P_{E_2}) \\ &\leq m \end{aligned}$$